Basic elements for simulations of standard-model physics with quantum annealers: Multigrid and clock states

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Basic Elements for Simulations of Standard Model Physics with Quantum Annealers: Multigrid and Clock States

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We explore the potential of D-Wave's quantum annealers for computing some of the basic components required for quantum simulations of Standard Model physics. By implementing a basic multigrid (including "zooming") and specializing Feynman-clock algorithms, D-Wave's `Advantage` is used to study harmonic and anharmonic oscillators relevant for lattice scalar field theories and effective field theories, the time evolution of a single plaquette of SU(3) Yang-Mills lattice gauge field theory, and the dynamics of flavor entanglement in four neutrino systems.

I. INTRODUCTION

Simulations of the dynamics of quantum matter, from neutron stars to materials, that are beyond the reach of classical computation, are expected to become possible through continued advances in quantum computation. While universal quantum computation [1–8] is essential in this quest to simulate Standard Model physics, the near-term devices that define the NISQ era [9], without high-fidelity qubits and error correction, will be challenged to provide results that can be quantitatively compared with experiment (see, for example, Refs. [10–12]). Much of the current research in this area is performed on gate-based quantum computers, and the alternative, adiabatic quantum computing [13–16], has not been explored with as much detail. Applications for such devices, such as D-Wave's quantum annealers (QAs) [17], are optimization [18–23], high energy physics [24, 25], machine learning [26–38], spin systems [39–49], quantum chemistry [50–55], biology [56–59], finance [60–62], graph [63–65], multivariate [66], integer [67] and linear equations [68], and factorization problems [69–74], and more, but a quantum advantage for scientific applications remains to be demonstrated.

The first steps toward simulating quantum field theories using QAs have been taken by finding the ground state of modest SU(2) plaquette systems and the time evolution of these systems [75] using the Feynman-clock algorithm [76]. Given that, modulo emergent fine-tunings, many Standard Model systems of interest are gapped, with finite correlation lengths, quantum circuits for universal quantum computers are expected to be able to have localized control structures for which domain decomposition will be effective. This suggests that QAs may provide efficient pre-conditioners for preparing parameterizations of ground states and excited states for universal quantum computers (e.g., fast-forwarding time evolution [77]) in the future. This, of course, remains to be demonstrated.

Using D-Wave's QAs (and simulators), we explore building blocks that are required for quantum simulations of Standard Model physics and its descendant low-energy effective field theories. Building upon the works of Refs. [68, 75], a "zooming" algorithm is used to converge coefficients of the basis states defining annealing problem instances for ground states. Second, we use a simple multigrid procedure that iteratively employs course grids to provide starting conditions for finer grids to converge wavefunctions. Third, we generalize a previously implemented Feynman clock algorithm [75] to arbitrary Hermitian matrices. These algorithms are used to simulate the harmonic (HO) and anharmonic oscillators (aHO), the time evolution of the SU(3) Yang-Mills plaquette [78], and neutrino flavor evolution [79], which have been previously simulated using IBM's superconducting quantum computers.

II. MAPPING A HAMILTONIAN ONTO A QUBO PROBLEM

In order to find the ground-state energy and wavefunction of a given Hamiltonian using D-Wave's QAs, a minimization problem is mapped onto a QUBO (quadratic unconstrained binary optimization) problem, $f_Q(q) = \sum_{ij} Q_{ij} q_i q_j$, where q_i are binary variables. Following techniques and protocols for using D-Wave's systems [80] and specific methods that users have developed [53, 75], an objective function of the form

$$F = \langle\Psi|\hat{\mathcal{H}}|\Psi\rangle - \eta\langle\Psi|\Psi\rangle \quad , \tag{1}$$

is minimized, where η is a parameter that is tuned to avoid the null solution ($\langle\Psi|\Psi\rangle = 0$). Expanding or approximating the wavefunction $|\Psi\rangle$ in a finite dimensional orthonormal basis $|\psi_\alpha\rangle$, $|\Psi\rangle = \sum_\alpha^{n_s} a_\alpha|\psi_\alpha\rangle$, with a_α being real numbers, F can be written as

$$\begin{aligned} F &= \sum_{\alpha\beta}^{n_s} a_\alpha a_\beta (\langle\psi_\alpha|\hat{\mathcal{H}}|\psi_\beta\rangle - \eta\langle\psi_\alpha|\psi_\beta\rangle) \\ &= \sum_{\alpha\beta}^{n_s} a_\alpha a_\beta (\hat{\mathcal{H}}_{\alpha\beta} - \eta\delta_{\alpha\beta}) = \sum_{\alpha\beta}^{n_s} a_\alpha a_\beta h_{\alpha\beta} \quad . \end{aligned} \tag{2}$$

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Mapping the minimization of F onto a QUBO problem appropriate for solution using an annealer requires expressing a_α in terms of binary variables. Following previous works, the fixed-point representation [53] of each a_α in terms of K bits q_i^α is used,

$$a_\alpha = -q_K^\alpha + \sum_{i=1}^{K-1} \frac{q_i^\alpha}{2^{K-i}} \quad , \tag{3}$$

where $a_\alpha \in [-1, 1)$. Finer digitizations of a_α, accomplished by the use of larger values of K, provide better resolution of the a_α and consequently higher precision and accuracy in solution, but are limited by device performance with increasing size of the QUBO matrix. This digitization of a_α puts the expression in Eq. (2) into QUBO form,

$$\begin{aligned} F &= \sum_{\alpha\beta,ij} Q_{\alpha,i;\beta,j} q_i^\alpha q_j^\beta \quad , \\ &\text{with} \quad Q_{\alpha,i;\beta,j} = 2^{i+j-2K}(-1)^{\delta_{iK}+\delta_{jK}} h_{\alpha\beta} \quad . \end{aligned} \tag{4}$$

The "QUBO matrix" $Q_{\alpha,i;\beta,j}$, with dimensions $Kn_s \times Kn_s$, is subsequently passed from the D-Wave API to a simulator or D-Wave's QAs.

An adaptive QA eigenvalue (AQAE) solver, implemented in Ref. [68, 75], incorporates an algorithmic improvement that reduces the required value of K to reach a given solve precision, and hence increases the size of problem instances that can be addressed using any given QA (a similar idea was applied in Ref. [81] for machine learning). After the initial solve for coefficients $a_\alpha^{(z=0)}$, the range of search values for $a_\alpha^{(z+1)}$ are systematically reduced, guided by the previously obtained $a_\alpha^{(z)}$. This permits not only a reduced value for K but also a reduced number of anneals at each zoom level. We implement a relation between successive zoom steps similar to Ref. [68], of the form

$$a_\alpha^{(z+1)} = a_\alpha^{(z)} - 2^{-z} q_K^\alpha + \sum_{i=1}^{K-1} \frac{q_i^\alpha}{2^{K-i+z}} \quad . \tag{5}$$

An example of the progressive decimation of a coefficient $a_\alpha^{(z+1)}$ with increasing zoom step is shown in Fig 1. In forming the QUBO matrix at each zoom step, the contributions that are naively linear in q_i^α in the product $a_\alpha^{(z+1)} a_\beta^{(z+1)}$ are changed to quadratic via $q_i^\alpha = (q_i^\alpha)^2$. The derivation of the QUBO matrix can be found in Appendix A, yielding

$$\begin{aligned} Q_{\alpha,i;\beta,j} =& 2^{i+j-2K-2z}(-1)^{\delta_{iK}+\delta_{jK}} h_{\alpha\beta} \\ &+ 2\delta_{\alpha\beta}\delta_{ij} 2^{i-K-z}(-1)^{\delta_{iK}} \sum_\gamma a_\gamma^{(z)} h_{\gamma\beta} \quad . \end{aligned} \tag{6}$$

As considered in Ref. [54], excited states can also be addressed with this same construction by including "chemical potentials" for the states that are lower in the spectrum. The N^{th} state in the spectrum can be obtained by including $N-1$ chemical potentials μ_n that give an energy shift to each of the n^{th} states that place them higher in the spectrum than the N^{th} state. To accomplish this, an effective Hamiltonian of the form

$$\hat{\mathcal{H}}^{(N)} = \hat{\mathcal{H}} + \sum_n^{N-1} \mu_n \, |\Psi_n\rangle\langle\Psi_n| \quad , \tag{7}$$

is used. The μ_n depend upon the energy eigenvalues of $|\Psi_{n<N}\rangle$, both of which are determined in earlier problem solves using $\mathcal{H}^{(n<N)}$ in the workflow, and an approximate knowledge of the energy of $|\Psi_N\rangle$, determined possibly in tuning, or from other approximate solves. Explicitly, to implement Eq. (7), and find the $n+1^{\text{th}}$ excited state, the wavefunctions of n^{th} lowest-lying states are used to to generate a matrix contribution from the outer-products $|\Psi_n\rangle\langle\Psi_n|$, which are multiplied by μ_n and added to the existing Hamiltonian (see Supplemental Material for examples).

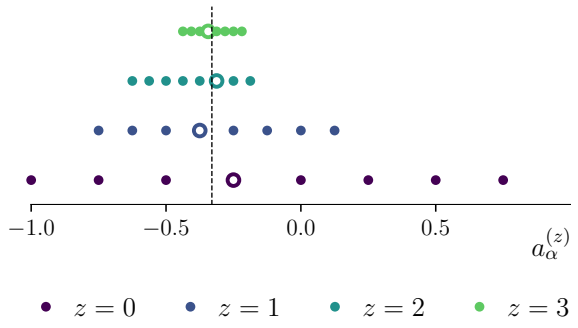

FIG. 1. An example of the convergence of $a_\alpha^{(z)}$ from the iterative zooming method discussed in the text using $K = 3$, for a true value of $a_\alpha = -0.33$ (vertical dashed line). The closed circles denote the sampled values of $a_\alpha^{(z)}$ at each zoom step. The white markers (open circles) correspond to the points closest to the true value of a_α, and are used as input for the subsequent zoom step.

A. Analyzing Results from Annealing Simulator and Quantum Devices

For any given QUBO matrix, annealing simulators or QAs perform N_A anneals to locate the lowest-energy configuration(s) and associated wavefunction(s). Outputs of the annealing workflow include an ensemble of N_A results, and generation of such ensembles can be repeated N_{run} times to provide estimates of associated uncertainties. As the lowest energy configuration in an ensemble provides the lowest upper bound to the true energy of the target state, N_{run} sets of such measurements can yield a global minimum energy and wavefunction, and also a mean and standard deviation, or a median and 68% confidence interval (for robustness). These estimators provide a measure of some uncertainties, including

those associated with zooming and fluctuations in the annealing process.

Systematic studies of uncertainties associated with D-Wave's QAs have been previously performed, e.g., Refs. [82–84]. These contain detailed sets of measurements and discussions of device configuration and noise. The D-Wave online documentation [85], in particular that related to `neal`, provides algorithms to simulate the annealers, discussions of the noise model, and provides codes.

To differentiate between results obtained with a noisy simulator and a QA that are shown in figures in the text, we will assign one of the icons introduced in Ref. [86] – the yellow square icon for results obtained using D-Wave's quantum simulator `neal` and the blue diamond icon for those obtained using D-Wave's QA `Advantage system 4.1` (which we will refer to simply as `Advantage`).

III. HARMONIC AND ANHARMONIC OSCILLATORS: EIGENSTATES AND ENERGIES

Perhaps the simplest quantum field theory to consider is $\lambda\phi^4$ scalar field theory. Jordan, Lee and Preskill have shown that state preparation and simulating S-matrix elements resides in the BQP-complete complexity class [87]. As a starting point for exploring lattice scalar field theory using QAs, we examine a single site harmonic oscillator with and without the non-linear $\lambda\phi^4$ interaction. The Hamiltonian for a single site has the form

$$\hat{\mathcal{H}} = \frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}m_0^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 \quad , \tag{8}$$

with the bare mass being m_0 and bare coupling being λ (all quantities are in lattice units, l.u.). In the Jordan-Lee-Preskill (JLP) basis [87–91], the field ϕ is digitized at each spatial site in a space spanned by n_s uniformly distributed states[1] with mapped values

$$\phi = -\phi_{\rm max} + \delta_\phi \beta_\phi \quad , \quad \delta_\phi = \frac{2\phi_{\rm max}}{n_s - 1} \quad , \tag{9}$$

where $\phi_{\rm max}$ is the maximum value of $\phi(\mathbf{x})$, and $\beta_\phi = 0, 1, \ldots, n_s - 1$. In ϕ-space, while two of the terms in Eq. (8) are diagonal, the conjugate momentum operator can be computed with a finite difference operator. However, this introduces polynomial δ_ϕ-discretization errors. A better way to compute it is to use Quantum Fourier Transforms into and out of conjugate momentum space [88, 89], since $\langle k_\phi|\hat{\Pi}^2|k'_\phi\rangle = k_\phi^2\delta_{k_\phi,k'_\phi}$, with

$$k_\phi = -k_\phi^{\rm max} + \left(\beta_\phi - \frac{1}{2}\right)\delta k_\phi \quad ,$$
$$k_\phi^{\rm max} = \frac{\pi}{\delta_\phi} \quad , \quad \delta k_\phi = \frac{2\pi}{\delta_\phi n_s} \quad . \tag{10}$$

This has been shown to eliminate power-law corrections to $\hat{\Pi}^2$, giving exponentially convergent digitization via the Nyquist-Shannon theorem [87–95]. A detailed comparison between this operator and finite-difference versions can be found in an appendix of Ref. [91]. For a given number of states n_s, the Hamiltonian is an $n_s \times n_s$ real matrix, from which the eigenstates and energies can be found via mappings to a QUBO-problem and annealing, as discussed in Section II.

A. Results from the Annealer Simulator `neal`

Available D-Wave annealer simulators were used to prepare for working with D-Wave's cloud-accessible QAs. In particular, for the 1-site system, the simulator was used to perform parameter tunings and calibrations, including of the maximum and minimum values of the field, $\phi_{\rm max}$, of the number of states over which the field is digitized, n_s, of η in the objective function (in Eq. (1)), of the chemical potentials μ_n (in Eq. (7)), of the number of qubits per coefficient, K, of the number of anneals per zoom step, N_A, and of the total number of zoom steps, $z^{\rm max}$. These identified values, or initial tunings, for these parameters, and measures of uncertainties, both systematic and statistical, are a subset of those that will be present for computations using QAs. Our workflow for the simulator and quantum hardware was implemented with `python` [96] using `jupyter` notebooks [97] after formulating the matrix problem with `Mathematica` [98].

1. *Tunings*

We present only highlights of parameter tunings as they generally behave as naively anticipated, or as determined previously. The left panel of Fig. 2 shows the systematic deviation from the true digitized ground state energy of the HO determined using the annealing simulator for $m_0 = 1$, $\lambda = 0$, $\phi_{\rm max} = 5$ digitized across $n_s = 32$ states as a function of η for $N_A = 10^3$ and $K = 3$. The solid lines with points correspond to minimum energy solutions, while the solid bands correspond to the 68% confidence intervals determined from $N_{\rm run} = 200$ samples. The accuracy in the energy is found to be optimized for $\eta \sim E_0$. The middle panel of Fig. 2 shows the deviations in energy for different levels of zooming as a function of N_A for $K = 3$ and $\eta = 0.51$. While the minimum energy estimate is improved with increasing number of anneals, increasing the zoom level leads to a more rapid convergence. Similarly, the right panel of Fig. 2 shows

[1] For a register of a universal quantum computer of n_Q qubits, $n_s = 2^{n_Q}$.

