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Parvinder Solanki, Noufal Jaseem, Michal Hajdušek, and Sai Vinjanampathy

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# Role of Coherence and Degeneracies in Quantum Synchronisation

Parvinder Solanki,[1,*] Noufal Jaseem,[1] Michal Hajdušek,[2] and Sai Vinjanampathy[1,3,†]

[1]*Department of Physics, Indian Institute of Technology-Bombay, Powai, Mumbai 400076, India.*
[2]*Keio University Shonan Fujisawa Campus, 5322 Endo, Fujisawa, Kanagawa 252-0882, Japan.*
[3]*Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543 Singapore, Singapore.*

Synchronisation in quantum systems has been largely driven by specific examples of frequency entrainment as well as mutual synchronisation. Here we study quantum synchronisation as a Liouville space perturbation theory. We begin by clarifying the role of centers, symmetries and oscillating coherences in the context of quantum synchronisation. We then analyse the eigenspectrum of the Liouville superoperator generating the dynamics of the quantum system and determine the conditions under which synchronisation arises. We apply our framework to derive a powerful relationship between energy conservation, degeneracies and synchronisation in quantum systems. Finally, we demonstrate our approach by analysing two mutually coupled thermal machines and prove that non-degenerate thermal networks cannot be simultaneously energy conserving and synchronous.

## INTRODUCTION

In analogy to classical synchronisation, quantum synchronisation stands for the adjustment of rhythms of self-sustained oscillators under the effect of weak coupling or an external drive [1]. Given the ubiquity of synchronisation in the classical world [1, 2], quantum synchronisation has garnered increasing interest. Initially, systems whose mean-field dynamics resembled classical dynamical systems were the primary focus of investigation [3–16]. Following this, the notion of synchronisation was soon extended to genuinely quantum systems with no classical counterparts [17–19] and experimentally observed in [20, 21]. The primary concern of the early studies of quantum synchronisation was to motivate a system-specific measure of synchronisation and demonstrate that such measure attains a finite value in some region of the parameter space, effectively demonstrating that synchronisation is possible even in the quantum regime. A typical first step to study synchronization in both classical and quantum systems involves establishing a valid limit cycle dynamics in the absence of perturbation (driving/coupling) [22]. A valid limit cycle possesses at least one neutral free phase (in sense of [17], for other definition see [23]) and the oscillations are robust to external perturbation.

What has been lacking thus far is a systematic study of the specific structure of perturbative driving or coupling that could bring about quantum synchronisation. In classical and hence in quantum dynamics, there are slightly different but related definitions of synchronisation [1, 24–26]. In this manuscript, we begin by clarifying various definitions of synchronisation in quantum systems and then construct a systematic perturbation theory in Liouville space to understand steady state synchronisation. We prove a powerful relationship between the energy structure of the physical systems and their interactions that may prevent the systems from synchronising. Finally, we apply our framework to the example of two coupled thermal machines and show that a network of non-degenerate thermal machines cannot be simultaneously energy conserving and synchronized.

We consider the Lindblad master equation given by

$$\dot{\rho} = -\mathrm{i}[H_0, \rho] + \sum_k \mathcal{D}[\mathcal{O}_k]\rho, \tag{1}$$

where $H_0$ is the bare Hamiltonian of the system and $\mathcal{D}[\mathcal{O}_k]\rho = \mathcal{O}_k\rho\mathcal{O}_k^\dagger - \frac{1}{2}\{\mathcal{O}_k^\dagger\mathcal{O}_k, \rho\}$ models the baths attached to the system with $\rho$ being the state of entire system. The steady state $\rho_{ss}$ of the evolution in Eq. (1) satisfies $\dot{\rho}_{ss} = 0$ which can either contain coherences or be diagonal in the eigenbasis of bare Hamiltonian $H_0$. We note that quasi-probability distribution functions corresponding to diagonal states are not localized whereas steady states with coherence are localized in phase space. We hence focus on the diagonal steady states though more general definitions of limit-cycles can be accommodated [22]. Diagonal limit cycles arise naturally in systems coupled to thermal baths [19]. The lack of off-diagonal terms in this basis signifies the absence of any phase-locking, either to an external drive or between systems. Perturbative driving or coupling to other systems may give rise to synchronisation as verified by phase-space based measures [10, 17, 18, 27, 28]. A common feature of all such measures is that they detect synchronisation for steady states which contain finite off-diagonal terms, pointing at a relationship between coherence generation and synchronisation. This relationship was solidified by the fact that the relative entropy of coherence $\mathcal{S}_{coh}$ [29] is a suitable measure of synchronisation [22].

