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Detecting Multipartite Classical States and their Resemblances

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We study various types of multipartite states lying near the quantum-classical boundary. The class of so-called classical states are precisely those in which each party can perfectly identify a locally held state without disturbing the global state, a task known as non-disruptive local state identification (NDLID). We show NDLID to be closely related local broadcasting and we introduce a new class of states called generalized-classical states which allow for both NDLID and multipartite broadcasting when the most general quantum measurements are permitted. Simple analytical methods and a physical criterion are given for detecting whether a multipartite state is classical or generalized-classical. For deciding the latter, a semi-definite programming algorithm is presented which may find use in other fields such as signal processing.

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Introduction.—There are many ways in which composite quantum systems can exhibit non-classical properties. The correlations between entangled states have generated some of the most puzzling paradoxes in quantum theory; however even unentangled, or *separable* states, possess correlations that *cannot* be simulated by classical systems and thus defy our intuition. Recently, much interest has been raised concerning the properties of these non-classical correlations with applications to a variety of fields [1–8].

In light of this, several measures have been designed to isolate and quantify precisely the non-classical nature of a quantum state such as quantum discord [9], quantum deficit [1], measurement induced disturbance [10], and similar quantities [2, 11–13]. One common feature of all these measures is that they vanish for *fully classical* states, i.e. those in which the shared correlations among all the parties can be simulated on a classical system. Thus, any such measure can be interpreted as quantifying how far away a given state is from the classical-quantum border, even within the class of separable states.

In this Letter, we take an alternative approach to the sharpening of the quantum-classical boundary region; instead of grouping states in this region according to some numerical distance away from the set of classical states, we identify a state as "nearly" classical if it possesses a well-defined trace of some purely classical property. Specifically, we address the following two questions: (i) in what physical ways can general quantum states resemble classical states, and (ii) how can one detect whether a given state is classical or at least resemblant to one in the sense of question (i)?

One answer to the first question, which we investigate below, involves a state's ability to undergo *non-disruptive local state identification* (NDLID). In the remainder of this letter, we will first give a precise description of NDLID and introduce the class of generalized-classical states which exhibit this property. We find that the ability for such states to undergo NDLID renders them suitable for the related task of probabilistic local broadcasting as studied by Piani *et al.* [5]. Since only fully classical states can be locally broadcast with probability 1 [5], the stochastic possibility for generalized-classical states to be locally broadcast further supports the strongly classical nature of these states. Generalized-classical states are found to occupy a measure zero volume of state space and belong to the class of so-called minimal length separable states. After that, we will proceed to answer question (ii) by providing computational and experimental methods for deciding whether or not a given multipartite state is classical or even just similar to one in its ability for NDLID. Our detection algorithm can be efficiently implemented which differs drastically from the best known methods of detecting separability.

NDLID and a Hierarchy of Separable States.—As a motivating example, consider the fully classical state $\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. Each party can perform a projective measurement in the computational basis and learn his/her local state to be either $|0\rangle$ or $|1\rangle$. When these results are not recorded or kept secret, the post-measurement state is still ρ, and the parties have thus identified their state without perturbing the overall state. The ability for each party to perform such an information-gathering process without failure is not particular to this example but, in fact, completely characterizes the set of fully classical states [9]. As a result, the possibility for a given state to undergo some sort of NDLID can be regarded as a signature of "classicalness."

In general, we will say a state ρ allows for NDLID by party k if there exists a decomposition $\rho = \sum_i p_i \rho_i^{(\overline{k})} \otimes |\phi_i^{(k)}\rangle\langle\phi_i^{(k)}|$ and local measurement $\{M_i^{(k)}\}_{i=1\dots n}$ with $\sum_{i=1}^n M_i^{(k)\dagger} M_i^{(k)} \leq I^{(k)}$ such that

$$M_i^{(k)}|\phi_j^{(k)}\rangle\langle\phi_j^{(k)}|M_i^{(k)\dagger} = \lambda\delta_{ij}|\phi_j^{(k)}\rangle\langle\phi_j^{(k)}| \qquad (1)$$

for some $0 < \lambda \leq 1$. Upon outcome i, party k can then conclude that his/her system is in state $|\phi_i^{(k)}\rangle$ among the ensemble $\{|\phi_j^{(k)}\rangle\}$, while the rest of the system is in state $\rho_i^{(\overline{k})}$. Furthermore, it can easily be seen that under the action of this measurement, the global state remains invariant: $\sum_{i=1}^{n}(I^{(\overline{k})}\otimes M_i^{(k)})\rho(I^{(\overline{k})}\otimes M_i^{(k)\dagger})=\lambda\rho$.

