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Entangled Matter-waves for Quantum Enhanced Sensing

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The ability to create and harness entanglement is crucial to the fields of quantum sensing and simulation, and ultracold atom-cavity systems offer pristine platforms for this undertaking. Here, we present a method for creating and controlling entanglement between solely the motional states of atoms in a cavity without the need for electronic interactions. We show this interaction arises from a general atom-cavity model, and discuss the role of the cavity frequency shift in response to atomic motion. This cavity response leads to many different squeezing interactions between the atomic momentum states. Furthermore, we show that when the atoms form a density grating, the collective motion leads to one-axis twisting, a many-body energy gap, and metrologically useful entanglement even in the presence of noise. Notably, an experiment has recently demonstrated this regime leads to an effective momentum-exchange interaction between atoms in a common cavity mode [1]. This system offers a highly tunable, many-body quantum sensor and simulator.

For the past century, our ever-growing understanding and control of quantum mechanics has changed the way we design crucial technologies such as atomic clocks [2, 3], biosensors [4], and inertial sensors [5–7]. These technologies have, in turn, set the stage for major advances in science, allowing for quantum mechanical tests of general relativity [8] and direct observation of gravitational waves [9–12]. In recent decades, it has been realized that the laws of quantum mechanics fundamentally constrain the levels of precision achievable in experiments on all scales [13], but also that these same laws may be used to benefit sensing experiments [9, 14–17]. In particular, experiments have demonstrated growing mastery over entanglement as a resource for precision measurement platforms [10, 18–20] and quantum simulation [21–24].

A foundational method to generate useful entanglement is that of spin squeezing [17] where atom-atom correlations decrease uncertainty in one phase quadrature by increasing noise in an orthogonal quadrature [16]. This allows sensing protocols to surpass the standard quantum limit (SQL) dictated by wavefunction collapse of unentangled particles, which bounds the uncertainty of an inferred parameter ϕ to $\Delta\phi^2 = 1/N$ for N constituent particles. Spin squeezing is primarily done using fine or hyperfine states of atoms [25–27], which require internal level dynamics to play a major role. Oftentimes, this means one must transfer the entanglement to another degree of freedom prior to sensing. This is often the case for atomic clocks [17] where one requires an optical clock transition, and in gravity gradiometry where one requires spatial or kinetic separation.

If one could directly create entanglement among matter-waves similar to a spin-squeezing process, it would create the opportunity for measurements of gravitational effects through matter-wave interferometry with

a true quantum advantage. So far however, generating entangled matter-waves has always involved atomic internal levels as a proxy for the necessary dynamics [19, 20]. Past theoretical works have suggested a myriad of techniques to couple spin squeezing dynamics to the momentum states, such as through the recoil due to absorption and emission of photons [28, 29], two counter-propagating modes of a ring cavity [30], or by using conditional measurements on electronic states to replicate spin squeezing on momentum states in a ring cavity [31].

In this Letter, we present a method for achieving momentum based entanglement in a manner that does not rely on interactions between electronic states of an atom. The mechanism enabling this entanglement generation is the shift in the cavity photon frequency in response to the atomic motion through the cavity. This cavity response is reflected back on the atoms as a dipole force, thereby allowing atoms to self-interact and creating the possibility for a different paradigm of entanglement enhanced matter-waves. We derive these interactions from a general atom-cavity model already realized in many experiments [32, 33]. Furthermore, we show that when the atoms form a density grating, their collective motion leads to a pseudo-spin exchange interaction that may be used to realize OAT dynamics and a many-body energy gap. In this limit, we derive the One-Axis Twisting (OAT) dynamics between momentum states only through cavity mediated exchange interactions which were recently observed in an experiment [1]. We show that these exchange interactions may be used to generate metrologically useful entanglement even in the presence of noise, which could be used to surpass the SQL in matter-wave interferometry. Notably, entanglement is generated between solely the momentum degrees of freedom, which differs from previous works [19, 28, 29, 31, 34, 35].

