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General Open-System Quantum Evolution in terms of Affine Maps of the Polarization Vector

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The operator-sum decomposition of a map from one density matrix to another has many applications in quantum information science. To this mapping there corresponds an affine map which provides a geometric description of the map of the density matrix in terms of the polarization vector representation. This has been thoroughly explored for qubits since the components of the polarization vector are measurable quantities (corresponding to expectation values of Hermitian operators) and also because it enables the description of map domains geometrically. Here we extend the operator-sum/affine map correspondence to qudits, briefly discuss general properties of the map, the form for some particular cases, and provide several explicit results for qutrit maps. We use the affine map and a singular-value-like decomposition to find positivity constraints that provide a symmetry for small polarization vector magnitudes (states which are closer to the maximally mixed state) which is broken as the polarization vector increases in magnitude (a state becomes more pure). The dependence of this symmetry on the magnitude of the polarization vector implies the polar decomposition of the map can not be used as it can for the qubit case. However, it still leads us to a connection between positivity and purity for general d-state systems.

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I. INTRODUCTION

The study of maps of density operators to density operators has become an active area of research in quantum information theory. They are used to describe open-system evolution, including noise, are used for describing methods to alleviate noise and, in some cases, as indicators of entanglement [1, 2]. All of these uses are very important in the study of quantum information processing.

General maps of density operators to density operators were studied by Sudarshan, Mathews and Rau (SMR) [3] in 1961. They provided a method for describing open-system evolution and referred to such maps as “dynamical maps.” This name is particularly appropriate for describing noise since detailed properties of the environment causing the noise are usually unknown. They showed that these maps can be written in what is called an operator-sum representation (OSR). Kraus later considered completely positive maps [4]. The assumption of complete positivity is sometimes a useful, but not a necessary assumption for open system evolution. The physical implications of the complete positivity assumption have recently been discussed at length in the literature [5–10]. Such work is not only important for quantum error correction and the description of noise in quantum systems [11], but also for quantum control more generally. (See Ref. [12, 13] for recent reviews of quantum control theory and applications.)

Studies of maps of qubits have been extensive. Some notable discussions are found in Refs. [14, 15] where they describe a geometric picture of the spaces of states of qubits and the ranges of maps of qubits. This is done using the Bloch vector, or polarization vector parameterization of the density operator [16–18] and an affine

mapping of the polarization vector. This affine map is essentially equivalent to the OSR, but is described geometrically, leading to constructions which are sometimes helpful tools for visualization as well as analysis.

The subject of noise in quantum systems governs most of the discussion of such maps in Ref. [14], and this will also be the case here. The discussions in Refs. [14, 15] are based on the Bloch vector parameterization of the density operator. The generalization of the Bloch vector is known as the generalized Bloch vector, coherence vector, or polarization vector. Since the dimension of this vector grows with the dimension of the Hilbert space as $d^2 - 1$, where d is the dimension of the Hilbert space, the simple three dimensional picture (for $d = 2$) becomes more difficult to use and one loses the ability to visualize the entire space. However, the polarization vector parameterization has several appealing properties. (1) The components of the polarization vector are measurable quantities; they are proportional to the expectation values of Hermitian operators. (2) The trace condition (the trace of the density operator must be one) and the hermiticity of the density operator become apparent in this picture. (3) The positivity of the density operator can be expressed in terms of the components of the density operator and the magnitude of the polarization vector is directly related to its purity ([19, 20]). (4) The Casimir invariants, quantities which are invariant under all unitary operators, are easily written down in terms of the polarization vector [19]. (5) A tensor product basis may be used so that subsystems of interest are manifest [17, 19]. Clearly these are not entirely independent properties. However, they are useful properties. For these reasons, this picture, which generalizes the Bloch equations, is becoming more widely used in both theory and experiment. This begs the question, which we partially answer. How can

we fully utilize this picture?

In this paper we discuss the generalization of affine maps of the polarization vector to systems with d dimensions. We provide explicit expressions for the affine map of the polarization vector in terms of the OSR elements. This representation of the map of the polarization vector is an affine transformation since it generally contains linear and translational terms. Our explicit calculation provides a direct link between the OSR, pioneered by Sudarshan, Mathews and Rau [3] and the affine map picture as is done in Refs. [14, 15] for qubits. We discuss the possibility of extending the description of the positivity domain for qubits [15] to d -dimensional systems and find a continuous symmetry breaking description which describes the inability of one to apply a simple singular value decomposition (SVD) to all density operators. This symmetry breaking provides a picture which complements the work of Kimura and Kossakowski describing the positivity domains for density operators of d -state systems [21] but treats it in a different way.

Specifically, in Section II, a brief review of the derivation of the general OSR from the dynamical map is given, along with a brief review of the polarization vector parameterization of the density operator. In Section III the form of the affine map is derived from the OSR and some properties of the affine map are given. In Section IV physical examples of noise are given to show how the polarization vector changes for some important types of noise. We then examine a particularly important basis for the operators comprising the OSR in Section V. Having established the relation between the OSR and affine map, we show how one can find the affine map directly from the corresponding dynamical map in Section VI. Section VII contains a review of the singular value decomposition of the affine map for qubits and our attempted generalization. Finally, concluding remarks and future directions are discussed in Section VIII.

II. BACKGROUND

In this section we provide background for what follows. This primarily follows the results of Ref. [3].

A. Dynamical Maps and the SMR Decomposition

As did Ref. [3] consider a general mapping from one Hermitian matrix to another of the form

$$\rho' = A\rho, \quad (1)$$

or more explicitly

$$\rho'_{r's'} = A_{r's',rs}\rho_{rs}. \quad (2)$$

Since the density matrix is required to be Hermitian $\rho = \rho^\dagger$, positive semi-definite, $\rho \geq 0$, and have trace one

$\text{Tr}\rho = 1$ the mapping A must have the following properties in order for it to map density operators to density operators:

$$A_{sr,s'r'} = (A_{rs,r's'})^*, \quad (3)$$

which ensures hermiticity, and

$$x_r^* x_s A_{sr,s'r'} y_{s'} y_{r'} \geq 0, \quad A_{rr,s'r'} = \delta_{s'r'}, \quad (4)$$

ensure positivity and the trace condition respectively. SMR then introduced a matrix B , related to A by relabeling, such that

$$B_{rr',ss'} \equiv A_{sr,s'r'}. \quad (5)$$

This matrix has the following properties:

$$B_{rr',ss'} = (B_{ss',rr'})^*, \quad (6)$$

corresponding to Eq. (3), and

$$z_{rr'}^* B_{rr',ss'} z_{ss'} \geq 0, \quad B_{rr',rs'} = \delta_{r's'}, \quad (7)$$

which corresponds to Eqs. (4).

In this article we will allow the map to be more general initially and only require the hermiticity condition, Eq. (3) or, equivalently, Eq. (6). Since B satisfies the hermiticity condition, it can be considered a Hermitian matrix and as such it is diagonalizable using a spectral decomposition, or eigenvector decomposition. Letting η_k be the eigenvalues of B the spectral decomposition can be written as

$$B_{rr',ss'} = \sum_k \eta_k \xi_{rr'}^{(k)} \xi_{ss'}^{(k)\dagger}. \quad (8)$$

This is also sometimes written with indices suppressed as follows:

$$B = \sum_k \eta_k C_k C_k^\dagger, \quad (9)$$

where $(C_k)_{rr'} = \xi_{rr'}^{(k)}$ are eigenvectors of B and η_k its eigenvalues. This is shown in detail in [7]. The C_k , as well as the density operator itself, may be written as a matrix or a vector.

If the η_k are all positive, they may be absorbed into the C 's to arrive at the familiar form of the operator-sum decomposition:

$$\rho' = \sum_k A_k \rho A_k^\dagger, \quad (10)$$

where $A_k = \sqrt{\eta_k} C_k$ [22].

