Anti-math xmlns="http://www.w3.org/1998/Math/MathML">mi mathvariant="script">PT/mi>/math>-symmetric qubit: Decoherence and entanglement entropy

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Anti-$\mathcal{PT}$-symmetric Qubit: Decoherence and Entanglement Entropy

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We investigate the dynamics of a general two-level based anti-parity-time (anti-$\mathcal{PT}$)-symmetric qubit and study its decoherence as well as entanglement entropy properties. We compare our findings with that of the corresponding parity-time ($\mathcal{PT}$)-symmetric and Hermitian qubits. To begin, we consider the time-dependent Dyson map to find the exact analytical dynamics for a general non-Hermitian qubit system weakly coupled with a thermal bath for pure dephasing, before specializing it to the case of a general anti-$\mathcal{PT}$-symmetric qubit. Basing comparison under the same coupling strength or increasing the non-Hermiticity, we observe that the decoherence function and entanglement entropy of the anti-$\mathcal{PT}$-symmetric qubit decays and grows more slowly, respectively, compared to the $\mathcal{PT}$-symmetric and Hermitian qubits. Similarly, the corresponding variance and area of Fisher information are much higher compared to the $\mathcal{PT}$-symmetric and Hermitian qubits. These results demonstrate that anti-$\mathcal{PT}$-symmetric qubits may be better suited for quantum computing and quantum information processing applications than conventional Hermitian or even $\mathcal{PT}$-symmetric qubits.

Introduction. One of the roadblocks for achieving viable quantum computing platforms is decoherence intrinsic to quantum systems [1]. This is particularly important for quantum information processing and storage [2]. Previously it was shown that $\mathcal{PT}$-symmetric qubits are better than Hermitian qubits from this perspective when the $\mathcal{PT}$-symmetric quantum system is coupled to a Hermitian environment (or bath) very weakly [3]. Very weak coupling ascertains that the system and the bath do not exchange any heat [4], which leads to what is known as pure decoherence or dephasing [5]. The choice of a Hermitian bath is only for the sake of simplicity. Here we explore whether another non-Hermitian realization, i.e. an anti-$\mathcal{PT}$-symmetric qubit, can further improve decoherence properties. We find that the answer is in the affirmative and it leads to substantial slowing down of decoherence.

Before exploring any properties, we need to find the dynamics of our qubit systems. The computation of the qubit's dynamics by the expectation value of the time evolved operator is usually performed from a Hermitian diagonalized qubit, which can be easily obtained for Hermitian and $\mathcal{PT}$-symmetric qubits [3, 6]. However, for the anti-$\mathcal{PT}$-symmetric qubit, we first need to provide a method to obtain the exact analytical expression for the evolved density matrix of a general non-Hermitian qubit system. This is accomplished by utilizing the time-dependent Dyson map to find a corresponding Hermitian system and as a result, obtain a reduced density matrix for a time-dependent Hermitian system [7, 8].

The decoherence is naturally revealed in the dynamics. We can subsequently also investigate further properties such as von Neumann entanglement entropy and Fisher information. Note that we are considering a unitary evolution under the anti-PT-symmetric Hamiltonian. Also, note that there are different bases in which one can compare decoherence and properties of the three types of qubits, including eigenvalue gaps, coupling strength and non-Hermiticity. Looking at the various options, we discuss how non-Hermiticity provides the most suitable comparison.

After the introduction of $\mathcal{PT}$-symmetry [9] and subsequently two decades of intensive research [10], the notion of anti-$\mathcal{PT}$ symmetry was introduced by Ge and Türeci [11] in optics by an appropriate spatial arrangement of the effective optical potential. For $\mathcal{PT}$ and anti-$\mathcal{PT}$-symmetries, the $\mathcal{PT}$ operator commutes $[H, \mathcal{PT}] = 0$ and anti-commutes $\{H, \mathcal{PT}\} = 0$ with the Hamiltonian, respectively. The anti-$\mathcal{PT}$ symmetry has been realized in spatially coupled atom beams [12], electrical circuit resonators [13], optical waveguides with imaginary couplings [14] and optical four-wave mixing in cold atoms [15]. In addition, constant refraction optical systems [16] and several experiments in atomic [17, 18] and optical [19–21] systems have realized the anti-$\mathcal{PT}$ symmetry. There are many other applications involving waveguide arrays [21], diffusive systems [22], phase transitions [23], spin chains [24], information flow [25] and non-Markovian aspects [26]. Possibly, an anti-$\mathcal{PT}$-symmetric qubit can be experimentally realized in recently demonstrated optical and microcavity settings [27]. A quantum circuit [28] and information flow [29] using a two-level system have also been recently discussed.

The Model. In this Letter, we will be considering the qubit system weakly coupled to a bath of bosonic systems described by the following Hamiltonian [30]

$$H = H_S + H_B + H_I, \tag{1}$$

where H_S denotes the system, $H_B = \sum_k \omega_k a_k^\dagger a_k$, the bath and $H_I = H_S \sum_k \left(g_k a_k^\dagger + g_k^* a_k\right)$, the interaction term between the system and bath with $V_B = \sum_k \left(g_k a_k^\dagger + g_k^* a_k\right)$. Here $a_k^\dagger$ and a_k denote bosonic creation and annihilation operators, respectively, ω_k are the eigenmodes of the bath and g_k are the coupling constants.