Self-Interaction via Cavity Frequency Modulation. —

In order to study the effects that drive entanglement, we consider an atomic cloud in an optical cavity with

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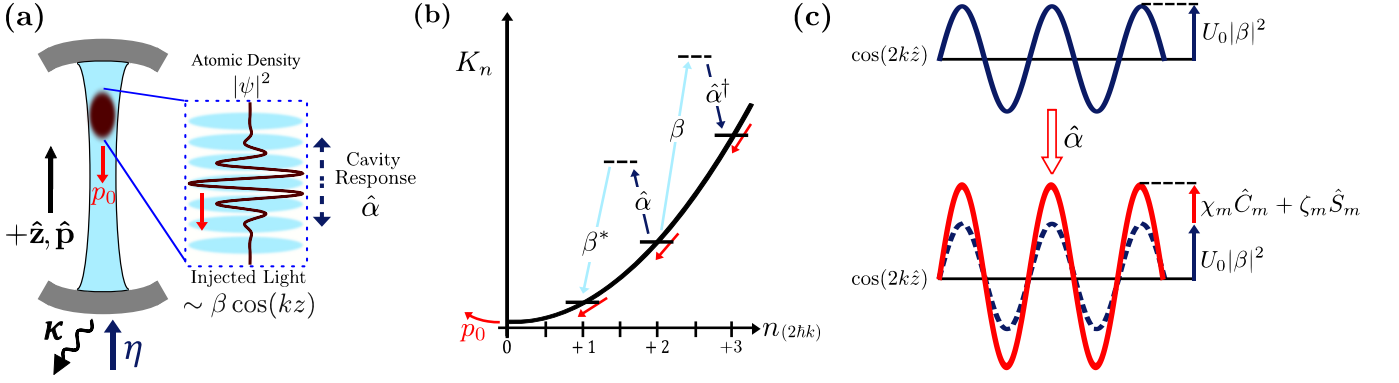


FIG. 1. Experimental scheme for direct generation of entanglement in momentum degrees of freedom for a matter-wave interferometer. (a) The atoms are well confined radially and allowed to move along z with a mean momentum of p_0 . The cavity is pumped by a coherent drive such that photons enter the cavity at a rate η and leak out at rate κ . As the atoms travel they interact with the injected field β and cause a frequency shift, which in turn leads to the cavity response $\hat{\alpha}$, given in Eq. 4. (b) The quadratic kinetic energy spectrum for momentum labeled by n , for $p = n2\hbar k$, and two example interactions with cavity photons. Two example processes are shown. Each transition labeled by the corresponding coherent field (or photon) response for absorption of a photon, β ($\hat{\alpha}$), and for emission, β^* ($\hat{\alpha}^\dagger$). These processes act to drive the cavity field response. (c) Where the atoms previously felt a standing field of strength $U_0|\beta|^2$, they now feel a standing field dependent on all other atomic momentum of strength $(U_0|\beta|^2 + \sum_m \chi_m \hat{C}_m + \zeta_m \hat{S}_m)$. The definition of $\hat{C}_m$ and $\hat{S}_m$ is given in text, above Eq. 3. The action of these operators are equivalent to the effects of a cosine and sine operator, respectively.

one relevant mode, as shown in Fig. 1(a). There are N atoms of mass m and two relevant excited and ground internal states, labeled $|e\rangle$ and $|g\rangle$ respectively, that are separated by an optical frequency ω_a . The atoms travel along the cavity with mean momentum p_0 , and are spatially dilute enough that atom-atom interactions may be ignored compared to the interactions with the common cavity mode. The cavity mode has frequency ω_c , which is far detuned from the atomic transition. The cavity mode has a wavenumber k and the atoms have a recoil frequency of $\omega_r = \hbar k^2/2m$ due to absorption and emission of cavity photons. At the anti-node, the atoms have a maximum coupling of g to the cavity. The cavity is pumped by a coherent laser with amplitude η and frequency ω_p , which is also far detuned from the atomic transition. The cavity decays at rate κ , while the atoms spontaneously emit light into free space at a rate γ .

The system's density matrix evolves under the Born-Markov master equation, $\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}]/\hbar + \sum_j \hat{\mathcal{D}}[\hat{L}_j]\hat{\rho}$, where $\hat{H}$ is the system Hamiltonian, $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator, the Lindbladian superoperator is given by $\hat{\mathcal{D}}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - (\hat{O}^\dagger\hat{O}\hat{\rho} + \hat{\rho}\hat{O}^\dagger\hat{O})/2$ summed over all jump operators $\hat{L}_j$. Prior to any simplifications, the system Hamiltonian is given by the Tavis-Cummings model [36, 37] with spatial dependence, $\hat{H}_{\text{org}} = \hat{H}_e + \hat{H}_c + \hat{H}_k$ where $\hat{H}_e = \hbar\Delta_a \sum_{\mu=1}^N \hat{\sigma}_\mu^z/2$ is the atomic energy and $\hat{H}_c = \hbar\Delta_c \hat{a}^\dagger \hat{a} + \hbar\eta (\hat{a} + \hat{a}^\dagger)$ is the cavity energy and pump contribution. Lastly, $\hat{H}_k$ is the term containing atomic motion and the atom-cavity coupling:

$$\hat{H}_k = \sum_{\mu=1}^N \frac{(\hat{p}_\mu - p_0)^2}{2m} + \hbar g \cos(k\hat{z}_\mu) (\hat{\sigma}_\mu^+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_\mu^-) \quad (1)$$

where $\hat{a}^\dagger$ ($\hat{a}$) is the photon creation (annihilation) operator of the cavity mode, $\hat{p}_\mu$ ($\hat{z}_\mu$) is the momentum (position) operator for the μ th atom along to the cavity axis, and $\hat{\sigma}_\mu^z = |e\rangle_\mu \langle e|_\mu - |g\rangle_\mu \langle g|_\mu$, $\hat{\sigma}_\mu^\pm = (\hat{\sigma}_\mu^\mp)^\dagger = |e\rangle_\mu \langle g|_\mu$ are Pauli matrices for the μ th atom. The frequencies, $\Delta_c \equiv \omega_c - \omega_p$ and $\Delta_a \equiv \omega_a - \omega_p$ are the pump frequency detuning from the cavity photon frequency and the atomic transition frequency, respectively. Decoherence is due to cavity decay and spontaneous emission: $\hat{L}_{\text{cav}} = \sqrt{\kappa}\hat{a}$, $\hat{L}_{\text{sp},\mu}(\theta) = \sqrt{\gamma} \hat{d}_\mu(\theta)\hat{\sigma}_\mu^-$, where $\hat{d}_\mu(\theta) = \sqrt{\mathcal{N}(\theta)} \exp[-ik\hat{z}_\mu \cos(\theta)]$ represents the atomic recoil following the dipole radiation pattern $\mathcal{N}(\theta)$, for θ being the angle between the emitted photon and the positive z -axis [38, 39], shown in the Supplemental Material (SM) [40].

To recover momentum-only dynamics, we assume that the atoms all start in the ground state and use the fact that the cavity photon frequency and the laser pump frequency are both far detuned from the atomic transition frequency, such that $|\omega_a - \omega_c|, |\Delta_a| \gg \sqrt{N}g$. This means that any dynamics involving the excited state are rapid compared to the ground state dynamics. As a result, we may consider the absorption of a photon and subsequent re-emission to be nearly instantaneous on the timescales of the atomic motion, thereby allowing us to adiabatically eliminate the excited state [40]. After elimination, the atom-field Hamiltonian is

$$\hat{H}_{\text{af}} = \hat{H}_c + \sum_{\mu=1}^N \left(\frac{[\hat{p}_\mu - p_0]^2}{2m} - \hbar U_0 \cos(2k\hat{z}_\mu) \hat{a}^\dagger \hat{a} \right) \quad (2)$$

where, $\Delta'_c \equiv \Delta_c - NU_0$ is now the pump detuning from the dressed cavity photon frequency, and $U_0 \equiv$

$(\Delta_a/2)g^2/(\Delta_a^2 + \gamma^2/4)$ is the effective coupling strength between the atomic motion and the cavity field. Now, spontaneous emission from atom μ leads to $\hat{L}(\theta)_\mu = \sqrt{2\gamma U_0/\Delta_a} \hat{a} \cos(k\hat{z}_\mu) \hat{d}_\mu(\theta)$. The spontaneous emission is now understood as a photon initially being absorbed from the cavity field through $\hat{a} \cos(k\hat{z}_\mu)$ at rate $2U_0/\Delta_a$, and subsequently emitted through $\hat{d}_\mu(\theta)$ at rate γ .

The relevant physical mechanism that leads to entanglement is the spatially dependent frequency shift on the cavity induced by the atomic motion. From Eq. (2), we can see that the rapid excited state dynamics cause a two-photon recoil, shown in Fig.1(b), which now contributes a frequency shift to the cavity photons. This means the frequency of the cavity photons now depends on the position of the atoms, via $(\Delta'_c - U_0 \sum_{\mu=1}^N \cos(2k\hat{z}_\mu)) \hat{a}^\dagger \hat{a}$. As the atoms move, the position dependent frequency shift oscillates between a red and blue shift on the cavity photons, whereupon this cavity response reflected back on the atoms as a dipole force.