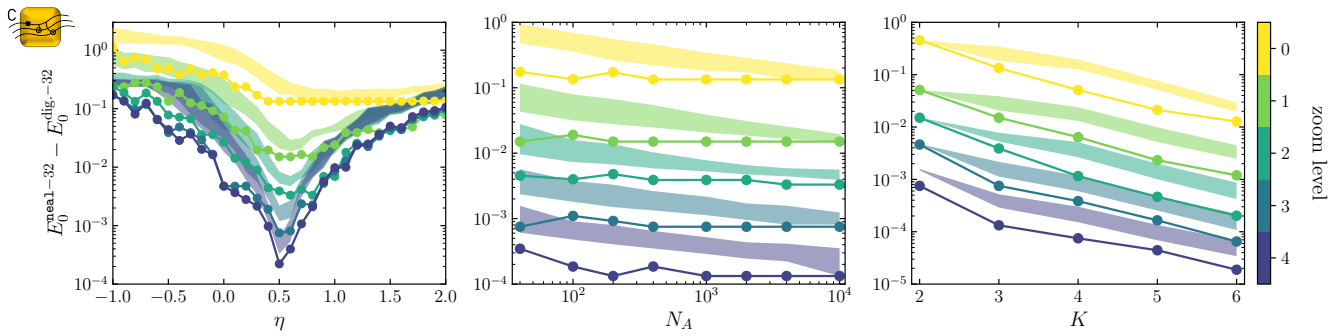

FIG. 2. The ground state energy (in l.u.) of the HO obtained with D-Wave's `neal` as a function of the η parameter (left panel), number of anneals N_A (center panel) and K parameter (right panel), for $m_0 = 1$, $\lambda = 0$, $\phi_{\max} = 5$ discretized across $n_s = 32$ states, setting $\eta = 0.51$, $N_A = 10^3$, and $K = 3$ when the corresponding parameter is not varied. The solid lines with points correspond to minimum energy solutions, while the solid bands correspond to the 68% confidence intervals determined from $N_{\text{run}} = 200$ independent runs of the annealing workflow.

the energy deviation as a function of K for different levels of zoom for $N_A = 10^3$ and $\eta = 0.51$. Although larger values of K can increase the precision of the ground-state energy, as is the case for N_A, using more zoom steps can also reach similar levels (without increasing the number of qubits). Overall conclusions from these explorations of parameter space are that η should be close to the energy of the ground state, and that the use of "sloppy solves", where a relatively small number of anneals N_A are used to iteratively estimate subsequent zoom intervals for QUBO parameters, can be used to make efficient use of computational resources.[2] We consider it to be somewhat unfortunate that the dependence on η is that shown in the left panel of Fig. 2, as ideally, quantities would be independent of η.

The systematic improvements in results with increasing N_A, K, and number of zoom levels using the annealer simulator do not persist indefinitely, which is attributed to the white noise intrinsic to `neal`.

2. *Harmonic Oscillator:* $V(\phi) = \frac{1}{2}\phi^2$

For demonstrative purposes, and through the use of appropriately tuned parameters, we present the results for the lowest six eigenstates of the HO with $V(\phi) = \frac{1}{2}\phi^2$ (i.e., $m_0 = 1$). The exact energies in the field-space continuum limit are $E_n^{\text{exact}} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, ...\}$, with eigenfunctions given by

$$\Psi_n(\phi) = \frac{1}{\sqrt{2^n n!}} \left(\frac{1}{\pi}\right)^{1/4} e^{-\frac{1}{2}\phi^2} \, \mathrm{H}_n(\phi) \quad , \tag{11}$$

where $\mathrm{H}_n(x)$ are the Hermite polynomials. A systematic study of the impact of digitization on the low-lying wavefunctions and energies has been performed previously [91], and we use that as a guide in selecting digitization parameters. We work with $\phi_{\max} = 5$ and $n_s = 64$, resulting in $\delta\phi = 0.1587$ and $k_{\max} = 19.7920$, and energies $E_{n\leq 6}^{\text{dig.}-64}$ that are the same as $E_{n\leq 6}^{\text{exact}}$ to better than $\sim 10^{-5}$. The value of η_n was set equal to the corresponding $E_n^{\text{exact}} + 0.01$, and the chemical potentials were set to $\mu_n = 10$ when appropriate, to move each of the previously determined states to an energy higher than the "next" ground state. The energies of the lowest states found using `neal` are given in Table I and displayed in the left panel of Fig. 3 (with $N_A = 10^3$), and recover the digitized values with accuracy $\sim 10^{-3} - 10^{-6}$.

n	E_n^{exact}	$\lvert\delta E_n^{\text{dig.}-64}\rvert$	$\lvert\delta E_n^{\texttt{neal}-64}\rvert_{z=0}$	$\lvert\delta E_n^{\texttt{neal}-64}\rvert_{z=8}$
0	1/2	3.5×10^{-11}	$(4.9^{+2.5}_{-1.8}) \times 10^{-6}$	$(4.0^{+2.6}_{-1.2}) \times 10^{-6}$
1	3/2	1.8×10^{-9}	$(9.1^{+409}_{-5.1}) \times 10^{-6}$	$(5.2^{+2.5}_{-1.7}) \times 10^{-6}$
2	5/2	4.1×10^{-8}	$(4.7^{+4.7}_{-2.0}) \times 10^{-6}$	$(4.4^{+2.2}_{-1.6}) \times 10^{-6}$
3	7/2	6.6×10^{-7}	$(0.7^{+200}_{-0.3}) \times 10^{-5}$	$(5.3^{+2.7}_{-1.6}) \times 10^{-6}$
4	9/2	6.6×10^{-6}	$(5.5^{+7.2}_{-2.0}) \times 10^{-6}$	$(4.4^{+2.1}_{-1.6}) \times 10^{-6}$
5	11/2	6.0×10^{-5}	$(0.7^{+100}_{-0.3}) \times 10^{-5}$	$(4.7^{+2.3}_{-1.6}) \times 10^{-6}$

TABLE I. The energies associated with the HO with $m_0 = 1$ (in l.u.). The exact energies are shown in the second column, the difference between the diagonalization of the digitized Hamiltonian and exact energies are shown in the third column, with $\phi_{\max} = 5$ and $n_s = 64$, and the differences between the digitized energies and the corresponding results obtained using D-Wave's annealer simulator, `neal`, are shown in the fourth (no MG) and fifth (with MG) columns, with the uncertainties showing the 68% confidence intervals, with $K = 3$ and $N_A = 10^3$.

In order to reduce the uncertainty bands, instead of increasing N_A or K, we work with a (somewhat basic) MultiGrid Adaptive QA Eigenvalue (MG-AQAE) solver, inspired by the wide success of multigrid algorithms [101], where a coarser (smaller) system is solved

[2] The latter technique is in the spirit of All-Mode Averaging techniques employed in some lattice QCD generations of light quark propagators, e.g., Refs. [99, 100].

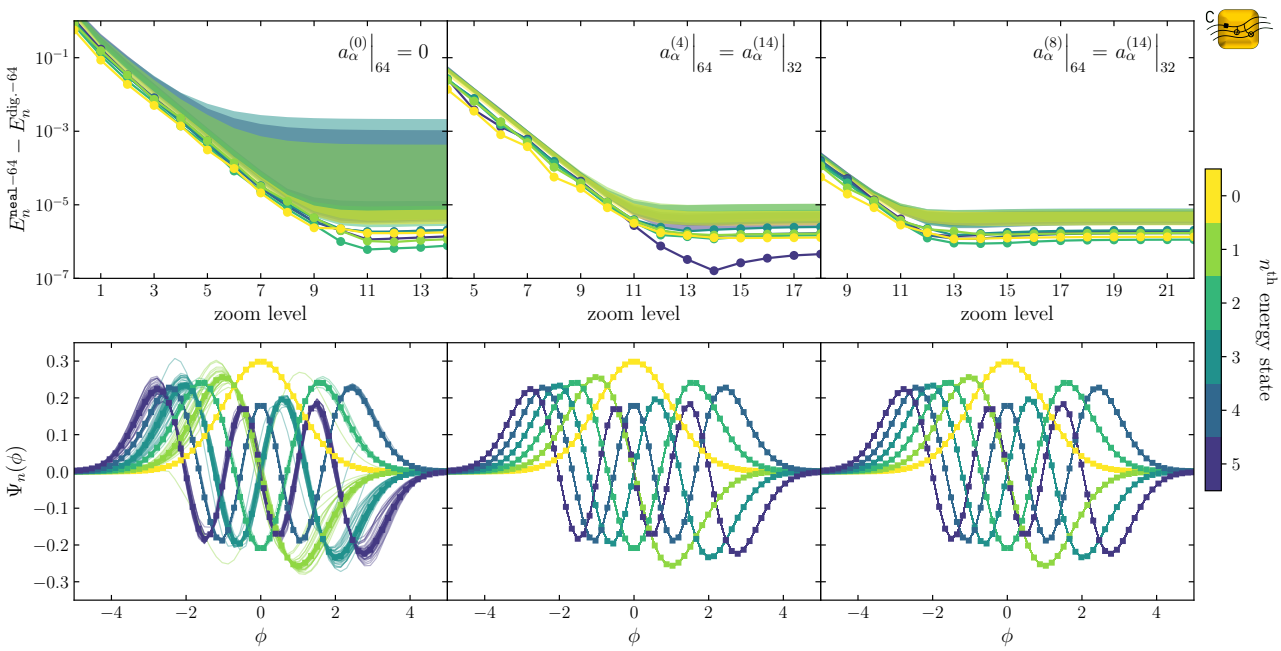

FIG. 3. The upper panels show the convergence of the energy (in l.u.) of the first six HO states as a function of the number of zoom steps, where the solid lines with points correspond to minimum energy solutions, while the solid bands correspond to the 68% confidence intervals determined from $N_{\rm run} = 200$ independent runs of the annealing workflow. The lower panels show the lowest six HO wavefunctions (multiplied by $(-1)^n$). The squares denote the exact values from the digitized Hamiltonian, while the lines show the $N_{\rm run} = 200$ independent runs obtained using D-Wave's annealer simulator `neal` with the maximal number of zoom steps. The results from `neal` are also sets of discrete points, and for display purposes we have shown them as joined line segments. The initial coefficients for the left panel are null coefficients, for the middle panels correspond to starting the $z^{\rm init} = 4$ zooming for $n_s = 64$ states from the $z^{\rm max} = 14$ values and their interpolations from $n_s = 32$, and the right panels correspond to staring with $z^{\rm init} = 8$. The maximum value of the field is $\phi_{\rm max} = 5$ digitized on $n_s = 64$ states, with $K = 3$ and $N_A = 10^3$.

first, and that solution is used as an input parameter for the finer (larger) system. Specific to our case, a coarser system can be defined by using a smaller value of n_s for the digitization of the field. With the coefficients $a_\alpha^{(z)}$ extracted from the $n_s = 32$ system, we interpolate (using cubic splines) to find starting points for the $n_s = 64$ system. To help guide the annealer, it has been found that starting with $z^{\rm init} \gtrsim 4$ reduces the uncertainty on the energies, as it limits the range of the values that $a_\alpha^{(z)}$ can assume. As shown in the center and right panels of Fig. 3, starting with $z^{\rm init} = 4$ or 8 significantly reduces the error compared to $z^{\rm init} = 0$.

To display the convergence obtained with `neal` for increasing number of zoom steps, the top panels of Fig. 3 show the deviation in the energy of the lowest six states as a function of zoom steps. The convergence is consistent with exponential in the number of zoom steps, as found in Ref. [68]. This result is encouraging as the HO is one of the simplest systems to consider.

The wavefunctions associated with the energies in Table I are shown in the lower panels of Fig. 3. The diagonalization of the HO employed $\phi_{\rm max} = 5$, $n_s = 64$, and $m_0 = 1$, and reproduced the values of the continuum-field wavefunctions with high precision – a well-known result. For the ground state, the digitized wavefunction (squares) reproduce the continuum wavefunction to better than $\sim 10^{-6}$, as shown in Fig. 4. Using `neal`, $K = 3$ and $\eta = 0.51$ were used with 14 levels of zoom. The wavefunctions, shown by the lines in Fig. 3, reproduce the digitized wavefunctions to better than $\sim 10^{-3}$, as shown for the ground state in Fig. 4. Similar fidelity is obtained for the other wavefunctions.

Overall, the HO is amenable to simulation with `neal`. With appropriate (and easy to identify) parameter tunings, the energies and wavefunctions of the lowest-lying states can be determined with precision. Results for coarser and finer digitizations behave in ways that are consistent with expectations. While we have not performed a systematic exploration, we expect that the primary limitation on the number of states of the HO that can be isolated with precision is the accumulation of errors through the iterative process of Eq. (7).

3. *Anharmonic Oscillator:* $V(\phi) = \frac{1}{2}\phi^2 + \frac{4}{3}\phi^4$

Depending upon the size of the non-linear interaction, the low-lying spectrum of the aHO can differ significantly

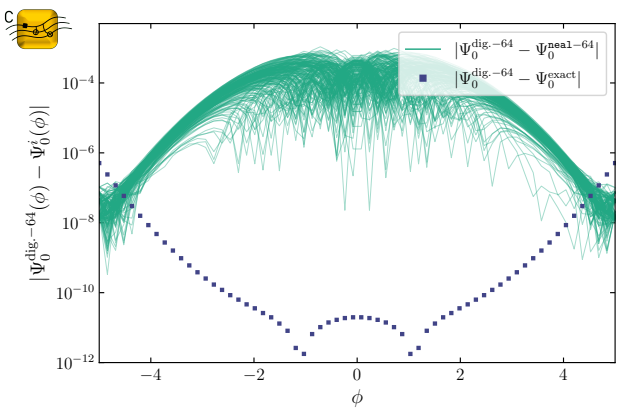

FIG. 4. Deviations between the digitized ground-state HO wavefunction and the i) analytic continuum-field expression (squares) and ii) the wavefunctions determined using `neal` (lines, showing the $N_{\mathrm{run}} = 200$ independent runs). The digitized simulation employed $\phi_{\max} = 5$, $n_s = 64$, $m_0 = 1$, $K = 3$, $\eta = 0.51$, $N_A = 10^3$, and $\{z^{\mathrm{init}}, z^{\max}\} = \{8, 22\}$.

n	E_n^{exact}	$\lvert\delta E_n^{\mathrm{dig.}-64}\rvert$	$\lvert\delta E_n^{\mathtt{neal}-64}\rvert_{z=0}$	$\lvert\delta E_n^{\mathtt{neal}-64}\rvert_{z=8}$
0	0.8597427	1.7×10^{-9}	$(0.9^{+223}_{-0.8}) \times 10^{-5}$	$(1.7^{+1.1}_{-0.7}) \times 10^{-6}$
1	2.9493637	2.4×10^{-8}	$(0.5^{+239}_{-0.5}) \times 10^{-4}$	$(1.8^{+1.6}_{-0.6}) \times 10^{-6}$
2	5.6096611	2.0×10^{-7}	$(0.2^{+161}_{-0.1}) \times 10^{-5}$	$(1.2^{+0.7}_{-0.6}) \times 10^{-6}$
3	8.6270258	1.5×10^{-6}	$(0.5^{+279}_{-0.3}) \times 10^{-5}$	$(2.7^{+1.6}_{-0.7}) \times 10^{-6}$
4	11.930637	8.0×10^{-6}	$(0.2^{+782}_{-0.1}) \times 10^{-5}$	$(1.4^{+0.7}_{-0.4}) \times 10^{-6}$
5	15.476155	4.4×10^{-5}	$(4.1^{+439}_{-4.1}) \times 10^{-4}$	$(3.0^{+1.6}_{-1.0}) \times 10^{-6}$

TABLE II. The energies associated with the aHO with $m_0 = 1$ and $\lambda = 32$ (in l.u.). The exact energies are shown in the second column, the difference between the diagonalization of the digitized Hamiltonian and exact energies are shown in the third column, with $\phi_{\max} = 2.6$ and $n_s = 64$, and the differences between the digitized energies and the corresponding results obtained using D-Wave's annealer simulator, `neal`, are shown in the fourth (no MG) and fifth (with MG) columns, with the uncertainties showing the 68% confidence intervals, with $K = 3$ and $N_A = 10^3$.

from those of the HO. For the coupling of $\lambda = 32$ that we have chosen for demonstrative purposes, important differences are present both in the energy and wavefunctions of the 1-site system. The value of η_n used to solve for n^{th} eigenstate was set, as in the HO case, to the corresponding $E_n^{\mathrm{exact}} + 0.01$, the chemical potentials were set to $\mu_n = 20$, $K = 3$, and the number of anneals is $N_A = 10^3$. Table II displays the exact energies, the difference between the exact and digitized energies for the system with $\phi_{\max} = 2.6$ and $n_s = 64$, and the difference between the results obtained using `neal` and the exact digitized energies.