Hermitian Qubit. To begin, let us review the dynamics for the Hermitian qubit as

$$H_S^h = \begin{pmatrix} \alpha+\nu & \xi+i\delta \\ \xi-i\delta & -\alpha+\nu \end{pmatrix}, \tag{2}$$

with $\alpha,\xi,\delta,\nu \in \mathbb{R}$ to satisfy the Hermiticity condition $H_S^h = \left(H_S^h\right)^\dagger$. We can think of this as two subsystems (e.g. oscillators) at an energy level represented by α. Here ν is just an overall energy shift. Taking the modulus of the off-diagonal terms with parameters ξ and δ gives the standard coupling strength between the subsystems. For convenience, we take a similarity transformation [31]

$$T = \begin{pmatrix} \omega_0-\alpha & -\xi-i\delta \\ \omega_0+\alpha & \xi+i\delta \end{pmatrix}, \tag{3}$$

where

$$\omega_0^h = \sqrt{\alpha^2+\delta^2+\xi^2}, \tag{4}$$

to obtain a diagonalized Hamiltonian $H_S^{Dh} = TH_S^hT^{-1}$ with eigenvalues $E_\pm = \nu \pm \omega_0^h$. The reduced density matrix for the system is recovered as a partial trace over the environment of the overall system plus bath evolution. Subsequently, it can be expressed in terms of the reduced density matrix for the diagonalized system under normalization as

$$\rho_S(t) = \frac{T^{-1}\rho_S^{Dh}(t)\left(T^{-1}\right)^\dagger}{Tr_S\left[T^{-1}\rho_S^{Dh}(0)\left(T^{-1}\right)^\dagger\right]}, \tag{5}$$

with diagonalized system's density matrix at time 0 and t being

$$\rho_S^{Dh}(0) = \begin{pmatrix} \rho_{11}^{Dh} & \rho_{12}^{Dh} \\ \rho_{21}^{Dh} & \rho_{22}^{Dh} \end{pmatrix}, \tag{6}$$

$$\rho_S^{Dh}(t) = \begin{pmatrix} \rho_{11}^{Dh} & 0 \\ 0 & \rho_{22}^{Dh} \end{pmatrix} + \begin{pmatrix} 0 & \rho_{12}^{Dh}(t) \\ \rho_{21}^{Dh}(t) & 0 \end{pmatrix} e^{-(\omega_0^h)^2\gamma(t)}, \tag{7}$$

respectively, where

$$\rho_{12}^{Dh}(t) = e^{2i\omega_0^h t}e^{-i\omega_0^h\Omega(t)}, \quad \rho_{21}^{Dh}(t) = \left[\rho_{12}^{Dh}(t)\right]^* \tag{8}$$

and

$$\Omega(t) = 4\nu\int_0^\infty dw J(w)\frac{wt-\sin(wt)}{w^2}, \tag{9}$$

$$\gamma(t) = 4\int_0^\infty dw J(w)\frac{1-\cos(wt)}{w^2}\coth\left(\frac{\beta w}{2}\right), \tag{10}$$

$$J(w) = \sum_k |g_k|^2\delta(w-w_k), \tag{11}$$

are respectively the function influencing phase evolution, decoherence function and spectral density of the bath [30].

$\mathcal{PT}$-symmetric Qubit. For the $\mathcal{PT}$-symmetric case, which has been studied in [3], let us take the Hamiltonian as

$$H_S^{\mathcal{PT}} = \begin{pmatrix} \alpha+i\theta & \xi+i\delta \\ \xi-i\delta & \alpha-i\theta \end{pmatrix}, \tag{12}$$

where $\alpha,\xi,\delta,\theta \in \mathbb{R}$, with θ being the amplification/decay rate. The parity operator $\mathcal{P}$, is taken as the Pauli matrix σ_x and time operator $\mathcal{T}$, being complex conjugation, then the commutation property $\left[\mathcal{PT}, H_S^{\mathcal{PT}}\right] = 0$ is satisfied. With the same similarity transformation (3) as for the Hermitian system, but where

$$\omega_0^{\mathcal{PT}} = \sqrt{\delta^2+\xi^2-\theta^2}, \tag{13}$$

leads to a diagonalized Hermitian Hamiltonian $h_S^{\mathcal{PT}} = TH_S^{\mathcal{PT}}T^{-1}$, with eigenvalues $E_\pm = \alpha \pm \omega_0^{\mathcal{PT}}$. In this Letter, we will be interested in the case when the eigenvalue gap is real, i.e. studying the parametric domain $\delta^2+\xi^2 \geq \theta^2$. The difference in the evolved reduced density matrix (7) is that ν is replaced by $-\theta$ in the expression for $\Omega(t)$.

Anti-$\mathcal{PT}$-symmetric Qubit. Let us introduce an anti-$\mathcal{PT}$-symmetric quantum system of the general form

$$H_S = \begin{pmatrix} \alpha+i\theta & \xi+i\delta \\ -\xi+i\delta & -\alpha+i\theta \end{pmatrix}, \tag{14}$$

where $\alpha,\xi,\delta,\theta \in \mathbb{R}$ and one can check that the anticommutation relation $\{\mathcal{PT}, H_S\} = 0$ is satisfied taking the same parity and time operators as above. The total system H, is diagonalizable by again taking the similarity transformation (3) with

$$\omega_0^{\mathcal{APT}} = \sqrt{\alpha^2-\xi^2-\delta^2}. \tag{15}$$

The resulting diagonalized Hamiltonian will be given by

$$H^D = THT^{-1} = \left(-\omega_0^{\mathcal{APT}}\sigma_z + i\theta\right)(1+V_B) + H_B. \tag{16}$$

This is equivalent to the eigenbasis representation $\sum_n E_n|n\rangle\langle n|$ with $E_n \in \mathbb{C}$, so H can be rewritten in a complete biorthonormal basis

$$H = \sum_n E_N|\psi_n^R\rangle\langle\psi_n^L| \tag{17}$$

by taking the set of right $|\psi_n^R\rangle = T^{-1}|n\rangle$ and left $\langle\psi_n^L| = \langle n|T$ eigenvectors satisfying the defining equations $\langle\psi_n^L|\psi_m^R\rangle = \delta_{nm}$ and $\sum_n|\psi_n^R\rangle\langle\psi_n^L| = \mathbb{I}$ [32–35]. The pair of eigenvalues of the anti-$\mathcal{PT}$-symmetric system (14) is given by $E_\pm = i\theta \pm \omega_0^{\mathcal{APT}}$.