This process couples discrete families of momentum states, where each state is by $2\hbar k$. The two-photon recoil now only couples neighboring states in these momentum families, i.e. $|p\rangle_\mu$ is only coupled to $|p \pm 2\hbar k\rangle_\mu$. As a result, we consider a generalized treatment of these momentum families through a second quantization approach [40] in order to study these dynamics. Momentum states are separated by integer multiples, $n2\hbar k$ for integer n , where the annihilation (creation) operator for an atom in the state $|n2\hbar k + p_0\rangle$ is $\hat{b}_n$ ($\hat{b}_n^\dagger$). This requires that the atoms initially have a narrow spread in momentum about p_0 [40] less than one $\hbar k$. On this discrete momentum family, the two photon recoil becomes quadratic in the mode operators between neighboring momentum states; $\hat{J}_n^\pm \equiv \hat{b}_{n\pm 1}^\dagger \hat{b}_n$ such that $\sum_\mu e^{\pm i2k\hat{z}_\mu} = \sum_n \hat{J}_n^\pm$. Furthermore, because cosine is the sum of two exponential terms we may rewrite it as a sum of raising and lowering operators; $\sum_\mu \cos(2k\hat{z}_\mu) = \sum_n \hat{C}_n$ where $\hat{C}_n \equiv (\hat{J}_n^+ + \hat{J}_{n+1}^-)/2$. Similarly, $\sum_\mu \sin(2k\hat{z}_\mu) = \sum_n \hat{S}_n$ where $\hat{S}_n \equiv (\hat{J}_n^+ - \hat{J}_{n+1}^-)/(2i)$. Lastly, the kinetic energy simplifies to $\sum_{\mu=1}^N (\hat{p}_\mu - p_0)^2 / (2\hbar m) = \sum_n K_n \hat{b}_n^\dagger \hat{b}_n$, where $K_n = 4\omega_r(n - n_0)^2$ is the kinetic energy for the n^{th} momentum state and n_0 is defined such that $p_0 = n_0 2\hbar k$. This allows us to simply express Eq. 2 as

$$\hat{H}_{\text{af}} = \hat{H}_c + \sum_n \hbar K_n \hat{b}_n^\dagger \hat{b}_n - \hbar U_0 \hat{C}_n \hat{a}^\dagger \hat{a}. \quad (3)$$

It is clear that the cavity dynamics depend only on the pump and the atomic motion. The pump injects a coherent field into the cavity, causing the cavity mode to become displaced by an amount $\beta \equiv \eta/(-\Delta'_c + i\kappa/2)$. The effects of atomic motion on the cavity mode are succinctly captured via the coupling $U_0 \sum_n \hat{C}_n \hat{a}^\dagger \hat{a}$ —which is the the cosine dependent frequency shift contributed per momentum mode n . This means the cavity response can be separated into two components, a classical field injected by the pump, β , and the photons that interact

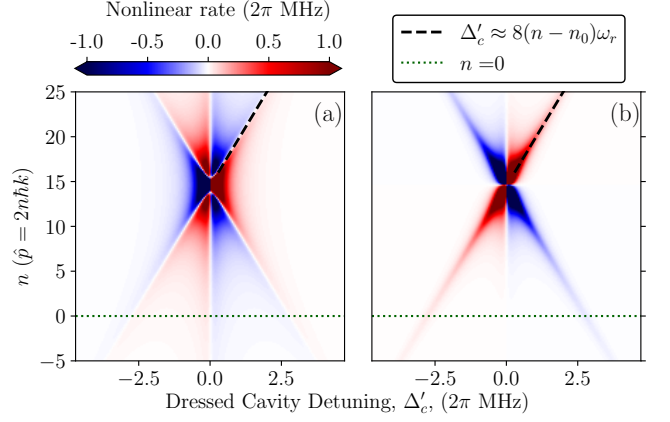


FIG. 2. (a) χ_n for all momentum states n . (b) ζ_n for all momentum states n . For both plots, $n_0 = 15$ sets the resonance. The black dashed line represents one of the resonances in χ_n for $\Delta'_c = 8(n - n_0)\omega_r$. There is a corresponding line of negative slope for the other resonance, not explicitly labeled. The green dotted line for $n = 0$, denotes the rate χ_0 , matching the one for the momentum states prepared in Ref. [1]. The cut off of 2π MHz was chosen to show contrast near $n = 0$, but not cut off the features near n_0 .