The convergence of the energy of the lowest-lying states with increasing numbers of zoom steps determined using `neal` are shown in the top panels of Fig. 5, with exponential convergence seen up to ~ 12 zoom steps, beyond which there are diminishing returns. The converged wavefunctions for each level are shown in the lower panels of Fig. 5. As in the case of the HO, the MG-AQAE method reduces the uncertainties in the extracted energies. Further, the wavefunctions converge well to the exact digitized wavefunctions (shown as the squares in Fig. 5). The lower panels of the figure show the wavefunctions resulting from $N_A = 10^3$ anneals compared to the exact result.

B. Implementations and Results from D-Wave Annealers

We have run the codes used in Sec. III A with `neal` on D-Wave's QA `Advantage`, which has 5627 physical qubits, and each qubit is connected with 15 other qubits (in a so-called Pegasus topology). The system is accessible through the cloud via D-Wave's website [102].

The mapping between the QUBO problem and the processor topology is performed automatically, via heuristics algorithms [103], and is the most time-consuming part of the simulation, as discussed in Appendix B (the embedding can be computed at the beginning, and it also can be reused for all the zoom steps). During this process, as there is no all-to-all connectivity, several physical qubits are chained together to form a logical qubit with the required connectivity. To enforce that the qubits in a certain chain have all the same value, an extra parameter, the chain-strength value (c_s), is fixed. As the elements of the QUBO matrix are re-scaled to lie in the range $[-1, 1]$ when they are passed to `Advantage`, if $c_s \gg \max(|Q|)$, the QUBO elements will be re-scaled closer to zero. We set $c_s = \omega \max(|Q|)$ and scanned over $\omega \in [0, 1]$, finding that $\omega = 0.2$ gives the lowest energies for these systems. Another parameter that can be tuned is the annealing schedule and annealing time (t_A), and we have used its default value of $t_A = 20\,\mu$s in our calculations. It has been previously observed, e.g., Ref. [61], that using different annealing schedules, like reverse annealing, can increase the success rate (finding the solution with minimum energy) of the QA. The exploration of such improvements is left for future work.

Due to `Advantage`'s intrinsic noise, extracting energies and wavefunctions with adequate precision for $n_s \geq 16$ required using MG-AQAE. Compared to the `neal` simulator, a smaller initial value of $n_s = 16$ was required for `Advantage` to provide meaningful results. These results were then used as starting values for the $n_s = 32$ and $n_s = 64$ anneals, with results shown in Fig. 6 for the HO and Fig. 7 for the aHO.[3] Interestingly, while a value of

[3] It is interesting to note that the ground-state wavefunction obtained with $n_s = 16$ when interpolated to $n_s = 64$ achieves 10^{-4} precision in the ground-state energy, without using MG-AQAE (for higher-energy states, the precision is reduced).

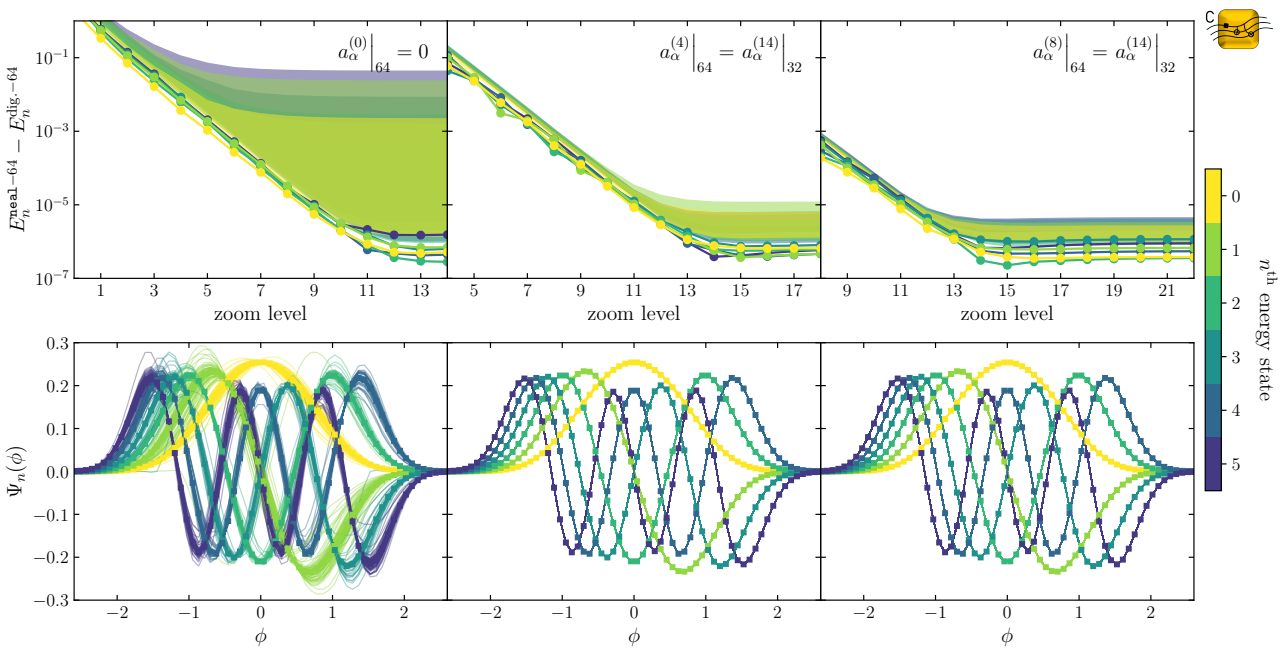

FIG. 5. The upper panels show the convergence of the energy (in l.u.) of the first six aHO states as a function of the number of zoom steps, where the solid lines with points correspond to minimum energy solutions, while the solid bands correspond to the 68% confidence intervals determined from $N_{\mathrm{run}} = 200$ independent runs of the annealing workflow. The lower panels show the lowest six aHO wavefunctions (multiplied by $(-1)^n$). The squares denote the exact values from the digitized Hamiltonian, while the lines show the $N_{\mathrm{run}} = 200$ independent runs obtained using D-Wave's annealer simulator `neal` with the maximal number of zoom steps. The initial coefficients for the left panel are null coefficients, for the middle panels correspond to starting the $z^{\mathrm{init}} = 4$ zooming for $n_s = 64$ states from the $z^{\mathrm{max}} = 14$ values and their interpolations from $n_s = 32$, and the right panels correspond to staring with $z^{\mathrm{init}} = 8$. The maximum value of the field is $\phi_{\mathrm{max}} = 2.6$ digitized on $n_s = 64$ states, with $K = 3$ and $N_A = 10^3$.

$K = 3$ was sufficient for the $n_s = 16$ and 32 systems, $K = 2$ was required for the $n_s = 64$ system to permit an embedding of the QUBO matrix into `Advantage`.

C. Delocalized Fields: $\lambda\phi^4$ with $m_0^2 < 0$ and Reflection Symmetry in Field Space

In the situation where $m_0^2 < 0$, corresponding to a double-well potential, the ground state with a symmetric wavefunction and the first-excited state with an antisymmetric wavefunction are nearly degenerate for a large region in mass-coupling space. For such parameters, the wavefunctions have support mainly in regions localized around the two minima of the potential, with exponential suppression of the energy difference as the minima become increasingly separated. Consequently, the results obtained with `neal` and `Advantage` are generally unable to uniquely identify the ground states of such systems. As the Hamiltonian has a reflection symmetry in field space, the near degeneracy of the lowest two states can be mitigated by solving the half-space using boundary conditions at the origin consistent with a symmetric or an antisymmetric wavefunction. Using such implementations, the ground state and first excited state of the systems with $m_0^2 < 0$ can be uniquely determined, as shown in Fig. 8. Results were obtained with `Advantage` using MG-AQAE (with $z^{\mathrm{init}} = 3$, and using the $n_s = 16$ system as the preconditioner). It is interesting to point out that the best solution was found to have a smaller energy than one found with `neal`. The use of boundary conditions at the origin reduces the dimensionality of the Hamiltonian that is sent to the annealer simulator or quantum hardware, and hence the problem itself, to one similar to that of a HO with $m_0^2 > 0$. Therefore, the implementation is essentially the same as described previously. However, the delocalization of the wavefunction, means that the digitization of the field requires an increased number of states to recover the same level of precision in, say, the ground state energy (by maintaining a fixed δ_ϕ).

The results obtained for these systems provide practical insights into the generic performance of a QA for simulating systems with (near-)degenerate ground states. Without the half-field truncation, each converged result of the system provided, in general, a different linear combination of the (two) degenerate states, as expected. While straightforward to perform, we did not undertake a study of the $m_0^2 - \lambda$ parameter space to identify regions where the energy gap was sufficient for `neal` and

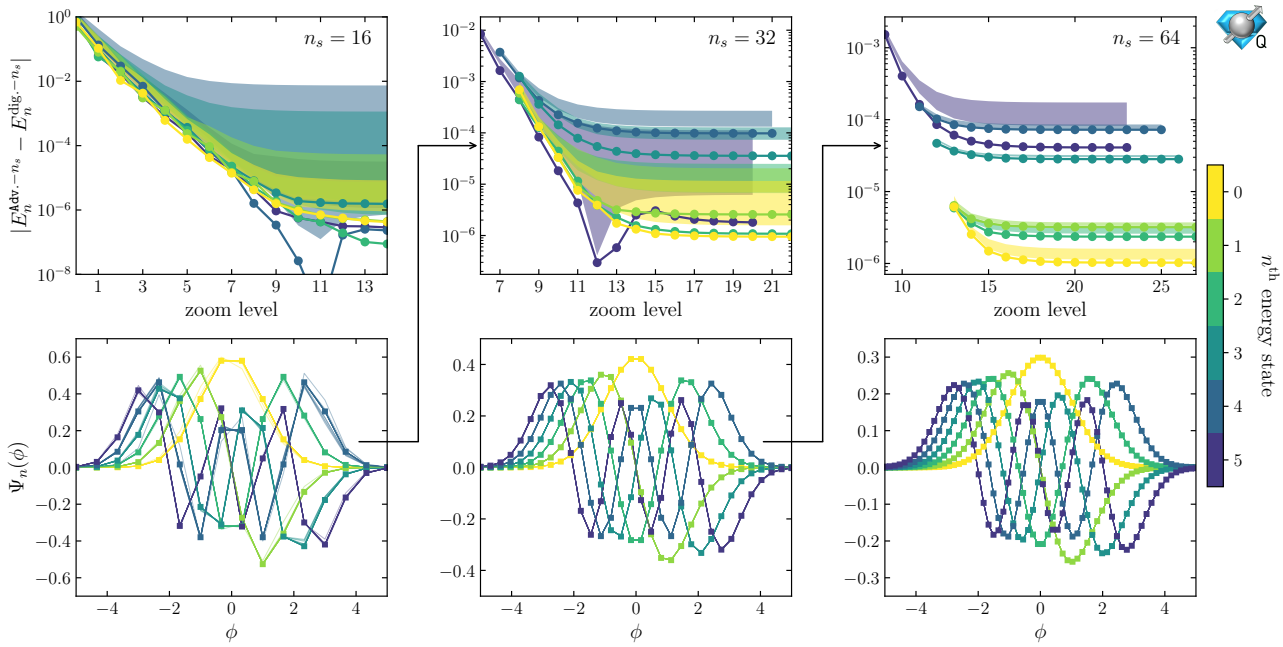

FIG. 6. The upper panels show the convergence of the energy (in l.u.) of the first six HO states as a function of the number of zoom steps, where the solid lines with points correspond to minimum energy solutions, while the solid bands correspond to the 68% confidence intervals determined from $N_{\text{run}} = 20$ independent runs of the annealing workflow. The lower panels show the lowest six HO wavefunctions (multiplied by $(-1)^n$). The squares denote the exact values from the digitized Hamiltonian, while the lines show the $N_{\text{run}} = 20$ independent runs obtained using D-Wave's `Advantage` with the maximal number of zoom steps. The maximum value of the field is $\phi_{\text{max}} = 5$ digitized on $n_s = \{16, 32, 64\}$ states, with $K = \{3, 3, 2\}$ and $N_A = 10^3$.

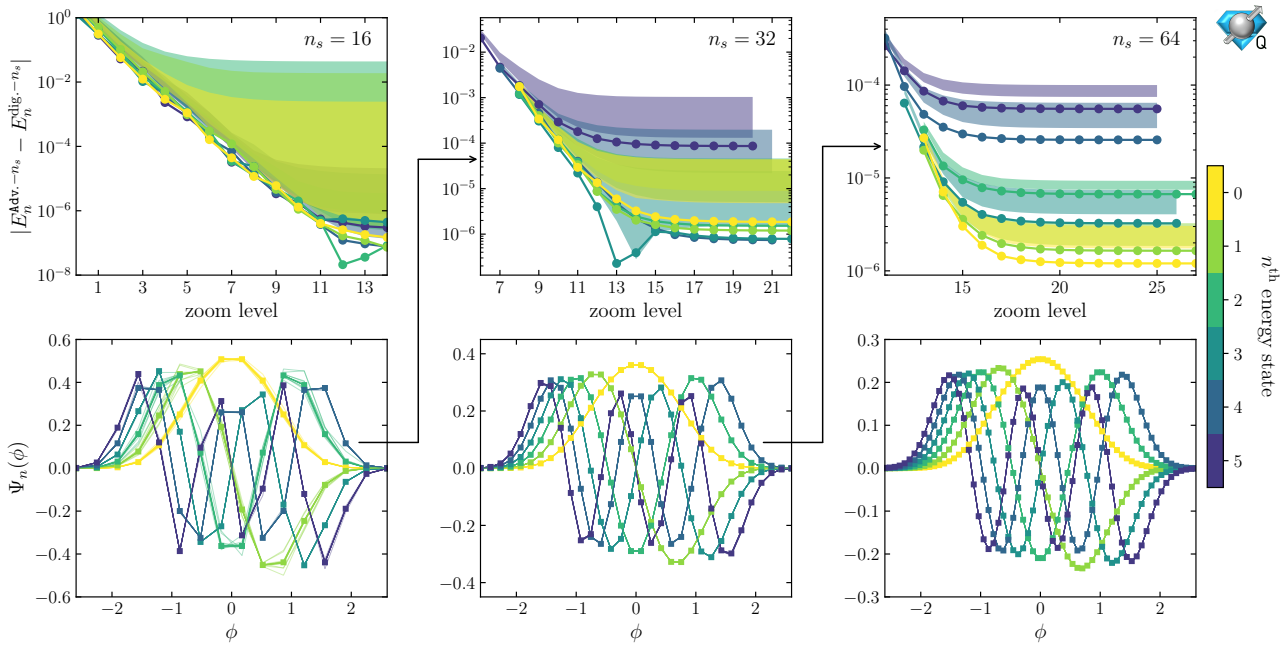

FIG. 7. The upper panels show the convergence of the energy (in l.u.) of the first six aHO states as a function of the number of zoom steps, where the solid lines with points correspond to minimum energy solutions, while the solid bands correspond to the 68% confidence intervals determined from $N_{\text{run}} = 20$ independent runs of the annealing workflow. The lower panels show the lowest six aHO wavefunctions (multiplied by $(-1)^n$). The squares denote the exact values from the digitized Hamiltonian, while the lines show the $N_{\text{run}} = 20$ independent runs obtained using D-Wave's `Advantage` with the maximal number of zoom steps. The maximum value of the field is $\phi_{\text{max}} = 2.6$ digitized on $n_s = \{16, 32, 64\}$ states, with $K = \{3, 3, 2\}$ and $N_A = 10^3$.