For a general non-Hermitian Hamiltonian H, it has been suggested in [36–38] that the Hamiltonian can also be viewed as a decomposition of real and imaginary parts $H = H_R + iH_I$ to give a complex extension of the Liouville-von Neumann equation. However, for non-Hermitian spin-boson models, calculations become quite involved. In what follows, we will present a scheme that makes computing the reduced density matrix of these systems more feasible.

Time-dependent Dyson Map for Density Matrix of a Non-Hermitian System. The key is to show that one can reformulate the non-Hermitian Hamiltonian in terms of a Hermitian one utilizing a time-dependent Dyson map [39–47]. Recently, this has been investigated for a $\mathcal{PT}$-symmetric bosonic system coupled to a bath of N bosonic systems [7] and $\mathcal{PT}$-symmetric Jaynes-Cummings Hamiltonian [8]. Here, we will present this method for a general non-Hermitian qubit system and in particular we will focus on the anti-$\mathcal{PT}$-symmetric case.

First, let us recall that for a Hermitian system, for example a diagonalized Hermitian Hamiltonian h^D, with density matrix ρ^{Dh}, the Liouville-von Neumann equation is given by

$$i\partial_t \rho^{Dh} = \left[h^D, \rho^{Dh}\right]. \tag{18}$$

In the non-Hermitian case, for example the diagonalized H^D, by taking the Schrödinger equation and its conjugate transpose, the corresponding non-Hermitian Liouville-von Neumann equation can be derived as

$$i\partial_t \rho^D = H^D \rho^D - \rho^D \left(H^D\right)^\dagger. \tag{19}$$

Let H^D and h^D be related by the Dyson relation

$$H^D = \eta^{-1} h^D \eta - i\eta^{-1}\left(\partial_t \eta\right), \tag{20}$$

where η relates the states $|\phi_i\rangle$, $|\psi_i\rangle$ of the Hamiltonians h^D, H^D respectively as $|\phi_i\rangle = \eta|\psi_i\rangle$.

Substituting the Dyson relation into (19) and comparing with (18) gives the relation of density matrices between the Hermitian and non-Hermitian Hamiltonians as $\rho^{Dh} = \eta\rho^D\eta^\dagger$. Then supposing $\rho^{Dh} = \sum_i P_i|\phi_i\rangle\langle\phi_i|$, we can check $\rho^D = \sum_i P_i|\psi_i\rangle\langle\psi_i|$, which shows that the mapping of density matrices is able to preserve the set of probabilities P_i.

On the other hand, taking the Hermitian Liouville-von Neumann equation (18) and substituting again the Dyson relation (20), it can be shown

$$i\partial_t \widetilde{\rho}^D = \left[H^D, \widetilde{\rho}^D\right] \tag{21}$$

under the relation $\widetilde{\rho}^D = \eta^{-1}\rho^{Dh}\eta = \rho^D M$, with $M = \eta^\dagger\eta$ being the metric such that for the quasi-Hermitian Hamiltonian

$$H^Q = H^D + i\eta^{-1}\left(\partial_t\eta\right), \tag{22}$$

it satisfies $\langle H^Q\psi_i|M\psi_i\rangle = \langle\psi_i|MH^Q\psi_i\rangle = \langle\phi_i|h^D\phi_i\rangle$ and $\left(H^Q\right)^\dagger M = MH^Q$. Looking at the anti-$\mathcal{PT}$-symmetric qubit of (16) which we now denote by H^D_S, we will see that the corresponding quasi-Hermitian qubit H^Q_S, is a two-level system with eigenvalues $E^Q_\pm = \pm\omega_0^{\mathcal{APT}}$. It follows that H^Q_S is interpreted as the *physical operator* that plays the role of energy for H^D_S [43, 44], and $\widetilde{\rho}^D$ is a Hermitian density matrix operator in the Hilbert space under the metric $M(t)$. We now find the corresponding Hermitian system from a time-dependent Dyson map.

Let us recall the diagonalized non-Hermitian Hamiltonian (16), if we take the Ansatz $\eta = \exp\left[\varphi(t)\left(1+V_B\right)\right]$ in the Dyson relation, h^D becomes

$$h^D = \left(-\omega_0^{\mathcal{APT}}\sigma_z + i\theta\right)\left(1+V_B\right) + H_B - \varphi\widetilde{V}_B - \varphi^2\Omega_k + i\partial_t\varphi, \tag{23}$$

where $\widetilde{V}_B = \sum_k \omega_k\left(g_k a_k^\dagger - g_k^* a_k\right)$ and $\Omega_k = \sum_k \omega_k|g_k|^2$. For h^D to be Hermitian $\left(h^D\right)^\dagger = h^D$, the constraining equation is $\partial_t\varphi = -\theta$, so we can take $\eta = \exp\left[-\theta t\left(1+V_B\right)\right]$, then the corresponding Hermitian Hamiltonian is

$$h^D = -\omega_0^{\mathcal{APT}}\sigma_z + H_B - \omega_0^{\mathcal{APT}}\sigma_z V_B + \theta t\widetilde{V}_B - \theta^2 t^2\Omega_k. \tag{24}$$

Note we can also represent h^D in terms of the quasi-Hermitian Hamiltonian H^Q [39, 48, 49], as

$$h^D = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}C_G^{(n)}\left(H^Q\right), \tag{25}$$

with $G = \theta t\left(1+V_B\right)$ and denoting

$$C_G^{(n)}\left(\mathcal{O}\right) = [\overbrace{G,[G,\cdots[G,\mathcal{O}]]}^{n}] \tag{26}$$

as the n-fold commutation for operators G and $\mathcal{O}$, then in terms of the non-Hermitian Hamiltonian H^D, h^D can be expressed as

$$h^D = -i\theta\left(1+V_B\right) + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}C_G^{(n)}\left(H^D\right). \tag{27}$$

Keeping the eigenvalue gap real, we will carry our investigations in the parametric domain of $\alpha^2 \geq \delta^2 + \xi^2$. The remaining step now is to find the density matrix of (24). Consequently, we also obtain the decoherence for our qubit system.

