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Phys. Rev. A **109**, L061503 — Published 27 June 2024

DOI: [10.1103/PhysRevA.109.L061503](https://doi.org/10.1103/PhysRevA.109.L061503)

Topological Winding Guaranteed Coherent Orthogonal Scattering

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(Dated: June 11, 2024)

Coherent control has enabled various novel phenomena in wave scattering. We introduce a new effect called coherent orthogonal scattering, where the output wave becomes orthogonal to the reference output state without scatterers. This effect leads to a unity extinction coefficient and complete mode conversion. We examine the conditions for this effect and reveal its topological nature by relating it to the indivisibility between the dimension and the winding number of scattering submatrices. These findings deepen our understanding of topological scattering phenomena.

Understanding wave scattering is crucial for various applications, such as imaging [1–12], sensing [13, 14], energy harvesting [15–31], and optical computing [32–36]. Coherent control of wave scattering [37–39] has been a significant advancement, enabling the tailoring of scattering behaviors by modifying the input wave profile. This approach has unveiled unique phenomena, particularly coherent perfect absorption [40–49]—the complete absorption of a tailored input wave profile—and related effects [50–52], including coherent virtual absorption [53–55] and reflectionless scattering modes [56–59].

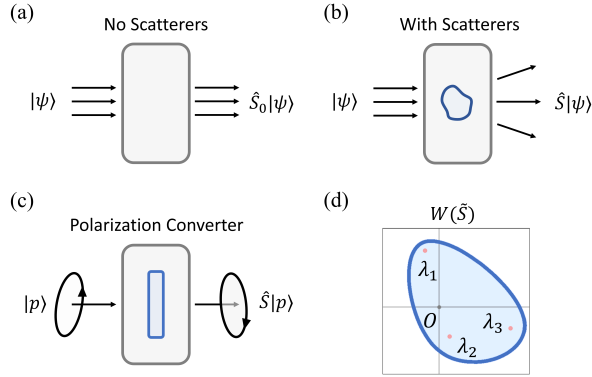


FIG. 1. The concept of coherent orthogonal scattering (COS). (a,b) A standard scattering setup with an input state $|\psi\rangle$. (a) In the absence of scatterers, the output state is $|\xi_0\rangle = \hat{S}_0 |\psi\rangle$. (b) In the presence of scatterers, the output state becomes $|\xi\rangle = \hat{S} |\psi\rangle$. COS occurs when $\langle \xi_0 | \xi \rangle = 0$. (c) In polarization optics, COS corresponds to complete polarization conversion. (d) The numerical range $W(\tilde{S})$ of a scattering submatrix $\tilde{S}$ is convex and contains its eigenvalues. COS occurs iff $0 \in W(\tilde{S})$.

In this Letter, we introduce a new phenomenon called *coherent orthogonal scattering*, which refers to the complete extinction of the output state with respect to the reference output state in the absence of scatterers. Consider a standard scattering setup [60, 61]. Let H be the Hilbert space of waves. We input a normalized state $|\psi\rangle$ into a system. In the absence of scatterers (Fig. 1a), the output state, termed as the reference state, is

$$|\xi_0\rangle = \hat{S}_0 |\psi\rangle, \quad (1)$$

where $\hat{S}_0$ is a unitary background scattering operator. In the presence of scatterers (Fig. 1b), the output state is

$$|\xi\rangle = \hat{S} |\psi\rangle, \quad (2)$$

where $\hat{S}$ is the total scattering operator which can be non-unitary. By orthogonal projection with respect to $|\xi_0\rangle$, the output state $|\xi\rangle$ can be uniquely decomposed as

$$|\xi\rangle = \alpha |\xi_0\rangle + \beta |\xi_0^\perp\rangle, \quad (3)$$

where $|\xi_0^\perp\rangle$ is a unit vector orthogonal to $|\xi_0\rangle$. The absorption coefficient is defined as

$$\chi_a = 1 - |\alpha|^2 - |\beta|^2 = 1 - \langle \psi | \hat{S}^\dagger \hat{S} | \psi \rangle. \quad (4)$$

The extinction coefficient is defined as

$$\chi_e = 1 - |\alpha|^2 = 1 - |\langle \xi_0 | \xi \rangle|^2 = 1 - |\langle \psi | \hat{S}_0^\dagger \hat{S} | \psi \rangle|^2, \quad (5)$$

which measures the deviation of $|\xi\rangle$ from $|\xi_0\rangle$. [62] For passive scatterers, χ_a and χ_e lie between 0 and 1.