`Advantage` to uniquely converge to the ground state. While there is significant importance in simulating

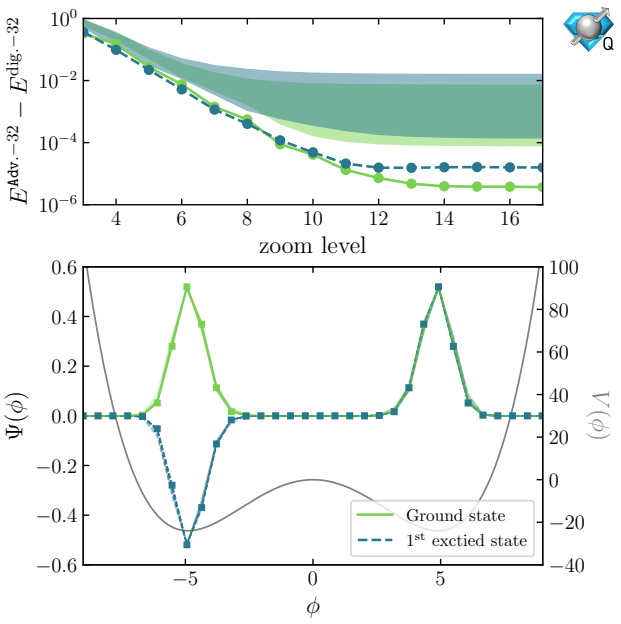

FIG. 8. Convergence of the ground state and first-excited state energies and wavefunctions of an aHO. The upper panel shows the convergence of the energies (in l.u.) and the lower panel shows the symmetric and anti-symmetric wavefunctions (with the maximal number of zoom steps, 17) obtained using `Advantage` ($N_{\rm run} = 20$) for the system with $m_0^2 = -4$, $\lambda = 1$, $\phi_{\rm max} = 9$ discretized across $n_s = 32$ states, with $K = 3$ and $N_A = 10^3$. The half field-space Hamiltonian employed appropriate boundary conditions at the origin to independently solve for wavefunctions with definite reflection symmetry.

scalar fields exhibiting spontaneous symmetry breaking in 3+1 dim in high-energy (Higgs field) and nuclear physics (σ-model and chiral perturbation theory), a detailed exploration is beyond the scope of the present work.

D. Scaling study

During this NISQ era, quantum processors are being characterized to determine their strengths and weaknesses. For this purpose, Fig. 9 shows the number of physical qubits required for the Hamiltonian in Eq. (8) with different numbers of basis states n_s. Specifically, we compute the QUBO matrix with $K = 2$ for different values of n_s, find the embedding using the `find_embedding` command from `minorminer` [85], as discussed in Sec. III B, and count the number of required physical qubits, that are shown in Fig. 9. As it uses a heuristic algorithm to find this mapping [103], the number of qubits is generally different each time this command is called, and the width of the bands represent 68% confidence interval determined from 20 different embeddings of the same problem. Fig. 9 also shows the number of qubits in the ideal case with the gray line, assuming

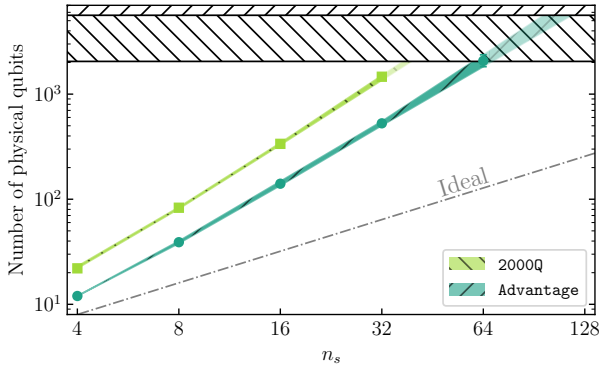

FIG. 9. The numbers of physical qubits required to map the QUBO matrix from the Hamiltonian in Eq. (8) as a function of n_s for a fixed $K = 2$. The bands show results for two available D-Wave QAs, `2000Q` and `Advantage`, with the lighter bands being extrapolations using the last two points. The gray dot-dashed line represents the ideal scaling, with Kn_s qubits.

an all-to-all connectivity, therefore requiring only Kn_s qubits.

It is interesting to note that the improvement over the previous D-Wave's QA, `2000Q`, with a total of 2048 qubits (and a connectivity of 6 qubits), to the current one, `Advantage`, allows for a reduction in the number of qubits by a factor of $2-3$ (directly related to the increase in connectivity between qubits, which increases from 6 to 15). However, the points for both QAs follow a line with a similar slope, which is larger than the ideal case.

Although the current limit to n_s is 64, without the zooming algorithm described in Sec. II, it would be smaller (assuming the same precision on the eigenenergies), as K would have to be larger. Further, using the symmetry properties of the wavefunction, as in Sec. III C, the value of n_s could be doubled.

IV. REAL-TIME EVOLUTION OF PLAQUETTES AND NEUTRINOS USING FEYNMAN CLOCKS

The real-time dynamics of physically interesting complex quantum systems is an expected capability of future quantum computers, which will advance the domain of sciences beyond what is possible with classical computing. While for universal gate-based quantum computers, the challenge to implement time evolution is determining efficient quantum circuits and mappings that can be executed on available devices, the challenge for D-Wave's QAs is in finding a viable QUBO matrix that can be implemented. First formalized for scientific applications in the context of quantum chemistry [76, 104], Feynman clock states [105, 106] provide a way to time-evolve quantum systems in a single run of a QA (its first implementation on quantum hardware can be found in Ref. [107]). The constraint of real entries in the QUBO matrix can be circumvented by an appropriate change

of basis that transforms the Hamiltonian into a purely imaginary form, rendering $\hat{U}_t = e^{-it\hat{\mathcal{H}}}$ real, for example, as used in Ref. [75]. Following the formulation of McClean *et al.* [76], called the Time-Embedded Discrete Variational Principle (TEDVP), the objective function to be minimized has the following form,

$$F = \sum_{t,t'} \langle t'| \langle \Psi_{t'}| \hat{\mathcal{C}} |\Psi_t\rangle |t\rangle - \eta \sum_{t,t'} \langle t'| \langle \Psi_{t'}|\Psi_t\rangle |t\rangle \quad , \tag{12}$$

where the η parameter has the same purpose as in Eq. (1), and $|\Psi_t\rangle|t\rangle$ are compound states formed of the physical wavefunction $|\Psi_t\rangle$ and the time register $|t\rangle$. The clock Hamiltonian[4] $\hat{\mathcal{C}}$ is defined as

$$\begin{aligned}\hat{\mathcal{C}} = \hat{C}_0 \, + \, \frac{1}{2} \sum_t (\hat{I} \otimes |t\rangle\langle t| - \hat{U}_{\delta t} \otimes |t+\delta t\rangle\langle t| \\ - \hat{U}^\dagger_{\delta t} \otimes |t\rangle\langle t+\delta t| + \hat{I} \otimes |t+\delta t\rangle\langle t+\delta t|) \quad , \end{aligned} \tag{14}$$

with $\hat{C}_0$ being a penalty term to select a particular (input) state at a time t. For our purposes, like previous works of others, we will use it to select the initial state $|\Psi_{\rm in}\rangle$ at $t = 0$ with $\hat{C}_0 = (I - |\Psi_{\rm in}\rangle\langle\Psi_{\rm in}|) \otimes |0\rangle\langle 0|$. One issue that arises in using this method with a QA is that the matrix elements of $\hat{U}_{\delta t}$ are required to be computed explicitly.

We extend the formulation of the Feynman clock to allow for complex values in the QUBO elements, following Ref. [108]. In such a situation, Eq. (2) becomes

$$F^{(\mathcal{C})} = \sum_{\alpha,\beta}^{n_T \times n_s} \overline{a}_\alpha a_\beta \; \mathcal{C}_{\alpha\beta} \quad , \tag{15}$$

where $\overline{a}_\alpha$ is the complex conjugate of a_α. $\mathcal{C}_{\alpha\beta}$ are matrix elements of $\hat{\mathcal{C}} - \eta\hat{I}$ between basis states spanning $|\Psi_t\rangle|t\rangle$, of which there are $n_T \times n_s$, where n_T is the number of time slices and n_s is the number of basis states of the Hamiltonian. Since $\hat{\mathcal{C}}$ is Hermitian, terms in Eq. (15) can be written as, after the summations,

$$\begin{aligned}\overline{a}_\alpha a_\beta \; \mathcal{C}_{\alpha\beta} = & \, a^{\rm Re}_\alpha a^{\rm Re}_\beta \mathcal{C}^{\rm Re}_{\alpha\beta} - a^{\rm Re}_\alpha a^{\rm Im}_\beta \mathcal{C}^{\rm Im}_{\alpha\beta} \\ & + a^{\rm Im}_\alpha a^{\rm Re}_\beta \mathcal{C}^{\rm Im}_{\alpha\beta} + a^{\rm Im}_\alpha a^{\rm Im}_\beta \mathcal{C}^{\rm Re}_{\alpha\beta} \quad , \end{aligned} \tag{16}$$

which involves only real numbers. With this form, together with Eq. (5) for the fixed-point representation of the real and imaginary parts of a_α, the elements of the QUBO matrix, $Q_{\alpha,i;\beta,j}$, become

$$Q_{\alpha,i;\beta,j} = \begin{cases} 2^{i+j-2K-2z}(-1)^{\delta_{iK}+\delta_{jK}} \mathcal{C}^{\rm Re}_{\alpha\beta} + 2\delta_{\alpha\beta}\delta_{ij} 2^{i-K-z}(-1)^{\delta_{iK}} \sum_\gamma \left(a^{{\rm Re},(z)}_\gamma \mathcal{C}^{\rm Re}_{\gamma\beta} + a^{{\rm Im},(z)}_\gamma \mathcal{C}^{\rm Im}_{\gamma\beta} \right) & 1 \le i,j \le K \, , \\ -2^{i+j'-2K-2z}(-1)^{\delta_{iK}+\delta_{j'K}} \mathcal{C}^{\rm Im}_{\alpha\beta} & 1 \le i,j' \le K \, , \\ 2^{i'+j-2K-2z}(-1)^{\delta_{i'K}+\delta_{jK}} \mathcal{C}^{\rm Im}_{\alpha\beta} & 1 \le i',j \le K \, , \\ 2^{i'+j'-2K-2z}(-1)^{\delta_{i'K}+\delta_{j'K}} \mathcal{C}^{\rm Re}_{\alpha\beta} + 2\delta_{\alpha\beta}\delta_{i'j'} 2^{i'-K-z}(-1)^{\delta_{i'K}} \sum_\gamma \left(a^{{\rm Im},(z)}_\gamma \mathcal{C}^{\rm Re}_{\gamma\beta} - a^{{\rm Re},(z)}_\gamma \mathcal{C}^{\rm Im}_{\gamma\beta} \right) & 1 \le i',j' \le K \, . \end{cases} \tag{17}$$

where, to accommodate both real and imaginary parts of a_α, the index i resides in the range $1 \le i \le 2K$, with $1 \le i \le K$ for $a^{\rm Re}_\alpha$ and $(K+1) \le i \le 2K$ for $a^{\rm Im}_\alpha$, with $i' \equiv i - K$ (the derivation of this expression can be found in Appendix A). The dimension of the QUBO matrix using n_T time slices with the TEDVP formalism, is $2Kn_Tn_s \times 2Kn_Tn_s$ (a factor of $(2n_T)^2$ times larger than that used to determine wavefunctions of the Hamiltonian, as described in Sec. II).

In the following subsections, we examine two systems of physical interest to Standard Model research, the time evolution of a single plaquette of SU(3) Yang-Mills lattice gauge theory and of a four neutrino system, using `Advantage`. Both of these systems have been simulated previously using IBM's superconducting quantum computers, with two and four qubits. In studying these systems with `Advantage`, the following values of parameters were found to be effective: $N_A = 10^3$, $N_{\rm run} = 20$, $K = 2$, $\eta = 0$, and $\{z^{\rm init}, z^{\rm max}\} = \{0, 14\}$. The chain strength coefficient is fixed to $\omega = 0.2$, and the default annealing-schedule parameters ($t_A = 20\,\mu$s) are used.

A. One-Plaquette in SU(3) Yang-Mills Lattice Gauge Theory

Quantum simulations of non-Abelian lattice gauge theories are anticipated to become increasingly important in Standard Model research. Progress toward this objective is at its earliest stages, with simulations of small systems in low-dimensions underway using the available NISQ-era devices, e.g., Refs. [10, 75, 78, 109–148]. The Kogut-Susskind Hamiltonian [149, 150] developed in the 1970s, provides one concrete framework for quantum simulations of lattice gauge theories, and is being actively pursued with superconducting devices, trapped ion systems,

[4] This is constructed in the context of superpositions of time-slices via projectors formed from

$$|t\rangle_- = \frac{1}{\sqrt{2}} \left(\, |t\rangle - |t+\delta t\rangle \, \right) \quad . \tag{13}$$

optical systems, superconducting radio frequency cavities, and QAs. Simulations of small systems have been performed in one and two spatial dimensions, with simulations of the smallest three-dimensional systems barely within reach of today's devices. Extensive efforts are underway to develop techniques to make simulations with this framework more practical, for instance, integrating over the gauge spaces at each lattice site [128, 134]. Other mappings of the gauge fields, for instance, quantum link models (e.g., Refs. [151–154]), spin systems, and the discrete sampling of gauge fields (e.g., Refs. [155, 156]) are under active exploration. While the formal construction for quantum simulations of non-Abelian gauge theories has been established for more than a decade, and concrete protocols for implementation on quantum devices known for a comparable period of time, first implementations appeared in 2016 using trapped-ion systems [122] and soon after using superconducting [129] and optical systems [131]. Last year, the first simulations of SU(3) Yang-Mills theories were performed [78] of one and two plaquettes, building upon previous simulations of SU(2) plaquette systems [134] and one-dimensional SU(2) chains [143]. These small systems can be simulated using D-Wave's annealers, as was first demonstrated for the SU(2) plaquette systems in the work of A Rahman and collaborators [75].

The time evolution of one- and two-plaquettes in SU(3) Yang-Mills gauge theory have been simulated using IBM's `Athens` quantum computer [78]. Both local basis and global bases were simulated, with the single plaquette a particularly simple system with a minimal qubit footprint in the global basis due to Gauss's Law restrictions. In this work, we focus on a one-plaquette system in the color parity basis, including the states $\{|\mathbf{1}\rangle, |\mathbf{3}^+\rangle, |\mathbf{6}^+\rangle, |\mathbf{8}\rangle\}$, which has the following Hamiltonian when mapped to two qubits,

$$\begin{aligned}\hat{\mathcal{H}} = g^2 &\left(\frac{23}{6}\hat{I}\otimes\hat{I} - \frac{5}{2}\hat{Z}\otimes\hat{I} - \frac{1}{2}\hat{I}\otimes\hat{Z} - \frac{5}{6}\hat{Z}\otimes\hat{Z}\right) \\ &- \frac{1}{2g^2}\left[\sqrt{2}\;\hat{I}\otimes\hat{X} + \sqrt{2}\;\hat{X}\otimes\left(\frac{\hat{I}-\hat{Z}}{2}\right)\right. \\ &+ \frac{1}{2}\hat{X}\otimes\hat{X} + \frac{1}{2}\hat{Y}\otimes\hat{Y} \\ &\left. + \frac{1}{4}\left(\hat{I}+\hat{Z}\right)\otimes\left(\hat{I}-\hat{Z}\right) - 6\;\hat{I}\otimes\hat{I}\right] \quad , \end{aligned} \tag{18}$$

where $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ are the Pauli matrices. The utility of color parity arises from the Hamiltonian containing only the symmetric combination of the plaquette operator $\square + \square^\dagger$ and the trivial vacuum being even under color parity transformation. Simulations performed with IBM's `Athens` used a strong coupling constant of $g = 1$. The system was time evolved using a Trotterized decomposition of the evolution operator to enable an efficient mapping onto quantum circuits [78]. Both first- and second-order Trotterizations were employed, using single step ($\delta t = t$) and multiple steps ($\delta t = t/2$) for both. Applying standard error mitigation techniques (for CNOT errors) and fitting systematic error estimation, the vacuum-to-vacuum probability $|\langle\mathbf{1}|\hat{U}_t|\mathbf{1}\rangle|^2$ (the vacuum is the plaquette in the $|\mathbf{1}\rangle \equiv |0\rangle\otimes|0\rangle$ state) and the expectation value of the electric energy (the g^2 terms in Eq. (18)) were computed as a function of time, as shown in Fig. 8 of Ref. [78].

