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Asymptotically Consistent Measures of General Quantum Resources: Discord, Non-Markovianity, and Non-Gaussianity

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Quantum resource theories provide a unified framework to quantitatively analyze inherent quantum properties as resources for quantum information processing. So as to investigate the best way for quantifying resources, desirable axioms for resource quantification have been extensively studied through axiomatic approaches. However, a conventional way of resource quantification by resource measures with such desired axioms may contradict rates of asymptotic transformation between resourceful quantum states due to an approximation in the transformation. In this paper, we analyze a novel axiom, *asymptotic consistency* of resource measures, and we investigate *asymptotically consistent resource measures*, which quantify resources without contradicting the rates of the asymptotic resource transformation. We prove that relative entropic measures are consistent with the rates for a broad class of resources, *i.e.*, all convex finite-dimensional resources, *e.g.*, entanglement, coherence, and magic, and even some non-convex or infinite-dimensional resources such as quantum discord, non-Markovianity, and non-Gaussianity. These results show that consistent resource measures are widely applicable to the quantitative analysis of various inherent quantum-mechanical properties.

I. INTRODUCTION

Quantum resource theories (QRTs) [1, 2] provide a unified framework for quantitatively analyzing quantum properties, such as entanglement [3] and magic [4–6], which underlies the advantage of quantum information processing over classical information processing. General frameworks to reveal the universal properties of quantum resources have been widely studied [2, 7–17], and quantification of resources is one of the major interests of QRTs [18–25]. To quantify resources, we use real-valued functions of states called *resource measures*. Resource measures can quantify resources without contradicting one-shot convertibility of resources; that is, the resource amount quantified by a resource measure does not increase when we transform a resource by free operations without error. For example, *the relative entropy of resource*, that is, the relative entropy between a given state and its closest free state, is conventionally used in various QRTs [4, 26–32]. With various resource measures proposed, desirable properties of resource measures have been extensively studied through axiomatic approaches [2, 19, 33–37] to seek the best resource measure.

On the other hand, it is known that resources are not necessarily comparable in terms of the exact convertibility. For example, in the QRT of magic for qutrits, there are two classes of states impossible to be exactly converted to each other using free operations [4]. Moreover, resource measures with conventionally adopted axioms, *i.e.,* asymptotic continuity and additivity, may contradict rates of asymptotic state conversion [2], where many copies of a given state are converted by free operations into many copies of a target state within a vanishingly small but nonzero error. Therefore, it is vital to investigate properties of resource measures associated with the asymptotic state conversion in addition to the exact conversion. While these two concepts, resource measures and asymptotic state conversion, are previously discussed separately, we proposed in Ref. [2] a concept of *asymptotically consistent resource measures*, or *consistent resource measures* for short, which quantify resources without contradicting the rates of the asymptotic state conversion as well as the exact conversion. Reference [2] also shows that a relative entropic measure called *the regularized relative entropy of resource* serves as a consistent resource measure in QRTs with particular restrictions, namely, convex and finite-dimensional QRTs with a full-rank free state.

However, physically well-motivated resources do not necessarily satisfy these restrictions. For example, the sets of states that have no quantum discord, *e.g.*, classical-classical states and classical-quantum states, are not convex [38]. In addition, the set of quantum Markov chains [39] and that of Gaussian states [40] are also known to be non-convex. Moreover, the Gaussian states are defined on an infinite-dimensional state space. Therefore, when we regard these properties as quantum resources, the existing technique for proving the consistency of the entropic measure is no longer directly applicable [2, 41], and a substantial breakthrough is needed for further development of general QRTs that cover such non-convex or infinite-dimensional resources.