In coherent control experiments, one varies $|\psi\rangle$ within an accessible input subspace $H_i \subseteq H$ to achieve desired outcomes [63–65]. Coherent perfect absorption refers to the case when $\chi_a = 1$. In contrast, coherent orthogonal scattering occurs when $\chi_e = 1$, which is equivalent to the output state being orthogonal to the reference state:

$$\langle \xi_0 | \xi \rangle = \langle \psi | \hat{S}_0^\dagger \hat{S} | \psi \rangle = 0. \quad (6)$$

For example, in polarization optics, H_i is the 2D polarization space, and coherent orthogonal scattering corresponds to complete polarization conversion (Fig. 1c). A related yet distinct effect, coherent perfect extinction [52], refers to the case when the output state is orthogonal to the reference output subspace $H_o = \hat{S}_0 H_i$:

$$\forall |\phi\rangle \in H_i, \quad \langle \phi | \hat{S}_0^\dagger \hat{S} | \psi \rangle = 0. \quad (7)$$

We can express Eqs. (6) and (7) in matrix forms. Suppose H_i is m -dimensional. Using a set of orthonormal bases, the compression of $\hat{S}_0^\dagger \hat{S}$ to H_i is represented by an $m \times m$ scattering submatrix $\tilde{S}$ [66]. (In the absence of scatterers, $\hat{S} = \hat{S}_0$, thus $\tilde{S} = I$.) Then Eq. (6) becomes

$$\exists \tilde{\mathbf{a}} \in \mathbb{C}^m \setminus \{\mathbf{0}\}, \quad \tilde{\mathbf{a}}^\dagger \tilde{S} \tilde{\mathbf{a}} = 0, \quad (8)$$

while Eq. (7) becomes

$$\exists \tilde{\mathbf{a}} \in \mathbb{C}^m \setminus \{\mathbf{0}\}, \quad \tilde{S} \tilde{\mathbf{a}} = \mathbf{0}. \quad (9)$$

The physical significance of coherent orthogonal scattering is two-fold. First, it corresponds to a unity extinction coefficient, a fundamental observable in scattering experiments with applications in diverse areas, including molecular spectroscopy [67–70], acoustic imaging [71], atmospheric science [72, 73], and astronomy [50, 74]. Understanding the conditions for achieving a unity extinction coefficient is important. Second, coherent orthogonal scattering enables complete mode conversion [75–82], such as complete polarization conversion [83–95], which plays a crucial role in various applications, including communications [79, 96–98], sensing [99–103], and quantum technology [104–107]. A key challenge in mode converter design is determining whether complete mode conversion can be achieved by adjusting specific design parameters, and efficiently identifying the parameter values that result in complete mode conversion [84, 91, 97, 108]. This again requires a deeper understanding of the conditions for coherent orthogonal scattering.

In this work, we investigate the conditions for coherent orthogonal scattering. A useful mathematical concept for this purpose is the *numerical range* of $\tilde{S}$ [109, 110]:

$$W(\tilde{S}) := \{\mathbf{x}^\dagger \tilde{S} \mathbf{x} : \mathbf{x} \in \mathbb{C}^m, \mathbf{x}^\dagger \mathbf{x} = 1\}. \quad (10)$$

$W(\tilde{S})$ is a compact *convex* subset of $\mathbb{C}$ that contains all the eigenvalues of $\tilde{S}$ [111, 112]. (See Supplemental Material (SM) [113], Sec. I for more details.) Figure 1d shows $W(\tilde{S})$ for an $\tilde{S} \in M_3$. The *Crawford number* of $\tilde{S}$ is the distance of $W(\tilde{S})$ from the origin [114–116]:

$$c(\tilde{S}) := \min\{|z| : z \in W(\tilde{S})\}. \quad (11)$$

From Eq. (8), coherent orthogonal scattering occurs iff

$$0 \in W(\tilde{S}), \quad \text{i.e.,} \quad c(\tilde{S}) = 0. \quad (12)$$

The condition (12) can be numerically checked using an approximate algorithm [109] (see SM, Sec. I D), but it cannot be determined analytically using the entries of $\tilde{S}$ when $m \geq 4$ [117, 118]. In contrast, coherent perfect extinction occurs iff [52]

$$\det \tilde{S} = 0, \quad (13)$$

which can be easily checked using the entries of $\tilde{S}$. [119] Hence, the key challenge is to find a simple analytical criterion for coherent orthogonal scattering.