Decoherence of an Anti-$\mathcal{PT}$-symmetric Qubit. We will consider the dynamics where the qubit is initially uncorrelated with a bath in thermal equilibrium i.e. the Gibbs state $\Omega_B = \exp\left[-\beta H_B\right]/Z$, where $Z = Tr_B\exp\left[-\beta H_B\right]$ is the partition function [50], so the initial density matrix of the total system becomes

$$\rho^{Dh}(0) = \rho_S^{Dh}(0)\otimes\Omega_B. \tag{28}$$

We take a general form for the reduced density matrix

$$\rho_S^{Dh}(0) = \begin{pmatrix}\frac{1}{2}\left(1+\langle\sigma_z\rangle\right) & \langle\sigma_-\rangle\\ \langle\sigma_+\rangle & \frac{1}{2}\left(1-\langle\sigma_z\rangle\right)\end{pmatrix}, \tag{29}$$

with $\sigma_\pm = \sigma_x \pm i\sigma_y$. The reduced system's density matrix at time t is given by $\rho_S^{Dh}(0) \to \rho_S^{Dh}(t)$, $\sigma_z \to \sigma_z(t)$, $\sigma_\pm \to \sigma_\pm(t)$ and the expectation value of a general time evolved operator $\mathcal{O}$ with Hermitian Hamiltonian h^D can be expressed as $\langle \mathcal{O}(t)\rangle = Tr\left[\mathcal{O}(t)\rho(0)\right]$ by defining

$$\mathcal{O}(t) = e^{i\int h^D dt}\mathcal{O}(0)\,e^{-i\int h^D dt}. \tag{30}$$

To proceed with computing the decoherence function, we want to find the expressions for the time-dependent qubit operators $\sigma_z(t)$ and $\sigma_\pm(t)$. To begin, consider (30) for the time-dependent bath operator $a_k(t)$. Noting the commutation relations $[H_B, a_k] = -\omega_k a_k$, $[V_B, a_k] = -g_k$, $\left[\widetilde{V}_B, a_k\right] = -\omega_k g_k$, this gives the equation of motion

$$\partial_t a_k = -i\omega_k\left[a_k - \omega_0^{\mathcal{APT}}\frac{g_k}{\omega_k}\sigma_z + \theta g_k t\right], \tag{31}$$

and the time-dependent bath operator expression is given by

$$a_k(t) = -\theta g_k t + e^{-i\omega_k t}\left[a_k - A_k(t)\sigma_z + B_k(t)\right], \tag{32}$$

where

$$A_k(t) = \omega_0^{\mathcal{APT}}\frac{g_k}{\omega_k}\left(1 - e^{i\omega_k t}\right), \tag{33}$$

$$B_k(t) = i\theta\frac{g_k}{\omega_k}\left(1 - e^{i\omega_k t}\right). \tag{34}$$

Utilizing this expression, we can find the expressions for the time-dependent qubit operators similarly, by using their equations of motion respectively, which are solved by

$$\sigma_z(t) = \sigma_z, \tag{35}$$

$$\sigma_\pm(t) = e^{\mp 2i\omega_0^{\mathcal{APT}}t}\,e_+^{\int_0^t \mp 2i\omega_0^{\mathcal{APT}}\sum_k\left[g_k a_k^\dagger(\tau) + g_k^* a_k(\tau)\right]d\tau}\sigma_\pm, \tag{36}$$

where $\exp_+[\cdots]$ denotes the time-ordered exponent. Now, we can express the reduced system's density matrix at time t, by computing the expectation values of $\sigma_\pm(t)$ and $\sigma_z(t)$

$$\langle\sigma_z(t)\rangle = \langle\sigma_z\rangle, \tag{37}$$

$$\langle\sigma_\pm(t)\rangle = \langle\sigma_\pm\rangle\,e^{\mp 2i\omega_0^{\mathcal{APT}}t}e^{\pm i\omega_0^{\mathcal{APT}}[\Omega_2(t) - \Omega_1(t)]}e^{-\left(\omega_0^{\mathcal{APT}}\right)^2\gamma(t)}, \tag{38}$$

where

$$\Omega_1(t) = 4\theta\int_0^\infty dw J(w)\frac{(1 - \cos(wt))}{w^2}, \tag{39}$$

$$\Omega_2(t) = 2\theta t^2\int_0^\infty dw J(w), \tag{40}$$

in the continuum limit of bath modes. The resulting density matrix of the reduced system at time t is given by

$$\rho_S^{Dh}(t) = \begin{pmatrix}\rho_{11}^{Dh} & 0\\ 0 & \rho_{22}^{Dh}\end{pmatrix} + \begin{pmatrix}0 & \rho_{12}^{Dh}(t)\\ \rho_{21}^{Dh}(t) & 0\end{pmatrix}e^{-\left(\omega_0^{\mathcal{APT}}\right)^2\gamma(t)}, \tag{41}$$