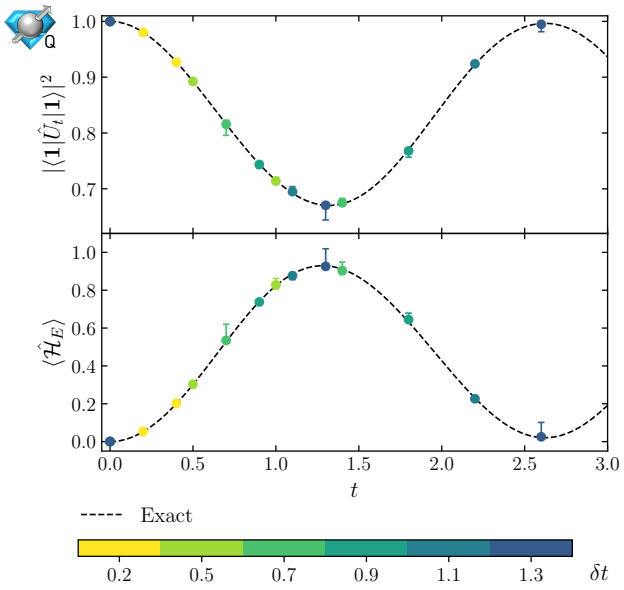

FIG. 10. The vacuum-to-vacuum probability $|\langle\mathbf{1}|\hat{U}_t|\mathbf{1}\rangle|^2$ (upper panel) and energy (in units of g^2) in the electric field (lower panel) of the one-plaquette system as a function of time (in units of $1/g^2$). The circles correspond to results obtained using D-Wave's QA `Advantage` with the maximal number of zoom steps ($z^{\max} = 14$), $K = 2$, $N_A = 10^3$ number of anneals, different time shifts δt, and $n_T = 3$ time steps $\{0, \delta t, 2\delta t\}$. The uncertainties correspond to the 68% confidence intervals determined from $N_{\rm run} = 20$ independent runs. The dashed curves correspond to the exact theoretical curves.

In the present work, using exact matrix exponentiation of the Hamiltonian to determine the evolution operator over the time interval δt, a QUBO matrix describing the Feynman clock evolution of this system was formed. Using the techniques described in previous sections to find eigenstates and energies, `Advantage` was used to evolve the SU(3) plaquette system forward in time from an initial state of the trivial vacuum. Results obtained for the vacuum-to-vacuum persistent probability and for the energy in the electric field using `Advantage` are shown in Fig. 10. These results are found to agree with exact theoretical curves within uncertainties. Further, the precision of the results are significantly better than those obtained previously using IBM's `Athens` [78].

B. Neutrino Flavor Dynamics in Beam-Beam Collisions

Neutrino flavor dynamics is a major focus of research in Standard Model physics. While neutrinos are rendered massless by dim-4 operators in the Standard Model, a result of the local gauge symmetries and particle content, in particular the absence of a right-handed neutrino field, the unambiguous observations of neutrino flavor dynamics, and non-zero mass differences, provides unique insight into aspects of physics beyond the Standard Model and the structure of higher-dimension operators. The connection between lepton-number violating Majorana neutrino masses and interactions that induce neutrinoless $\beta\beta$-decay of nuclei is a strong theoretical motivation driving the current experimental program(s) searching for such processes (for a recent review, see, e.g., Ref. [157]). Non-zero neutrino masses, when combined with Standard Model electroweak interactions, have implications for matter under the extreme conditions of density and temperature that are found in the early universe, e.g., Refs. [158–160], and core-collapse supernova, e.g., Refs. [159, 161] (for recent works, see, e.g., Ref. [162]). Decades of work on this subject continue to uncover new phenomena in neutrino dynamics in these environments, including the recent identification of dynamical phase transitions in collective dynamics and correlations with quantum entanglement [163–171]. With their importance in transport from within the core, high-precision simulations of the evolution of supernova require the inclusion of 3-dim neutrino distributions with detailed quantum kinetics, a problem that has been estimated to lie beyond classical exascale computing. This has prompted the increasing number of explorations of neutrino dynamics using quantum simulations [79, 172, 173] and modern theoretical tools using entanglement as an essential ingredient, e.g., tensor networks [170, 171, 174], and with, for example, classical simulations utilizing symmetries and matrix sparsity [175] to study systems currently beyond reach of matrix product states. These works constitute important explorations of the roles of quantum information, entanglement and real-time dynamics in dense neutrino systems, and make inroads into unifying previously identified collective phenomena while searching for new behaviors.

One such recent detailed study, which we parallel, by Hall *et al.* [79], performed quantum simulations of systems of $N = 4$ neutrinos, restricted to two active flavors and without the inclusion of electroweak interactions with matter (such as $e^{\pm}$, $\mu^{\pm}$ and $q, \overline{q}$'s) but with self-interactions. Using the known mapping of the two-flavor neutrino system to quantum spin models, it was simulated using IBM's `Vigo` superconducting quantum computer for a selection of parameters, including mass differences and neutrino densities, using the effective Hamiltonian,

$$\hat{\mathcal{H}} = \frac{1}{2}\sum_{i}^{N}(-\Delta_i \cos 2\theta_v \ \sigma_i^z + \Delta_i \sin 2\theta_v \ \sigma_i^x) + \ \kappa \ \sum_{i<j}^{N}(1-\cos\theta_{ij}) \ \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \quad , \tag{19}$$

where θ_v is the flavor mixing angle, that is set to $\theta_v = 0.195$ for this model, $\Delta_i = \delta m^2/(2E_i)$ is the strength of the one-body term determined by the difference in neutrino squared masses δm^2 and the energy of each neutrino, E_i. The strength of the two-body term, κ, depends on the neutrino density and electroweak couplings, and θ_{ij} is the angle between the momenta of the i^{th} and j^{th} neutrinos. The spin operators $\boldsymbol{\sigma}_i$ act in the two-dimensional neutrino flavor space, $\boldsymbol{\nu}_i = (\nu_{i,e}, \nu_{i,\mu})^T$.

For the test-case model simulation presented in Ref. [79], a monochromatic neutrino beam is assumed, with $E_i = \delta m^2/(4\kappa)$, and with an anisotropic distribution of momentum directions, $\theta_{ij} = \arccos(\zeta) \times |i-j|/(N-1)$, with $\zeta = 0.9$.[5] The time evolution of the system was determined by first-order Trotterization of the evolution operator derived from the Hamiltonian separated into neutrino-pair terms (as opposed to one- and two-body operators) [79]. One of the observables examined was the probability of the i^{th} neutrino transforming between flavors $\nu_e \leftrightarrow \nu_\mu$,

$$P_i(t) = \frac{1}{2}\langle\Psi_t|1 \mp \sigma_i^z|\Psi_t\rangle \quad , \tag{20}$$

starting with $|\Psi_0\rangle = |\nu_e\nu_e\nu_\mu\nu_\mu\rangle$, and with the sign depending on the initial state of the system, $\nu_{i,e}$ $(-)$ or $\nu_{i,\mu}$ $(+)$. The results of those simulations can be found in their Figs. 3 and 4 of their paper [79]. The evolution of flavor entanglement in the four-neutrino system was also studied in Ref. [79]. The single-neutrino entanglement entropy is given by

$$S_i(t) = -\text{Tr}[\rho_i(t)\log_2(\rho_i(t))] \quad , \tag{21}$$

where $\rho_i(t)$ is the reduced density matrix for the i^{th} neutrino, with $\rho_i(t) = \text{Tr}_{j\neq i}[\ |\Psi_t\rangle\langle\Psi_t|\ ]$. The concurrence was also studied [79], and in this work we consider the logarithmic negativity,

$$\mathcal{N}_{ij}(t) = \log_2 ||\rho_{ij}^{\Gamma}(t)||_1 \quad , \tag{22}$$

where $\rho_{ij}(t)$ is the two-neutrino reduced density matrix for the ij neutrino pair, Γ indicates the partial transposition of ρ, and $||\cdot||_1$ is the trace norm. The logarithmic negativity, related to the concurrence, is an upper bound on the distillable entanglement. The neutrino Hamiltonian in Eq. (19) is invariant under neutrino exchanges

[5] i.e., $\theta_{12} = \theta_{23} = \theta_{34} = \frac{1}{3}\arccos(\zeta)$, $\theta_{13} = \theta_{24} = \frac{2}{3}\arccos(\zeta)$ and $\theta_{14} = \arccos(\zeta)$.

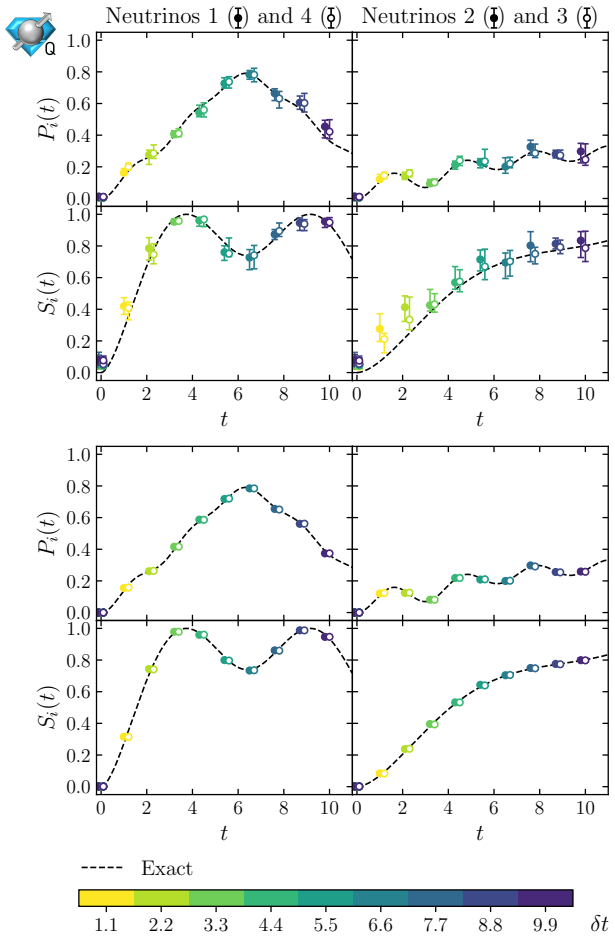

FIG. 11. The probability of flavor transitions (panel rows 1 and 3) and the single-neutrino entanglement entropy (panel rows 2 and 4) for the 1st and 4th (left panels) and 2nd and 3rd (right panels) neutrinos as a function of time (in units of $1/\kappa$). The results are obtained using D-Wave's QA `Advantage` with the maximal number of zoom steps, $K = 2$, $N_A = 10^3$ number of anneals, different time shifts δt, and $n_T = 2$ time steps $\{0, \delta t\}$ (they have been shifted slightly along the x-axis for clarity). The upper four panels show the raw results, while the lower four panels show the results after two iterations of the procedure described in the main text. The uncertainties correspond to the 68% confidence intervals determined from $N_{\rm run} = 20$ independent runs. The dashed curves correspond to the exact theoretical curves.

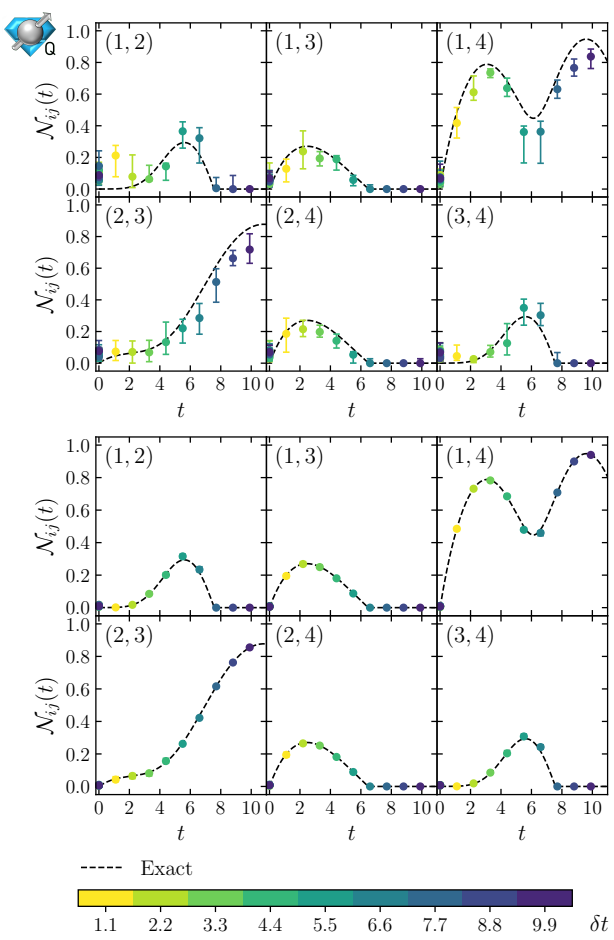

FIG. 12. The logarithmic negativity of pairs of neutrinos as a function of time (in units of $1/\kappa$). The results are obtained using D-Wave's QA `Advantage` with the maximal number of zoom steps, $K = 2$, $N_A = 10^3$ number of anneals, different time shifts δt, and $n_T = 2$ time steps $\{0, \delta t\}$.The upper six panels show the raw results, while the lower six panels show the results after two iterations of the procedure described in the main text. The uncertainties correspond to 68% confidence intervals determined from $N_{\rm run} = 20$ independent runs. The dashed curves correspond to the exact theoretical curves.

$1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ [79]. This gives rise to relations between observables, such as $P_1(t) = P_4(t)$, $S_1(t) = S_4(t)$, and $\mathcal{N}_{12}(t) = \mathcal{N}_{34}(t)$.

The results of our quantum simulations obtained using `Advantage` are shown in Figs. 11 and 12. The implementation of the clock state using the matrix representation of the exact evolution operator between time-slices, without Trotterization into products of unitaries associated with neutrino pairs, eliminates a significant source of ("theory") systematic error imposed by circuit-volume limitations of available devices with different architectures, as can be seen by comparing the results shown in Figs. 11 and 12 and the results presented in Figs. 3, 4 and 6 of Ref. [79]. Further, and equally important, the absence of systematic errors associated with device performance in the simulation, dominated by CNOT gates and subsequent mitigation procedures, improves the accuracy of the simulations of this system that are possible with `Advantage` compared with other quantum devices.

The uncertainties associated with the dynamics of four neutrinos are considerably larger than for those associated with the single plaquette of SU(3) Yang-Mills lattice gauge theory, discussed in the previous subsection. This is due to a larger QUBO matrix that is passed to the annealer, and is one indication of the scaling of the capabilities of `Advantage` with increasing system size.

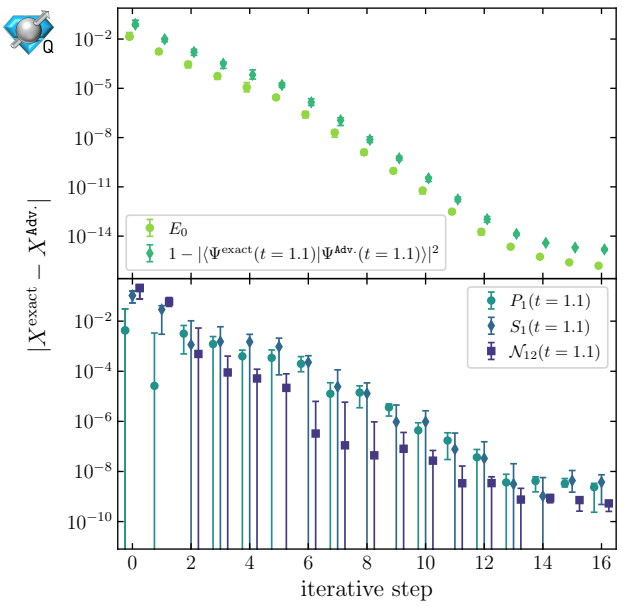

FIG. 13. Convergence of the ground-state energy and wavefunction (upper panel), and the flavor transition probability, single-neutrino entanglement, and logarithmic negativity (lower panel) as a function of the number of steps of the iterative procedure described in the main text. The results are obtained using D-Wave's QA `Advantage` with the maximal number of zoom steps, $K = 2$, and $N_A = 10^3$ number of anneals. The uncertainties correspond to 68% confidence intervals determined from $N_{\rm run} = 20$ independent runs.