Here, we present a simple sufficient analytical criterion for coherent orthogonal scattering. Our idea is to examine not a single, but a *loop* of scattering submatrices (Fig. 2a). Consider a continuous map

$$\tilde{S} : [0, 1] \rightarrow M_m, \quad \tau \mapsto \tilde{S}(\tau), \quad (14)$$

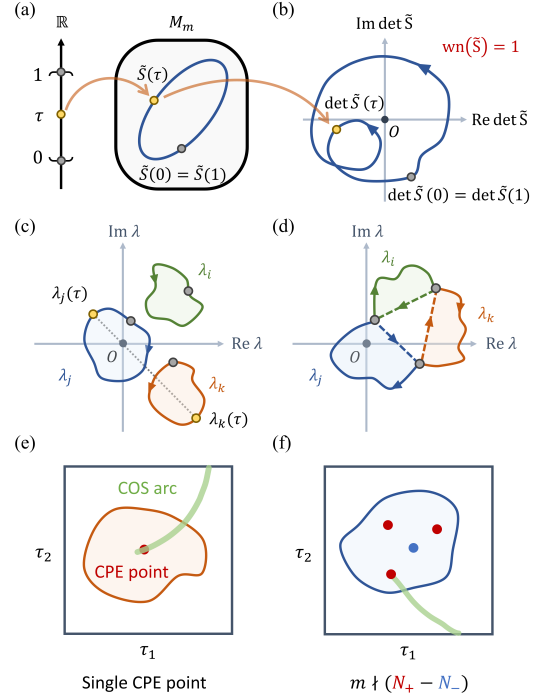


FIG. 2. Topological winding guaranteed coherent orthogonal scattering. (a) A loop of $\tilde{S}(\tau) \in M_m$. (b) $\det \tilde{S}(\tau)$ has a winding number $\text{wn}(\tilde{S})$. If m does not divide $\text{wn}(\tilde{S})$, denoted as $m \nmid \text{wn}(\tilde{S})$, coherent orthogonal scattering must occur for some $\tilde{S}(\tau_0)$. (c, d) The path of eigenvalues when (c) $\nu(\tilde{S}) = \text{id}$, (d) $\nu(\tilde{S}) \neq \text{id}$. (e, f) CPE points and COS arcs in a 2D parameter space. (e) A topological COS arc exists associated with each CPE point. (f) A topological COS arc exists across a simple loop that surrounds multiple CPE points if $m \nmid (N_+ - N_-)$. COS arcs within the loop are not shown.

where $\tilde{S}(0) = \tilde{S}(1)$. We ask whether

$$\exists \tau_0 \in [0, 1], \quad 0 \in W(\tilde{S}(\tau_0)), \quad (15)$$

motivated by the question whether coherent orthogonal scattering can be achieved by tuning a certain parameter.

Our criterion is as follows. We calculate $\det \tilde{S}(\tau)$ and check whether

$$\exists \tau_0 \in [0, 1], \quad \det \tilde{S}(\tau_0) = 0. \quad (16)$$

If true, then $\tilde{S}(\tau_0)$ exhibits coherent perfect extinction, thus also coherent orthogonal scattering. If false, then $\det \tilde{S}$ traces a closed path in $\mathbb{C} \setminus \{0\}$ (Fig. 2b), which has a well-defined winding number around the origin [52, 120]:

$$\text{wn}(\tilde{S}) := \varphi(1) - \varphi(0) \in \mathbb{Z}, \quad (17)$$

where $2\pi\varphi(\tau)$ is the continuous polar angle of $\det \tilde{S}(\tau)$. If m does not divide $\text{wn}(\tilde{S})$, denoted as $m \nmid \text{wn}(\tilde{S})$, then

$$\exists \tau_0 \in [0, 1], \quad 0 \in W(\tilde{S}(\tau_0)). \quad (18)$$

If m divides $\text{wn}(\tilde{S})$, denoted as $m \mid \text{wn}(\tilde{S})$ [121], such τ_0 may or may not exist. See SM, Sec. II for examples.