where

$$\rho_{12}^{Dh}(t) = e^{2i\omega_0^{\mathcal{APT}}t}e^{-i\omega_0^{\mathcal{APT}}[\Omega_2(t) - \Omega_1(t)]}, \tag{42}$$

$$\rho_{21}^{Dh}(t) = \left[\rho_{12}^{Dh}(t)\right]^* \tag{43}$$

and the *decoherence function* reads

$$D(t) = e^{-\left(\omega_0^{\mathcal{APT}}\right)^2\gamma(t)}, \tag{44}$$

which quantifies the loss of quantum information to the environment. There are various options one can take as a basis to compare decoherence of the three qubits. Taking eigenvalue gap $2\omega_0$, as the basis of comparison, the decoherence factor is found to be the same in all three cases. However, the problem with this comparison is that the eigenvalue gap in the Hermitian case cannot be made non-trivially zero, but can in the non-Hermitian cases. Moreover, eigenvalue gaps do not represent energy gaps. Another basis of comparison that solves these problems is taking the magnitude of coupling strength for the three qubits to be $\sqrt{\xi^2 + \delta^2}$. The spectral density, as an example, will be taken to be $J(w) = J_0 w^{1+\mu}\exp[-w/w_c]$. All the parameters in our qubits, α, δ, θ, ν and ξ are real. As the decoherence function evolves with time, we are able to discern that the anti-$\mathcal{PT}$-symmetric (14) qubit decays more gradually compared to a Hermitian (2) or $\mathcal{PT}$-symmetric (12) qubit, as shown in the left panel of Fig. 1.

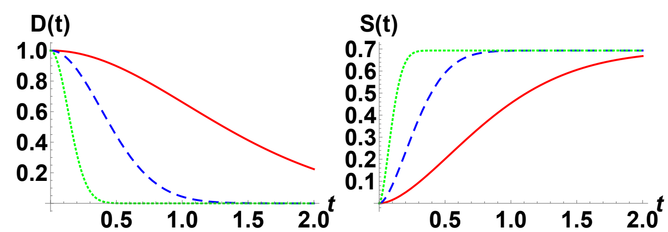

FIG. 1. Decoherence function (left) and von Neumann entropy (right) for anti-$\mathcal{PT}$-symmetric (red, solid line), $\mathcal{PT}$-symmetric (blue, dashed line) and Hermitian (green, dotted line) qubits with $J_0 = w_c = 1$, $\beta = 0.5$, $\delta = 0.56$, $\mu = -0.5$, $\xi = 0.81$, $\theta = 0.86$, $\nu = 0.86$ and $\alpha = 1$.

The most obvious and suitable method of comparison is using non-Hermiticity as the basis of comparison. This method starts with all Hamiltonians being Hermitian, by setting all non-Hermitian terms in the Hamiltonians to be zero. As we increase non-Hermiticity from the sets of non-Hermitian terms in each case, we observe a progressively more gradual decrease in decoherence for the anti-$\mathcal{PT}$-symmetric case compared with the $\mathcal{PT}$-symmetric case, compared with the Hermitian case. The slowing of decoherence is the result of increase of non-Hermiticity. The effect of having $\mathcal{PT}$-symmetry is contribution of one non-Hermitian term, θ. The effect of having anti-$\mathcal{PT}$-symmetry is contribution of two non-Hermitian terms, δ and ξ. Hence, we conclude that non-Hermiticity reduces decoherence, so the anti-$\mathcal{PT}$-symmetric qubit is better than the $\mathcal{PT}$-symmetric, which is better than the Hermitian qubit.

Entanglement Entropy. Let us now study the entanglement entropy, which is a common measure of entanglement and quantum information. In particular, for our system, this will be a measure of entanglement between our qubit and environment. To do this, we first take the initial state of the reduced system (29) as a pure state $Tr\left[\left(\rho_S^{Dh}\right)^2(0)\right] = 1$, with equal populations of ground and excited state, i.e. $\rho_{11}^{Dh} = \rho_{22}^{Dh}$. Recalling the time-evolved reduced density matrix, the eigenvalues of the reduced density matrix are given by

$$\lambda_{1,2}^{Dh} = \frac{1}{2}\left[1 \pm \sqrt{4\langle\sigma_+(t)\rangle\langle\sigma_-(t)\rangle + \langle\sigma_z\rangle^2}\right]. \tag{45}$$

Consequently, the (von Neumann) entropy can be calculated as

$$\begin{aligned} S(t) &= -\lambda_1 \ln\lambda_1 - \lambda_2\ln\lambda_2, \\ &= \ln 2 - \frac{1}{2}\left[1 + D(t)\right]\ln\left[1 + D(t)\right] \\ &\quad - \frac{1}{2}\left[1 - D(t)\right]\ln\left[1 - D(t)\right]. \end{aligned} \tag{46}$$

At $t = 0$, $S(0) = 0$ and as $t \to \infty$, $S(\infty) = \ln 2$, as shown in the right panel of Fig. 1. What is also evident is that the entropy with anti-$\mathcal{PT}$-symmetric qubit increases more gradually compared with Hermitian or $\mathcal{PT}$-symmetric qubit. This result shows the anti-$\mathcal{PT}$-symmetric qubit entangles with the environment at a slower rate compared to Hermitian or $\mathcal{PT}$-symmetric qubit, which is equivalent to the ability to preserve quantum information for a longer time. We note that this calculation complements the analysis of decoherence function.