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In this paper, we investigate the regularized relative entropy of resource as a prospective candidate for a consistent resource measure even for the general classes of resources. We prove the consistency of the regularized relative entropy of resource in *all* finite-dimensional convex QRTs even without full-rank free states. Moreover, as physically well-motivated non-convex or infinite-dimensional resources, we study discord, non-Markovianity, and non-Gaussianity. Even though the proof technique of the previous work cannot be straightforwardly applied due to the non-convexity and the infinite-dimensionality, we analyze the consistency of the regularized relative entropy of resource for these three resources. A more detailed overview of our results is at the beginning of Sec. III. Here, we emphasize that the concept of consistent resource measures accomplishes the fundamental and physically intuitive property that resources do not increase under free operations in both asymptotic and one-shot state conversions. Even though the regularized relative entropy of resource is a measure asymptotically defined, which may not be extracted with one-shot manipulation, the regularized relative entropy still qualifies as a valid resource measure with monotonicity under one-shot state conversion.

Our results show the existence of a consistent resource measure even for physically important classes of non-convex or infinite-dimensional resources, namely, discord, non-Markovianity, and non-Gaussianity, which are not covered in our initial work [2] introducing consistent resource measures. Our analysis should be an essential first step for the breakthrough in general QRTs to broadly cover the general class of resources. We believe that our investigation for consistent resource measures leads to further understandings of quantifications of resources useful for studying a larger class of physically well-motivated quantum phenomena.

II. CONSISTENT RESOURCE MEASURE

In this section, we provide a brief review of consistent resource measures. For more details, see Sections 6.3 and 6.4 of Ref. [2]. Throughout this paper, we let $\mathcal{D}(\mathcal{H})$ denote the set of states on a quantum system $\mathcal{H}$, and we consider a subset $\mathcal{S}(\mathcal{H}) \subseteq \mathcal{D}(\mathcal{H})$ to be the set of states of interest in QRTs. In a QRT, free operations are defined as a subclass of quantum operations (linear completely positive and trace-preserving maps) that is closed under composition and tensor product, and includes the identity map and the (partial) trace [1, 2]. Free states on $\mathcal{H}$ are states in $\mathcal{S}(\mathcal{H})$ into which an arbitrary state can be converted by free operations. To begin with, we provide a definition of a resource measure.

Definition 1 (Resource Measure)**.** A resource measure $R_{\mathcal{H}}$ is a family of real functions from $\mathcal{S}(\mathcal{H})$ to $\mathbb{R}$ satisfying *monotonicity*: for any states $\phi \in \mathcal{S}(\mathcal{H})$ and $\psi \in \mathcal{S}(\mathcal{H}')$,

$$\exists \mathcal{N} : \text{free operation s.t. } \mathcal{N}(\phi) = \psi \Rightarrow R_{\mathcal{H}}(\phi) \geqq R_{\mathcal{H}'}(\psi).$$

We may omit the subscript of $R_{\mathcal{H}}$ to write R for brevity when the system $\mathcal{H}$ is obvious.

In Ref. [2], a concept of *consistency* of a resource measure is introduced, which we investigate in this paper. As implied in the definition, a resource measure R quantifies resources consistently with the exact state conversion under the free operations. By also considering the consistency with the rate of the asymptotic state conversion, we give the definition of a consistent resource measure.

Definition 2 (Asymptotically Consistent Resource Measure [2])**.** For quantum systems $\mathcal{H}$ and $\mathcal{H}'$, a resource measure R is called *asymptotically consistent*, or *consistent* for short, if for any states $\phi \in \mathcal{S}(\mathcal{H})$ and $\psi \in \mathcal{S}(\mathcal{H}')$, it holds that

$$R_{\mathcal{H}'}(\psi)\, r(\phi \to \psi) \leqq R_{\mathcal{H}}(\phi), \tag{1}$$

In Eq. (1), $r(\phi \to \psi)$ is the rate of the asymptotic state conversion between ϕ and ψ definied as

$$r(\phi \to \psi) := \inf \Big\{ r \geqq 0 : \exists (\mathcal{N}_n : \text{free operation})_{n \in \mathbb{N}}, \\ \liminf_{n \to \infty} \left\| \mathcal{N}_n \left(\phi^{\otimes n}\right) - \psi^{\otimes \lceil rn \rceil} \right\|_1 = 0 \Big\},$$

where $\|\cdot\|_1$ is the trace norm, and $\lceil \cdots \rceil$ is the ceiling function.