The criterion (18) is our main result. It can be proved using a theorem established in Refs. [122, 123]. Here, we outline the essential ideas. Consider the m eigenvalues of $\tilde{S}(\tau)$, denoted as $\lambda_1(\tau)$, ..., $\lambda_m(\tau)$. Since $\tilde{S}(\tau)$ is a continuous map from $[0, 1]$ to GL_m , we can choose all $\lambda_j(\tau)$'s to be continuous functions from $[0, 1]$ to $\mathbb{C} \setminus \{0\}$ [124, 125] (see SM, Sec. III.). Since $\tilde{S}(0) = \tilde{S}(1)$, the set of $\lambda_j(1)$'s must coincide with the $\lambda_j(0)$'s, up to a permutation:

$$\lambda_j(1) = \lambda_{\nu_j}(0); \quad \nu(\tilde{S}) = \begin{pmatrix} 1 & 2 & \dots & m \\ \nu_1 & \nu_2 & \dots & \nu_m \end{pmatrix}. \quad (19)$$

For simplicity, let us first consider the special case when the permutation is the identity ($\nu(\tilde{S}) = \text{id}$). Then, each $\lambda_j(\tau)$ traces out a loop in the punctuated complex plane with a well-defined winding number about the origin:

$$\text{wn}(\lambda_j) \in \mathbb{Z}, \quad j = 1, \dots, m. \quad (20)$$

There is a simple yet important relation:

$$\text{wn}(\tilde{S}) = \sum_{j=1}^m \text{wn}(\lambda_j), \quad (21)$$

which follows from the fact that [120]

$$\det \tilde{S}(\tau) = \prod_{i=1}^m \lambda_i(\tau). \quad (22)$$

Combining Eq. (21) with our premise $m \nmid \text{wn}(\tilde{S})$, there must exist a pair of eigenvalue loops, say λ_j and λ_k , that have different winding numbers. As τ runs over $[0, 1]$, the line segment $\overline{\lambda_j(\tau)\lambda_k(\tau)}$ connecting these two eigenvalues must sweep across the origin at least once, implying that

$$\exists \tau_0 \in [0, 1], \quad 0 \in \overline{\lambda_j(\tau_0)\lambda_k(\tau_0)} \subseteq W(\tilde{S}(\tau_0)). \quad (23)$$

Here we used the fact that the numerical range contains the convex hull of the eigenvalues [111, 112]. This completes the proof for the $\nu(\tilde{S}) = \text{id}$ case. To prove the $\nu(\tilde{S}) \neq \text{id}$ cases (Fig. 2d), we construct $\tilde{S}' : [0, 1] \rightarrow \text{GL}_m$ that undoes the permutation $\nu(\tilde{S})$ along the line segments $\overline{\lambda_{\nu_j}(0)\lambda_j(0)}$. The concatenation of $\tilde{S}$ and $\tilde{S}'$ becomes a loop with the identity permutation, allowing the previous analysis to complete the proof [123].

With the mathematical groundwork established, we now discuss the physical implications of criterion (18). First, our criterion establishes a deeper connection between coherent orthogonal scattering and coherent perfect extinction. Consider an $\tilde{S} \in M_m$ with $m \geq 2$ that depends continuously on two parameters $\tau = (\tau_1, \tau_2) \in \Omega$, where Ω is a compact and simply connected subset of $\mathbb{R}^2$. Coherent perfect extinction generically occurs in Ω at isolated points, known as CPE points [52]. Along a simple closed curve that encloses a single generic CPE point (Fig. 2e),

$$\text{wn}(\tilde{S}) = \pm 1. \quad (24)$$

Since $m \nmid \pm 1$, according to (18), coherent orthogonal scattering must occur somewhere along the loop. We can deform the loop continuously and deduce that coherent orthogonal scattering must occur along at least one arc, referred to as a COS arc. Such a COS arc is topologically protected. We can extend this analysis to the case of multiple CPE points (Fig. 2f). Suppose there are N_+ and N_- CPE points with winding numbers $+1$ and -1 , respectively, enclosed by a simple closed curve. A COS arc must exist across the loop if

$$m \nmid (N_+ - N_-). \quad (25)$$

Second, we investigate the topological winding of scattering submatrices due to a resonance. We consider a single-resonance scattering submatrix [126–128]