Adding one more neutrino to the system renders the problem intractable for `Advantage` as the QUBO matrix will not fit onto its QPU.[6] Additionally, it can be seen that the uncertainties for the logarithmic negativity in Fig. 12 are larger than those of the single-neutrino entanglement entropy in Fig. 11, which are in turn larger than the neutrino flavor transition probability, also in Fig. 11. The study of such quantum correlations requires high-precision calculations. In some cases, the wavefunctions $|\Psi_t\rangle$ are determined with $10^{-1} - 10^{-2}$ precision, which is seen to be insufficient.

One way of improving the results (and reduce uncertainties) is to increase the number of anneals N_A by a factor ξ. As this leads to only a $1/\sqrt{\xi}$ reduction in the uncertainty in energies, this (brute force) approach demands excessive computational resources. A better method for reducing the uncertainties is analogous to the multigrid method used in Sec. III. The challenge here is that it is not straightforward to define the Hamiltonian for a smaller system to provide interpolating wavefunctions for the larger system (the four-neutrino case is not continuously connected to the 2,3 (or 5) neutrino systems). We have found that using the solution from the QA as a starting point for a subsequent anneal, but with $z \neq 0$ (to narrow the window of a_α that the QA can explore), leads to approximately a one-order magnitude reduction in the uncertainty in the energy (while only doubling the number of anneals). As an example, the results obtained after two steps of this iterative procedure are shown (in panels below the raw results) in Figs. 11 and 12, and show clear reductions in the uncertainties.

This iterative procedure can be repeated several times until no further improvement is obtained. For the four-neutrino system, we obtain an ultimate precision of 10^{-16} in the energy, and $10^{-8} - 10^{-10}$ in the flavor transition probability, single-neutrino entanglement, and logarithmic negativity (as shown in Fig. 13 for $t = 1.1$, in units of $1/\kappa$). It appears that the success of this iterative method is due, in part, to the ground-state energy of the TEDVP objective function being *a priori* known to vanish.

In contrast to the calculations with IBM's superconducting hardware, where scaling to larger problems is limited by qubit and gate fidelity, along with connectivity, for increasing qubit requirements, the qubit footprint on the annealing devices naively scales exponentially with the number of neutrinos. This is expected to be mitigated using techniques that have enabled classical computing to provide a series of precision calculations in these model systems.

V. CONCLUSIONS

We have explored the potential of D-Wave's quantum annealers for simulating some key basic aspects of Standard Model physics. In particular, the eigenstates and energies of the lowest-lying states of the harmonic oscillator and anharmonic oscillator were studied using zooming and basic coordinate-space multigrid (MG-AQAE). Deviations in the extracted energies were less than $\sim 10^{-4}$. These simulations are the basic elements of lattice scalar field theory of importance, for instance, low-energy chiral nuclear physics or high-energy Higgs physics. The time evolution of a single plaquette of SU(3) Yang-Mills gauge theory truncated to $\{|\mathbf{1}\rangle, |\mathbf{3}^+\rangle, |\mathbf{6}^+\rangle, |\mathbf{8}\rangle\}$ in the color parity basis, and of neutrino flavor in a monochromatic beam with angular dispersion were also studied through a refinement of the Feynman clock algorithm, with deviations that can be reduced below 10^{-8}. The results of our quantum simulations of the plaquette and neutrino evolution are found to compare favorably with previous quantum simulations performed using IBM's superconducting quantum computers.

Except for the cases in which the Hamiltonian of interest directly maps (or can be efficiently transformed) onto the annealer, which is optimal for transverse-field Ising models (e.g., Ref. [43]), with the current formulation, the qubit requirements on the device scale with size of the

[6] We attempted to study the $N = 5$ neutrino system with `Advantage` by setting $K = 1$, but the results (and uncertainty estimations) obtained were unreliable. One of the issues is that a_α, with $K = 1$, only takes two values, $\{-1, 0\}$, requiring a large value of η to prevent the null solution.

Hilbert space. Therefore, the maximum dimensionality of the systems that we have considered in this work that can be addressed with `Advantage` remains small. For the harmonic and anharmonic oscillators, the maximum number of states in the decimation of the field wavefunction that we could reliably simulate was $n_s = 64$. For time evolution, due to the extra $2n_T$ factor in the dimensions of the QUBO matrix, this decimation is reduced to 16. Further, there is a somewhat unfortunate dependence on the η parameter, although this can be partially mitigated by performing a more structured scan, as shown in Ref. [54]. It seems that paths forward for simulating quantum field theories requires efficiently utilizing the innate Ising-Hamiltonian architectures of the annealers, including the dynamics of the annealing process. A step in this direction has been taken in Refs. [176, 177], that utilizes the JLP field mapping to qubits but not the quantum Fourier Transform onto conjugate momentum space, and has the potential to improve the scaling of ground-state preparation. With the current formulation, the main application can be the preparation of states which are later used in universal quantum computers to perform time evolution, acting as preconditioners to speed up the process of finding ground states using domain decomposition techniques to reduce the size of the problem (for example, by computing the angles to set up the wavefunction for a scalar field theory [178]). A somewhat different implementation employing Floquet engineering also has promise [179].

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Appendix A: The QUBO Matrix

In this appendix, an outline is presented of the derivation of expressions for the QUBO matrix used in Sec. II to compute eigenstates and energies of a given Hamiltonian, and in Sec. IV to determine the time-evolution of a SU(3) Yang-Mills plaquette and a system of four neutrinos.

Starting from the objective function in Eq. (2) for the eigenstates and energies of a given Hamiltonian, and using the fixed-point representation for the zoom coefficients in Eq. (5),

$$F = \sum_{\alpha\beta} a_\alpha a_\beta h_{\alpha\beta} = \sum_{\alpha\beta} \left[a_\alpha^{(z)} + \sum_i 2^{i-K-z} (-1)^{\delta_{iK}} q_i^\alpha \right] \left[a_\beta^{(z)} + \sum_j 2^{j-K-z} (-1)^{\delta_{jK}} q_j^\beta \right] h_{\alpha\beta} \quad , \tag{A1}$$

which consists of three different types of terms. The first type is the product of $a_\alpha^{(z)} a_\beta^{(z)}$, without q_i^α variables. It provides a constant off-set to F, which does not modify the position of the minimum, and thus can be omitted from the QUBO matrix. The second type comes from the product of terms with q_i^α and q_j^β, which is of the desired form in Eq. (4), and multiplied with a 2^{-2z} zooming factor. The last type originates from the product of $a_\alpha^{(z)}$ with q_j^β, which, as it involves a single q_j^β, naively has the potential to be problematic for forming a viable QUBO matrix. The difficulty is averted by using a defining property of binary variables, $q_j^\beta = (q_j^\beta)^2$, leading to, for example,

$$\begin{aligned} \sum_{\alpha\beta} a_\alpha^{(z)} \sum_j (-1)^{\delta_{jK}} 2^{j-K-z} q_j^\beta h_{\alpha\beta} &= \sum_{\beta,j} (-1)^{\delta_{jK}} 2^{j-K-z} q_j^\beta q_j^\beta \underbrace{\left(\sum_\alpha a_\alpha^{(z)} h_{\alpha\beta} \right)}_{\alpha\to\gamma} \\ &= \sum_{\alpha\beta,ij} \delta_{\alpha\beta}\delta_{ij} (-1)^{\delta_{jK}} 2^{j-K-z} q_i^\alpha q_j^\beta \sum_\gamma a_\gamma^{(z)} h_{\gamma\beta} \quad . \end{aligned} \tag{A2}$$

As $h_{\alpha\beta}$ is symmetric, the two contributions of this form give the same contribution, leading to a QUBO matrix of the form, as given in Eq. (6),

$$Q_{\alpha,i;\beta,j} = 2^{i+j-2K-2z} (-1)^{\delta_{iK}+\delta_{jK}} h_{\alpha\beta} + 2\delta_{\alpha\beta}\delta_{ij} 2^{i-K-z} (-1)^{\delta_{iK}} \sum_\gamma a_\gamma^{(z)} h_{\gamma\beta} \quad . \tag{A3}$$

In order to reduce this four-index array into a more manageable matrix form, and following Ref. [75], the indices α and i are combined into $n = K(\alpha-1)+i$.

To construct the QUBO matrix for time evolution using the Feynman clock method, discussed in Sec. IV, the starting point is to use Eq. (16) to write the objective function in Eq. (15) as a real function, as required for implementation on D-Wave's QA,

$$F^{(\mathfrak{C})} = \sum_{\alpha\beta} \left(a_\alpha^{\mathrm{Re}} a_\beta^{\mathrm{Re}} \mathfrak{C}_{\alpha\beta}^{\mathrm{Re}} - a_\alpha^{\mathrm{Re}} a_\beta^{\mathrm{Im}} \mathfrak{C}_{\alpha\beta}^{\mathrm{Im}} + a_\alpha^{\mathrm{Im}} a_\beta^{\mathrm{Re}} \mathfrak{C}_{\alpha\beta}^{\mathrm{Im}} + a_\alpha^{\mathrm{Im}} a_\beta^{\mathrm{Im}} \mathfrak{C}_{\alpha\beta}^{\mathrm{Re}} \right) \quad . \tag{A4}$$

The rows and columns of the QUBO matrix are doubled in length to accommodate the real and imaginary parts of each coefficient $a_\alpha = a_\alpha^{\mathrm{Re}} + i a_\alpha^{\mathrm{Im}}$. This is accomplished simply by increasing the range of the i index from $[1,K]$ to $[1,2K]$, with $1 \le i \le K$ used for a_α^{Re}, and $(K+1) \le i \le 2K$ for a_α^{Im}. Obtaining the expression for the QUBO matrix associated with $F^{(\mathfrak{C})}$ follows straightforwardly from its derivation given above for the corresponding QUBO matrix for eigenstates and energies. For example, the first term in Eq. (A4) is analogous to that in Eq. (6),

$$\sum_{\alpha\beta} a_\alpha^{\mathrm{Re}} a_\beta^{\mathrm{Re}} \mathfrak{C}_{\alpha\beta}^{\mathrm{Re}} = \sum_{\alpha\beta} \sum_{i,j=1}^{K} \left[2^{i+j-2K-2z} (-1)^{\delta_{iK}+\delta_{jK}} \mathfrak{C}_{\alpha\beta}^{\mathrm{Re}} + 2\delta_{\alpha\beta}\delta_{ij} 2^{i-K-z} (-1)^{\delta_{iK}} \sum_\gamma a_\gamma^{\mathrm{Re},(z)} \mathfrak{C}_{\gamma\beta}^{\mathrm{Re}} \right] q_i^\alpha q_j^\beta \quad , \tag{A5}$$

where the sum over the indices i,j is over the range $[1,K]$. The last term in Eq. (A4) is also analogous, but with i,j summed over $[K+1,2K]$,

$$\sum_{\alpha\beta} a_\alpha^{\mathrm{Im}} a_\beta^{\mathrm{Im}} \mathfrak{C}_{\alpha\beta}^{\mathrm{Re}} = \sum_{\alpha\beta} \sum_{i,j=K+1}^{2K} \left[2^{i'+j'-2K-2z} (-1)^{\delta_{i'K}+\delta_{j'K}} \mathfrak{C}_{\alpha\beta}^{\mathrm{Re}} + 2\delta_{\alpha\beta}\delta_{i'j'} 2^{i'-K-z} (-1)^{\delta_{i'K}} \sum_\gamma a_\gamma^{\mathrm{Im},(z)} \mathfrak{C}_{\gamma\beta}^{\mathrm{Re}} \right] q_i^\alpha q_j^\beta \quad , \tag{A6}$$

where i' is defined by $i' \equiv i - K$.

For the second and third terms in Eq. (A4), we employ the same binary identity, $q_i^\alpha = (q_i^\alpha)^2$, but keep in mind that the indices i, j run over different values. For example, for the $a_\alpha^{\mathrm{Re}} a_\beta^{\mathrm{Im}}$ term,

$$-\sum_{\alpha\beta} a_\alpha^{\mathrm{Re}} a_\beta^{\mathrm{Im}} \mathcal{C}_{\alpha\beta}^{\mathrm{Im}} = -\sum_{\alpha\beta} \left[\sum_{i=1}^{K} \sum_{j=K+1}^{2K} 2^{i+j'-2K-2z} (-1)^{\delta_{iK}+\delta_{j'K}} \mathcal{C}_{\alpha\beta}^{\mathrm{Im}} + \sum_{i,j=K+1}^{2K} \delta_{\alpha\beta}\delta_{i'j'} 2^{j'-K-z} (-1)^{\delta_{j'K}} \sum_\gamma a_\gamma^{\mathrm{Re},(z)} \mathcal{C}_{\gamma\beta}^{\mathrm{Im}} + \sum_{i,j=1}^{K} \delta_{\alpha\beta}\delta_{ij} 2^{i-K-z} (-1)^{\delta_{iK}} \sum_\gamma a_\gamma^{\mathrm{Im},(z)} \mathcal{C}_{\alpha\gamma}^{\mathrm{Im}} \right] q_i^\alpha q_j^\beta \,. \tag{A7}$$

Collecting together the contributions, the following QUBO matrix for the Feynman clock is obtained,

$$Q_{\alpha,i;\beta,j} = \begin{cases} 2^{i+j-2K-2z}(-1)^{\delta_{iK}+\delta_{jK}} \mathcal{C}_{\alpha\beta}^{\mathrm{Re}} & \\ \quad +\delta_{\alpha\beta}\delta_{ij} 2^{i-K-z}(-1)^{\delta_{iK}} \sum_\gamma \left(2a_\gamma^{\mathrm{Re},(z)} \mathcal{C}_{\gamma\beta}^{\mathrm{Re}} - a_\gamma^{\mathrm{Im},(z)} \mathcal{C}_{\alpha\gamma}^{\mathrm{Im}} + a_\gamma^{\mathrm{Im},(z)} \mathcal{C}_{\gamma\beta}^{\mathrm{Im}} \right) & 1 \le i,j \le K\,, \\ -2^{i+j'-2K-2z}(-1)^{\delta_{iK}+\delta_{j'K}} \mathcal{C}_{\alpha\beta}^{\mathrm{Im}} & 1 \le i,j' \le K\,, \\ 2^{i'+j-2K-2z}(-1)^{\delta_{i'K}+\delta_{jK}} \mathcal{C}_{\alpha\beta}^{\mathrm{Im}} & 1 \le i',j \le K\,, \\ 2^{i'+j'-2K-2z}(-1)^{\delta_{i'K}+\delta_{j'K}} \mathcal{C}_{\alpha\beta}^{\mathrm{Re}} & \\ \quad +\delta_{\alpha\beta}\delta_{i'j'} 2^{i'-K-z}(-1)^{\delta_{i'K}} \sum_\gamma \left(2a_\gamma^{\mathrm{Im},(z)} \mathcal{C}_{\gamma\beta}^{\mathrm{Re}} - a_\gamma^{\mathrm{Re},(z)} \mathcal{C}_{\gamma\beta}^{\mathrm{Im}} + a_\gamma^{\mathrm{Re},(z)} \mathcal{C}_{\alpha\gamma}^{\mathrm{Im}} \right) & 1 \le i',j' \le K\,. \end{cases} \tag{A8}$$

This can be somewhat simplified because $\mathcal{C}_{\alpha\beta}^{\mathrm{Im}}$ is antisymmetric while $\mathcal{C}_{\alpha\beta}^{\mathrm{Re}}$ is symmetric, leading to the expressions given in Eq. (16),

$$Q_{\alpha,i;\beta,j} = \begin{cases} 2^{i+j-2K-2z}(-1)^{\delta_{iK}+\delta_{jK}} \mathcal{C}_{\alpha\beta}^{\mathrm{Re}} + 2\delta_{\alpha\beta}\delta_{ij} 2^{i-K-z}(-1)^{\delta_{iK}} \sum_\gamma \left(a_\gamma^{\mathrm{Re},(z)} \mathcal{C}_{\gamma\beta}^{\mathrm{Re}} + a_\gamma^{\mathrm{Im},(z)} \mathcal{C}_{\gamma\beta}^{\mathrm{Im}} \right) & 1 \le i,j \le K\,, \\ -2^{i+j'-2K-2z}(-1)^{\delta_{iK}+\delta_{j'K}} \mathcal{C}_{\alpha\beta}^{\mathrm{Im}} & 1 \le i,j' \le K\,, \\ 2^{i'+j-2K-2z}(-1)^{\delta_{i'K}+\delta_{jK}} \mathcal{C}_{\alpha\beta}^{\mathrm{Im}} & 1 \le i',j \le K\,, \\ 2^{i'+j'-2K-2z}(-1)^{\delta_{i'K}+\delta_{j'K}} \mathcal{C}_{\alpha\beta}^{\mathrm{Re}} + 2\delta_{\alpha\beta}\delta_{i'j'} 2^{i'-K-z}(-1)^{\delta_{i'K}} \sum_\gamma \left(a_\gamma^{\mathrm{Im},(z)} \mathcal{C}_{\gamma\beta}^{\mathrm{Re}} - a_\gamma^{\mathrm{Re},(z)} \mathcal{C}_{\gamma\beta}^{\mathrm{Im}} \right) & 1 \le i',j' \le K\,. \end{cases} \tag{A9}$$

Appendix B: More on our Results

The (average) time it takes to perform $N_A = 10^3$ anneals on both the `neal` simulator as well as on `Advantage` for the HO and aHO Hamiltonian for a single zoom step from Sec. III is reported in Table III. Regarding the time specific to the quantum processor, it takes into account two separate steps. The first one is the programming time, during which the values of the QUBO matrix are transferred to the quantum processor (same for all problem sizes, $8 - 15$ ms). The next step is the annealing phase, repeated N_A times, which is further decomposed into three steps: the annealing time ($t_A = 20$ μs), the readout time (size-dependent, varying between $100 - 200$ μs), and the delay time (same for all problem sizes, 20.54 μs), during which the system is allowed to cool and reinitialized. For `Advantage`, the time spent finding the embedding should also be taken into account (although this part is not computed on the quantum processor, but locally), therefore its contribution is shown in a separate column.