We here note that from the definition (1), consistent resource measures are weakly additive [2]; that is, $R(\phi^{\otimes n}) = nR(\phi)$ for all states ϕ and all positive integers n. In Ref. [2], a sufficient condition for the *regularization* R^{∞} of a resource measure R, defined as $R^{\infty}(\phi) := \lim_{n\to\infty} R(\phi^{\otimes n})/n$, to be consistent was provided. The regularization of a resource measure also serves as a resource measure. The sufficient condition consists of the following conventionally considered properties for a resource measure:

1. *Asymptotic continuity*: For any sequence of positive integers $(n_i)_{i\in\mathbb{N}}$, and any sequences of states $\left(\phi_{n_i} \in \mathcal{S}\left(\mathcal{H}^{\otimes n_i}\right)\right)_i$ and $\left(\psi_{n_i} \in \mathcal{S}\left(\mathcal{H}^{\otimes n_i}\right)\right)_i$ satisfying $\lim_{i\to\infty} \|\phi_{n_i} - \psi_{n_i}\|_1 = 0$, it holds that $\lim_{i\to\infty} |R_{\mathcal{H}^{\otimes n_i}}(\phi_{n_i}) - R_{\mathcal{H}^{\otimes n_i}}(\psi_{n_i})|/n_i = 0$.

2. *Subadditivity*: For any states ϕ and ψ, it holds that $R(\phi \otimes \psi) \leqq R(\phi) + R(\psi)$.

Note that, in Ref. [42], another sufficient condition for consistency of resource measures is given; in this paper, we investigate the sufficient condition shown in the following lemma [2].

Lemma 3 (Sufficient Condition for Consistency)**.** *The regularization R^{∞} of a resource measure R is consistent if R is asymptotically continuous and subadditive.*

From this lemma, a promising way for constructing a consistent resource measure is to consider a subadditive resource measure and then to check its asymptotic continuity. As a subadditive resource measure, we here consider the relative entropy of resource R_{R}, which is widely studied in various QRTs [4, 26–31]. For our purpose, it may not be suitable to use non-subadditive measures such as robustness-based measures [5, 19, 43–45].

Definition 4 (Relative Entropy of Resource)**.** The relative entropy of resource R_{R} is defined as

$$R_{\mathrm{R}}(\phi) \coloneqq \min_{\psi:\mathrm{free}} \mathrm{D}(\phi\|\psi) \tag{2}$$

for all states ϕ, where $D(\cdot\|\cdot)$ is the relative entropy $\mathrm{D}(\rho||\sigma) \coloneqq \mathrm{Tr}(\rho\log_2\rho - \rho\log_2\sigma)$.

The regularization of the relative entropy of resource R_R^∞ is called *the regularizaed relative entropy of resource*. Reference [46] gives an operational interpretation of the regularlized relative entropy of resource; the regularized relative entropy of resource is considered as the optimal rate of asymptotic hypothesis testing where one aims to distinguish a given state and its closest free state.

III. MAIN RESULTS

In this section, we investigate the consistency of the regularized relative entropy of resource for general resources of physical importance. Before we go through the further details, we briefly overview the backgrounds and our contributions.

In Ref. [2], it was shown that the regularized relative entropy of resource is consistent in all finite-dimensional convex QRTs with a full-rank free state. The proof was based on asymptotic continuity of the relative entropy of resource in such QRTs [36, 47].