Fisher Information. Quantum Fisher information is an important quantity in quantum metrology [51]. It quantifies the precision that can be achieved in estimating a parameter for a given quantum state. Thus, it can be regarded as a measure of reliability of a quantum system. A higher value of Fisher information equates to higher precision of estimating a parameter. One can compute from the relative entropy (i.e. the Kullback-Leibler divergence [52, 53])

$$\begin{aligned} D_{KL}(K,t) &= Tr\left[\rho_S^{Dh}\left(\widetilde{K},t\right)\ln\rho_S^{Dh}\left(\widetilde{K},t\right)\right] \\ &\quad - Tr\left[\rho_S^{Dh}\left(\widetilde{K},t\right)\ln\rho_S^{Dh}\left(K,t\right)\right], \end{aligned} \tag{47}$$

the Fisher information with respect to (the inverse) temperature parameter, $K = \beta$ as

$$\begin{aligned} S_f(\beta,t) &= \frac{\partial^2}{\partial\widetilde{\beta}^2} D_{KL}\left(\widetilde{\beta},t\right)\bigg|_{\widetilde{\beta}=\beta}, \\ &= \frac{\omega_0^4}{2}\left\{\coth\left[\omega_0^2\gamma(\beta,t)\right] - 1\right\}\left[\frac{\partial}{\partial\beta}\gamma(\beta,t)\right]^2. \end{aligned} \tag{48}$$

From the left panel of Fig. 2, we see that although the maximum Fisher information for the three types of qubit are roughly equal, the anti-$\mathcal{PT}$-symmetric case is still slightly higher, as seen from numerical values given in Table I. However, the *variance* and area of Fisher information are visibly much larger for the anti-$\mathcal{PT}$-symmetric qubit. This may be interpreted that we need a larger interval of time in estimating β accurately.

Data	**parameter β**		**parameter ω_0**	
	S_f^{max}	S_f^{area}	S_f^{max}	S_f^{area}
Hermitian qubit	0.611	0.115	0.343	0.064
$\mathcal{PT}$-qubit	0.613	0.219	1.222	0.437
Anti-$\mathcal{PT}$-qubit	0.619	0.506	5.887	4.804

TABLE I. Fisher information data with respect to β and ω_0 for anti-$\mathcal{PT}$-symmetric, $\mathcal{PT}$-symmetric and Hermitian Hamiltonians, with parameters $J_0 = w_c = 1$, $\mu = -0.5$, $\beta = \delta = 0.5$, $\xi = 0.8$, $\theta = 0.6$, $\nu = 0.6$ and $\alpha = 1$.

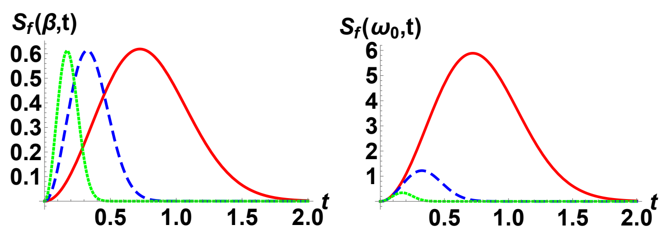

FIG. 2. Fisher entropy with respect to inverse temperature β (left) and the combined qubit parameter ω_0 (right) for anti-$\mathcal{PT}$-symmetric (red, solid line), $\mathcal{PT}$-symmetric (blue, dashed line) and Hermitian (green, dotted line) qubits with $J_0 = w_c = 1$, $\mu = -0.5$, $\beta = \delta = 0.5$, $\xi = 0.8$, $\theta = 0.6$, $\nu = 0.6$ and $\alpha = 1$.

Similarly, the Fisher information depending on what we will call the 'combined qubit parameter' ω_0 can be computed to be

$$S_f(\omega_0,t) = 2\omega_0^2\left\{\coth\left[\omega_0^2\gamma(t)\right] - 1\right\}\gamma^2(t). \tag{49}$$

The right panel of Fig. 2 clearly shows that both the maximum and area of Fisher information with respect to ω_0 for the anti-$\mathcal{PT}$-symmetric qubit are much greater compared with the Hermitian and $\mathcal{PT}$-symmetric cases. In particular, this represents that a higher accuracy can be obtained in measuring the parameter ω_0.

Conclusions. We have studied the decoherence and entanglement (via von Neumann entropy and Fisher information) properties of an anti-$\mathcal{PT}$-symmetric qubit comprising a two-level system. To this end, we utilized the time-dependent Dyson map to find the metric that provides us with the well-defined inner product for a Hilbert space of the non-Hermitian system and as a result, also transforms our problem of computing the reduced density matrix from a non-Hermitian to a more feasible Hermitian one. We found *superior* decoherence properties as compared to the $\mathcal{PT}$-symmetric and Hermitian qubits. We also found a *slower* growth for the entanglement entropy, further implying that the anti-$\mathcal{PT}$-symmetric qubit is better able to fight the effects of being destroyed by the environment, and much *higher* Fisher information for the anti-$\mathcal{PT}$-symmetric qubit, which means greater precision of estimating parameters. These results are obtained under the same coupling strength.

The key mechanism responsible for such improved properties is the result of the form of combined qubit parameter ω_0, since our properties depend on it. At first sight, the decoherence factor may seem to be identical in all three cases as a function of the eigenvalue gap, $2\omega_0$. So one may conclude that symmetries do not play a role in improved properties, but we have explained the problems of using this type of basis as comparison. The most suitable basis of comparison is non-Hermiticity. There are three real parameters in ω_0 for all three qubit cases. In the Hermitian case, all three parameters (α, δ, ξ), being Hermitian, contribute to deterioration of properties. Having $\mathcal{PT}$ or anti-$\mathcal{PT}$-symmetry is the result of non-Hermitian terms being introduced into the system. We start all cases as Hermitian (by taking non-Hermitian terms of the $\mathcal{PT}$ and anti-$\mathcal{PT}$-symmetric models to be zero). As one increases the values of the set of non-Hermitian terms in each case, improvement of properties such as reduction of decoherence in the system is seen to be progressively better in the anti-$\mathcal{PT}$-symmetric case compared with the $\mathcal{PT}$-symmetric case, compared with the Hermitian case.