Problem size	`neal`	`Advantage` (without embedding)	`Advantage` (with embedding)
$n_s = 16,\ K = 3$	0.7 s	0.15 s	10.15 s
$n_s = 32,\ K = 3$	2.0 s	0.20 s	100.20 s
$n_s = 64,\ K = 2$	2.6 s	0.25 s	200.25 s

TABLE III. Comparison of the (average) time (seconds) taken to perform $N_A = 10^3$ anneals for a specific problem size (fixed n_s and K) and for a single zoom step between the simulator `neal`, run on a 2.40 GHz Intel Core i9-9980HK CPU, and the quantum processor `Advantage` ($t_A = 20$ μs), with and without including the time for the embedding.

Although the time required to find the configuration with minimum energy is around an order of magnitude smaller for `Advantage` than `neal` (increasing t_A would reduce this difference), the overhead time spent computing the embedding (not needed for the simulator) inverts the situation. As mentioned in Sec. III B, the embedding can be reused for the multiple zoom steps. This is possible because the required connectivity between the different logic qubits does not change, only the entries of the QUBO matrix change. More specifically, and looking at Eq. (6), the additional term obtained from the zooming is added in the diagonal part of the QUBO matrix, where $\alpha = \beta$ and $i = j$ (the same argument can be used for Eq. (16)). Table III does not take into account the time spent building the QUBO matrix or analyzing the results, since these steps are the same for both cases.

All of the results shown in the main text can be found in HDF5 format [182] in the file `SSMQA_data.h5`, where each set is labeled by the figure number, with additional metadata to specify the parameters used during its production and the dimension of the array (like the value of N_{run} or the number of zoom levels). For the results related to Figs. 10, 11 and 12, only the compound states $|\Psi_t\rangle|t\rangle$ are included.

Additionally, in the following tables we provide the values plotted in the figures from the main text. The results shown in Figs. 6 and 7 are given in Table IV. The results shown in Fig. 8 are given in Table V. The results shown in Fig. 9 are given in Table VI. The results shown in Fig. 10 are given in Table VII. The results shown in Fig. 11 are given in Table VIII (raw results) and IX (after two steps of the iterative procedure), and the results shown in Fig. 12 are given in Table X (raw results) and XI (after two steps of the iterative procedure).

n	$\lvert\delta E_n^{\text{Adv.}-16}\rvert_{\text{HO}}$	$\lvert\delta E_n^{\text{Adv.}-32}\rvert_{\text{HO}}$	$\lvert\delta E_n^{\text{Adv.}-64}\rvert_{\text{HO}}$	$\lvert\delta E_n^{\text{Adv.}-16}\rvert_{\text{aHO}}$	$\lvert\delta E_n^{\text{Adv.}-32}\rvert_{\text{aHO}}$	$\lvert\delta E_n^{\text{Adv.}-64}\rvert_{\text{aHO}}$
0	$(2.4^{+6.1}_{-1.5}) \times 10^{-6}$	$(3.9^{+7.8}_{-2.3}) \times 10^{-6}$	$(1.2^{+0.3}_{-0.1}) \times 10^{-6}$	$(3.4^{+22}_{-3.4}) \times 10^{-4}$	$(1.5^{+3.0}_{-1.0}) \times 10^{-5}$	$(1.9^{+1.1}_{-0.2}) \times 10^{-6}$
1	$(3.6^{+52}_{-1.9}) \times 10^{-6}$	$(9.6^{+11}_{-2.6}) \times 10^{-6}$	$(3.6^{+0.2}_{-0.2}) \times 10^{-6}$	$(2.6^{+186}_{-2.6}) \times 10^{-4}$	$(1.9^{+2.4}_{-1.1}) \times 10^{-5}$	$(2.7^{+0.5}_{-0.9}) \times 10^{-6}$
2	$(1.9^{+5.5}_{-1.2}) \times 10^{-6}$	$(9.8^{+15}_{-3.1}) \times 10^{-6}$	$(2.8^{+0.4}_{-0.2}) \times 10^{-6}$	$(0.7^{+4487}_{-0.7}) \times 10^{-5}$	$(2.2^{+2.4}_{-1.7}) \times 10^{-5}$	$(8.3^{+1.0}_{-0.9}) \times 10^{-6}$
3	$(0.8^{+117}_{-0.6}) \times 10^{-5}$	$(9.7^{+1.5}_{-0.3}) \times 10^{-5}$	$(30^{+1.8}_{-0.2}) \times 10^{-6}$	$(1.7^{+1.8}_{-0.9}) \times 10^{-6}$	$(0.7^{+1.8}_{-0.8}) \times 10^{-5}$	$(6.0^{+1.4}_{-2.0}) \times 10^{-6}$
4	$(0.5^{+744}_{-0.5}) \times 10^{-5}$	$(1.6^{+1.1}_{-0.3}) \times 10^{-4}$	$(8.0^{+0.6}_{-0.7}) \times 10^{-5}$	$(0.6^{+14}_{-0.5}) \times 10^{-6}$	$(6.3^{+14}_{-4.1}) \times 10^{-5}$	$(5.3^{+1.1}_{-1.8}) \times 10^{-5}$
5	$(2.9^{+29}_{-1.8}) \times 10^{-6}$	$(4.7^{+6.6}_{-5.3}) \times 10^{-5}$	$(1.3^{+0.5}_{-0.5}) \times 10^{-4}$	$(1.5^{+20}_{-0.7}) \times 10^{-6}$	$(6.1^{+4.2}_{-4.9}) \times 10^{-4}$	$(8.1^{+1.9}_{-0.6}) \times 10^{-5}$

TABLE IV. Differences between the digitized energies and the corresponding results obtained using D-Wave's `Advantage` for the HO with $m_0 = 1$, $\phi_{\text{max}} = 5$, and the aHO with $m_0 = 1$, $\lambda = 32$, $\phi_{\text{max}} = 2.6$, shown in Figs. 6 and 7, with $n_s = \{16, 32, 64\}$, $K = \{3, 3, 2\}$, $N_A = 10^3$, and the maximum number of zoom steps. The uncertainties correspond to 68% confidence intervals determined from $N_{\text{run}} = 20$ independent runs.

n	$\lvert\delta E_n^{\text{Adv.}-32}\rvert$
0	$(2.5^{+5.0}_{-2.4}) \times 10^{-3}$
1	$(1.2^{+16}_{-1.1}) \times 10^{-3}$

TABLE V. Differences between the digitized energies and the corresponding results obtained using D-Wave's `Advantage` for the aHO with $m_0^2 = -4$, $\lambda = 1$, $\phi_{\text{max}} = 9$, shown in Fig. 8, with $n_s = 32$, $K = 3$, $N_A = 10^3$, and the maximum number of zoom steps. The uncertainties correspond to 68% confidence intervals determined from $N_{\text{run}} = 20$ independent runs.

n_s	`2000Q`	`Advantage`
4	22^{+1}_{-0}	12^{+0}_{-0}
8	83^{+1}_{-2}	39^{+1}_{-1}
16	337^{+8}_{-9}	141^{+8}_{-6}
32	1464^{+39}_{-109}	528^{+24}_{-26}
64	–	2077^{+293}_{-254}

TABLE VI. Number of physical qubits required to map the QUBO matrix for the two available D-Wave's QAs, shown in Fig. 9, with $K = 2$. The uncertainties correspond to 68% confidence intervals determined from 20 different embedding with the same problem.

δt	t	$\lvert\langle 00\lvert\hat{U}_t\rvert 00\rangle\rvert^2$	$\langle\hat{\mathcal{H}}_E\rangle$
0.2	0	0.999988^{+10}_{-46}	$(0.6^{+2.5}_{-0.5}) \times 10^{-4}$
	0.2	0.9802^{+13}_{-3}	0.0537^{+43}_{-14}
	0.4	0.9271^{+23}_{-6}	0.2018^{+97}_{-33}
0.5	0	0.999956^{+42}_{-64}	$(2.0^{+4.2}_{-1.8}) \times 10^{-4}$
	0.5	0.8921^{+47}_{-42}	0.3030^{+95}_{-71}
	1.0	0.7139^{+66}_{-56}	0.828^{+34}_{-20}
0.7	0	0.99995^{+3}_{-81}	$(0.3^{+4.4}_{-0.2}) \times 10^{-3}$
	0.7	0.816^{+7}_{-20}	0.535^{+85}_{-14}
	1.4	0.6750^{+78}_{-51}	0.903^{+45}_{-20}
0.9	0	0.99999^{+1}_{-13}	$(0.9^{+7.9}_{-0.8}) \times 10^{-4}$
	0.9	0.7435^{+18}_{-66}	0.738^{+11}_{-8}
	1.8	0.768^{+2}_{-11}	0.645^{+34}_{-9}
1.1	0	0.999980^{+17}_{-68}	$(0.9^{+4.4}_{-0.7}) \times 10^{-4}$
	1.1	0.6949^{+91}_{-21}	0.877^{+8}_{-21}
	2.2	0.9239^{+20}_{-20}	0.226^{+7}_{-10}
1.3	0	0.99990^{+9}_{-73}	$(0.4^{+4.1}_{-0.3}) \times 10^{-3}$
	1.3	0.670^{+5}_{-26}	0.926^{+93}_{-17}
	2.6	0.995^{+2}_{-13}	0.025^{+76}_{-9}

TABLE VII. The vacuum-to-vacuum probability $\lvert\langle 00\lvert\hat{U}_t\rvert 00\rangle\rvert^2$ and energy in the electric field $\langle\hat{\mathcal{H}}_E\rangle$ of the one-plaquette system, as shown in Fig. 10, with $K = 2$ and $N_A = 10^3$. The uncertainties correspond to 68% confidence intervals determined from $N_{\rm run} = 20$ independent runs.

δt	t	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	$S_1(t)$	$S_2(t)$	$S_3(t)$	$S_4(t)$
1.1	0	0.0065^{+36}_{-11}	0.0073^{+48}_{-26}	0.0053^{+27}_{-34}	0.0054^{+46}_{-18}	0.052^{+15}_{-8}	0.054^{+40}_{-16}	0.041^{+19}_{-21}	0.044^{+33}_{-10}
	1.1	0.163^{+26}_{-15}	0.121^{+31}_{-8}	0.144^{+22}_{-18}	0.200^{+24}_{-19}	0.420^{+54}_{-52}	0.277^{+95}_{-80}	0.211^{+36}_{-86}	0.408^{+38}_{-75}
2.2	0	0.0092^{+50}_{-31}	0.0077^{+27}_{-21}	0.0062^{+28}_{-33}	0.0070^{+15}_{-27}	0.071^{+25}_{-23}	0.057^{+23}_{-17}	0.051^{+23}_{-25}	0.056^{+6}_{-19}
	2.2	0.276^{+30}_{-61}	0.143^{+22}_{-23}	0.159^{+23}_{-23}	0.285^{+53}_{-32}	0.785^{+68}_{-74}	0.413^{+72}_{-90}	0.334^{+143}_{-64}	0.746^{+67}_{-59}
3.3	0	0.0054^{+31}_{-25}	0.0057^{+37}_{-21}	0.0057^{+36}_{-24}	0.0067^{+29}_{-36}	0.041^{+7}_{-18}	0.042^{+33}_{-10}	0.049^{+19}_{-21}	0.051^{+25}_{-20}
	3.3	0.407^{+21}_{-21}	0.095^{+25}_{-13}	0.101^{+21}_{-12}	0.411^{+31}_{-20}	0.953^{+18}_{-15}	0.426^{+100}_{-37}	0.431^{+68}_{-50}	0.958^{+14}_{-10}
4.4	0	0.0078^{+32}_{-29}	0.0075^{+15}_{-39}	0.0056^{+14}_{-18}	0.0058^{+8}_{-22}	0.058^{+19}_{-16}	0.059^{+8}_{-27}	0.043^{+10}_{-14}	0.048^{+4}_{-14}
	4.4	0.542^{+47}_{-26}	0.212^{+24}_{-27}	0.240^{+26}_{-32}	0.559^{+45}_{-44}	0.961^{+17}_{-36}	0.569^{+100}_{-42}	0.575^{+75}_{-65}	0.968^{+13}_{-46}
5.5	0	0.0088^{+71}_{-41}	0.0092^{+37}_{-44}	0.0060^{+52}_{-24}	0.0066^{+28}_{-30}	0.051^{+55}_{-14}	0.058^{+26}_{-26}	0.051^{+32}_{-24}	0.043^{+25}_{-12}
	5.5	0.727^{+36}_{-29}	0.229^{+23}_{-35}	0.234^{+76}_{-29}	0.737^{+32}_{-40}	0.760^{+41}_{-51}	0.715^{+59}_{-73}	0.670^{+110}_{-83}	0.752^{+99}_{-18}
6.6	0	0.0085^{+46}_{-35}	0.0086^{+55}_{-41}	0.0092^{+37}_{-26}	0.0092^{+38}_{-33}	0.064^{+19}_{-23}	0.066^{+26}_{-28}	0.059^{+20}_{-15}	0.067^{+27}_{-20}
	6.6	0.782^{+26}_{-29}	0.201^{+56}_{-42}	0.220^{+41}_{-26}	0.782^{+41}_{-44}	0.727^{+50}_{-77}	0.695^{+61}_{-101}	0.703^{+69}_{-93}	0.741^{+62}_{-85}
7.7	0	0.0123^{+73}_{-60}	0.0133^{+69}_{-56}	0.0066^{+53}_{-35}	0.0083^{+22}_{-43}	0.087^{+41}_{-33}	0.082^{+46}_{-27}	0.054^{+39}_{-26}	0.051^{+23}_{-14}
	7.7	0.665^{+29}_{-46}	0.326^{+42}_{-53}	0.286^{+57}_{-28}	0.632^{+45}_{-60}	0.873^{+30}_{-33}	0.803^{+87}_{-103}	0.750^{+41}_{-63}	0.895^{+51}_{-34}
8.8	0	0.009^{+13}_{-4}	0.0083^{+67}_{-34}	0.0081^{+69}_{-49}	0.0065^{+68}_{-31}	0.065^{+45}_{-25}	0.059^{+24}_{-16}	0.055^{+49}_{-27}	0.053^{+39}_{-24}
	8.8	0.606^{+41}_{-42}	0.280^{+25}_{-24}	0.273^{+33}_{-22}	0.603^{+61}_{-57}	0.945^{+32}_{-47}	0.813^{+37}_{-32}	0.793^{+44}_{-41}	0.939^{+27}_{-42}
9.9	0	0.0104^{+49}_{-28}	0.0106^{+66}_{-29}	0.0106^{+24}_{-43}	0.0106^{+45}_{-40}	0.078^{+23}_{-20}	0.079^{+34}_{-25}	0.075^{+16}_{-24}	0.077^{+28}_{-27}
	9.9	0.455^{+39}_{-72}	0.298^{+51}_{-73}	0.246^{+99}_{-36}	0.422^{+76}_{-45}	0.954^{+16}_{-38}	0.834^{+57}_{-107}	0.787^{+106}_{-85}	0.949^{+30}_{-24}

TABLE VIII. The (raw) probability of flavor transitions $P_i(t)$ and single-neutrino entanglement entropy $S_i(t)$, as shown in Fig. 11, with $K = 2$ and $N_A = 10^3$. The uncertainties correspond to 68% confidence intervals determined from $N_{\rm run} = 20$ independent runs.