Here, we consider relaxing the assumption. First, in Sec. III A, we study the necessity of full-rank free states. Indeed, by refining the definition of consistency, we show that the regularized relative entropy of resource is consistent in finite-dimensional convex QRTs even without full-rank free states. Next, we investigate physically well-motivated non-convex or infinite-dimensional resources. We will overview the physical significance of these resources at the beginning of each subsection. For non-convex resources, namely, discord (Sec. III B) and non-Markovianity (Sec. III C), despite the existence of a counterexample for asymptotic continuity of the relative entropy of resource [41], we prove that the regularized relative entropy of resource is consistent. Moreover, we analyze an infinite-dimensional resource, non-Gaussianity, in Sec. III D. Contrary to the statement in the previous work [31], we prove that the relative entropy of non-Gaussinity is not continuous even with a reasonable energy constraint. On the other hand, if we take the convex hull of the set of Gaussian states, we show that the relative entropy of non-Gaussianity is asymptotically continuous under an appropriate energy constraint; therefore, the regularized relative entropy can be employed as a consistent resource measure under the energy constraint in this convex but infinite-dimensional QRT.

A. Finite dimensional convex QRTs

In this section, we show that the regularized relative entropy of resource is indeed consistent in all finite-dimensional convex QRTs even without full-rank free states. This result contributes to removing an arguably artificial restriction of imposing full-rankness of free states in considering general QRTs. Here, the problem is that if no full-rank free state exists, the regularized relative entropy of resource R_{R} in Eq. (2) may become infinite, and the definition of consistency may become an indeterminate form. To resolve these problems, we consider the inequality (1) to be also true when $R(\phi) = \infty$ or when $R(\phi) < \infty$, $R(\psi) = \infty$, and $r(\phi \to \psi) = 0$. If the set of free states contains a full-rank state, the regularized relative entropy of resource becomes consistent as shown in the previous work. As shown in Supplemental Materials, we show that these two cases cover all the cases where R diverges to infinity. Therefore, under this refined definition to take infinities into account, the regularized relative entropy of resources can be consistent even in the QRTs without full-rank free states.

B. Discord

In this section, we investigate consistent resource measures for discord. Discord is conceptually understood as a form of quantum correlation; it is defined as the difference between total (quantum and classical) correlation and classical correlation [38]. While entanglement is widely appreciated as a quantum correlation enhancing various quantum information processing tasks [3], it has been revealed that discord serves as a resource for several tasks even without the existence of entanglement [48–54]. In addition, discord also has operational meanings in several tasks [55, 56]. From such operational significance of discord, it is essential to establish a consistent way for quantitative analysis of discord.

To analyze discord for this purpose, one may consider a QRT of discord by choosing, as free operations, a class of operations that preserve a set of states with zero discord. As explained in Sec. II, each choice of free operations determines the corresponding set of free states. For discord, there have been three kinds of sets of states conventionally considered as free states. Here, we only consider the bipartite case for simplicity of presentation, while it is also possible to consider multipartite cases [29]. Hereafter, we consider two parties A and B with finite-dimensional quantum systems $\mathcal{H}^A$ and $\mathcal{H}^B$ respectively. Then, let D denote the dimension of the composit system $\mathcal{H}^A \otimes \mathcal{H}^B$. The first set of free states is the set of classical-classical states c-c $\coloneqq \left\{\sum_k p_k \left|a_k\right\rangle\left\langle a_k\right|^A \otimes \left|b_k\right\rangle\left\langle b_k\right|^B\right\}$, where $\{|a_k\rangle^A\}$ is an orthonormal basis of $\mathcal{H}^A$ and $\{|b_k\rangle^B\}$ is an orthonormal basis of $\mathcal{H}^B$. Note that these bases are not necessarily fixed to some standard bases in contrast to the QRT of coherence [57]. The second one is the set of quantum-classical states q-c $:= \left\{\sum_k p_k \rho_k^A \otimes |b_k\rangle\langle b_k|^B\right\}$, and the last one is the set of classical-quantum states c-q, where the roles of A and B in q-c are interchanged. Due to the symmetry of the definitions of q-c and c-q, we only consider c-c and q-c in the rest of this section. Since the bases $\{|a_k\rangle^A\}$ and $\{|b_k\rangle^B\}$ are not fixed but arbitrary, it is clear that none of these three sets are convex. Then, for each of these sets of free states, discord can be analyzed in the framework of QRTs. Due to the non-convexity of these sets, however, these QRTs of discord are non-convex QRTs.