Classical dynamics is strongly affected by constraints placed on Hamiltonian or on dissipative evolution. For instance, the constraint that phase space volume is conserved implies that Hamiltonian systems cannot have attractors. In the quantum context, a natural question that arises is how constraints placed on Hamiltonian or Lindbladian evolution affect quantum synchronisation. The question is important because quantum information theoretic tasks often are accompanied by a fixed resource constraint, which then affects the underlying dynamics in a non-trivial manner. An example of such a resource constraint is the demand that the coupling between thermal machines should be energy conserving. We begin by clarifying the role of the quantum analogues of fixed points and centers to quantum synchronisation, following which we present a Liouville space perturbation analysis of synchronisation. We then present a theorem relating the structure of the synchronising perturbation to energy conservation. We study an example of coupled thermal machines to exemplify our theorem.

## LIOUVILLE EIGENSPECTRUM & SYNCHRONISATION

Consider the Lindblad master equation in Liouville form namely

$$|\dot{\rho}(t)\rangle\rangle = \mathcal{L}|\rho(t)\rangle\rangle, \tag{2}$$

where $\mathcal{L}$ is the Liouville superoperator and $|\rho(t)\rangle\rangle$ is the corresponding Liouville state [30–32]. Eigenvalues of Liouville superoperators are complex numbers $\lambda_\mu = \alpha_\mu + i\beta_\mu$ with physical density matrices corresponding to $\alpha_\mu \leq 0$. Eigenvalues having $\alpha_\mu < 0$ have been useful in discussing transient sychronisation [33–35]. Whereas the steady state of this master equation is given by $\alpha_\mu = 0$, with non-zero $\beta_\mu$ indicating oscillating coherences. Before we discuss steady state synchronisation in the context of the Liouvillian eigenspectrum, we note that a prerequisite for phase-space based measure of synchronisation is the existence of a self-sustained oscillator with a valid limit-cycle corresponding to an observable free phase. This disqualifies linear quantum systems and also precludes a discussion of qubit synchronisation [18]. In classical mechanics, a stable limit-cycle is first established for a non-linear system possessing free phase with neutral stability, which is then synchronised even by a perturbative drive or coupling [1]. Following these lines, the limit-cycle in the quantum context can be established by proving that the unperturbed Liouvillian $\mathcal{L}_0$ has a diagonal steady state [18, 19, 22]. Once the limit cycle is established, the system is typically perturbed by an external drive or coupling to another system, which we represent by $\varepsilon\mathcal{L}_V$. Such a perturbation can either produce steady state coherence ($\lambda_k = 0$ with the corresponding eigenket having non-zero coherences) or oscillating coherences ($\lambda_\mu = \pm i\beta_\mu$). As discussed below, we show that while steady state coherence can be used to define a phase space based measure of synchronisation, the case of oscillating coherences is more subtle.

Alternative to the phase space-based measures, one could start with two or more systems whose uncoupled dynamics have steady state oscillating coherences and demand that the oscillations determined by the imaginary eigenvalues adjust their rhythms due to coupling [36]. This phenomenon, shown in Fig. 1 restricts the underlying limit cycles to themselves be oscillating coherences [37, 38] and hence can include certain qubit models. We note that the classical analogue of oscillating coherences is centers in classical two-dimensional flows [25], which are also complex conjugate pairs of imaginary eigenvalues. Since centers do not represent either a stable limit cycle or a neutral free phase, they fall outside the scope of this manuscript.

Another issue that arises in defining synchronisation around centers relates to the information preserving nature of the dynamics. Genuine limit cycle dynamics involves a basin of attraction, wherein any dynamics that starts in the basin ends up on the limit cycle. Thus classical limit cycle dynamics does not preserve initial conditions and flows to a unique asymptotic state. This is not the case for quantum dynamics which lead to oscillating coherences, which can arise with strong or weak symmetries [39] or when the Lindbladian pumps the quantum system to a subspace with Hamiltonian dynamics [40]. For Lindblad dynamics with strong symmetries (i.e., existence of unitary $S$ such that $[S, H] = 0$ and $[S, \mathcal{O}_k] = 0$), the dimensionality of the Liouville subspace $\mathbb{L}_{ss}$ is determined by the number of unique eigenvalues of $S$ and subspaces with $\mathbb{L}_{ss} \geq 2$ are known to be *information preserving* [41]. For all of these reasons we refer to the phenomenon involving multiple oscillating coherences as undergoing *coherence synchronisation* in general [42] and use the phrase *phase synchronisation* for models where the underlying dynamics exhibits a stable limit cycle and a neutral free phase. We note that the general theory of phase ordering of oscillating coherences remains an open problem.