B. Polarization Vector Representation of the Density Operator

Before we present the polarization representation of density operators, it is important to provide our conventions. These are contained in Ref. [19] which follow Refs. [18, 23]. However, they are not completely

standard; see for example [16, 17, 20] for other conventions. A density operator on a d -dimensional Hilbert space \mathcal{H}_d will be represented using the identity matrix plus a set of traceless, Hermitian matrices $\{\lambda_i\}$, $i = 1, 2, \dots, d^2 - 1$. These have the following normalization condition $\text{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}$, the commutation relations $[\lambda_i, \lambda_j] = 2if_{ijk}\lambda_k$, where the f_{ijk} are the totally antisymmetric structure constants, and anticommutation relations $\{\lambda_i, \lambda_j\} = \frac{4}{d}\mathbb{1}\delta_{ij} + 2d_{ijk}\lambda_k$ where d_{ijk} are the totally symmetric d -tensor components. The sum over repeated indices is to be understood unless otherwise stated. However, in some cases the sum is displayed explicitly for emphasis. These three relations may be summarized using the following equation

$$\lambda_i \lambda_j = \frac{2}{d}\mathbb{1}\delta_{ij} + d_{ijk}\lambda_k + if_{ijk}\lambda_k. \quad (11)$$

The density operator can now be written as

$$\rho = \frac{1}{d} \left(\mathbb{1} + b\vec{n} \cdot \vec{\lambda} \right), \quad (12)$$

where $b = \sqrt{d(d-1)/2}$. The “dot product” is a sum over repeated indices,

$$\vec{a} \cdot \vec{b} = a_i b_i = \sum_{i=1}^{d^2-1} a_i b_i. \quad (13)$$

It is worth noting that any complete set of $d^2 - 1$ mutually trace-orthogonal, Hermitian matrices can serve as a basis and can be chosen to satisfy the conditions given here. One may consult, for example, Refs.[24, 25] for other conventions. The components of the polarization vector \vec{n} are proportional to the expectation values of the set of Hermitian observables $\vec{\lambda}$:

$$n_i = \frac{d}{2b} \text{Tr}(\lambda_i \rho). \quad (14)$$

Therefore these are directly measurable quantities which can be used to completely specify any state, pure or mixed.

Pure states have the properties that

$$\vec{n} \cdot \vec{n} = 1, \quad \text{and} \quad \vec{n} \star \vec{n} = \vec{n}, \quad (15)$$

where the “star” product is defined by

$$(\vec{a} \star \vec{b})_k = \frac{1}{d-2} \sqrt{\frac{d(d-1)}{2}} d_{ijk} a_i b_j. \quad (16)$$

For later use, a “cross” product between two coherence vectors can also be defined by

$$(\vec{a} \times \vec{b})_k = f_{ijk} a_i b_j. \quad (17)$$

The set of mixed states can be specified in terms of a set of positivity conditions [19, 20].

III. AFFINE MAPS FROM THE OSR

In this section we obtain the affine map of the polarization vector in terms of the components of the OSR of the dynamical map. However, we begin with a general case, as is done in Ref. [15], and restrict to particular classes of maps which are often physically relevant. Furthermore, we do not restrict to completely positive maps. The affine map provides a final connection between these three different forms of the map: the dynamical map A (or equivalently B), the OSR, and the affine map.

A. Explicit form for the Affine Map

Let the operator-sum decomposition for a map from one density operator to another be given by

$$\rho' = \Phi(\rho) = \sum_k \eta_k C_k \rho C_k^\dagger. \quad (18)$$

Following Ref. [15] which treats the qubit case, the C_k can be represented by (complex) linear combinations of the $\{\lambda_i\}$, as

$$C_k = v_{0k}\mathbb{1} + \vec{v}_k \cdot \vec{\lambda}, \quad (19)$$

where $v_{0k}, v_{ik} \in \mathbb{C}$.

The following identity will be used repeatedly in our derivations

$$(a\mathbb{1} + \vec{u} \cdot \vec{\lambda})(b\mathbb{1} + \vec{w} \cdot \vec{\lambda}) = \left(ab + \frac{2}{d}\vec{u} \cdot \vec{w}\right)\mathbb{1} + \left(a\vec{w} + b\vec{u} + i\vec{u} \times \vec{w} + \frac{1}{c}\vec{u} \star \vec{w}\right) \cdot \vec{\lambda}, \quad (20)$$

where the definitions Eqs. (16) and (17) were used along with the following definition,

$$c = \sqrt{\frac{d(d-1)}{2}} \frac{1}{d-2}. \quad (21)$$

Two special classes of maps are particularly important, unital maps and trace preserving maps. They are defined by

1. Φ is called *unital* if $\Phi(\mathbb{1}) = \mathbb{1}$,
2. Φ is called *trace preserving* if $\forall \rho, \text{Tr}(\Phi(\rho)) = \text{Tr}(\rho)$.

If the map is unital, then by explicit calculation using Eq. (18),

$$\sum_k \eta_k \left[|v_{0k}|^2 + \frac{2}{d} \vec{v}_k \cdot \vec{v}_k^* \right] = 1, \quad (22)$$

and

$$\sum_k \eta_k \left(v_{0k} \vec{v}_k^* + v_{0k}^* \vec{v}_k + i\vec{v}_k \times \vec{v}_k^* + \frac{1}{c} \vec{v}_k \star \vec{v}_k^* \right) = 0. \quad (23)$$

Similarly, if the map is trace preserving, then an explicit calculation shows that

$$\sum_k \eta_k \left[|v_{0k}|^2 + \frac{2}{d} \vec{v}_k \cdot \vec{v}_k^* \right] = 1, \quad (24)$$

and

$$\sum_k \eta_k \left(v_{0k} \vec{v}_k^* + v_{0k}^* \vec{v}_k + i \vec{v}_k^* \times \vec{v}_k + \frac{1}{c} \vec{v}_k^* \star \vec{v}_k \right) = 0. \quad (25)$$

Note that if the map is either unital or trace-preserving, then the condition given in Eq. (22) (equivalently Eq. (24)) holds.

To provide the explicit form of the map, we first use Eqs. (18), (19), and (20), so that

$$\begin{aligned} \Phi(\rho) &= \sum_k \eta_k C_k \rho C_k^\dagger \\ &= \sum_k \eta_k (v_{0k} \mathbb{1} + \vec{v}_k \cdot \vec{\lambda}) \frac{1}{d} (\mathbb{1} + b \vec{n} \cdot \vec{\lambda}) (v_{0k}^* \mathbb{1} + \vec{v}_k^* \cdot \vec{\lambda}), \end{aligned} \quad (26)$$

can be rewritten and the identity and $\vec{\lambda}$ parts treated separately.

First, consider the identity part. The coefficient of the identity can be written in the following form

$$\begin{aligned} \sum_k \eta_k \left(\frac{1}{d} \left[|v_{0k}|^2 + \frac{2}{d} \vec{v}_k \cdot \vec{v}_k^* \right] \right. \\ \left. + \frac{2b}{d^2} \left(v_{0k}^* \vec{v}_k + v_{0k} \vec{v}_k^* + i \vec{v}_k^* \times \vec{v}_k + \frac{1}{c} \vec{v}_k^* \star \vec{v}_k \right) \cdot \vec{n} \right), \end{aligned} \quad (27)$$

If the map is unital, using Eqs. (22) and (23), the coefficient of the identity reduces to

$$\frac{1}{d} + \frac{4ib}{d^2} \sum_k \eta_k (\vec{v}_k^* \times \vec{v}_k) \cdot \vec{n}. \quad (28)$$

If the map is also trace-preserving, subtracting Eq. (23) from Eq. (25) implies that the coefficient of the identity is $1/d$.