$$\tilde{S}(\omega) = \tilde{C} + \frac{\tilde{\mathbf{d}}\tilde{\mathbf{\kappa}}^T}{-i(\omega - \omega_0) + \gamma} \in M_m, \quad (26)$$

where ω_0 and γ are the resonant frequency and decay rate, respectively. The column vectors $\tilde{\mathbf{\kappa}}$ and $\tilde{\mathbf{d}}$ represent the coupling rates between the resonator and the input and output waves in the ports, respectively. $\tilde{C} \in M_m$ describes the background scattering. We can prove that $\det \tilde{S}(\omega)$ traces out a circle in the complex plane with

$$\text{wn}(\tilde{S}) = \begin{cases} 0, & \text{Re } \tilde{\rho} < \frac{1}{2}, \\ 1, & \text{Re } \tilde{\rho} > \frac{1}{2}, \end{cases} \quad \tilde{\rho} := -\frac{\tilde{\mathbf{\kappa}}^T \tilde{C}^{-1} \tilde{\mathbf{d}}}{2\gamma} \in \mathbb{C}. \quad (27)$$

When $\text{Re } \tilde{\rho} = \frac{1}{2}$, the circle passes through the origin, and $\text{wn}(\tilde{S})$ is undefined. (See the proof in SM, Sec. IV–VI.) Combining (27) and (18), we conclude that when $m \geq 2$,

$$\text{Re } \tilde{\rho} \geq \frac{1}{2} \implies \exists \omega_c \in (-\infty, \infty), \quad 0 \in W(\tilde{S}(\omega_c)). \quad (28)$$

We illustrate our theory with two numerical examples. In both cases, coherent orthogonal scattering corresponds to complete polarization conversion in the transmission. The effect of coherent orthogonal scattering can also be demonstrated in other (e.g., spatial) degrees of freedom. Our theoretical analysis applies to these cases as well.

The first example is a 1D dielectric grating taken from Ref. [129]. The structure has a periodicity a . Each unit cell has a central rod of width $w = 0.45a$ and height $h = 1.5a$ with a dielectric constant $\varepsilon_0 = 2.1025$, and two additional pieces of width $(a - w)/2 = 0.275a$ and height $h_d = 0.1a$ with a dielectric constant $\varepsilon_d = 1.21$ (Fig. 3a). This grating supports a band of p -polarized guided resonances near $\omega = 0.8 \times 2\pi c/a$ (Fig. 3b). We consider an incident plane wave with an incident angle $\theta = \pm 5^\circ$, and calculate the 2×2 transmission matrix

$$\tilde{S}(\omega) = \begin{pmatrix} t_{pp} & 0 \\ 0 & t_{ss} \end{pmatrix}, \quad (29)$$

which is diagonal due to the xz mirror symmetry. Consequently, we have $\lambda_1 = t_{pp}$, $\lambda_2 = t_{ss}$, $\det \tilde{S} = t_{pp} t_{ss}$, and $W(\tilde{S})$ is the line segment connecting t_{pp} and t_{ss} .