Finally we note that the presence of degeneracies has a profound effect on the entire dynamics that has to be carefully reconciled. Lindblad-type master equations are derived under the Born-Markovian Secular (BMS) approximation [43] which may not hold in the presence of degeneracies and can hence modify the underlying master equation dramatically [44, 45].

## PERTURBATION OF LIMIT CYCLE OSCILLATORS

We add a perturbation whose strength is moderated by $\varepsilon$ to make the total Liouville super-operator take the form $\mathcal{L} = \mathcal{L}_0 + \varepsilon\mathcal{L}_V$, where $\mathcal{L}_0 = \mathcal{L}_{H_0} + \mathcal{L}_D$ is the unperturbed Liouville super-operator and $\mathcal{L}_V$ is an arbitrary perturbation in Liouville space. In general $\mathcal{L}$ is not Hermitian ($\mathcal{L}^\dagger \neq \mathcal{L}$) and has different left ($|l_\mu\rangle\rangle$) and right ($|r_\mu\rangle\rangle$) eigenvectors [46] such that

$$\mathcal{L}|r_\mu\rangle\rangle = \lambda_\mu|r_\mu\rangle\rangle, \qquad \mathcal{L}^\dagger|l_\mu\rangle\rangle = (\lambda_\mu)^*|l_\mu\rangle\rangle. \tag{3}$$

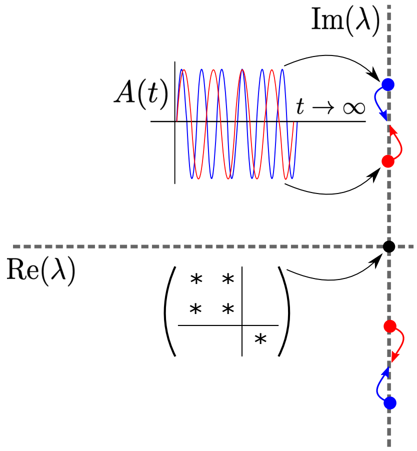


FIG. 1. Two different dynamical phenomena that can cause adjustment of rhythms: The first phenomenon is phase synchronisation caused by the development of steady state coherences due to driving or coupling and is indicated by the matrix whose corresponding eigenvalue is zero. The second is a phase ordering phenomenon which we refer to as coherence synchronisation, caused by the adjustment of oscillating coherences (blue and red sinusoidal waves indicates the oscillation of ) indicated by the arrows pulling the imaginary eigenvalues closer to each other. $A(t)$ depicts the time dependence of a local observable of the system under investigation.

These eigenvectors do not form an orthonormal basis. For small $\varepsilon$ we can expand the eigenvectors ($|r_\mu\rangle\!\rangle, |l_\mu\rangle\!\rangle$) and eigenvalues ($\lambda_\mu$) in term of the unperturbed eigenvectors ($|r_\mu^{(0)}\rangle\!\rangle, |l_\mu^{(0)}\rangle\!\rangle$) and eigenvalues ($\lambda_\mu^{(0)}$) as follows:

$$\lambda_\mu = \lambda_\mu^{(0)} + \varepsilon\lambda_\mu^{(1)} + \varepsilon^2\lambda_\mu^{(2)} + \dots \tag{4}$$
$$|r_\mu\rangle\!\rangle = |r_\mu^{(0)}\rangle\!\rangle + \varepsilon|r_\mu^{(1)}\rangle\!\rangle + \varepsilon^2|r_\mu^{(2)}\rangle\!\rangle + \dots, \tag{5}$$

and likewise for $|l\mu\rangle\!\rangle$. Since perturbation strength $\varepsilon$ need to be small, we restrict ourselves to the first order perturbation by only retaining linear terms in $\varepsilon$. Therefore, using the above set of equations in Eq. (3) and equating first order terms in $\varepsilon$ we get

$$\mathcal{L}_0|r_\mu^{(1)}\rangle\!\rangle + \mathcal{L}_V|r_\mu^{(0)}\rangle\!\rangle = \lambda_\mu^{(0)}|r_\mu^{(1)}\rangle\!\rangle + \lambda_\mu^{(1)}|r_\mu^{(0)}\rangle\!\rangle. \tag{6}$$