Based on these free states, we can define the relative entropy of discord as

$$D_{\mathrm{rel}}^{\mathcal{F}}(\phi^{AB}) := \min_{\psi^{AB}\in\mathcal{F}} \mathrm{D}(\phi^{AB}\|\psi^{AB}), \tag{3}$$

where $\mathcal{F}$ is either c-c or q-c. Indeed, we derive a simple expression of the relative entropic measure for any choice of free states. In more detail, for all states $\phi^{AB} \in \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B)$, it holds that

$$D_{\mathrm{rel}}^{\mathcal{F}}(\phi^{AB}) = \min_{\chi_\phi^{\mathcal{F}}} \mathrm{H}(\chi_\phi^{\mathcal{F}}) - \mathrm{H}(\phi^{AB}), \tag{4}$$

with the von Neumann entropy $\mathrm{H}(\rho) := -\operatorname{Tr}(\rho\log_2\rho)$. The minimization in Eq. (4) is taken over the free states $\chi_\phi^{\text{c-c}} := \sum_k (\langle a_k, b_k|)\phi^{AB}(|a_k, b_k\rangle)\, |a_k, b_k\rangle\langle a_k, b_k|^{AB}$ or $\chi_\phi^{\text{q-c}} := \sum_k (\mathbb{1}^A \otimes \langle b_k|)\phi^{AB}(\mathbb{1}^A \otimes |b_k\rangle) \otimes |b_k\rangle\langle b_k|^B$ with the identity operator $\mathbb{1}$. See also Ref. [29] for derivation of this expression in the case $\mathcal{F} = $ c-c.

With this simplified expression of the relative entropy of discord, we prove that this relative entropic measure is asymptotically continuous by repeatedly using Fannes-Audenaert inequality [58] on asymptotic continuity of the von-Neumann entropy. Indeed, we show that for all states $\phi^{AB}, \psi^{AB} \in \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B)$, it holds that

$$\begin{aligned} &|D_{\mathrm{rel}}^{\mathcal{F}}(\phi^{AB}) - D_{\mathrm{rel}}^{\mathcal{F}}(\psi^{AB})| \\ &\leqq \|\phi^{AB} - \psi^{AB}\|_1 \log_2 D + 2h_2\left(\frac{\|\phi^{AB} - \psi^{AB}\|_1}{2}\right), \end{aligned} \tag{5}$$

where h_2 is the binary entropy function defined as $h_2(x) := -x\log_2 x - (1-x)\log_2(1-x)$ for $x \in [0,1]$. Then, Eq. (5) implies asymptotic continuity of the relative entropy of discord. The rigorous proof is shown in Supplemental Materials. Considering Lemma 3 and Eq. (5) together, we conclude that the regularization of the relative entropy of discord is consistent for all the choices of free states.

On the other hand, we also prove that another discord measure, *measurement-based quantum discord* [59], is subadditive and asymptotically continuous. Therefore, we can also employ its regularization as a consistent resource measure for discord. The detailed definition and proofs are shown in supplemental Materials.

C. Non-Markovianity

In this section, we prove that the regularized relative entropy of non-Markovianity is consistent. Quantum Markov chain is a quantum extension of the classical Markov chain, originally formulated in Ref. [39]. In the classical case, a sequence of random variables XYZ is said to be a Markov chain if Z conditioned on Y is independent of X, which is indeed equivalent to $I(X:Z|Y) = 0$, where $I(X:Z|Y)$ is the conditional mutual information [60]. Analogously, a quantum state $\phi^{ABC} \in \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C)$ is said to be a quantum Markov chain if C conditioned on B is independent of A [39], which is indeed equivalent to the condition $I(A:C|B)_\phi = 0$ [61] in terms of the quantum conditional mutual information [62]. Non-Markovianity serves as a resource in various tasks. For example, while classical non-Markov chain enables the secret key agreement [63], quantum non-Markovianity is exploited as a resource in quantum one-time pad [64], which is a protocol to ensure secure communication between two parties.