From Eq. (1), it immediately follows that the task of NDLID is equivalent to unambiguous state discrimination among the states $|\phi_j^{(k)}\rangle$ with a post-selection rate of λ. A well-known necessary and sufficient condition for accomplishing this feat is that the $|\phi_j^{(k)}\rangle$ are linearly independent [14]. In this case, the measurement operators take the form $M_i^{(k)}=|\psi_{i_k}^{(k)}\rangle\langle\phi_{i_k}^{(k)\perp}|$ where $\langle\phi_{j_k}^{(k)}|\phi_{i_k}^{(k)\perp}\rangle=\delta_{ij}\lambda$ for some $0<\lambda\leq 1$ and $|\psi_{i_k}^{(k)}\rangle$ form an orthonormal basis (see [14] for details). Furthermore, we have $\lambda=1$ if and only if the $|\phi_j^{(k)}\rangle$ are orthogonal and the NDLID can be performed by a complete projective measurement. These facts motivate the following classifications of multipartite separable states.

Definition 1 *Let* $\{|\phi(\vec{i})\rangle\}=\{|\phi_{i_1}^{(1)}\phi_{i_2}^{(2)}\dots\phi_{i_N}^{(N)}\rangle\}$ *denote a product state basis.*

a. *A multipartite state ρ is called separable if it is diagonal in some product state basis; i.e.* $\rho=\sum_{\vec{i}}p_{\vec{i}}|\phi(\vec{i})\rangle\langle\phi(\vec{i})|$,

b. *The state ρ is called generalized-classical for the k^{th} party if it is diagonal in some product state basis in which the states $\{|\phi_{i_k}^{(k)}\rangle\}$ are linearly independent.*

c. *The state ρ is called classical for the k^{th} party if it is diagonal in some product state basis in which the states $\{|\phi_{i_k}^{(k)}\rangle\}$ are orthogonal.*

d. *The state ρ is called fully generalized-classical or fully classical if it is diagonal in some product state basis in which statements b or c are true respectively for all parties.*

From the discussion preceding Definition 1, generalized-classical states are nearly classical in the following sense:

> *A state is classical (resp. generalized-classical) with respect to party k if and only if party k can perform NDLID by a projective (resp. generalized) measurement.*

We observe another important physical interpretation of generalized classical states which relies on a fundamental connection between unambiguous state discrimination and probabilistic cloning [15]. As introduced in Ref. [5], a multipartite state $\rho^{(12\dots N)}$ allows for local broadcasting if there exists local maps $\Lambda^{(i)}:\mathcal{H}^{(i)}\to\mathcal{H}^{(i)}\otimes\mathcal{H}^{(i)}$ such that the state $\sigma^{(11'22'\dots NN')}=[\Lambda^{(1)}\otimes\dots\otimes\Lambda^{(N)}]\rho$ has reduced states $\sigma^{(12\dots N)}=\sigma^{(1'2'\dots N')}=\rho$. The task of probabilistically cloning a set of states $\{|\phi_1\rangle,...,|\phi_m\rangle\}$ involves a general CP map Λ with action $\Lambda:|\phi_i\rangle\mapsto\sqrt{q_i}|\phi_i\rangle|\phi_i\rangle$ $\forall i$ with $\sum_{i=1}^{m}q_i\leq 1$. The nonzero number q_i is known as the *efficiency* for cloning the state $|\phi_i\rangle$, and the set can be probabilistically cloned if and only if the $|\phi_i\rangle$ are linearly independent [15]. Moreover, if a probabilistic cloner exists with different efficiencies, then one can always be constructed with all efficiencies equal to the smallest one q_0, and so the action of such a map on the mixed state $\frac{1}{m}\sum|\phi_i\rangle\langle\phi_i|$ is simply $\frac{q_0}{m}\sum|\phi_i\phi_i\rangle\langle\phi_i\phi_i|$. Therefore, since generalized-classical states have locally linearly independent states $|\phi_j^k\rangle$, each party can apply a local probabilistic cloner that transforms $\rho=\sum_{\vec{i}}p_{\vec{i}}|\phi(\vec{i})\rangle\langle\phi(\vec{i})|\to\lambda\sum_{\vec{i}}p_{\vec{i}}|\phi(\vec{i})\phi(\vec{i})\rangle\langle\phi(\vec{i})\phi(\vec{i})|$, which is a local broadcast state. The authors of [5] show that $\lambda=1$ if and only if ρ is fully classical, and we thus see the unification of tasks:

$$\text{Deterministic Local Broadcasting}\Leftrightarrow\text{Deterministic NDLID}\Leftrightarrow\rho\text{ Fully Classical.}$$

While for the "generalized form" of this statement we have shown that Probabilistic Local Broadcasting $\Leftarrow$ NDLID $\Leftrightarrow\rho$ Generalized-Classical, it is not clear whether the converse of the first relation is true, and we leave this as an open research question.