with the atoms through a small coupling frequency U_0 . We solve the equations of motion for these photons by adiabatically eliminating [40] the cavity response following Ref. [41], which yields the photon motion exclusively in terms of the operators describing the atomic motion. Beginning to end, this process yields the effective replacement $\hat{a} \rightarrow \beta + \hat{\alpha}(t)$ where $\hat{\alpha}(t)$ replaces the annihilation operator in order to describe the photon motion, and only depends on the operators for atomic motion:

$$\hat{\alpha}(t) \equiv \beta \sum_n \left(A_n^+ \hat{J}_n^+ + A_n^- \hat{J}_n^- \right) / 2, \quad (4)$$

where the amplitude $A_n^\pm \equiv U_0/(\Delta'_c + (K_{n\pm 1} - K_n) - i\kappa/2)$ depends on the required change in kinetic energy due to the two photon recoil. This yields a Hamiltonian that only involves the classical injected field, and quantum momentum degrees of freedom:

$$\begin{aligned} \frac{\hat{H}_a}{\hbar} = & \sum_n \left(K_n \hat{b}_n^\dagger \hat{b}_n - U_0 |\beta|^2 \hat{C}_n \right) \\ & - \sum_{n,m} \left[\chi_m \hat{C}_m + \zeta_m \hat{S}_m \right] \hat{C}_n, \end{aligned} \quad (5)$$

where $\chi_n \equiv \frac{U_0 |\beta|^2}{2} \Re(A_n^+ + A_{n+1}^-)$ and $\zeta_n \equiv \frac{U_0 |\beta|^2}{2} \Im(A_{n+1}^- - A_n^+)$ are the rates for the cosine and sine non-linear interactions, with $\Re(Z)$ and $\Im(Z)$ being the real and imaginary parts of Z , respectively.

The jump operators are now given by the same collective atomic recoil due to cavity decay, now $\hat{L}_c = \sqrt{\kappa} \hat{a}$, and a single atomic recoil term, which is not simply expressed in this formalism, but is treated explicitly in the

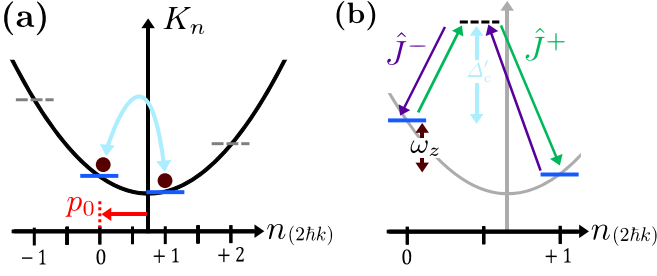


FIG. 3. (a) The kinetic energy spectrum, with the translation due to p_0 shown in red. In order to prepare the atoms into a collective density grating, a Bragg pulse is applied driving the transition from the $0\hbar k - p_0$ to the $2\hbar k - p_0$ state prior to the cavity pump being turned on, so each atom starts in $|\psi(t=0)\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. (b) The kinetic energy spectrum zoomed in to only show the two-level approximation. Here, we've shown $p_0 \approx \frac{1}{3}2\hbar k$. In the two-level approximation, the effects of the pseudo-spin raising (green) and lowering (purple) operators are shown with energy ω_z between 0 and +2, and the meta-state (dotted line) with the energy of a dressed cavity photon, Δ'_c .

SM [40]. The two decay processes are understood simply as follows: the collective atomic recoil occurs when a photon entangled to the atomic motion is lost out of the cavity, through $\hat{\alpha}$ with rate κ . The individual atomic recoil, meanwhile, occurs when an atom directly scatters a photon from the coherent field, at rate $2U_0|\beta|/\Delta_a$, out of the cavity at angle θ , at rate $\mathcal{N}(\theta)\gamma$.

The two non-linear terms, $\chi_m \hat{C}_m \hat{C}_n$ and $\zeta_m \hat{S}_m \hat{C}_n$, generate generalized squeezing dynamics [42]. The cosine-cosine non-linearity, $\chi_m \hat{C}_m \hat{C}_n$, is the modification of the optical lattice due to spatially dependent frequency shift of the cavity mode, i.e. what was previously $U_0 \hat{C}_n \hat{a}^\dagger \hat{a}$ in Eq. 3. The sine-cosine non-linearity, $\zeta_m \hat{S}_m \hat{C}_n$, is due to the fact that this frequency shift causes the cosine and sine quadrature to no-longer be perfectly orthogonal, and we note ζ_m is nearly zero unless near a resonance in χ_m . As a function of momentum and pump detuning from the dressed cavity photon frequency, the two rates χ_n and ζ_n are shown in Fig. 2 with the physical parameters given in the SM [40], where one of the two resonances is shown as a black dashed line.

In the rest of this paper we study the entanglement generation in a simple regime of Eq. (5), where we restrict to only two momentum states. There, the non-linear interactions in Eq. (5) yield OAT dynamics on the two-level manifold, which we show leads to an effective momentum exchange [1] and metrologically useful entanglement.