The QRT of non-Markovianity has been established in Refs. [65, 66] to analyze this quantum version of Markov property in the QRT framework. Hereafter, we consider the tripartite system $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C$ with dimension D in total. The set of quantum Markov chains $\mathcal{D}_{\mathrm{Markov}}$ are defined as $\mathcal{D}_{\mathrm{Markov}} := \{\psi \in \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C) : I(A:C|B)_\psi = 0\}$. Here, note that $\mathcal{D}_{\mathrm{Markov}}$ is not convex [65]; the QRT of non-Markovianity is a non-convex QRT. Then, the relative entropy of non-Markovianity is defined as

$$\Delta(\phi^{ABC}) := \min_{\psi\in\mathcal{D}_{\mathrm{Markov}}} \mathrm{D}(\phi^{ABC}\|\psi^{ABC}). \tag{6}$$

We prove that the relative entropy of non-Markovianity is asymptotically continuous. We show that for all states $\phi, \psi \in \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C)$ satisfying $\|\phi - \psi\|_1/2 \leqq 1/3$, it holds that

$$\begin{aligned} &|\Delta(\phi^{ABC}) - \Delta(\psi)^{ABC}| \\ &\leqq 2(\|\phi^{ABC} - \psi^{ABC}\|_1 \log_2 D + h_2(\|\phi^{ABC} - \psi^{ABC}\|_1)). \end{aligned} \tag{7}$$

Then, Eq. (7) implies that the relative entropy of non-Markovianity is asymptotically continuous. Our proof is based on a simple expression of the relative entropy of non-Markovianity, in a similar form to Eq. (4), as a gap of two von Neumann entropies [30]. Then, we explicitly show asymptotic continuity of the relative entropy of non-Markovianity using this characterization. In the proof, we further decompose the characterization and repeatedly exploit Fannes-Audenaert inequality [58]. See Supplemental Materials for more details of the proof. Therefore, from Lemma 3 and Eq. (7), it follows that the regularized relative entropy of non-Markovianity is consistent.

D. Non-Gaussianity

In this section, we investigate the consistency of the relative entropic measure in the QRT of non-Gaussianity [67–70]. Gaussian quantum information [40], based on Gaussian states and Gaussian operations, plays a central role in continuous-variable (CV) quantum information processing since Gaussian states and operations are experimentally implementable with high-precision control in quantum-optical setups [71]. In addition, despite the infinite-dimensionality, the theoretical analysis of Gaussian states is relatively tractable due to the fact that their characterization functions are in Gaussian forms. Indeed, various quantum information processing protocols, such as quantum teleportation [72–74], noisy quantum cloning [75, 76], quantum illumination [77], quantum reading [78], and quantum key distribution [79–83], can be implemented by Gaussian states and operations. However, it was shown that non-Gaussianity is an essential resource for several tasks including entanglement distillation [84–87], quantum error correction [88], and universal quantum computation using CV systems [89–93]. Non-Gaussianity also reveals implementation cost of CV fault-tolerant quantum computation [70, 93].

Here, we briefly review the basic concepts for the QRT of non-Gaussianity. For further details, see Refs. [40, 94]. Consider an N-mode bosonic system and define real quadratic field operators $\hat{x} := (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)$. A quantum state ϕ is described by its Wigner characteristic function $\chi(\xi, \phi) = \mathrm{Tr}[\phi \hat{D}(\xi)]$. Here, $\xi = (\xi_1, \ldots, \xi_{2N}) \in \mathbb{R}^{2N}$ is a real-valued vector, and $\hat{D}(\xi) := \exp(i\hat{x}^{\mathrm{T}}\Omega\xi)$ is the Weyl displacement operator with $\Omega := i\bigoplus_{k=1}^{N} Y$, where Y is the Pauli-Y matrix.