The perturbation $\varepsilon\mathcal{L}_V$ can have three distinct possible effects on the unperturbed steady state which we discuss for completeness. The first effect of perturbation is that it can change the eigenvectors keeping the corresponding eigenvalues same [46]. Assuming $\lambda_\mu^{(1)} = 0$ and collecting terms for $|r_\mu^{(1)}\rangle\!\rangle$, Eq. (6) reduces to

$$(\mathcal{L}_0 - \lambda_\mu^{(0)})|r_\mu^{(1)}\rangle\!\rangle = -\mathcal{L}_V|r_\mu^{(0)}\rangle\!\rangle. \tag{7}$$

Multiplying the above equation with the $(\mathcal{L}_0 - \lambda_\mu^{(0)})^+$ from left gives the first order correction to corresponding eigenvector

$$|r_\mu^{(1)}\rangle\!\rangle = -(\mathcal{L}_0 - \lambda_\mu^{(0)})^+\mathcal{L}_V|r_\mu^{(0)}\rangle\!\rangle, \tag{8}$$

where $\mathcal{O}^+$ is the Moore-Penrose pseudo-inverse[47] of operator $\mathcal{O}$. Now the first order perturbation to the steady state (lets $|\rho_{ss}^{(1)}\rangle\!\rangle \equiv |r_0^{(1)}\rangle\!\rangle$) having eigenvalue $\lambda_0^{(0)} = 0$, reduces to

$$|\rho_{ss}^{(1)}\rangle\!\rangle = (-\mathcal{L}_0^+\mathcal{L}_V)|\rho_{ss}^{(0)}\rangle\!\rangle, \tag{9}$$

where $|\rho_{ss}^{(0)}\rangle\!\rangle \equiv |r_0^{(0)}\rangle\!\rangle$ is the steady state of $\mathcal{L}_0$. Depending on the form of $\mathcal{L}_V$ new steady state $|\rho_{ss}\rangle\!\rangle = |\rho_{ss}^{(0)}\rangle\!\rangle + |\rho_{ss}^{(1)}\rangle\!\rangle$ can contain coherences and we can observe steady synchronisation for such perturbation.

Alternatively, the perturbation $\varepsilon\mathcal{L}_V$ can change the eigenvalue keeping the eigenvector unchanged. Since we are interested in steady state synchronisation, we restrict ourselves to the case where $\lambda_\mu^{(1)}$ can only take on an imaginary contribution, which we prove to be unphysical. Using $|r_\mu^{(1)}\rangle\!\rangle = 0$ in Eq. (9) we will get

$$\mathcal{L}_V|r_\mu^{(0)}\rangle\!\rangle = \lambda_\mu^{(1)}|r_\mu^{(0)}\rangle\!\rangle. \tag{10}$$

First order correction to the eigenvalue can be calculated by multiplying Eq. (10) with $\sum_\nu\langle\!\langle l_\nu^{(0)}|$ to get

$$\lambda_\mu^{(1)} = \frac{\sum_\nu\langle\!\langle l_\nu^{(0)}|\mathcal{L}_V|r_\mu^{(0)}\rangle\!\rangle}{\sum_\nu\langle\!\langle l_\nu^{(0)}||r_\mu^{(0)}\rangle\!\rangle}. \tag{11}$$

Since the underlying limit-cycle states are diagonal, when complex eigenvalues arise in the Liouville spectrum, they appear as complex-conjugate pairs and hence at least two steady states with eigenvalues $i\beta$ and $-i\beta$ will arise with corresponding Liouville eigenvectors $|\rho^+\rangle\!\rangle$ and $|\rho^-\rangle\!\rangle$. Note that the steady state is going to be a linear combination of both the states i.e. $|\rho\rangle\!\rangle = e^{i\beta t}|\rho^+\rangle\!\rangle + e^{-i\beta t}|\rho^-\rangle\!\rangle$. For $\rho$ to be a valid density matrix it needs to be Hermitian, implying $(\rho^-)^\dagger \equiv \rho^+$ and since $\rho^-$ and $\rho^+$ are diagonal density matrices, $\rho^- \equiv \rho^+$. This cannot be true as one eigenvector cannot have two different eigenvalues, which concludes the proof. A third possible effect of perturbation can be change in both eigenvalues and eigenvectors simultaneously. This in general leads to oscillating coherences, which are outside the scope of this manuscript.