A plausible physical explanation is that when gain and loss parts of the $\mathcal{PT}$-symmetric qubit interact with the environment, they tend to balance each other's effect thus reducing decoherence. In the anti-PT symmetric qubit, the gain (or equivalently loss) is asymmetric on the two parts of the qubit. This fact combined with an imaginary coupling can further reduce loss of coherence due to the environment.

These findings suggest advantages of the utility of anti-$\mathcal{PT}$-symmetric qubits for quantum information processing and storage. It would be desirable to have a possible experimental realization (e.g. in optical waveguides [14] and microcavity systems [27]) of the anti-$\mathcal{PT}$-symmetric qubit to observe the predicted superior properties. Very recently, a possible experimental realization of the anti-$\mathcal{PT}$-symmetric qubit has been discussed using certain rare-earth elements (Tm, Er) based compounds, e.g. phosphors, embedded in a polymer matrix [54].

In this Letter, we have considered the qubit systems weakly coupled with a thermal bath and a simple system-bath interaction. Naturally, it would be very interesting to extend investigations to multi-qubits or other baths and more complex interactions, even with anti-$\mathcal{PT}$-symmetries.

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[1] W. H. Zurek, Decoherence, einselection, and the quantum origins of the classical, Reviews of Modern Physics **75**(3), 715 (2003).

[2] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe and J. L. O'Brien, Quantum computers, Nature **464**(7285), 45 (2010).

[3] B. Gardas, S. Deffner and A. Saxena, $\mathcal{PT}$-symmetric slowing down of decoherence, Physical Review A **94**(4), 040101 (2016).

[4] D. V. Averin, K. Xu, Y. P. Zhong, C. Song, H.Wang and S. Han, Suppression of Dephasing by Qubit Motion in Superconducting Circuits, Physical Review Letters **116**(1), 010501 (2016).

[5] A. W. Cummings and S. Roche, Effects of Dephasing on Spin Lifetime in Ballistic Spin-Orbit Materials, Physical Review Letters **116**(8), 086602 (2016).

[6] J. Łuczka, Spin in contact with thermostat: Exact reduced dynamics. Physica A: Statistical Mechanics and Its Applications, **167**(3), 919 (1990).

[7] A. Fring and T. Frith, Eternal life of entropy in non-Hermitian quantum systems, Physical Review A **100**(1), 010102 (2019).

[8] T. Frith, Exotic entanglement for non-Hermitian Jaynes-Cummings Hamiltonians, Journal of Physics A: Mathematical and Theoretical, **53** (48), 485303 (2020).

[9] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry, Physical Review Letters **80**(24), 5243 (1998).

[10] C. M. Bender, P. E. Dorey, C. Dunning, A. Fring, D. W. Hook, H. F. Jones, S. Kuzhel, G. Levai, and R. Tateo, $\mathcal{PT}$-Symmetry: In Quantum and Classical Physics, World Scientific, Singapore (2019).

[11] L. Ge and H. E. Türeci, Antisymmetric $\mathcal{PT}$-photonic structures with balanced positive- and negative-index materials, Physical Review A **88**(5), 053810 (2013).

[12] P. Peng, W. Cao, C. Shen, W. Qu, J. Wen, L. Jiang and Y. Xiao, Anti-parity-time symmetry with flying atoms, Nature Physics **12**(12), 1139 (2016).

[13] Y. Choi, C. Hahn, J. W. Yoon and S. H. Song, Observation of an anti-PT-symmetric exceptional point and energy difference conserving dynamics in electrical circuit resonators, Nature Communications **9**, 2182 (2018).

[14] X.-L. Zhang, T. Jiang and C. T. Chan, Dynamically encircling an exceptional point in anti-parity-time symmetric systems: asymmetric mode switching for symmetry-broken modes, Light: Science & Applications **8**, 88 (2019).

[15] Y. Jiang, Y. Mei, Y. Zuo, Y. Zhai, J. Li, J. Wen and S. Du, Anti-Parity-Time Symmetric Optical Four-Wave Mixing in Cold Atoms, Physical Review Letters **123**(19), 193604 (2019).

[16] F. Yang, Y.-C. Liu and L. You, Anti-$\mathcal{PT}$ symmetry in dissipatively coupled optical systems. Physical Review A **96**(5), 053845 (2017).

[17] X. Wang and J.-H. Wu, Optical $\mathcal{PT}$-symmetry and $\mathcal{PT}$-antisymmetry in coherently driven atomic lattices, Optics Express **24**(4), 4289 (2016).

[18] Y.-L. Chuang, Ziauddin and R.-K. Lee, Realization of simultaneously parity-time-symmetric and parity-time-antisymmetric susceptibilities along the longitudinal direction in atomic systems with all optical controls, Optics Express **26**(17), 21969 (2018).

[19] V. V. Konotop and D. A. Zezyulin, Odd-Time Reversal $\mathcal{PT}$ Symmetry Induced by an Anti-$\mathcal{PT}$-Symmetric Medium, Physical Review Letters **120**(12), 123902 (2018).

[20] Q. Li et al., Experimental simulation of anti-parity-time symmetric Lorentz dynamics, Optica **6**(1), 67 (2019).

[21] S. Ke, D. Zhao, J. Liu, Q. Liu, Q. Liao, B. Wang and P. Lu, Topological bound modes in anti-PT-symmetric optical waveguide arrays, Optics Express **27**(10), 13858 (2019).

[22] Y. Li et al., Anti-parity-time symmetry in diffusive systems, Science **364**(6436), 170 (2019).