A quantum state ψ is a Gaussian state if its characteristic function $\chi(\xi, \psi)$ is expressed as a Gaussian function whose form is determined only by the mean and covariance matrix of $\hat{x}$ with respect to ψ [40, 94]. We define $\mathcal{G}$ as the set of Gaussian states. Since the sum of Gaussian functions are not necessarily Gaussian, the set $\mathcal{G}$ is not convex. The relative entropy of non-Gaussianity is defined as

$$\delta[\phi] := \min_{\psi \in \mathcal{G}} \mathrm{D}(\phi \| \psi). \tag{8}$$

It is known that the relative entropy of non-Gaussianity can be written as $\delta[\phi] = \mathrm{H}(\phi_{\mathrm{G}}) - \mathrm{H}(\phi)$, where ϕ_{G} is the Gaussification of ϕ, that is, the Gaussian state with the same mean and covariance matrix as ϕ [31]. Despite this simple characterization, we cannot directly apply the same strategy for asymptotic continuity as in the QRTs of discord and non-Markovianity because Gaussian states are defined on an infinite-dimensional state space.

While continuity of the relative entropy of non-Gaussianity is claimed in Ref. [31], we discover a counterexample showing that the relative entropy of non-Gaussianity is not continuous even under a reasonable energy constraint. Note that this discovery may also affect some of the arguments in other existing literature citing Ref. [31]. Consider a single-mode system with Hamiltonian $H = \hbar\omega a^\dagger a$. Take two Fock-diagonal states $\rho = \epsilon \left|E/\epsilon\right\rangle \left\langle E/\epsilon\right| + (1-\epsilon)\left|0\right\rangle\left\langle 0\right|$ and $\sigma = \left|0\right\rangle\left\langle 0\right|$ with a small positive number ϵ and fixed positive number E. Then, these states satisfy the energy constraint $\mathrm{Tr}(\rho H) \leqq E$ and $\mathrm{Tr}(\sigma H) \leqq E$. However, while $\|\rho - \sigma\|_1 \approx \epsilon$, we have $|\delta[\rho] - \delta[\sigma]| \approx (E+1)\log_2(E+1) - E\log_2 E$ even for infinitesimal ϵ. Therefore, the relative entropy of non-Gaussianity is not continuous. More detailed discussion and proof are shown in Supplemental Materials.

We also show that we can avoid this discontinuity of the relative entropy of non-Gaussianity, by considering the convex QRT of non-Gaussianity [67, 68]. In the convex QRT of non-Gaussianity, we take, as free operations, Gaussian operations that may be conditioned on outcomes of homodyne detections. That is, we use the convex hull of the set of Gaussian operations as free operations. The set of free state is the convex hull of the set of Gaussian states $\mathrm{conv}(\mathcal{G})$. Then, we consider the following modified version of relative entropy of non-Gaussianity, which we call the relative entropy of convex non-Gaussianity:

$$\delta_{\mathrm{conv}}[\phi] := \min_{\psi \in \mathrm{conv}(\mathcal{G})} \mathrm{D}(\phi \| \psi). \tag{9}$$