We further simplify Eq. (9) to understand the coherence generation by expanding $\mathcal{L}_0^+$ as a sum of $\mathcal{L}_{H_0}^+$ and another term we denote by $X_{(H_0,\mathcal{D})}$. The pseudoinverse of the sum of two matrices $A$ and $B$ can be used to define a new operator $X_{(A,B)}$, given by the relation

$$(A+B)^+ = A^+ + X_{(A,B)}. \tag{12}$$

Using the property of pseudoinverse we can write,

$$(A^+ + X_{(A,B)})(A+B) = I, \tag{13}$$

where $I$ is the identity operator. Using $A^+A = I_A$, the identity in the non-singular subspace of $A$, Eq. (13) can be rewritten as

$$I = I_A + A^+B + X_{(A,B)}(A+B). \tag{14}$$

After rearranging Eq. (14) we get

$$X_{(A,B)} = (\delta I - A^{+}B)(A+B)^{+}, \tag{15}$$

where $\delta I = I - I_A$. Using Eq. (12) and Eq. (15) we can write $\mathcal{L}_0^{+} = \mathcal{L}_{H_0}^{+} + X_{(H_0,\mathcal{D})}$ where $X_{(H_0,\mathcal{D})} \equiv (\delta I - \mathcal{L}_{H_0}^{+}\mathcal{L}_{\mathcal{D}})(\mathcal{L}_{H_0} + \mathcal{L}_{\mathcal{D}})^{+}$. Hence Eq. (9) can be re-written as follows

$$\begin{aligned}|\rho_s^{(1)}\rangle\!\rangle &= -(\mathcal{L}_{H_0}^{+} + X_{(H_0,\mathcal{D})})\mathcal{L}_V|\rho_s^{(0)}\rangle\!\rangle \\ &= -\mathcal{L}_{H_0}^{+}\mathcal{L}_V|\rho_s^{(0)}\rangle\!\rangle - X_{(H_0,\mathcal{D})}\mathcal{L}_V|\rho_s^{(0)}\rangle\!\rangle. \end{aligned} \tag{16}$$

This equation forms the basis of our analysis of the relationship between coherence, degeneracies and given constraints on the system. Since the unperturbed steady state has the interpretation of the underlying limit cycle state, we demand that $\mathcal{L}_0$ does not generate any coherences.

## ENERGY CONSERVATION & DEGENERACIES

Typical constraints on coupled quantum systems in a thermodynamic setting is to demand energy conserving interactions so that the coupling is not associated with the work output of the thermal machine. To understand role of energy conservation in coherence generation, we constrain the generic $\mathcal{L}_V$ to be a Hamiltonian coupling $V$. Energy conservation requires $[H_0, V] = 0$. We now prove a theorem relating coherence generation, energy conservation and degeneracy of the system for open quantum system dynamics. We also present the closed system analogue of this theorem for pedagogical clarity in Section A of the Supplementary Material [48], which includes Refs.[49–54].

**Theorem.** *In an open quantum system whose dynamics is described by a Markovian master equation with a non-degenerate energy spacings, energy conserving interactions cannot generate phase synchronisation.*

The proof follows from the fact that coherence generation costs energy in non-degenerate systems [55] and it has been shown that $l_1$-norm of coherence is directly related to synchronisation [19]. This can also be seen structurally using Liouvillian perturbation theory discussed in last section, specifically following Eq. (16).