Now let us consider the non-identity part of the map. Denoting the result of the map by ρ' , we have

$$\Phi(\rho) = \rho' = \frac{1}{d} (\mathbb{1} + b \vec{n}' \cdot \vec{\lambda}). \quad (29)$$

Viewing the map as an affine map of the coherence vector \vec{n} ,

$$\vec{n} \mapsto \vec{n}' = T \vec{n} + \vec{t}, \quad (30)$$

so the components of \vec{n}' are given by

$$n'_q = T_{pq} n_p + t_q. \quad (31)$$

Thus, after some rearrangement, the map can be written as

$$\frac{1}{d} \sum_k \eta_k \left[v_{0k}^* \vec{v}_k + v_{0k} \vec{v}_k^* + i \vec{v}_k \times \vec{v}_k^* + \frac{1}{c} \vec{v}_k^* \star \vec{v}_k \right] \cdot \vec{\lambda} \quad (32)$$

$$\begin{aligned} + \frac{1}{d} \sum_k \eta_k \left[b |v_{0k}|^2 \vec{n} + \frac{2b}{d} (\vec{v}_k \cdot \vec{n}) \vec{v}_k^* + i b (v_{0k}^* \vec{v}_k \times \vec{n} + v_{0k} \vec{n} \times \vec{v}_k^*) + \frac{b}{c} (v_{0k} \vec{n} \star \vec{v}_k^* + v_{0k}^* \vec{n} \star \vec{v}_k) \right. \\ \left. + \frac{ib}{c} \{ (\vec{v}_k \star \vec{n}) \times \vec{v}_k^* + (\vec{v}_k \times \vec{n}) \star \vec{v}_k^* \} - b (\vec{v}_k \times \vec{n}) \times \vec{v}_k^* + \frac{b}{c^2} (\vec{v}_k \star \vec{n}) \star \vec{v}_k^* \right] \cdot \vec{\lambda}, \end{aligned} \quad (33)$$

where $T = \sum_k \eta_k T_k$ and $\vec{t} = \sum_k \eta_k \vec{t}_k$, which is zero for a unital map. The k^{th} term, Eq. (32), is

$$\vec{t}_k = \frac{1}{b} \left[v_{0k}^* \vec{v}_k + v_{0k} \vec{v}_k^* + i \vec{v}_k \times \vec{v}_k^* + \frac{1}{c} \vec{v}_k^* \star \vec{v}_k \right]. \quad (34)$$

The k^{th} term (33) represents the action of the *real* $(d^2 - 1) \times (d^2 - 1)$ matrix T . (The proof of the reality of T is given below.) The components of the linear part of the map, which we refer to as the T matrix, may then be

written as

$$\begin{aligned} T_{pq} &= \frac{1}{d} \sum_k \eta_k \left[\left(|v_{0k}|^2 - \frac{2}{d} v_{rk}^* v_{rk} \right) \delta_{pq} \right. \\ &\quad + \frac{2}{d} (v_{pk} v_{qk}^* + v_{pk}^* v_{qk}) + i f_{rpq} (v_{0k}^* v_{rk} - v_{0k} v_{rk}^*) \\ &\quad + d_{rpq} (v_{0k}^* v_{rk} + v_{0k} v_{rk}^*) - i d_{rpq} f_{srt} v_{tk}^* v_{sk} \\ &\quad \left. + (d_{qsr} d_{rtp} + d_{qtr} d_{spr} - d_{qpr} d_{str}) v_{tk} v_{sk}^* \right]. \end{aligned} \quad (35)$$

It is worth emphasizing that the T matrix and the

vector \vec{t} specify an affine map of the polarization vector. We also note that the map is linear when $\vec{t} = 0$. This is equivalent to the dynamical map, A (or B) specified in Section II and also the operator-sum decomposition. Thus we have provided a mapping of a vector to a vector which corresponds to a general mapping of a density operator to a density operator. However, the positivity of the result has not yet been established. The next question which we will answer in part is the following. What are the properties of this map? We are able to provide partial answers to this broad question for both the general map and for some particular cases of interest.

B. Properties of the Associated Linear Map

In this subsection we show that the linear term in the affine map, i.e., the T matrix, is real. Although this must be true, it is shown explicitly since it is not obvious from the expression Eq. (35). In addition, we provide general conditions for the matrix to be symmetric.

1. The T -matrix is real

It is not clear that the third to the last term in the curly brackets $(\{\})$, (33), which mixes star and cross products, is real. If we can show that this is real, then we will have shown that T is real since the other terms are obviously real. Let us consider whether

$$+ \frac{ib}{c} \sum_k \eta_k [((\vec{v}_k \star \vec{n}) \times \vec{v}_k^*) \cdot \vec{\lambda} + ((\vec{v}_k \times \vec{n}) \star \vec{v}_k^*) \cdot \vec{\lambda}] \quad (36)$$

is equal to its complex conjugate.

To provide a sufficient condition, let us first consider a term with a given k . From the identity, Eq. (A3):

$$d_{ijm}f_{mln} + d_{jlm}f_{min} + d_{lim}f_{mjn} = 0 \quad (37)$$

we can form the following identities with any vectors $\vec{a}, \vec{b}, \vec{c}, \vec{\lambda}$. (Here we will use $\vec{\lambda}$ as the basis for the Lie algebra, but in these identities, it could be any vector.) First, we contract the LHS of Eq. (37) with $a_i b_j c_l \lambda_n$ and find:

$$[(\vec{a} \star \vec{b}) \times \vec{c}] \cdot \vec{\lambda} + [(\vec{b} \star \vec{c}) \times \vec{a}] \cdot \vec{\lambda} + [(\vec{c} \star \vec{a}) \times \vec{b}] \cdot \vec{\lambda} = 0. \quad (38)$$

For the next identity, we contract Eq. (37) with $a_j b_l c_n \lambda_i$ to obtain:

$$[(\vec{b} \times \vec{c}) \star \vec{a}] \cdot \vec{\lambda} + \vec{\lambda} \cdot [\vec{c} \times (\vec{a} \star \vec{b})] + [(\vec{a} \times \vec{c}) \star \vec{b}] \cdot \vec{\lambda} = 0. \quad (39)$$

For our purposes, it is relevant to note the following symmetries in the indices. The identity is invariant under the interchange of the following pairs of indices (l, i) , (i, j) , (l, j) , so that these are the only two different types of identities when we distinguish $\vec{\lambda}$ as a set of basis elements.

We will use this second identity Eq. (39) to rewrite the first of the two terms in Eq. (36). First, note that

$$[\vec{v}_k^* \times (\vec{v}_k \star \vec{n})] \cdot \vec{\lambda} + [(\vec{n} \times \vec{v}_k^*) \star \vec{v}_k] \cdot \vec{\lambda} + [(\vec{v}_k \times \vec{v}_k^*) \star \vec{n}] \cdot \vec{\lambda} = 0. \quad (40)$$

Now the first term in this equation is the same (up to sign) as the first of the two terms in Eq. (36). This allows us to rewrite Eq. (36), less a multiplicative factor of ib/c , as

$$[(\vec{n} \times \vec{v}_k^*) \star \vec{v}_k] \cdot \vec{\lambda} + [(\vec{v}_k \times \vec{v}_k^*) \star \vec{n}] \cdot \vec{\lambda} + [(\vec{v}_k \times \vec{n}) \star \vec{v}_k^*] \cdot \vec{\lambda} \quad (41)$$

We now want to show that the complex conjugate of the coefficient of $\vec{\lambda}$ is real by showing it is equal to itself ($z^* = z \Rightarrow z$ is real). Let us take the complex conjugate of the coefficient recalling that \vec{n} is real:

$$[(\vec{n} \times \vec{v}_k) \star \vec{v}_k^*] \cdot \vec{\lambda} + [(\vec{v}_k \times \vec{v}_k) \star \vec{n}] \cdot \vec{\lambda} + [(\vec{v}_k^* \times \vec{n}) \star \vec{v}_k] \cdot \vec{\lambda}. \quad (42)$$

The first term is the negative of the third term above, the second is the negative of the second above and the third is the negative of the first due to the change of sign of the factor ib/c . Therefore the two are equal due to the overall minus sign in this latter expression and so the coefficient is real and this part of the map is real. To prove that this sufficient condition is also necessary, we use the fact that there exists a minimal decomposition of the map such that each T_k is independent and can always be put in minimal form [26]. Then, since the η_k are real, this condition is also necessary when the map is in minimal form. \square

2. Conditions for T to be a symmetric matrix

The importance of this stems from the fact that a symmetric matrix can be decomposed in a polar decomposition [38]

$$S = \mathcal{O} D \mathcal{O}, \quad (43)$$

where S is a real symmetric matrix and \mathcal{O} is orthogonal. We will show this to be true for a given k and the generalization follows from the sum $\sum_k \eta_k T_k$.