Here, to avoid discontinuity due to the infinite-dimensionality of the state space, we prove asymptotic continuity under an energy constraint. Hereafter, let H be the Hamiltonian of a given system. We suppose that the smallest eigenvalue of H is fixed to zero. Then, let $E \geqq 0$ represent an upper bound of the energy of the system. Let $\mathcal{D}_{H,E}(\mathcal{H}) := \{\rho \in \mathcal{D}(\mathcal{H}) : \mathrm{Tr}(H\rho) \leqq E\}$ denote a subset of the set of density operators that contains all states satisfying the energy constraint, and we here take the set of states of interest as $\mathcal{S}(\mathcal{H}) = \mathcal{D}_{H,E}(\mathcal{H})$. The relative entropy of convex non-Gaussianity δ_{conv} is indeed asymptotically continuous in the following sense: if the Hamiltonian H satisfies the condition from Ref. [95]

$$\lim_{\lambda \to \infty} [\mathrm{Tr}(\mathrm{e}^{-\lambda H})]^{\lambda} = 0, \tag{10}$$

for any sequences $\{\phi_n \in \mathcal{D}_{H_n, nE}(\mathcal{H})\}_n$ and $\{\psi_n \in \mathcal{D}_{H_n, nE}(\mathcal{H})\}_n$ such that $\lim_{n\to\infty} \|\phi_n - \psi_n\|_1 \to 0$ with the Hamiltonian $H_n = H \otimes I \otimes \cdots \otimes I + I \otimes H \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes H$, then it holds that $\lim_{n\to\infty} |\delta_{\mathrm{conv}}[\phi_n] - \delta_{\mathrm{conv}}[\psi_n]|/n = 0$.

Here, we note that the assumtion (10), on which asymptotic continuity is based, is satisfied by representative choices of Hamiltonians such as those of harmonic oscillators representing light. Our proof of the asymptotic continuity is based on a modified version of Proposition 3 in Ref. [95] showing that for the Hamiltonian H satisfying Eq. (10) and for all states $\phi, \psi \in \mathcal{D}_{H,E}(\mathcal{H})$ such that $\|\phi - \psi\| \leqq \epsilon \leqq 1/2$, it holds that $|\delta_{\mathrm{conv}}[\phi] - \delta_{\mathrm{conv}}[\psi]| \to 0$ as $\epsilon \to 0$. Then, letting $r^{(H,E)}(\phi \to \psi)$ denote the asymptotic conversion rate under the energy constraint, we have the following inequality showing the consistency of $\delta_{\rm conv}^{\infty}$

$$\delta_{\rm conv}^{\infty}[\psi] r^{(H,E)}\left(\phi \to \psi\right) \leqq \delta_{\rm conv}^{\infty}[\phi]. \tag{11}$$

See Supplemental Materials for the rigorous statement and proof. Therefore, from Eq. (11), it follows that the regularized relative entropy of convex non-Gaussianity is consistent in terms of the asymptotic state conversion under the energy constraint.

IV. CONCLUSION

We investigated the relative entropy of resource for general resources, including discord, non-Markovianity, and non-Gaussianity to show the existence of asymptotically consistent resource measures in QRTs of these resources. In contrast with the analysis in our initial paper [2], which showed the consistency of the regularized relative entropy of resource by assuming the finite-dimensionality, convexity, and existence of full-rank free states, we here showed that the regularized relative entropy of resource is consistent in all finite-dimensional convex QRTs even *without assuring the existence of full-rank free states*. Furthermore, we proved that the regularized relative entropy of resource serves as a consistent resource measure for some physically well-motivated *non-convex* quantum resources, namely, discord and non-Markovianity. For an *infinite-dimensional* resource, non-Gaussianity, we showed that the relative entropy of non-Gaussinity is indeed discontinuous even though the continuity is argued in the previous research [31]. At the same time, by considering the convex hull of Gaussian states, we proved that the convex relative entropy of non-Gaussinaity is asymptotically continuous under an energy constraint and that its regularization is consistent. Thus, we disclosed the consistency of the regularized relative entropy of resource for general resources beyond those covered in the previous work. These results pave the way for the quantification of resources consistent with the asymptotic state conversion, which fuels further investigation of resources not exactly comparable with each other. The results in this paper establish a theoretical foundation for quantitative studies of a wide variety of quantum resources that may not satisfy restrictive mathematical assumptions such as convexity and finite-dimensionality.

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