If the bare Hamiltonian is non-degenerate, both $H_0$ and $V$ need to be diagonal in the same basis to commute. On the other hand, if $H_0$ has degeneracies, the interaction $V$ can be off-diagonal in the degenerate subspace of $H_0$ and still be energy conserving. Hence the corresponding super-operator $\mathcal{L}_V$ will also be off-diagonal only in the degenerate subspace of $\mathcal{L}_{H_0}$ and hence $[\mathcal{L}_{H_0}, \mathcal{L}_V] = 0$. This means that though $\mathcal{L}_V$ can be off-diagonal, $\mathcal{L}_{H_0}$ and $\mathcal{L}_V$ can share the eigenbasis of $\mathcal{L}_V$. As $\mathcal{L}_V$ is off-diagonal, $\mathcal{L}_V|\rho_{ss}^{(0)}\rangle\!\rangle \equiv |\tilde{\rho}_{ss}\rangle\!\rangle$ can create coherences. Since $\mathcal{L}_{H_0}$ is diagonal, the pseudoinverse $\mathcal{L}_{H_0}^{+} = \sum_j \Lambda_j^{+}|\Lambda_j\rangle\!\rangle\langle\!\langle\Lambda_j|$ is also diagonal where $\Lambda_j^{+} = 0$ for $\Lambda_j = 0$ and $\Lambda_j^{+} = \Lambda_j^{-1}$ for non-zero $\Lambda_j$. Therefore the term $\mathcal{L}_{H_0}^{+}|\tilde{\rho}_s\rangle\!\rangle$ can affect coherences only up to a multiplicative factor, whereas $-X_{(H_0,\mathcal{D})}\mathcal{L}_V|\rho_{ss}^{(0)}\rangle\!\rangle$ can generate coherences. This shows that both terms in Eq. (16) are capable of generating coherences. Hence the presence of degeneracies in the bare Hamiltonian can impact coherence generation and synchronisation.

Our analysis can be extended to study any other constraint besides the energy conservation. Let us consider that bare Hamiltonian $H_0$ commutes with an additional symmetry operator $S$. Now conservation of such symmetry requires the perturbation $V$ to commute with the operator $S$. Let $\mathcal{L}_{H_0}, \mathcal{L}_S$ and $\mathcal{L}_V$ be the Liouvillian superoperators corresponding to $H_0, S$ and $V$ respectively. Following from Eq. (16), generation of coherence depends on whether $\mathcal{L}_{H_0}$ commutes with $\mathcal{L}_V$ or not. Specifically there can be only two cases: First is where $[\mathcal{L}_{H_0}, \mathcal{L}_S] = 0$ and both share the same eigenbasis. This implies that either both of them are diagonal and non-degenerate or they have degeneracy in the same subspace. In such a case $[\mathcal{L}_{H_0}, \mathcal{L}_V] = 0$ and the same analysis applies as before. In second a case, $[\mathcal{L}_{H_0}, \mathcal{L}_S] = 0$ but the two Liouville superoperators do not share the same eigenbasis. This could happen when both $\mathcal{L}_{H_0}$ and $\mathcal{L}_S$ are diagonal but have degeneracy in different subspaces so that a given off-diagonal operator $\mathcal{L}_V$ will commute with either one of them. Commutation of $\mathcal{L}_S$ requires $[\mathcal{L}_S, \mathcal{L}_V] = 0$ which leads to $[\mathcal{L}_{H_0}, \mathcal{L}_V] \neq 0$ in our case. In such case coherences will always be generated following Eq. (16). An example for such a case will be where two spins having bare Hamiltonian $H_0 = \sigma_Z^A + \sigma_Z^B$ conserve the symmetry $S = \sigma_Z^A \otimes \sigma_Z^B$ such that $[H_0, S] = 0$. Here $\sigma_X = |1\rangle\langle 2| + H.c.$ is the usual Pauli matrix and likewise $\sigma_Z$. Furthermore, $H_0$ and $S$ both are diagonal but have degeneracy in different subspace. Now interaction term $V = \sigma_X^A \otimes \sigma_X^B$ will conserve the symmetry $S$ but will not commute with bare Hamiltonian $H_0$ as a result of which coherences will be generated in the eigenbasis of $H_0$ following Eq. (16).

## SYNCHRONISATION OF COUPLED THERMAL MACHINES

Consider the synchronization of two mutually coupled thermal machines shown in Fig. 2(a). The two 3-level systems are described by bare Hamiltonian $H_0 = \sigma_{22}^A + (1+\Omega)\sigma_{33}^A + (\Omega+\Delta)\sigma_{22}^B + (1+\Omega)\sigma_{33}^B$ where $\sigma_{ij} = |i\rangle\langle j|$. They are mutually coupled with each other by interaction Hamiltonian of the form $V = \sigma_{23}^A\sigma_{21}^B + \sigma_{12}^A\sigma_{32}^B + h.c.$ which is energy conserving interaction for $\Delta = 0$. We note that the eigenvalues and eigenvectors of total Hamiltonian $H = H_0 + \varepsilon V$ change abruptly as soon as we turn