There exists an even broader class of separable states still hovering close to the quantum-classical border. An N-partite state ρ of rank r will be called a *minimal length separable state* if it has a decomposition $\rho=\sum_{i=1}^{r}\lambda_i|\phi_1^{(1)}\cdots\phi_i^{(N)}\rangle\langle\phi_1^{(1)}\cdots\phi_i^{(N)}|$ [16]. It is quite easy to see from the following lemma that any fully generalized-classical state is also a minimal length separable state.

Lemma 2 *For some multi-index $(i_1,...,i_N)$, if up to repetition of states the $|\phi_{i_j}^{(j)}\rangle$ are linearly independent for all parties j, then the product states $|\phi_{i_1}^{(1)}\cdots\phi_{i_N}^{(N)}\rangle$ are also linearly independent.*

By this lemma and Definition 1, if ρ is fully generalized-classical, it has a decomposition $\rho=\sum_{i=1}^{\delta}|\phi_{i_1}^{(1)}\cdots\phi_{i_N}^{(N)}\rangle\langle\phi_{i_1}^{(1)}\cdots\phi_{i_N}^{(N)}|$ with $\delta\geq r$ and each $|\phi_{i_1}^{(1)}\cdots\phi_{i_N}^{(N)}\rangle$ linearly independent. This last property implies that $r=\delta$ and so we see that each fully generalized-classical state is a minimal length state. Furthermore, in the bipartite case, if a state is generalized-classical with respect to just one of the parties, it will be of minimal length. The following chain of inclusions summarizes the main parsings described in this letter:

$$\text{separable}\supset\text{minimal length}\supset\text{fully generalized-classical}\supset\text{fully classical}\supset\text{product.}$$

Here, product states refer to states of the form $\rho=\rho_1\otimes\cdots\otimes\rho_N$.

There are two reasons to consider minimal length states as also lying near the quantum-classical border. First, it is known that only *non*-minimal length states constitute the opposite end of the spectrum at the separable/non-separable boundary [16]. While this alone does not imply a closeness between minimal length and classical states, such an interpretation becomes further justified when considering the volumes of each set in state space. Separable states possess a nonzero volume [17] while minimal length states are of measure zero [18]. This final point has an even greater relevance to our discussion since it implies that fully generalized-classical states are also of measure zero. In other words, nearly all multipartite quantum states lack the property of non-disruptive local state identification. Also note that this provides an alternative proof for the result in Ref. [19] which shows a generic state to have a nonzero discord (i.e. is non-classical).

Decision Algorithms for Classical and Generalized-Classical States.— In the last portion of this Letter we address the question of deciding whether a given multipartite state is classical or generalized-classical. Our results, discovered independently, generalize the recent works on this topic [7, 20–23] in which necessary and sufficient conditions have been provided for deciding the non-classical bipartite states. The techniques we use are similar to those in Ref. [22] in that both our algorithms involve checking commutation relations. Interestingly, we find that deciding whether a state is generalized-classical reduces to a problem similar in nature to those well-studied in the field of signal processing [24, 25]. Hence, our use of semi-definite programming (SDP) in detecting generalized-classical states may be of interest to researchers in that subject, as well as the linear algebra community at large.

We first make an easy but important observation.

Lemma 3 *The state ρ is fully generalized-classical (resp. classical) if it is generalized-classical (resp. classical) for all parties.*

Proof. We will prove this for the bipartite case, but the idea immediately generalizes to arbitrary number of parties. Suppose $\rho = \sum_i \rho_i \otimes |b_i\rangle\langle b_i| = \sum_i |a_i\rangle\langle a_i| \otimes \sigma_i$ where the $|b_i\rangle$ and $|a_i\rangle$ are linearly independent (resp. orthonormal). Then we see that each ρ_i is a linear combination of the $|a_i\rangle\langle a_i|$ so that $|a_i\rangle\otimes|b_j\rangle$ is a product basis in which ρ is diagonal. □