The only antisymmetric term in the expression for T is

$$T_{pq}^{(a)} = \sum_k \eta_k i f_{rpq} (v_{0k}^* v_{rk} - v_{0k} v_{rk}^*) \quad (44)$$

Therefore, for T to be symmetric, we require this term to vanish.

Again, let us first consider the necessary condition that each term given by a fixed k vanish independently. Since the matrices $(f_{pq})_r$ form a representation of the Lie algebra of $SU(d)$, the only way for this to happen is if all vector components, indexed by r , vanish independently. This implies that we must have

$$v_{0k}^* v_{rk} = v_{0k} v_{rk}^*. \quad (45)$$

There are several ways in which this can happen. Any of the following is sufficient to ensure that T is symmetric:

1. For each k , the v_{0k} and v_{rk} are all real (implying all C_k are Hermitian) or all imaginary (implying all C_k are anti-Hermitian).
2. For a given k either $v_{0k} = 0$, or $v_{rk} = 0$, $\forall r$. (Note $\vec{v}_{rk} = 0$ for all r is rather trivial because then $C_k \propto \mathbb{1}$.)
3. $v_{rk} = v_{sk}$, $\forall k, \forall r, s$ and $v_{0k} = v_{rk}$. Note that the vectors can be different for each k , but all have the same components for each k . They are also not required to be real.

We also believe that, except for cancellation of terms with different k , this set to be exhaustive, meaning some combination of these is required for the symmetry of the T matrix, at least for the minimal decomposition.

IV. EXAMPLE CHANNELS

Here we consider some examples of maps which are of special interest. These are far from exhaustive and we will consider other examples in future applications.

A. C_k hermitian

When the C_k are hermitian, the T matrix is symmetric as shown in the previous section. Here we provide explicit expressions.

Consider a map with $C_k = C_k^\dagger$ so that the C_k are Hermitian, or sometimes called self-adjoint. Then C_k can be expressed as

$$C_k = (v_{0k}\mathbb{1} + v_{ik}\lambda_i), \quad (46)$$

with $v_{0k}, v_{ik} \in \mathbb{R}$.

Since the components of T , Eq. (35), are real, the matrix components of the affine map for Hermitian C_k , are

$$\begin{aligned} (T_k)_{pq} &= |v_{0k}|^2 \delta_{pq} + \frac{2}{d} (\vec{v}_k)_p (\vec{v}_k)_q + 2v_{0k} d_{rpq} (\vec{v}_k)_r \\ &\quad - f_{tpr} f_{rsq} (\vec{v}_k)_t (\vec{v}_k)_s \\ &\quad + d_{tpr} d_{rsq} (\vec{v}_k)_t (\vec{v}_k)_s, \end{aligned} \quad (47)$$

where the third and fifth terms in Eq. (35) vanish since v_{0k}, \vec{v}_k are real. This is symmetric (in p and q) since the first two terms clearly are symmetric and the latter two can be shown to be symmetric by renaming the dummy indices s and t , and cyclicly permuting the indices on c and d , viz.

$$\begin{aligned} f_{tpr} f_{rsq} (\vec{v}_k)_t (\vec{v}_k)_s &= f_{spr} f_{rtq} (\vec{v}_k)_s (\vec{v}_k)_t \\ &= f_{rsp} f_{tqr} (\vec{v}_k)_s (\vec{v}_k)_t \\ &= f_{tqr} f_{rsp} (\vec{v}_k)_t (\vec{v}_k)_s. \end{aligned} \quad (48)$$

Therefore if the C_k are Hermitian, then the map T_{pq} is real and symmetric in p and q .

The affine part of the transformation reduces to

$$\vec{t}_k = \frac{1}{b} (2v_{0k}\vec{v}_k + \frac{1}{c} \vec{v}_k \star \vec{v}_k). \quad (49)$$

B. C_k Unitary

This case will be explored further in the next subsection where numerous reasons are given for the special consideration. Here we provide an explicit expression for the T matrix and note that \vec{t} is zero for this case. One case of interest is when the map is given by $\Phi(\rho) = \sum_k \eta_k U_k \rho U_k^\dagger$ where $\sum_k \eta_k = 1$ and $\eta_k > 0$.

Consider a map with $C_k C_k^\dagger = \mathbb{1} = C_k^\dagger C_k$. Then C_k can be expressed as

$$C_k = (v_{0k}\mathbb{1} + v_{ik}\lambda_i), \quad (50)$$

with the following constraints

$$\left(|v_{0k}|^2 + \frac{2}{d} \vec{v}_k \cdot \vec{v}_k^* \right) = 1, \quad (51)$$

and

$$\begin{aligned} 0 &= \left(v_{0k} \vec{v}_k^* + v_{0k}^* \vec{v}_k + i \vec{v}_k \times \vec{v}_k^* + \frac{1}{c} \vec{v}_k \star \vec{v}_k^* \right) \\ &= \left(v_{0k} \vec{v}_k^* + v_{0k}^* \vec{v}_k + i \vec{v}_k^* \times \vec{v}_k + \frac{1}{c} \vec{v}_k \star \vec{v}_k^* \right). \end{aligned} \quad (52)$$

This implies separately that

$$0 = \left(v_{0k} \vec{v}_k^* + v_{0k}^* \vec{v}_k + \frac{1}{c} \vec{v}_k \star \vec{v}_k^* \right) \quad (53)$$

and

$$0 = \vec{v}_k \times \vec{v}_k^*. \quad (54)$$

This implies that T_k reduces to

$$\begin{aligned} (T_k)_{pq} &= |v_{0k}|^2 \delta_{pq} + \frac{2}{d} (\vec{v}_k)_p (\vec{v}_k^*)_q \\ &\quad + i [v_{0k}^* f_{rpq} (\vec{v}_k)_r - v_{0k} f_{rpq} (\vec{v}_k^*)_r] \\ &\quad + [v_{0k} d_{rpq} (\vec{v}_k^*)_r + v_{0k}^* d_{rpq} (\vec{v}_k)_r] \\ &\quad + i [d_{spr} f_{rtq} (\vec{v}_k)_s (\vec{v}_k^*)_t + d_{rsq} f_{tpr} (\vec{v}_k)_t (\vec{v}_k^*)_s] \\ &\quad - f_{tpr} f_{rsq} (\vec{v}_k)_t (\vec{v}_k^*)_s + d_{tpr} d_{rsq} (\vec{v}_k)_t (\vec{v}_k^*)_s. \end{aligned} \quad (55)$$

C. Qutrits

In this section we provide explicit example channels which are relevant for physical problems involving three-state systems. See also Refs. [25, 27] and references therein.

1. Qutrit Depolarizing Channel

In general the effect of the depolarizing channel on a density operator is to cause a uniform shrinking of the polarization vector \vec{n} . This is written as

$$\rho = \frac{1}{d} (\mathbb{1} + b \vec{n} \cdot \vec{\lambda}) \rightarrow \rho' = \frac{1}{d} (\mathbb{1} + b p \vec{n} \cdot \vec{\lambda}), \quad (56)$$

where p is the shrinking factor. (See for example [28] and references therein.)