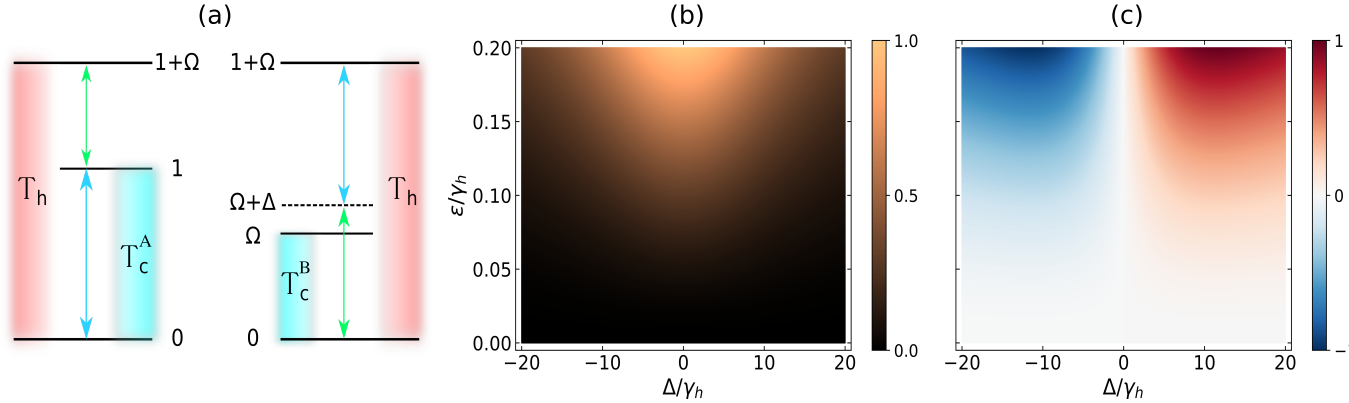


FIG. 2. (a) Two coupled heat engines described by Eq. (17) with parameter values given as $\Omega/\gamma_h = 40$, $\gamma_h = \gamma_h^{(A,B)} = 0.01$, $\gamma_c = \gamma_c^{(A,B)} = 10\gamma_h$, $\bar{n}_c^{(A,B)} \approx 10^{-3}$ and $\bar{n}_h^{(A,B)} = 1$. Plots for steady state (b) synchronization measure $S_{coh}(\rho)/\max[S_{coh}(\rho)]$ and (c) output power $P/\max[|P|]$ for a given range of coupling strength $\varepsilon$ and detuning $\Delta$ where $\max[S_{coh}(\rho)] \approx 0.001$ and $\max[|P|] \approx 1.8 \times 10^{-6}$. It can be seen that coherence uncouples from power at the point of degeneracy.

on the perturbative interaction $\varepsilon$. Since it is well known [56, 57] that local master equations describe the thermodynamics of such systems more faithfully, we describe the dynamics of the coupled thermal machine in Fig. 2 by the master equation

$$\dot{\rho} = -i[H_0 + \varepsilon V, \rho] + \sum_{i=A,B} (\mathbf{D}_h^i[\rho] + \mathbf{D}_c^i[\rho]), \quad (17)$$

where $\mathbf{D}_{h(c)}^i[\rho] \equiv \gamma_{h(c)}^i \bar{n}_{h(c)}^i \mathcal{D}[\sigma_{32}^i]\rho + \gamma_{h(c)}^i (1 + \bar{n}_{h(c)}^i)\mathcal{D}[\sigma_{23}^i]\rho$ represents the system $i = (A, B)$ coupled to hot (cold) bath at temperature $T_{h(c)}^i$. The individual systems do not have any degeneracies and the full bare Hamiltonian has a degeneracy of degree 3 corresponding to the eigenvalue $1 + \Omega$. The interaction Hamiltonian $V$ is off-diagonal only in the degenerate subspace, hence $[H_0, V] = 0$ which results in $[\mathcal{L}_{H_0}, \mathcal{L}_V] = 0$. It can be inferred from Eq. (16) that steady state will contain coherences and hence phase localization will be observed in this case. For $\Delta \neq 0$ the degeneracy is lifted and $[\mathcal{L}_{H_0}, \mathcal{L}_V] \neq 0$, resulting in energy not being conserved anymore. Hence a direct consequence of our theorem is the observation that coherence generation in this case costs energy.