By Lemma 3, it will be sufficient to only consider bipartite systems in the following discussion. So introduce Alice and Bob and let d_A and d_B denote the dimensions of their subsystems respectively. If ρ is classical or generalized-classical with respect to Bob, there exists some basis $|b_i\rangle$ such that

$$\rho = \sum_i p_i \rho_i \otimes |b_i\rangle\langle b_i|, \tag{2}$$

while for classical states, the $|b_i\rangle$ are orthogonal. Note that in both cases, the contraction $\langle\phi_1^{(A)}|\rho|\phi_2^{(A)}\rangle$ will be diagonal in the basis $|b_i\rangle$ for any two states $|\phi_1^{(A)}\rangle, |\phi_2^{(A)}\rangle \in \mathcal{H}_A$. This fact leads to the following theorem.

Theorem 4 *Let $\{|\phi_i^{(A)}\rangle\}$ be any orthonormal basis for $\mathcal{H}_A$. Then ρ is generalized-classical (resp. classical) if and only if*

$$\rho_{ij}^{(B)} := \langle\phi_i^{(A)}|\rho|\phi_j^{(A)}\rangle \tag{3}$$

is diagonal in the same (resp. orthonormal) basis $\{|b_i\rangle\}$ for all i, j.

Proof. Necessity follows from the above observation. For sufficiency, suppose that $\rho_{ij}^{(B)} = \sum_m b_{ijm}|b_m\rangle\langle b_m|$ where $\{|b_m\rangle\}$ is any linearly independent (resp. orthonormal) set spanning $\mathcal{H}_B$. From the general expansion $\rho = \sum_{ijmn} c_{ijmn}|\phi_i^{(A)}\rangle\langle\phi_j^{(A)}| \otimes |b_m\rangle\langle b_n|$, we see that $c_{ijmn} = \delta_{mn} b_{ijm}$ and so

$$\rho = \sum_{ijm} b_{ijm}|\phi_i^{(A)}\rangle\langle\phi_j^{(A)}| \otimes |b_m\rangle\langle b_m| = \sum_m \rho_m \otimes |b_m\rangle\langle b_m| \tag{4}$$

where $\rho_m = \sum_{ij} b_{ijm}|\phi_i^{(A)}\rangle\langle\phi_j^{(A)}| = \langle b_m^\perp|\rho|b_m^\perp\rangle$ and $|b_m^\perp\rangle$ are vectors such that $\langle b_i|b_j^\perp\rangle = \delta_{ij}$. The last equation implies that ρ_m is semidefinite positive. Hence the state ρ is generalized-classical (resp. classical) as defined in Eq. 2. □

Theorem 4 implies that to decide whether ρ is generalized-classical for Bob, we need to check if the $\frac{1}{2}d_A(d_A-1)$ matrices $\{\langle\phi_i^{(A)}|\rho|\phi_j^{(A)}\rangle\}_{1\leq i\leq j\leq d_A}$ of size $d_B \times d_B$ are simultaneously congruent to diagonal matrices. In a more general form, this problem asks to decide whether for some set $\{A_i\}_{i=0\ldots m}$ of $n\times n$ matrices there exists an invertible matrix P such that $PA_iP^\dagger = \Lambda_i$ is diagonal for all i. We thank Yaoyun Shi for his assistance in constructing the following decision algorithm. To our knowledge, SDP is a novel approach to solving the described problem.

Since a general square matrix can always be expressed as a complex combination of hermitian matrices, without loss of generality we can assume the A_i are hermitian. Then the $PA_iP^\dagger$ are also hermitian, and if $PA_iP^\dagger = \Lambda_i$, the $PA_iP^\dagger$ are simultaneously diagonalized which means $[PA_iP^\dagger, PA_jP^\dagger] = 0$ for all i, j. Conversely, if this latter condition holds, then there exists a unitary U such that $UPA_iP^\dagger U^\dagger = \tilde{P}A_i\tilde{P}^\dagger = \Lambda_i$ for all i. So the question is whether $PA_iP^\dagger PA_jP^\dagger = PA_jP^\dagger PA_iP^\dagger$ for all i, j. Or in other words, $A_iWA_j = A_jWA_i$ where W is a positive-definite matrix. Note that if W is positive-definite, then we can scale appropriately so that $W \geq I$. Thus, we have the SDP feasibility problem:

$$\begin{aligned} &\text{Find} && W \\ &\text{subject to} && A_iWA_j = A_jWA_i \text{ for all } i,j \\ &&& W - I \geq 0. \end{aligned} \tag{5}$$

Known algorithms based on the ellipsoid and interior-point methods can efficiently solve this problem [26].