δt	t	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	$S_1(t)$	$S_2(t)$	$S_3(t)$	$S_4(t)$
1.1	0	0.00013^{+11}_{-7}	0.00015^{+6}_{-7}	0.00010^{+9}_{-3}	0.00011^{+8}_{-4}	0.0018^{+12}_{-9}	0.0020^{+8}_{-8}	0.0013^{+11}_{-4}	0.0013^{+10}_{-4}
	1.1	0.1553^{+35}_{-27}	0.1205^{+25}_{-29}	0.1242^{+44}_{-24}	0.1586^{+39}_{-36}	0.3154^{+92}_{-146}	0.0843^{+102}_{-94}	0.0832^{+100}_{-75}	0.3143^{+69}_{-82}
2.2	0	0.00013^{+4}_{-5}	0.00012^{+13}_{-6}	0.00008^{+4}_{-5}	0.00011^{+8}_{-4}	0.0016^{+5}_{-7}	0.0015^{+11}_{-7}	0.0011^{+5}_{-6}	0.0014^{+7}_{-5}
	2.2	0.2610^{+47}_{-42}	0.1240^{+23}_{-53}	0.1256^{+29}_{-54}	0.2639^{+32}_{-60}	0.7437^{+132}_{-88}	0.2377^{+32}_{-96}	0.2397^{+128}_{-84}	0.7408^{+76}_{-67}
3.3	0	0.00007^{+7}_{-4}	0.00007^{+9}_{-4}	0.00009^{+5}_{-4}	0.00008^{+3}_{-3}	0.0009^{+8}_{-4}	0.0010^{+9}_{-5}	0.0012^{+8}_{-4}	0.0010^{+4}_{-3}
	3.3	0.4159^{+31}_{-14}	0.0811^{+13}_{-34}	0.0810^{+25}_{-24}	0.4164^{+22}_{-22}	0.9785^{+16}_{-11}	0.3958^{+70}_{-119}	0.3933^{+102}_{-85}	0.9788^{+16}_{-14}
4.4	0	0.00013^{+5}_{-3}	0.00012^{+10}_{-4}	0.00010^{+5}_{-3}	0.00010^{+9}_{-4}	0.0018^{+5}_{-8}	0.0016^{+11}_{-6}	0.0012^{+6}_{-3}	0.0014^{+9}_{-5}
	4.4	0.5871^{+72}_{-51}	0.2185^{+38}_{-47}	0.2185^{+45}_{-58}	0.5859^{+48}_{-67}	0.9593^{+31}_{-28}	0.5328^{+100}_{-83}	0.5325^{+159}_{-73}	0.9603^{+43}_{-46}
5.5	0	0.00010^{+6}_{-4}	0.00011^{+6}_{-3}	0.00009^{+6}_{-3}	0.00014^{+7}_{-4}	0.0013^{+5}_{-4}	0.0014^{+3}_{-4}	0.0013^{+8}_{-4}	0.0017^{+4}_{-8}
	5.5	0.7174^{+35}_{-61}	0.2088^{+45}_{-29}	0.2101^{+17}_{-48}	0.7202^{+50}_{-33}	0.7996^{+53}_{-34}	0.6434^{+80}_{-93}	0.6396^{+57}_{-72}	0.7967^{+72}_{-69}
6.6	0	0.00011^{+9}_{-6}	0.00013^{+7}_{-6}	0.00011^{+9}_{-3}	0.00013^{+7}_{-7}	0.0016^{+7}_{-8}	0.0016^{+7}_{-7}	0.0013^{+7}_{-5}	0.0017^{+10}_{-8}
	6.6	0.7846^{+36}_{-28}	0.1996^{+17}_{-48}	0.2012^{+57}_{-41}	0.7843^{+49}_{-49}	0.7339^{+78}_{-61}	0.7041^{+87}_{-94}	0.7057^{+117}_{-80}	0.7356^{+105}_{-65}
7.7	0	0.00014^{+4}_{-6}	0.00014^{+4}_{-7}	0.00006^{+7}_{-2}	0.00010^{+3}_{-6}	0.0017^{+6}_{-6}	0.0018^{+4}_{-8}	0.0009^{+7}_{-3}	0.0013^{+4}_{-7}
	7.7	0.6550^{+42}_{-48}	0.2978^{+37}_{-44}	0.2911^{+54}_{-48}	0.6510^{+44}_{-70}	0.8612^{+60}_{-52}	0.7497^{+88}_{-89}	0.7477^{+56}_{-93}	0.8592^{+62}_{-66}
8.8	0	0.00009^{+10}_{-2}	0.00012^{+5}_{-4}	0.00011^{+7}_{-5}	0.00010^{+8}_{-5}	0.0012^{+9}_{-4}	0.0015^{+7}_{-5}	0.0013^{+9}_{-6}	0.0012^{+11}_{-5}
	8.8	0.5607^{+58}_{-53}	0.2562^{+40}_{-52}	0.2540^{+48}_{-29}	0.5614^{+68}_{-46}	0.9881^{+15}_{-36}	0.7755^{+63}_{-90}	0.7732^{+75}_{-82}	0.9874^{+18}_{-30}
9.9	0	0.00011^{+6}_{-7}	0.00011^{+8}_{-6}	0.00009^{+6}_{-4}	0.00008^{+11}_{-4}	0.0015^{+8}_{-8}	0.0015^{+8}_{-8}	0.0012^{+5}_{-5}	0.0011^{+16}_{-4}
	9.9	0.3740^{+36}_{-34}	0.2593^{+40}_{-41}	0.2576^{+70}_{-45}	0.3740^{+29}_{-36}	0.9464^{+37}_{-23}	0.7991^{+143}_{-47}	0.7980^{+136}_{-38}	0.9466^{+27}_{-22}

TABLE IX. The probability of flavor transitions $P_i(t)$ and single-neutrino entanglement entropy $S_i(t)$, as shown in Fig. 11, with $K = 2$ and $N_A = 10^3$, after two steps of the iterative procedure. The uncertainties correspond to 68% confidence intervals determined from $N_{\rm run} = 20$ independent runs.

δt	t	$\mathcal{N}_{12}(t)$	$\mathcal{N}_{13}(t)$	$\mathcal{N}_{14}(t)$	$\mathcal{N}_{23}(t)$	$\mathcal{N}_{24}(t)$	$\mathcal{N}_{34}(t)$
1.1	0	0.107^{+32}_{-29}	0.063^{+44}_{-42}	0.053^{+52}_{-33}	0.048^{+44}_{-32}	0.078^{+62}_{-47}	0.054^{+13}_{-20}
	1.1	0.212^{+64}_{-134}	0.128^{+61}_{-83}	0.417^{+98}_{-95}	0.073^{+70}_{-56}	0.186^{+99}_{-116}	0.045^{+70}_{-27}
2.2	0	0.103^{+67}_{-23}	0.068^{+97}_{-37}	0.088^{+39}_{-49}	0.048^{+49}_{-30}	0.046^{+46}_{-37}	0.028^{+46}_{-17}
	2.2	0.078^{+137}_{-69}	0.238^{+130}_{-110}	0.611^{+103}_{-51}	0.071^{+69}_{-70}	0.215^{+56}_{-47}	0.026^{+10}_{-22}
3.3	0	0.056^{+34}_{-26}	0.033^{+45}_{-16}	0.038^{+48}_{-27}	0.049^{+44}_{-35}	0.064^{+78}_{-43}	0.085^{+44}_{-36}
	3.3	0.063^{+87}_{-12}	0.194^{+43}_{-47}	0.735^{+24}_{-31}	0.068^{+76}_{-59}	0.197^{+43}_{-33}	0.071^{+42}_{-32}
4.4	0	0.120^{+31}_{-38}	0.057^{+36}_{-21}	0.045^{+22}_{-12}	0.066^{+9}_{-28}	0.036^{+25}_{-23}	0.052^{+56}_{-25}
	4.4	0.143^{+19}_{-88}	0.188^{+23}_{-68}	0.637^{+64}_{-52}	0.133^{+127}_{-77}	0.142^{+42}_{-46}	0.126^{+124}_{-74}
5.5	0	0.089^{+57}_{-33}	0.062^{+56}_{-47}	0.059^{+62}_{-38}	0.051^{+44}_{-39}	0.064^{+54}_{-43}	0.042^{+30}_{-29}
	5.5	0.365^{+60}_{-106}	0.058^{+29}_{-36}	0.361^{+38}_{-196}	0.220^{+57}_{-93}	0.053^{+19}_{-53}	0.349^{+55}_{-109}
6.6	0	0.056^{+67}_{-32}	0.062^{+29}_{-18}	0.074^{+102}_{-53}	0.078^{+39}_{-47}	0.081^{+42}_{-41}	0.075^{+49}_{-53}
	6.6	0.321^{+66}_{-157}	0.000^{+27}_{-0}	0.363^{+66}_{-200}	0.285^{+92}_{-102}	0.000^{+28}_{-0}	0.303^{+75}_{-65}
7.7	0	0.146^{+96}_{-43}	0.047^{+29}_{-34}	0.061^{+33}_{-44}	0.042^{+58}_{-19}	0.055^{+40}_{-23}	0.040^{+36}_{-29}
	7.7	0.006^{+68}_{-6}	0	0.631^{+57}_{-58}	0.513^{+84}_{-128}	0	0.000^{+67}_{-0}
8.8	0	0.076^{+49}_{-33}	0.056^{+48}_{-36}	0.072^{+33}_{-35}	0.083^{+60}_{-59}	0.065^{+46}_{-32}	0.031^{+43}_{-18}
	8.8	0.000^{+86}_{-0}	0	0.766^{+56}_{-52}	0.663^{+50}_{-46}	0	0
9.9	0	0.088^{+45}_{-43}	0.069^{+45}_{-43}	0.067^{+90}_{-27}	0.076^{+20}_{-49}	0.071^{+43}_{-28}	0.068^{+61}_{-50}
	9.9	0	0	0.837^{+47}_{-76}	0.718^{+100}_{-87}	0.000^{+28}_{-0}	0

TABLE X. The (raw) logarithmic negativity for the different neutrino pairs $\mathcal{N}_{ij}(t)$, as shown in Fig. 12, with $K = 2$ and $N_A = 10^3$. The uncertainties correspond to 68% confidence intervals determined from $N_{\rm run} = 20$ independent runs.

δt	t	$\mathcal{N}_{12}(t)$	$\mathcal{N}_{13}(t)$	$\mathcal{N}_{14}(t)$	$\mathcal{N}_{23}(t)$	$\mathcal{N}_{24}(t)$	$\mathcal{N}_{34}(t)$
1.1	0	0.0164^{+66}_{-70}	0.0085^{+76}_{-50}	0.0068^{+40}_{-41}	0.0073^{+49}_{-25}	0.0069^{+83}_{-53}	0.0063^{+39}_{-33}
	1.1	0.0015^{+48}_{-13}	0.1948^{+133}_{-114}	0.4851^{+66}_{-109}	0.0420^{+181}_{-124}	0.1945^{+150}_{-134}	0.0007^{+6}_{-7}
2.2	0	0.0089^{+99}_{-30}	0.0053^{+51}_{-24}	0.0076^{+53}_{-35}	0.0053^{+62}_{-17}	0.0077^{+97}_{-29}	0.0070^{+73}_{-17}
	2.2	0.0173^{+36}_{-17}	0.2683^{+143}_{-75}	0.7316^{+41}_{-61}	0.0649^{+136}_{-166}	0.2650^{+61}_{-81}	0.0190^{+17}_{-34}
3.3	0	0.0051^{+46}_{-23}	0.0058^{+44}_{-22}	0.0027^{+59}_{-10}	0.0064^{+49}_{-38}	0.0036^{+47}_{-19}	0.0089^{+74}_{-43}
	3.3	0.0847^{+53}_{-40}	0.2495^{+110}_{-48}	0.7827^{+86}_{-56}	0.0816^{+65}_{-169}	0.2515^{+68}_{-53}	0.0851^{+37}_{-33}
4.4	0	0.0154^{+72}_{-23}	0.0076^{+68}_{-37}	0.0076^{+43}_{-42}	0.0071^{+36}_{-25}	0.0075^{+39}_{-38}	0.0065^{+73}_{-41}
	4.4	0.2013^{+127}_{-91}	0.1805^{+93}_{-56}	0.6847^{+68}_{-37}	0.1563^{+163}_{-113}	0.1816^{+66}_{-66}	0.2038^{+164}_{-103}
5.5	0	0.0066^{+25}_{-20}	0.0088^{+26}_{-34}	0.0101^{+38}_{-57}	0.0054^{+62}_{-29}	0.0105^{+91}_{-50}	0.0074^{+76}_{-35}
	5.5	0.3155^{+137}_{-97}	0.0869^{+25}_{-38}	0.4790^{+114}_{-81}	0.2623^{+54}_{-83}	0.0888^{+18}_{-54}	0.3076^{+147}_{-101}
6.6	0	0.0082^{+56}_{-52}	0.0081^{+64}_{-44}	0.0107^{+95}_{-60}	0.0105^{+25}_{-43}	0.0101^{+44}_{-61}	0.0070^{+79}_{-31}
	6.6	0.2342^{+173}_{-125}	0	0.4603^{+115}_{-182}	0.4220^{+59}_{-81}	0	0.2423^{+142}_{-126}
7.7	0	0.0174^{+39}_{-55}	0.0066^{+54}_{-37}	0.0077^{+40}_{-49}	0.0076^{+54}_{-25}	0.0091^{+34}_{-69}	0.0073^{+38}_{-25}
	7.7	0	0	0.7085^{+89}_{-42}	0.6158^{+61}_{-87}	0	0
8.8	0	0.0088^{+27}_{-63}	0.0066^{+69}_{-31}	0.0082^{+34}_{-37}	0.0065^{+48}_{-34}	0.0076^{+60}_{-44}	0.0070^{+65}_{-37}
	8.8	0	0	0.8992^{+20}_{-25}	0.7624^{+42}_{-47}	0	0
9.9	0	0.0098^{+119}_{-55}	0.0054^{+32}_{-26}	0.0073^{+26}_{-41}	0.0065^{+51}_{-22}	0.0082^{+33}_{-21}	0.0071^{+64}_{-27}
	9.9	0	0	0.9397^{+20}_{-34}	0.8554^{+66}_{-48}	0	0

TABLE XI. The logarithmic negativity for the different neutrino pairs $\mathcal{N}_{ij}(t)$, as shown in Fig. 12, with $K = 2$ and $N_A = 10^3$, after two steps of the iterative procedure. The uncertainties correspond to 68% confidence intervals determined from $N_{\text{run}} = 20$ independent runs.