Formation of a Density Grating. — When the atoms occupy just the two momentum states, $n = 0$ and $n = +1$. they form a density grating. This may be achieved physically by selecting an appropriate p_0 and applying a Bragg pulse. This process is shown schematically in Fig. 3 (a), and symmetrically prepares each atom in the initial state $|\psi(t=0)\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. The choice of p_0 translates the kinetic energy spectrum in momen-

tum, shown in Fig. 3 (a), thereby allowing the momentum states $0\hbar k$ and $+2\hbar k$ to be addressed independently by tuning the dressed cavity-pump detuning, Δ'_c , as is shown in Fig. 3 (b).

By restricting to these two-levels, we are left with only two-level operators shown schematically in Fig. 3 (b). For brevity, we drop the subscripts; $\hat{J}^+ \equiv \hat{J}_0^+$ and $\hat{J}^- \equiv \hat{J}_1^-$, and $\hat{J}_z \equiv \frac{1}{2}(\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_0^\dagger \hat{b}_0)$. These two momentum states are separated by a kinetic energy $\hbar\omega_z \equiv \hbar(K_{+1} - K_0)$. The interference pattern of the two momentum states forms a density grating composed of all N atoms. The motion of this density grating modifies the dressed cavity frequency, and drives scattered photons according to $\hat{\alpha}(t) \approx \frac{\beta}{2} (A_0^+ \hat{J}^+ + A_{+1}^- \hat{J}^-)$, with the now collective amplitudes $A^\pm = U_0/(\Delta'_c \pm \omega_z - i\kappa/2)$. These substitutions give a typical OAT Hamiltonian;

$$\begin{aligned} \frac{\hat{H}_{\text{eff}}}{\hbar} &= \omega_z \hat{J}_z - U_0 |\beta|^2 \hat{J}_x - \chi \hat{J}_x^2 - \frac{\zeta}{2} (\hat{J}_x \hat{J}_y + \hat{J}_y \hat{J}_x), \\ \hat{L}_c &= \sqrt{\kappa |\beta|^2 / 4} (A_0^+ \hat{J}^+ + A_{+1}^- \hat{J}^-), \end{aligned} \quad (6)$$

where $\hat{J}_x \equiv (\hat{J}^+ + \hat{J}^-)/2$ and $\hat{J}_y \equiv (\hat{J}^+ - \hat{J}^-)/2i$ have taken the place of cosine and sine, respectively. For clarity, non-linear rates on this two-level manifold simplify to $\chi \equiv \chi_0 = U_0 |\beta|^2 \Re(A_0^+ + A_{+1}^-)/2$ and $\zeta \equiv \zeta_0 = U_0 |\beta|^2 \Im(A_{+1}^- - A_0^+)/2$. Notably, χ is the same as the spin exchange rate given in Ref. [1], and matches the slice at $n = 0$ of Fig. 2 (a). The two non-linearities in Eq. (6) arise from the cosine-cosine and cosine-sine modulation in Eq. (5), where now the motion of the atomic ensemble acts in unison rather than in many different momentum states, leading to an effective collective enhancement of N on the non-linear rates.

With these rates defined, we now discuss the conditions for energetically isolating these two momentum states from the rest of the momentum basis. We require that higher momentum states aren't excited via either Bragg scattering off the coherent field, $\hat{C}_n$, or non-linear interactions, $(\chi_m \hat{C}_m + \zeta_m \hat{S}_m) \hat{C}_n$ in Eq. (5). These restrictions ensure the dynamics remain in the two-level manifold, and give limits on the rates $U_0 |\beta|^2$, χ and ζ , in Eq. (6);

$$|U_0 |\beta|^2| \ll \frac{4kp_0}{m}, \quad \text{and} \quad N|\chi|, N|\zeta| \ll 8\omega_r. \quad (7)$$

The explicit calculation of these is given in the SM [40], and it is important to note that the inequalities in Eq. (7) only guarantees unitary dynamics won't drive higher momentum states—quantum jumps may still drive higher momentum states. The first of these terms is the condition that the atoms are well into the conduction band of the optical lattice—meaning they translate smoothly over the lattice. The second term is the condition that the non-linear effects aren't driven so strongly that they couple to higher momentum states.