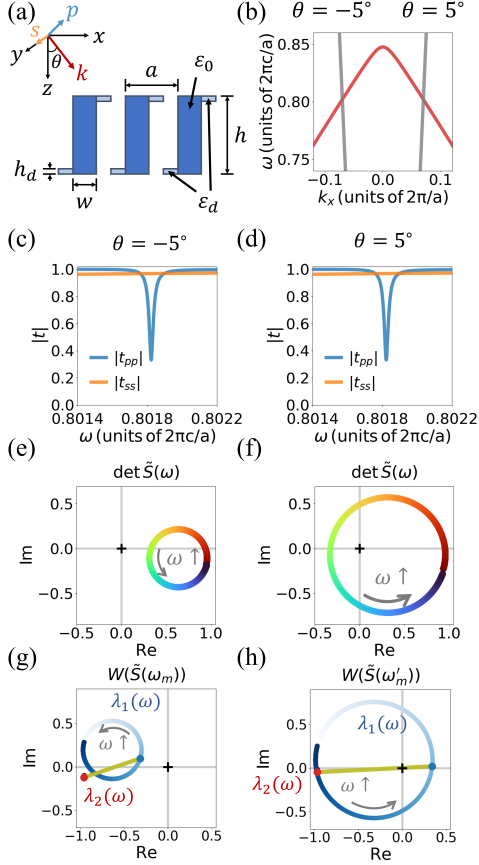


FIG. 3. A grating example. (a) Geometry. (b) Band structure. (c,e,g) Results for $\theta = -5^\circ$. (c) $|t_{pp}|(\omega)$ and $|t_{ss}|(\omega)$. (e) $\det \tilde{S}(\omega)$. (g) Trajectories of $\lambda_1(\omega)$ and $\lambda_2(\omega)$. The red and blue dots denote $\lambda_1(\omega_m)$ and $\lambda_2(\omega_m)$, respectively. The line segment denotes $W(\tilde{S})$ at $\omega_m = 0.80182 \times 2\pi c/a$. (d,f,h) Corresponding results for $\theta = 5^\circ$. In (h), $\omega'_m = 0.80220 \times 2\pi c/a$.

Figs. 3(c,e,g) present the results for $\theta = -5^\circ$. Fig. 3c shows that $|t_{pp}(\omega)|$ exhibits a single guided resonance [130] while $|t_{ss}(\omega)|$ almost remains constant. We fit the complex spectra with Eq. (26) and obtain $\tilde{\rho} = 0.334 - 0.009i$. Since $\text{Re } \tilde{\rho} < \frac{1}{2}$, Eq. (27) predicts that $\text{wn}(\tilde{S}) = 0$. Indeed, Fig. 3e shows that $\det \tilde{S}(\omega)$ traces out a circle that does not enclose the origin. Fig. 3g shows that $\lambda_1(\omega)$ forms a circle that does not enclose the origin, while $\lambda_2(\omega)$ almost remains constant. Fig. 3g also depicts $W(\tilde{S})$ at $\omega_m = 0.80182 \times 2\pi c/a$, where $c(\tilde{S})$ reaches minimum. $W(\tilde{S}(\omega_m))$ forms a line segment connecting $\lambda_1(\omega_m)$ and $\lambda_2(\omega_m)$ with $0 \notin W(\tilde{S}(\omega_m))$. Thus, coherent orthogonal scattering does not occur in the entire frequency range.

Figs. 3(d,f,h) present the results for $\theta = 5^\circ$. The transmission amplitude spectra in Fig. 3d are similar to those in Fig. 3c; the differences are in the phases. We fit the complex spectra and obtain $\tilde{\rho} = 0.666 - 0.009i$. Since $\text{Re } \tilde{\rho} > \frac{1}{2}$, Eq. (27) states that $\text{wn}(\tilde{S}) = 1$, and (28) predicts that coherent orthogonal scattering must occur

at some frequency. Indeed, Fig. 3f shows that $\det \tilde{S}(\omega)$ traces out a circle that winds around the origin once counterclockwise. Fig. 3h shows that $\lambda_1(\omega)$ winds around the origin once, while $\lambda_2(\omega)$ almost remains constant. Fig. 3h depicts $W(\tilde{S})$ at $\omega'_m = 0.80220 \times 2\pi c/a$, which is the line segment connecting $\lambda_1(\omega'_m)$ and $\lambda_2(\omega'_m)$ with $0 \in W(\tilde{S}(\omega'_m))$. Explicit calculation shows that coherent orthogonal scattering occurs at ω'_m for the incident polarization $(0.86\hat{\rho} + 0.51e^{i\eta}\hat{s})$ with η an arbitrary phase.