Let us study the mutual synchronization for this system. When the thermal engines are not coupled ($\varepsilon = 0$) then the steady states of individual systems under the effects of their thermal bath are diagonal and hence both systems are in a corresponding limit cycle state. We use the relative entropy of synchronisation [22] $\Omega_R(\rho) = S_{coh}(\rho) = S(\rho_{diag}) - S(\rho)$ which measures the distance to the nearest diagonal limit cycle state using relative entropy measure [22]. The synchronization measure $S_{coh}(\rho)$ in Fig. 2(b) displays the typical Arnold tongue behaviour which confirms that steady state mutual synchronization exists between two thermal engines.

Power is given by $P = -i\mathrm{Tr}([H, \rho]H_0)$ [53]. Hence power for the given system having a steady state $\rho^{ss} = \sum_{i,j=1}^{9} \rho_{ij}^{ss} |i\rangle\langle j|$ is given by $P = 2\varepsilon\Delta\ \mathrm{Im}[\rho_{35}^{ss} + \rho_{75}^{ss}]$, where $|i\rangle$ denotes the global basis states, see Section B of the Supplementary Material [48]. Steady state is given by $\rho^{ss} = \rho_{ss}^{(0)} + \rho_{ss}^{(1)}$ where $\rho_{ss}^{(0)}$ is the eigenstate corresponding to zero eigenvalue of unperturbed Liouvillian superoperator $\mathcal{L}_0$ and $\rho_{ss}^{(1)}$ is the first order correction due to $\mathcal{L}_V$ which is obtained from Eq. (16). The heat current from hot (cold) bath is given by $J_{h(c)} = \sum_{i=A,B} \mathrm{Tr}(\mathbf{D}_{h(c)}^i[\rho]H_0)$, see Section B of the Supplementary Material [48]. Power is plotted as a function of coupling strength $\varepsilon$ and detuning parameter $\Delta$ in Fig. 2(c). For the values of parameters given in Fig. 2, $J_h \geq 0$ and $J_c \leq 0$ where equality holds only in the absence of coupling $\varepsilon = 0$.

Now from Fig. 2, one can observe that for energy conserving case given by $\Delta = 0$, the power output is zero while the synchronization measure is non-zero which means that coherences are generated in the absence of power. For finite detuning, energy is no longer conserved and coherences are generated at the cost of power. For $\Delta < 0$, power is negative while $J_h \geq 0$ and $J_c \leq 0$ which means that coupled system behaves like a thermal engine and power is generated. Power is positive for positive detuning $\Delta > 0$ which means that power is being consumed by the system while coherence is being generated as a result of energy non-conservation. For $\Delta > 0$ the coupled system acts as a dissipator or heater. This example demonstrates our argument that non-degenerate quantum thermal machines cannot be synchronised by energy conserving interactions. We emphasize that coherent power uncouples from quantum synchronisation in the presence of degeneracies since there is zero work cost to create degenerate coherences. Several other examples of coupled thermal machines and quantum synchronous systems can be understood within this framework, as presented in Section C of the Supplementary Material.

## CONCLUSIONS

The literature on quantum synchronisation thus far has been lead by system specific examples. In this manuscript, we go beyond such an approach and discuss a Liouville space perturbation theoretic approach to study phase-space based measures of steady state synchronisation. Unlike previous approaches, we highlight methods to detect emergent synchronicity by analysing the different parts of the Liouville superoperator. While quantum synchronisation of underlying limit cycle oscillators can be understood in terms of steady state coherences under coupling, we also clarified the role of centers, symmetries and oscillating coherences in this context. Finally, we show that degeneracies have a strong role to play in the relationship between thermodynamic quantities such as coherent power and quantum synchronisation. While our local master equation approach is well motivated, the analysis needs to be reconsidered if the BMS condition no longer applies [58–62]. Our example illustrates that while coupled thermal machines can always be synchronised, there is a finite cost to doing so outside of a degenerate manifold. This method can further be used to understand and subsequently design quantum thermal machines and quantum synchronising systems. Our approach can be applied to systematically study quantum synchronisation in the perturbative regime and will find applications in future quantum technologies.

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* psolanki@phy.iitb.ac.in
† sai@phy.iitb.ac.in

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