The effect of the depolarizing channel on the coherence vector is such that it uniformly shrinks each component of \vec{n} by a common factor. It can be readily verified that if we let

$$C_0 = \sqrt{1-x} \mathbb{1}, \quad (57)$$

and

$$C_k = \sqrt{3x/16} \lambda_k, \quad k = 1, 2, \dots, 8 \quad (58)$$

we then have $\sum_{k=0}^8 C_k C_k^\dagger = \mathbb{1}$, $\vec{t} = 0$, $(T_0)_{pq} = (1-x)\delta_{pq}$, and $(T_k)_{pq} = (x/8)\delta_{kp}\delta_{kq} + (3x/16)(d_{kpr}d_{rkq} - f_{kpr}f_{rkq})$ for $k = 1, 2, \dots, 8$. The coherence vector is thus transformed according to

$$\vec{n} \mapsto T\vec{n} = \sum_k T_k \vec{n} = (1-9x/8)\vec{n}. \quad (59)$$

Thus the shrinking factor is $p = 1 - 9x/8$.

It is interesting to note that the depolarizing channel, for pure state inputs, has a Schmidt decomposition for all dimensions d [28].

2. Qutrit Phase Damping Channel

Analogous to the qubit phase damping channel, the effect of the phase damping channel on a three state system is such that it leaves the diagonal components of \vec{n} unchanged while uniformly shrinking the off-diagonal components. When the operators C_k are chosen to be

$$C_0 = \sqrt{1-x} \mathbb{1}, \quad (60)$$

$$C_1 = \sqrt{3x/20} (\mathbb{1} + \lambda_3), \quad C_2 = \sqrt{3x/20} (\mathbb{1} - \lambda_3), \quad (61)$$

and

$$C_3 = \sqrt{3x/20} (\mathbb{1} + \lambda_8), \quad C_4 = \sqrt{3x/20} (\mathbb{1} - \lambda_8), \quad (62)$$

it can be shown that $\sum_{k=0}^4 C_k C_k^\dagger = \mathbb{1}$, $\vec{t} = 0$, $(T_0)_{pq} = (1-x)\delta_{pq}$, $(T_1)_{pq} = \mu_3^+$, $(T_2)_{pq} = \mu_3^-$, $(T_3)_{pq} = \mu_8^+$, and $(T_4)_{pq} = \mu_8^-$, where

$$\mu_l^\pm \equiv \frac{3x}{20} \left[\delta_{pq} + \frac{2}{3} \delta_{pl} \delta_{ql} \pm d_{lpq} - f_{lpr} f_{lqr} + d_{lpr} d_{lqr} \right]. \quad (63)$$

When $\eta_k = 1$ for $\forall k$ the matrix T shrinks the off-diagonal components of the coherence vector by a factor of $1 - 3x/5$, i.e.,

$$n_i \mapsto \begin{cases} n_i, & \text{for } i = 3, 8 \\ (1 - 3x/5)n_i, & \text{otherwise} \end{cases} \quad (64)$$

In the last expression we have taken the usual convention of labeling the diagonal elements of the basis as 3 and 8.

3. Off-Diagonal Channel

Let us now define

$$C_0 = \sqrt{1-x} \mathbb{1}, \quad (65)$$

and

$$C_k = \sqrt{x/4} \lambda_k, \quad \text{for } k = 1, 2, 4, 5, 6, 7. \quad (66)$$

These values of k correspond to the off-diagonal generators of $SU(3)$. It can be verified that this particular OSR transforms the coherence vector according to

$$n_i \mapsto \begin{cases} (1 - 3x/2)n_i, & \text{for } i = 3, 8 \\ (1 - x)n_i, & \text{otherwise} \end{cases} \quad (67)$$

when $\eta_k = 1$ for $\forall k$. In this case, the diagonal components of \vec{n} shrink faster with x than the off-diagonal components. If instead $\eta_k = (-1)^k$, the mapping given by $T = \sum_k (-1)^k T_k$ (again $\vec{t} = \sum_k (-1)^k \vec{t}_k = \vec{0}$) affects the components of \vec{n} in such a way as to shrink them according to

$$n_i \mapsto \begin{cases} (1 - 3x/2)n_i, & \text{for } i = 1, 4, 6 \\ (1 - x/2)n_i, & \text{for } i = 2, 5, 7 \\ (1 - x)n_i, & \text{for } i = 3, 8 \end{cases} \quad (68)$$

The components of \vec{n} associated with the real off-diagonal elements (λ_i , $i = 1, 4, 6$) shrink with x faster than those associated with the imaginary off-diagonal elements (λ_i , $i = 2, 5, 7$). The diagonal components are reduced in magnitude at an intermediate rate. The values assigned to the η_k have a significant effect on the evolved state, as clearly illustrated by this example.

4. Trit Flip Channel

For qubits, the bit flip channel acts in such a way as to flip the basis state $|1\rangle$ to $|2\rangle$ and vice versa. For qutrits there are three basis states which can be labeled as $|1\rangle$, $|2\rangle$, and $|3\rangle$. To describe the effect of flipping one of these basis states to another in terms of an affine mapping of the polarization vector let us first define the following pure state density matrices

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (69)$$

Now the affect of flipping from the state $|1\rangle$ to $|2\rangle$ is equivalent to the density operator ρ_1 evolving to ρ_2 . This can be achieved by the transformation $C_{(1,2)} \rho_1 C_{(1,2)}^\dagger = \rho_2$, where

$$C_{(1,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (70)$$

Notice that $C_{(1,2)}$ is an element of the Gell-Mann basis for $SU(3)$. The affine map associated with this transformation has a linear part as well as a nonzero translational part, specifically

$$T_{(1,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \end{pmatrix}, \quad (71)$$

and

$$t_{(1,2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{pmatrix}. \quad (72)$$

Note that since $C_{(1,2)}\rho_2 C_{(1,2)}^\dagger = \rho_1$, the affine map associated with the flip $|2\rangle \rightarrow |1\rangle$ is the same as above. Similarly one can show that the transformation $C_{(1,3)}\rho_1 C_{(1,3)}^\dagger = \rho_3$, with

$$C_{(1,3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (73)$$

is equivalent to the affine mapping

$$T_{(1,3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\sqrt{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{3} & 0 & 0 & 0 & 0 & -2/3 \end{pmatrix}, \quad (74)$$

and

$$t_{(1,3)} = \begin{pmatrix} 0 \\ 0 \\ 1/2\sqrt{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ -1/6 \end{pmatrix}. \quad (75)$$

Since $C_{(1,3)}\rho_3 C_{(1,3)}^\dagger = \rho_1$, the mapping associated with $T_{(1,3)}$ and $t_{(1,3)}$ will yield the transition $|3\rangle \rightarrow |1\rangle$. For

the flips $|2\rangle \leftrightarrow |3\rangle$, the associated OSR decomposition has the term $C_{(2,3)}$ with

$$C_{(2,3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (76)$$

This leads to an affine mapping of the form

$$T_{(2,3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1/\sqrt{3} & 0 & 0 & 0 & 0 & -2/3 \end{pmatrix}, \quad (77)$$

and

$$t_{(2,3)} = \begin{pmatrix} 0 \\ 0 \\ -1/2\sqrt{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ -1/6 \end{pmatrix}, \quad (78)$$

V. A PREFERRED BASIS: C_k UNITARY AND ORTHOGONAL

It is often the case that a preferred basis for the C_k is chosen. One important case, is when the C_k are both orthogonal and unitary. This choice corresponds to a nice error basis [29] and applies to the so-called generalized Pauli matrices which are formed from C_k given by $U_{m,n} = X^m Z^n$, $m, n = 0, 1, 2$ where $X|j\rangle = |j+1 \bmod d\rangle$, $Z|j\rangle = w^j|j\rangle$, $w = e^{2\pi i/3}$ and $\{|j\rangle\}_{j=0}^2$ is an orthonormal basis. We also note that it can be applied to decoupling controls which form a discrete subgroup of the unitary group, but in that case the Hamiltonian is modified rather than the density operator.