[23] T. E. Lee, F. Reiter and N. Moiseyev, Entanglement and Spin Squeezing in Non-Hermitian Phase Transitions, Physical Review Letters **113**(25), 250401 (2014).

[24] R. Couvreur, J. L. Jacobsen and H. Saleur, Entanglement in Nonunitary Quantum Critical Spin Chains, Physical Review Letters **119**(4), 040601 (2017).

[25] S. Chakraborty and D. Chruściński, Information flow versus divisibility for qubit evolution, Physical Review A **99**(4), 042105 (2019).

[26] S. Haseli, G. Karpat, S. Salimi, A. S. Khorashad, F. F. Fanchini, B. Çakmak, G. H. Aguilar, S. P. Walborn, and P. H. Souto Ribeiro, Non-Markovianity through flow of information between a system and an environment, Physical Review A **90**(5), 052118 (2014).

[27] F. Zhang, Y. Feng, X.Chen, L. Ge, and W. Wan, Synthetic Anti-PT Symmetry in a Single Microcavity, Physical Review Letters **124**(5), 053901 (2020).

[28] C. Zheng, Duality quantum simulation of a generalized anti-PT-symmetric two-level system, Europhysics Letters **126**(3), 30005 (2019).

[29] J. Wen, G. Qin, C. Zheng, S. Wei, X. Kong, T. Xin and G. Long, Observation of information flow in the anti-$\mathcal{PT}$-symmetric system with nuclear spins, npj Quantum Information **6**, 28 (2020).

[30] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, New York, (2002).

[31] G. Strang, Linear Algebra and Its Applications, Fourth Edition, Brooks/Cole, Belmont, CA, (2006).

[32] J. Wong, Results on Certain Non-Hermitian Hamiltonians, Journal of Mathematical Physics **8**(10), 2039 (1967).

[33] J. Wong, On a Condition for Completeness, Journal of Mathematical Physics **10**(8), 1438 (1969).

[34] A. Mostafazadeh, Pseudo-Hermiticity versus PT-symmetry. II. A complete characterization of non-Hermitian Hamiltonians with a real spectrum, Journal of Mathematical Physics, **43**(5), 2814 (2002).

[35] D. C. Brody, Biorthogonal quantum mechanics, Journal of Physics A: Mathematical and Theoretical **47**(3), 035305 (2014).

[36] D. C. Brody and E.-M. Graefe, Mixed-state Evolution in the Presence of Gain and Loss, Physical Review Letters **109**(23), 230405 (2012).

[37] A. Sergi and K. G. Zloshchastiev, Non-Hermitian quantum dynamics of a two-level system and models of dissipative environments, International Journal of Modern Physics B **27**(27), 1350163 (2013).

[38] Y. Y. Wang and M. F. Fang, Quantum speed limit time of a non-Hermitian two-level system, Chinese Physics B **29**(3), 030304 (2020).

[39] C. Figueira de Morisson Faria and A Fring, Time evolution of non-Hermitian Hamiltonian systems, Journal of Physics A: Mathematical and General **39**(29), 9269 (2006).

[40] A. Mostafazadeh, Time-dependent pseudo-Hermitian Hamiltonians defining a unitary quantum system and uniqueness of the metric operator, Physics Letters B **650**(2), 208 (2007).

[41] M. Znojil, Time-dependent version of crypto-Hermitian quantum theory, Physical Review D **78**(8), 085003 (2008).

[42] J. Gong and Q.-H. Wang, Time-dependent $\mathcal{PT}$-symmetric quantum mechanics, Journal of Physics A: Mathematical and Theoretical **46**(48), 485302 (2013).

[43] A. Fring and M. H. Y. Moussa, Unitary quantum evolution for time-dependent quasi-Hermitian systems with nonobservable Hamiltonians, Physical Review A **93**(4), 042114 (2016).

[44] A. Fring and T. Frith, Exact analytical solutions for time-dependent Hermitian Hamiltonian systems from static unobservable non-Hermitian Hamiltonians, Physical Review A **95**(1), 010102(R) (2017).

[45] A. Fring and T. Frith, Mending the broken PT-regime via an explicit time-dependent Dyson map, Physics Letters A **381**(29), 2318 (2017).

[46] A. Mostafazadeh, Energy observable for a quantum system with a dynamical Hilbert space and a global geometric extension of quantum theory, Physical Review D **98**(4), 046022 (2018).

[47] J. Cen, A. Fring and T. Frith, Time-dependent Darboux (supersymmetric) transformations for non-Hermitian quantum systems, Journal of Physics A: Mathematical and Theoretical **52**(11), 115302 (2019).

[48] A. Mostafazadeh, $\mathcal{PT}$-symmetric cubic anharmonic oscillator as a physical model, Journal of Physics A: Mathematical and General, **38**(29), 6557 (2005).

[49] B. Gardas, S. Deffner and A. Saxena, Repeatability of measurements: Non-Hermitian observables and quantum Coriolis force, Physical Review A **94**(2), 022121 (2016).

[50] H. B. Callen, Thermodynamics and an Introduction to Thermostatistics, John Wiley and Sons, New York, (1985).

[51] G. Tóth and D. Petz, Extremal properties of the variance and the quantum Fisher information, Physical Review A **87**(3), 032324 (2013).

[52] S. Kullback, Information Theory and Statistics, Dover Publications, Mineola, NY (1978).

[53] Y. Bao, S. Choi, and E. Altman, Theory of the phase transition in random unitary circuits with measurements, Phys. Rev. B **101**, 104301 (2020).

[54] D. N. Patel, S. S. Sarkisov, and A. M. Darwish, Quantum computers based on rare-earth compounds and $\mathcal{PT}$- and anti-$\mathcal{PT}$ symmetric qubits, Proc. SPIE **11844**, Photonics for Quantum 2021, 118440X (2021).