To decide whether ρ is classical for Bob, the situation is easier since P must be a unitary matrix. Thus, the A_i themselves must commute which in our case amounts to the vanishing of at most $\frac{1}{2}d_A^2(d_A^2-1)$ commutation relations.

Physical detection of classical states—Theorem 4 can be experimentally implemented by a set of projective operations and quantum state tomography. A direct reconstruction of the elements in Eq. 3 is not possible since they are not Hermitian and therefore do not correspond to anything physical. However, these terms can be computed indirectly if Alice makes a set of linearly independent projective operations (observables) that span her Hilbert-Schmidt space: $\mathcal{L} = \{|\phi_i^{(A)}\rangle\langle\phi_i^{(A)}|, |\psi_{ij}^{(A)}\rangle\langle\psi_{ij}^{(A)}|, |\chi_{ij}^{(A)}\rangle\langle\chi_{ij}^{(A)}|\}$ where $|\chi_{ij}^{(A)}\rangle = \frac{1}{\sqrt{2}}\left(|\phi_i^{(A)}\rangle - i|\phi_j^{(A)}\rangle\right)$ and $|\psi_{ij}^{(A)}\rangle = \frac{1}{\sqrt{2}}\left(|\phi_i^{(A)}\rangle + |\phi_j^{(A)}\rangle\right)$ for $i > j$. With that we have the elements of Eq. 3: $\langle\phi_i^{(A)}|\rho|\phi_j^{(A)}\rangle = \langle\psi_{ij}^{(A)}|\rho|\psi_{ij}^{(A)}\rangle + i\langle\chi_{ij}^{(A)}|\rho|\chi_{ij}^{(A)}\rangle - \frac{1+i}{2}(\langle\phi_i^{(A)}|\rho|\phi_i^{(A)}\rangle + \langle\phi_j^{(A)}|\rho|\phi_j^{(A)}\rangle)$.

According to Theorem 4 a state ρ is classical if and only if it has the same orthonormal basis for $\langle\phi_i^{(A)}|\rho|\phi_j^{(A)}\rangle$ for all i, j. It is clear that if $\mathrm{Tr}_A[P\rho]$ is diagonal for all $P \in \mathcal{L}$ then ρ is classical. Conversely, $\langle\phi_i^{(A)}|\rho|\phi_j^{(A)}\rangle$ diagonal in some orthonormal basis for all i, j implies that $\mathrm{Tr}_A[P\rho]$ is diagonal in same basis for all $P \in \mathcal{L}$. As the elements of $\mathcal{L}$ span Alice's space, any POVM she can perform will have operator elements with each being a linear combination of these projectors. Furthermore, if we consider "Alice's" system as the joint system of $N-1$ parties, then any local POVM performed by the $N-1$ parties will have product operators $E_{\vec{i}} = \bigotimes_{j=1}^{N-1} E_{i_j}^{(j)}$ also being a linear combination of projectors from $\mathcal{L}$, and conversely any element of $\mathcal{L}$ can be expressed as a linear combination of product operators constituting complete local measurements on the $N-1$ subsystems. Thus we obtain the following:

Theorem 5 *An N-partite state ρ is classical with respect to party k if and only if for any local POVM performed by the other parties,*

$$[\rho_{\vec{i}}, \rho_{\vec{i'}}] = 0 \ \textit{for all} \ \vec{i}, \vec{i'} \tag{6}$$

where $\rho_{\vec{i}} = Tr_{\overline{k}}[E_{\vec{i}}\rho]$.

Hence, we see that the non-classical nature of a state can be detected precisely by the non-commutativity of reduced states after some local POVM is locally implemented on all but one of the subsystems.

Conclusion.—We have introduced a class of states called generalized-classical which permit the purely classical task of non-disruptive local state identification when general quantum measurements are used. In this sense, generalized-classical states can be said to hover near the quantum-classical boundary. We have provided methods, both analytic and physical, which decide if a state is classical or generalized-classical. The interpretation of generalized-classical states lying near the quantum-classical border is further supported by the fact that generalized-classical states can be probabilistically locally broadcast, a task known to be deterministically possible only for fully classical states. It is interesting how this ability to locally broadcast can be seen as a consequence of probabilistic cloning among linearly independent states, and an intriguing question is whether the no-cloning theorem is even more fundamental than the no-broadcasting theorem of Ref. [5].

We have also shown the set of generalized classical states to occupy zero volume

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