Since the case where the C_k are unitary and orthogonal is a particularly important case, we show that when the C_k are orthogonal, unitary, and belong to a discrete subgroup of the unitary group, the trace of pairs of unequal T_k must be -1 between elements of the set.

Let $\{U_k\}$ be such a set which also forms a subset of a fundamental irreducible representation of a discrete subgroup of the unitary group, i.e., $U_k^\dagger U_k = \mathbf{1}$, and the set of U_k can be extended to form a discrete subgroup of the unitary group, and $\text{Tr}(U_i U_j^\dagger) = 0$ for $i \neq j$. Then, since $U_j^\dagger U_i$ is a group element, the trace corresponds to the character of the product. The character, for an irreducible representation, is the same for each element in a class and, in this case, the class has character equal

to zero. However, it can be different for different representations. Note that the corresponding T matrix is an orthogonal matrix since it belongs to the adjoint representation of the group which is a real $(d^2 - 1)$ -dimensional representation. This follows from the equation

$$\begin{aligned} U_k \rho U_k^\dagger &= U_k \frac{1}{d} \left(\mathbb{1} + b \vec{n} \cdot \vec{\lambda} \right) U_k^\dagger \\ &= \frac{1}{d} \left(\mathbb{1} + b \sum_{ij} n_i (R_k)_{ij} \lambda_j \right), \end{aligned} \quad (79)$$

where $U_k \in U(d)$ and R_k is in the adjoint representation of the group. In this case T_k corresponds to R_k , and U_k is the same group element as R_k but in a different representation.

Since $\text{Tr}(U_i^\dagger U_j) = 0$, and $U_i \otimes U_j$ is in the tensor product of the representation of $U(d) \otimes U(d)$, we can write $\text{Tr}(U_i^\dagger \otimes U_j^\dagger U_k \otimes U_l) = 0$ and this is the product of the two representations. If we take the complex conjugate of the first of these two representations (to obtain the conjugate representation) we find that the product, which is equal to the direct sum of the real $(d^2 - 1)$ -dimensional representation and the trivial one, must be zero. Since the trace of the sum is the sum of the traces, and the trivial representation has character (trace) one, the other must have trace -1 . Therefore, if the C_k are unitary, orthogonal, and the set of these can be extended to form a group, then the corresponding T_k have the traces of the products all equal to -1 , viz., $\text{Tr}(C_i C_j^\dagger) = 0 \Leftrightarrow \text{Tr}(T_i^T T_j) = -1, \forall i \neq j$.

As mentioned above, this result is important for error prevention and control methods. It is also generally important for translating between the OSR and the affine map when this basis is chosen.

We now briefly discuss how one would obtain the affine map directly from the dynamical map, showing that the OSR is in fact not a necessary intermediate step.

VI. AFFINE MAP FROM THE DYNAMICAL MAP

It is fairly straight-forward to show that the affine map can be obtained directly from the dynamical map. In this case, a convenient basis is the set of matrices which form a basis for $\text{GL}(n, \mathbb{C})$. Such a basis is defined by the set of matrices with a 1 in the i th column and j th row and zeros everywhere else, denoted $\{E_{ij}\}$. From Eq. (2), the two bases may be converted from one to another using, for example, the correspondence between the raising and lower operators of the group and the hermitian basis. To be specific, one would expand $\rho = \sum_{r,s} \rho_{rs} E_{rs}$, and use the corresponding hermitian basis $\Lambda_{rs} = E_{rs} + E_{sr}$ and $\Lambda'_{rs} = i(E_{rs} - E_{sr})$ as well as linear combinations of the identity and Cartan algebra elements to expand the diagonal matrices. (See for example [30] and references therein.)

Thus we have given the following correspondences: dynamical maps \rightarrow OSR \rightarrow affine map and dynamical map \rightarrow affine map. Note, however, that the OSR is not unique – not for completely positive matrices [14] nor for those that are not completely positive [26]. Therefore, the set of T_k matrices is also not unique and thus converting from one picture to another is not a one-to-one transformation. The dynamical map is essentially unique as is the T matrix as well as the minimal decomposition of the map [39].

VII. SINGULAR VALUE DECOMPOSITION

In the analysis of maps of qubit density operators, sufficient conditions for the positivity of maps follows from positivity conditions described by the Bloch-sphere [15, 31]. The maps, represented as an affine map of the vector, can be decomposed in terms of a singular value decomposition (SVD) which greatly simplifies the analysis [15]. Here we discuss a decomposition which is similar to the singular value decomposition of the affine map for qubits, but applies to a d -dimensional system when $d > 2$. This turns out to be much more complicated for d -state systems than it is for a two-state system. The reason, as explained in detail below, is that there are restrictions on the “rotation” part of the map (that part which does not change the magnitude of \vec{n}). The restrictions are determined by the “shrinking” part of the map (that part which can reduce the magnitude of one or more components of \vec{n}) and differ for different shrinking matrices. Thus we are seeking an answer to the following question. To what extent can we generalize the results of qubits to qudits with $d > 2$? Our results provide conditions under which a SVD may be performed as well as restrictions to its use. This provides some insight into the geometry of the space.

A. SVD For Affine Qubit Maps

Before presenting our results for qudits, we will review the basic idea for qubits as used in Ref. [15]. Let an arbitrary initial density matrix be given by

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{n} \cdot \vec{\sigma}), \quad (80)$$

and acted on by

$$\Phi(\rho) = \frac{1}{2}(\mathbb{1} + \vec{n}' \cdot \vec{\sigma}). \quad (81)$$

This map can be described by an affine map of the polarization vector,

$$\vec{n}' = T\vec{n} + \vec{t}. \quad (82)$$

In what follows we will let $\vec{t} = 0$ and discuss T . The translation may be treated separately.

For two-state systems, maps can be diagonalized by means of a SVD. This is because any real $N \times N$ matrix M can be written as

$$M = \mathcal{O}_1 D \mathcal{O}_2, \quad (83)$$

where $\mathcal{O}_i \in SO(N)$ and D is a diagonal matrix with the entries being called the singular values. For qubit maps $N = d^2 - 1$ is 3 and an action of $U \in U(2)$ on the density operator is equivalent to an $SO(3)$ matrix acting on the polarization vector \vec{n} . To see this, we write

$$U \rho U^\dagger = \frac{1}{2} \left(\mathbb{1} + \sum_{ij} n_i R_{ij} \sigma_j \right), \quad (84)$$

where $R \in SO(3)$. Thus acting before and after a map enables one to choose a preferred basis for the map. This basis can be chosen so that T is diagonal since choosing R and R' appropriately enables us to put T into a diagonal form using $D = R T R'$. This set of diagonalized maps can be considered a double coset space of the elements of the set of maps.

For future reference it is relevant to note that for a positive map, the matrix D cannot increase the magnitude of \vec{n} or any component of \vec{n} . (Again, we are taking $\vec{t} = 0$.) In general, for the SVDs we will consider here, we will refer to the D part (diagonal part) as the “shrinking factor” since, for a positive map, it can decrease, but not increase, the magnitude of the polarization vector. The parts \mathcal{O} will be called “rotations” although the maps can clearly be physically achieved in many different ways.

B. SVD For Qutrits

Here we will discuss the case of qutrits. The generalization to qudits is qualitatively quite similar. We know that for a mapping from pure states to pure states, the map is a unitary transformation. That is, any map from a pure state to a pure state can be written as a unitary transformation. The unitary transformation acts on the density operator to produce an affine map which has the form $T \in \text{Ad}(\text{SU}(3)) \subset \text{SO}(8)$. That is, the T matrix is an element of the adjoint representation of $\text{SU}(3)$ which is a subset of $\text{SO}(8)$. It clearly does not span the space of $\text{SO}(8)$ in the way that $\text{SU}(2)$ covers $\text{SO}(3)$ since the dimension of $\text{SU}(3)$ is $3^2 - 1 = 8$ and the dimension of $\text{SO}(8)$ is $8 \cdot 7/2 = 28$, see also Ref. [32].