Lastly, we consider the regime where the atoms are fast moving, or the pump is far detuned from the cavity. In Fig. 2 we see that either of these conditions, large

p_0 or large Δ'_c , will yield a small χ and ζ . Here, ω_z becomes large compared to all other frequencies, such as $U_0|\beta|^2$, $N\chi$, and $N\zeta$. We take an interaction picture with $R = \exp(i\omega_z \hat{J}_z t)$ and drop terms which are fast rotating. Mathematically, this is a rotating wave approximation and physically, this means the rate of single atom interactions with the optical lattice is small compared to the kinetic energy. This means the raising operators transform to $\hat{J}^\pm e^{\pm i\omega_z t}$, so that $R(\hat{J}_x \hat{J}_y + \hat{J}_y \hat{J}_x)R^\dagger \approx 0$, and $4R(\hat{J}_x)^2 R^\dagger \approx \hat{J}^+ \hat{J}^- + \hat{J}^- \hat{J}^+ = 2(\hat{J}_z)^2 - 2\hat{J}^2$. Lastly, the collective decay in the Born Markov master equation splits such that $\hat{D}[R\hat{L}_c R^\dagger] = \hat{D}[\hat{L}_c^+] + \hat{D}[\hat{L}_c^-]$, where $\hat{L}_c^\pm = \sqrt{\Gamma_\pm} \hat{J}^\pm$, with $\Gamma_\pm = \kappa|\beta|^2|A^\pm|^2/4$.

This regime represents an effective momentum exchange between the atoms, and this interaction drives the effects studied in Ref. [1], where the Hamiltonian simplifies to

$$\hat{H}_{\text{OAT}} = \hbar\chi \left[\hat{J}^2 - (\hat{J}_z)^2 \right] / 2. \quad (8)$$

In this regime, only the energy conserving interactions of $\hat{J}^+ \hat{J}^- + \hat{J}^- \hat{J}^+$ affect the unitary dynamics, where one atom exchanges its momentum state with another via a dispersive cavity photon. Through this energy conservation, a strong bias appears against effects which would take the atoms out of a symmetric configuration. This leads to a large many-body energy gap of the form $\hat{J}^2$, which is a pseudo-spin length that biases against dynamics between states with different $\hat{J}^2$ eigenvalues, i.e., between states of different permutational symmetry. This many-body energy gap leads to a further suppression of single atom effects by slowing down the effective rate of single particle decoherence effects such as Doppler broadening [1, 43].

Spin Squeezing with Decoherence. — To calculate the usefulness in the presence of noise we consider the atomic species ^{87}Rb . We use the numerical parameters given in the SM [40]. Collective emission through the cavity is the biggest barrier to achieving a quantum advantage through this momentum squeezing, unless the pump is far detuned cavity, whereupon single particle emission becomes relevant. This causes atoms to be lost from the two-level manifold at a rate $\gamma_l \equiv (\gamma U_0 |\beta|^2) / (2\Delta_a)$, which is explicitly calculated in the SM [40]. In order to study the generation of entanglement in the presence of decoherence, we calculate the Dicke spin squeezing parameter [44];

$$\xi^2 = \min_{\theta} \frac{\text{var}(\hat{J}_\theta^\perp)}{|\langle \mathbf{J} \rangle|^2 / 2}, \quad (9)$$

where $\text{var}(\hat{A}) = \langle \hat{A}^2 \rangle_\psi - \langle \hat{A} \rangle_\psi^2$ is the variance, $\hat{J}_\theta^\perp$ is the pseudo-spin tilted at angle θ and perpendicular to the average spin length, $\mathbf{J}$. Notably, $\Delta\phi^2 = \xi^2/N$ and therefore, $\xi^2 < 1$ indicates better than SQL certainty and the presence of metrologically useful entanglement [29].

We assume that the squeezing is done for short times, such that $N\Gamma_+ t$, $N\Gamma_- t \ll 1$ and collective decoherence

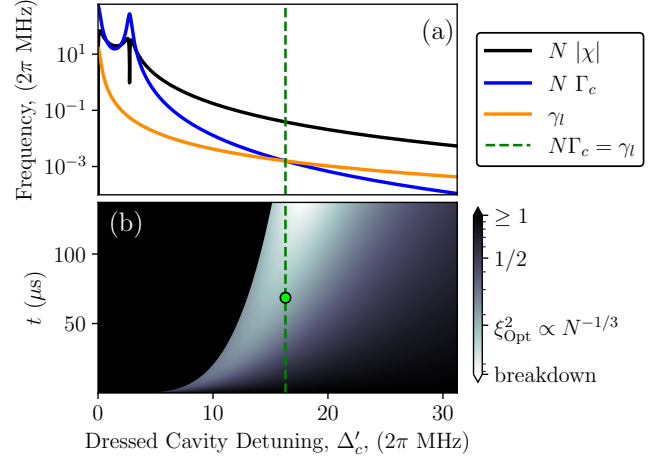


FIG. 4. (a) Collective rates that appear in the optimized spin squeezing parameter. (b) Eq. (9) as a function of squeezing time, t , and Dressed cavity detuning, Δ'_c . We note that while values of ξ^2 lower than ξ_{opt}^2 are found, these values are in the regime where the assumption that noise adds in quadrature is close to breaking down, i.e. where the assumptions that $N\Gamma_+ t$, $N\Gamma_- t \ll 1$ are close to breaking down.