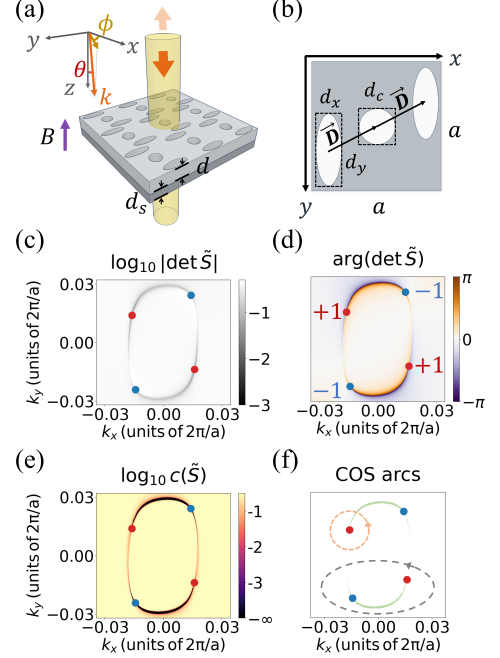


FIG. 4. A photonic crystal slab example. (a) Geometry. (b) Unit cell. (c-f) Results for $\tilde{S}(k_x, k_y)$ at $\omega = 0.386 \times 2\pi c/a$. (c) $\log_{10}|\det \tilde{S}(k_x, k_y)|$. (d) $\arg[\det \tilde{S}(k_x, k_y)]$. (e) $\log_{10} c[\tilde{S}(k_x, k_y)]$. (f) The green region: the COS arcs. The red and blue dots: CPE points with winding numbers +1 and -1, respectively.

The second example is a 2D magneto-optical photonic crystal slab. The structure consists of two layers (Fig. 4a). The first layer is a photonic crystal slab with a lattice constant a and a thickness $d = 0.3a$. It has a square lattice, with each unit cell having a circular hole at the center with a diameter $d_c = 0.28a$, and two elliptical holes displaced by $\pm \vec{D} = \pm(0.375a, -0.18a)$ from the center, where the major axis is $d_y = 0.56a$ and the minor axis is $d_x = 0.20a$ (Fig. 4b). The second layer is a uniform slab with a thickness $d_s = 0.2a$. Both slabs are made of a magneto-optical material with a relative permittivity tensor

$$\varepsilon = \begin{pmatrix} \varepsilon_r & i\varepsilon_a & 0 \\ -i\varepsilon_a & \varepsilon_r & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix}, \quad (30)$$

where $\varepsilon_r = 12.0$, $\varepsilon_a = 5.0$. We consider a plane wave with a fixed frequency $\omega = 0.386 \times 2\pi c/a$ incident from

the top side, and calculate the 2×2 transmission matrix

$$\tilde{S}(k_x, k_y) = \begin{pmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{pmatrix}, \quad (31)$$

where (k_x, k_y) are the transverse wavevector components. Unlike that in Eq. (29), here $\tilde{S}$ is a generic 2×2 matrix. In general, $W(\tilde{S})$ forms an elliptical disk with foci at λ_1 and λ_2 (see SM, Sec. I C.)

Fig. 4c plots

$$\log_{10} |\det \tilde{S}(k_x, k_y)|, \quad (32)$$

which exhibits sharp variations near the isofrequency contour of the guided resonances. There are four CPE points where $\det \tilde{S} = 0$. Fig. 4d shows that $\det \tilde{S}(k_x, k_y)$ exhibits a 2π (-2π) phase winding around each red (blue) CPE point, indicating a winding number of $+1$ (-1). The two red (blue) CPE points are related by the C_2 symmetry. Fig. 4e and 4f show that $c[S(k_x, k_y)] = 0$ occurs in two disjoint COS arcs. Each COS arc occupies a two-dimensional domain that contains one red and one blue CPE point in its interior. These COS arcs are topologically protected. For instance, the orange loop encircles a single CPE point, so there must exist a COS arc that intersects the orange loop. In contrast, the grey loop encircles two CPE points with a total winding number $N_+ - N_- = 1 - 1 = 0$. Since $2 \mid 0$, a COS arc may or may not cross the grey loop. Indeed, in this case, the COS arc is contained entirely inside the loop. (See SM, Sec. VII for additional numerical results.) These numerical examples confirm our theoretical predictions.

In conclusion, we have introduced the phenomenon of coherent orthogonal scattering, which occurs when the output wave becomes orthogonal to the reference output state in the absence of scatterers. This effect leads to a unity extinction coefficient and complete mode conversion. We have revealed the topological nature of this effect by establishing a connection between the winding number of scattering submatrices and the existence of coherent orthogonal scattering. We have discovered topologically protected coherent orthogonal scattering arcs in a two-dimensional parameter space, which connect coherent perfect extinction points with opposite winding numbers. These findings provide a deeper understanding of the interplay between topology and scattering phenomena, paving the way for novel approaches to manipulate coherent wave interactions in various physical systems.

This work is funded by a Simons Investigator in Physics grant from the Simons Foundation (Grant No. 827065), by a Multidisciplinary University Research Initiative (MURI) grant from the U. S. Office of Naval Research (Grant No. N00014-20-1-2450), and by U. S. Department of Energy (Grant No. DE-FG02-07ER46426).

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