For an even simpler case, consider a map which acts as a depolarizing channel together with a “rotation.” In this case there is a uniform shrinking factor such that each component of the polarization vector is reduced by the same factor $p < 1$. (See for example [28] and references therein.) For qutrits, when $p \leq 1/2$, the map has a SVD with

$$T = \mathcal{O}_1 D \mathcal{O}_2, \quad (85)$$

where $\mathcal{O}_i \in \text{SO}(8)$. Any matrix in $\text{SO}(8)$ is allowed since any direction for the polarization vector leads to a positive density matrix output as long as the shrinking factor is $p \leq 1/2$ and uniform. This follows from the positivity conditions in Refs. [19, 20].

For there to exist a SVD for maps which have $1/2 < p < 1$ the SVD can not either be the SVD for $p < 1/2$ nor can it be the SVD for pure states. The SVD for these intermediate values of p must have some elements $\mathcal{O}_i \in \text{SO}(8)$ that can not be arbitrary elements of $\text{SO}(8)$, but come from a restricted set which depend on the shrinking factor. However, they are also not restricted only to the set of matrices in $\text{SU}(3) \subset \text{SO}(8)$, they form a larger set. The question is, what is that set and how do we find the values?

Whereas the general problem is quite complicated, we can make some general statements about the allowable rotations. Let us consider the algebra of $\text{SU}(3) \subset \text{SO}(8)$. For pure states, the set of allowable rotations are only those which are in $\text{SU}(3)$ and at the other extreme, all rotations in $\text{SO}(8)$ are allowable when the magnitude of the polarization vector $|\vec{n}| \leq 1/2$ as discussed above. Thus the $\text{SO}(8)$ symmetry is much larger than the $\text{SU}(3)$ symmetry and it could be said that the symmetry is broken continuously by the increasing magnitude of the polarization vector. Such considerations are important for a variety of reasons. Most notably, the “good quantum numbers” are labels for states which are carriers of an irreducible representation of a group. (See for example [33].) Thus symmetries determine simultaneously measurable quantities for a quantum system and thus enable us to specify a state as well as we can. When a symmetry is broken, the symmetry for the system is reduced and the set of good quantum numbers changes, indicating a change in allowed states of the system. In this description of the allowable states, one possible symmetry breaking construction which could be used for broken symmetries is the following subgroup chain:

$$\text{SO}(8) \supset \text{SO}(6) \times \text{SO}(2) \cong \text{SU}(4)/\mathbb{Z}_2 \times \text{U}(1) \supset \text{SU}(3).$$

Note, however, that the restricted symmetry arises from positivity conditions on the density operator. This positivity condition is associated with purity/mixedness of the system and thus also its entropy. This relation is only direct for two-state systems, however, since there are two positivity constraints for three-level systems and more for higher-dimensional systems, whereas the entropy for a d -state system is a single function which depends on $d - 1$ parameters.

The explicit parameterization of the rotation matrices can be obtained from the exponentiation of the algebraic elements given in the appendix. There we have expressed the algebraic elements of $\text{SO}(8)$ in terms of the algebraic elements of $\text{SU}(3)$. This embedding of the $\text{SU}(3)$ rotations into the set of $\text{SO}(8)$ rotations specifies the unitary part of the transformation. We will not provide the explicit parameterizations in each of the regimes, but do discuss an example.

As an illustrative example we consider the case where a qutrit is initially in the pure state $|2\rangle$ (let $|1\rangle, |2\rangle$, and $|3\rangle$ represent the basis states of a three-state system). The coherence vector associated with the density matrix $|2\rangle\langle 2|$ is given by $\vec{n} = (0, 0, -\sqrt{3}/2, 0, 0, 0, 0, 1/2)^T$, where the third and eighth components are associated with the diagonal generators of $SU(3)$, i.e., $|1\rangle\langle 1| - |2\rangle\langle 2|$ and $1/\sqrt{3}(|1\rangle\langle 1| + |2\rangle\langle 2| - 2|3\rangle\langle 3|)$ respectively. If the components of \vec{n} experience a rotation due to, say, the orthogonal matrix f_{28} (see Appendix B), and shrink in magnitude by the factor p , the evolved state $\vec{n}' = p \exp(-i\theta f_{28})\vec{n}$ may or may not correspond to a positive-semidefinite matrix. Since the initial state was pure, the necessary condition for positivity is $S_3 = 1 - 3\vec{n}' \cdot \vec{n}' + 2(\vec{n}' \star \vec{n}') \cdot \vec{n}' \geq 0$, see Ref. [19, 20].

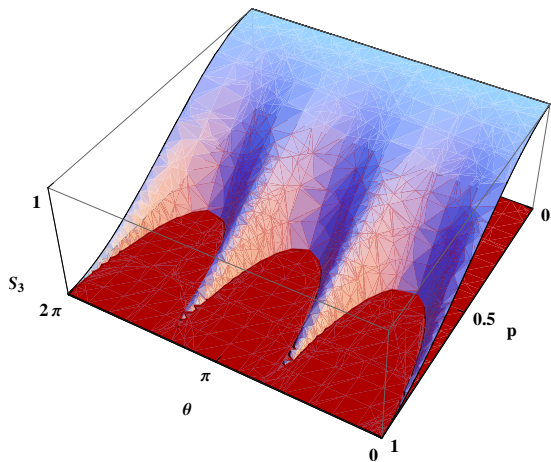


FIG. 1: (Color online) Plot showing the region of positivity when the initial pure state $|2\rangle$ is allowed to experience a uniform shrinking p and rotation of angle θ by the matrix f_{28} (see Appendix B). The quantity $S_3 = 1 - 3\vec{n}' \cdot \vec{n}' + 2(\vec{n}' \star \vec{n}') \cdot \vec{n}'$ is graphed vertically since for this initial pure state the condition $S_3 \geq 0$ is necessary and sufficient for positivity.

In Fig. 1 we plot the region of positivity as a function of the parameters θ and p . The figure shows that while $S_3 = 0$ initially ($p = 1, \theta = 0$), this quantity quickly becomes negative as the state begins to rotate without shrinking. The figure also shows how the initial state becomes positive again as θ increases to multiples of $2\pi/3$ while no shrinking has occurred. When the components of the initial coherence vector uniformly shrink such that $p \leq 1/2$ we see that any rotation about the matrix f_{28} leads to a positive density operator.

C. Arbitrary dimensions

The general case is straight-forward to infer. There is a ball at the center of the space of density operators which has a spherical symmetry in which all density operators are positive semi-definite. The radius of this ball is $1/(d-1)$. At the other extreme are the pure states which can only be acted upon by a rotation part consisting of

unitary transformations. The intermediate points have “rotation” matrices which are restricted and depend on the magnitude and direction of the polarization vector.

VIII. CONCLUDING REMARKS

We have obtained an expression for an arbitrary affine mapping of the polarization vector associated with a d -dimensional density operator in terms of the components of the operator-sum representation of the dynamical map. This provides a geometric picture, albeit somewhat abstract, via the polarization vector components which are measurable quantities. We also described a direct method for expressing the affine map in terms of the dynamical map. Some example channels were provided in order to highlight the connection between particular terms in our expression and specific operators appearing in the OSR. For the important case that the OSR components are unitary and orthogonal, we have shown that the corresponding affine components are real and have trace -1 between pairs of unequal elements. This is particularly useful for the generalization of the Bloch equations since this basis is used for error prevention methods.

To generalize the methods used for qubit maps, we discussed the possible generalization of the singular value decomposition of qubit affine maps to higher dimensional systems. We have found that a symmetry-breaking occurs due to positivity constraints. Thus we have given a relation between the physical symmetry, the purity and the entropy of the physical system. In this particular case we have provided the example of the depolarizing channel, but the generalization is also discussed. In the general case the positivity conditions are dependent upon both the magnitude and direction of the polarization vector and the rotation matrices and shrinking matrix are dependent upon one another.