doesn't drive atoms out of the two-level manifold. This assumption allows dissipative dynamics to be treated independently of the OAT dynamics, so we add uncertainty due to collective emission in quadrature to variance of the squeezed state. We calculate the mean-field dynamics [40], where we find

$$\xi^2 = \frac{1}{(N\chi t/2)^2} + t\gamma_l + Nt\Gamma_c, \quad (10)$$

where $\Gamma_c = 2(\Gamma_+ + \Gamma_-)$ is the contribution due to collective decoherence. In Fig. 4 (a) we show these four rates, for $N = 3000$. The spin squeezing parameter is minimized as a function of squeezing time, t , and detuning of the pump from the dressed cavity detuning, Δ'_c .

In order to achieve quantum enhanced sensitivity, we consider the far detuned regime where collective decay is on the order of single particle effects. By balancing the single particle decoherence and collective decoherence, we find Δ'_c such that $N\Gamma_c = \gamma_l$, shown in Fig. 4(a) and (b) by the green dashed vertical line. To highest order in N , we find an optimal squeezing time and dressed cavity detuning of

$$t_{\text{opt}} \approx 4 \left(\frac{2N^2 \Delta_a^5 \kappa^4}{U_0 \gamma^5 \eta^6} \right)^{1/3}, \quad \Delta'_{c,\text{opt}} \approx \pm \sqrt{\frac{2NU_0 \Delta_a \kappa}{\gamma}} \quad (11)$$

and an optimal spin squeezed parameter of

$$\xi_{\text{opt}}^2 \approx 3 \left(\frac{2\gamma\kappa}{NU_0 \Delta_a} \right)^{1/3}, \quad (12)$$

which corresponds to $\Delta\phi^2 \propto 1/N^{4/3}$ and is better than SQL scaling. Lastly, if using the fact that $\Delta_a^2 + \gamma^2/4 \approx$

Δ_a^2 , we have $U_0 = g^2/(2\Delta_a)$, and

$$t_{\text{opt}} \approx \frac{8\kappa\Delta_a^2}{\gamma^2\eta^2} \left(\frac{2N^2}{C_1} \right)^{1/3}, \quad \Delta'_{c,\text{opt}} \approx \pm \frac{\kappa}{2} \sqrt{NC_1}, \quad (13)$$

$$\xi_{\text{opt}}^2 \approx 3(NC_1)^{-1/3},$$

where $C_1 \equiv 4g^2/(\gamma\kappa)$ is an effective single atom cooperativity. In Fig. 4 (b), we show that for an atomic detuning of $2\pi \times 150$ MHz, and a pump rate of $\eta = 2\pi \times 10$ MHz, the optimally squeezed state is reached in a time of 150 μs .

Conclusion. — In this work, we provided a detailed derivation of the momentum exchange interaction in Ref. [1], and discussed its connection to the frequency shift of an optical cavity mode due to atomic motion. We have introduced a method which may be used to realize momentum based entanglement independent of electronic degrees of freedom with many possible squeezing interactions, shown in Eq. (5). These squeezing interactions are generated from the cavity field's response to the atomic motion. We showed that, in proper limits, the momentum states may be truncated to form an effective two-level system to the momentum states $0\hbar k - p_0$ and $+2\hbar k - p_0$, which form a density grating inside the cavity. Using this effective two-level system, we have demonstrated that entanglement is generated even in the presence of noise.

Here, we've limited the discussion of entanglement generation to an effective two-level truncation, but one could

go further to realize multi-level squeezing dynamics from Eq. (5). Of particular interest is the use of these additional states to realize multi-level dynamics via the use of additional dressing lasers to drive higher order interactions. Furthermore, these momentum states provide a promising opportunity for realizing spin squeezing via two-axis counter twisting effects, which can be achieved by periodically modulating the amplitude injected field at a parametric resonance [45] or injecting a second pump [46]. Future work could include using measurements on the light field exiting the cavity. This light is highly entangled to the atoms, and measurements would, in principle, allow one to induce a back-action on the atomic cloud. This creates the opportunity for quantum non-demolition measurements similar to those presented in Ref. [19], or even for extending methods of continuous phase tracking [47] directly to matter-wave interferometry.

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