While we have provided some particular expressions for physically motivated quantum channels (maps), applications of this work range from information theory to practical experiments as is the case with the OSR. One example of the utility is given in Ref. [34], where the robust simulation of quantum systems with quantum systems was considered. In addition, the preferred basis given here provides a connection between quantum error correcting codes and direct application thereof. This is due, in part, to the ability to more directly determine the input and output, and therefore the map itself, for quantum systems undergoing noisy evolution. In future work we will use this to try to provide more practical methods for determining the best combination of error prevention methods. (See [35] for a review.)

More generally, we anticipate applications of this could be more wide spread in quantum control. For example, classical control systems utilizing affine maps may be found in Jurdjevic’s book, “Geometric Control Theory,” chapter 4, [36]. This includes Linear systems reachability, and constraints. We also hope that our work will be

useful in ways not yet foreseen by us.

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Appendix A: Identities

In this appendix we provide identities that were used in the derivations of the expressions above.

The commutation relations, anti-commutation relations, and normalization of the matrices representing the basis for the Lie algebra can be summarized by the following equation:

$$\lambda_i \lambda_j = \frac{2}{d} \delta_{ij} + i f_{ijk} \lambda_k + d_{ijk} \lambda_k, \quad (\text{A1})$$

where here, and throughout this appendix, a sum over repeated indices is to be understood.

As with any Lie algebra we have the Jacobi identity:

$$f_{ilm} f_{jkl} + f_{jlm} f_{kil} + f_{klm} f_{ijl} = 0. \quad (\text{A2})$$

There is also a Jacobi-like identity,

$$f_{ilm} d_{jkl} + f_{jlm} d_{kil} + f_{klm} d_{ijl} = 0, \quad (\text{A3})$$

which was given by Macfarlane, et al. [37].

The following identities, are also provided in [37],

$$d_{iik} = 0, \quad (\text{A4})$$

$$d_{ijk} f_{ljk} = 0, \quad (\text{A5})$$

$$f_{ijk} f_{ljk} = d \delta_{il}, \quad (\text{A6})$$

$$d_{ijk} d_{ljk} = \frac{d^2 - 4}{d} \delta_{il}, \quad (\text{A7})$$

and

$$f_{ijm} f_{klm} = \frac{2}{d} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (d_{ikm} d_{jlm} - d_{jkm} d_{ilm}) \quad (\text{A8})$$

and finally

$$f_{piq} f_{qjr} f_{rkp} = - \left(\frac{d}{2} \right) f_{ijk}, \quad (\text{A9})$$

$$d_{piq} f_{qjr} f_{rkp} = - \left(\frac{d}{2} \right) d_{ijk}, \quad (\text{A10})$$

$$d_{piq} d_{qjr} f_{rkp} = \left(\frac{d^2 - 4}{2d} \right) f_{ijk}, \quad (\text{A11})$$

$$d_{piq} d_{qjr} d_{rkp} = \left(\frac{d^2 - 12}{2d} \right) d_{ijk}. \quad (\text{A12})$$

The proofs of these are fairly straight-forward and are omitted.

Appendix B: Algebra of rotation matrices

In this section we provide the basis elements for the symmetry breaking from the SO(8) symmetry to the SU(3) symmetry. A basis for the algebra of SO(8) is given by one set of matrices and the SU(3) algebraic basis elements are expressed in terms of that basis using the structure constants which provide the algebra for the adjoint representation of the group. This provides the aforementioned embedding of the group SU(3) into SO(8) by the exponentiation of the algebra. We emphasize that our choice of basis matrices, for both SU(3) and SO(8), are not unique. Our choices are simply the most common. Another choice, which is particularly useful for studying geometric phases is given in Ref. [32].

The structure constants for SU(3) are f_{ijk} and we write them in matrix form as $(f_i)_{jk}$, i.e. the matrix f_i has elements j, k . Consider the basis for a matrix representation of SO(8) given by the set of all

$$m_{jk} = i(|j\rangle\langle k| - |k\rangle\langle j|),$$

for which $j < k = 2, \dots, 8$.

For SU(3) we label the matrices 1-8, with elements j, k corresponding to the eight Gell-Mann matrices. The basis is

$$\begin{aligned}
f_{1jk} &= i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & f_{2jk} &= i \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
f_{3jk} &= i \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & f_{4jk} &= i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \end{pmatrix} \\
f_{5jk} &= i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \end{pmatrix} & f_{6jk} &= i \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix} \\
f_{7jk} &= i \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix} & f_{8jk} &= i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

These are clearly basis elements for the adjoint representation of the algebra which is a subset of the $SO(8)$ algebra.

We now make the connection between the two by writing the basis elements for $SU(3)$ in terms of the $SO(8)$ basis elements.

$$f_1 = m_{23} + \frac{1}{2}(m_{47} - m_{56}), \quad f_2 = -m_{13} + \frac{1}{2}(m_{46} + m_{57}) \quad (B1)$$

$$f_3 = m_{12} + \frac{1}{2}(m_{45} - m_{67}), \quad f_4 = -\frac{1}{2}(m_{17} + m_{26} + m_{35}) + \frac{\sqrt{3}}{2}m_{58} \quad (B2)$$

$$f_5 = \frac{1}{2}(m_{16} - m_{27} + m_{34}) - \frac{\sqrt{3}}{2}m_{48}, \quad f_6 = \frac{1}{2}(-m_{15} + m_{24} + m_{37}) + \frac{\sqrt{3}}{2}m_{78} \quad (B3)$$

$$f_7 = \frac{1}{2}(m_{14} + m_{25} - m_{36}) - \frac{\sqrt{3}}{2}m_{68}, \quad f_8 = \frac{\sqrt{3}}{2}(m_{45} + m_{67}) \quad (B4)$$

Let us now complete the basis by finding a complete set of 28 matrices which are orthogonal in the Hilbert-Schmidt sense and have f_i , $i = 1, \dots, 8$ as members. Considering the Hilbert-Schmidt to be an inner product, we can immediately write down a complete set. We will number them in no particular order. (We can always choose the matrices that they are normalized to $\text{Tr}(m_{ij}m_{kl}) = \delta_{ik}\delta_{jl}$.) Let us consider a set of matrices which span the space m_{23}, m_{47}, m_{56} when f_1 is included. The set is (for f_1)

$$f_9 = \frac{1}{2}(m_{47} + m_{56}), \quad f_{10} = -(m_{47} - m_{56}).$$

Similarly, we will group the remaining matrices: (f_2)

$$f_{11} = m_{13} + (m_{46} + m_{57}), \quad f_{12} = (m_{46} - m_{57}),$$

(f_3 and f_8)

$$f_{13} = m_{12} - (m_{45} - m_{67}),$$

(f_4)

$$f_{14} = (m_{17} + m_{26} + m_{35}) + \sqrt{3} m_{58}, \quad f_{15} = m_{17} - 2m_{26} + m_{35}, \quad f_{16} = m_{17} - m_{35}$$

(f_5)

$$f_{17} = (m_{16} - m_{27} + m_{34}) + \sqrt{3} m_{48}, \quad f_{18} = m_{16} + 2m_{27} + m_{34}, \quad f_{19} = m_{16} - m_{34}$$

(f_6)

$$f_{20} = (-m_{15} + m_{24} + m_{37}) - \sqrt{3} m_{78}, \quad f_{21} = m_{24} + 2m_{15} + m_{37}, \quad f_{22} = m_{24} - m_{37}$$

(f_7)

$$f_{23} = (m_{14} + m_{25} - m_{36}) + \sqrt{3} m_{68}, \quad f_{24} = m_{14} + 2m_{36} + m_{25}, \quad f_{25} = m_{14} - m_{25}.$$

Finally, we have the following three:

$$f_{26} = m_{18}, \quad f_{27} = m_{28}, \quad f_{28} = m_{38}.$$

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