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P. R. Berman and J.-L. Le Gouët

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Pulsed field transmission by atomic frequency combs and random spike media: the prominent role of dispersion

P. R. Berman

Physics Department, University of Michigan,

450 Church Street, Ann Arbor, Michigan 48109-1040

J.-L. Le Gouët

Laboratoire Aimé Cotton, CNRS, Univ. Paris Sud,

bâtiment 505, campus universitaire, 91405 Orsay, France

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Abstract

A theory of pulse transmission is presented in which the medium through which the pulse propagates is characterized by an inhomogeneous distribution that is either an atomic frequency comb (AFC) or a medium of randomly spaced frequency spikes (RSM). For an AFC, we obtain analytic expressions for the transmitted field amplitude, which is composed of the (partially) transmitted incident pulse, plus a train of equally spaced echoes. For RSM, we derive expressions for the average transmitted field amplitude and field intensity. In the limit that the spike width is much less than the spike separation, normally encountered with AFC, the overall atom-field dynamics is dominated by dispersion - absorption plays a negligible role. The importance of the average group time delay of the pulse is discussed.

I. INTRODUCTION

For the past twenty years or so, a strong interest in optically carried quantum information has stimulated research on light-atom interactions. For quantum information storage and retrieval, various schemes have been proposed and demonstrated, such as Electromagnetically Induced Transparency (EIT) [1, 2] and Atomic Frequency Combs (AFC). [3–5]

In EIT, the incoming signal pulse travels through a transparency window that a control field has opened within the absorption line. Because of the control field, EIT works in three-level systems, but a similar process can be observed in two-level systems, when a spectral hole is burned in the inhomogeneously broadened profile of an absorption line. [6, 7] The resulting refractive index dispersion may strongly reduce the group velocity inside that window, far beneath the phase velocity. Provided the signal bandwidth is sufficiently small, all the spectral components are confined to the middle of the transparency window and experience the same group delay. Hence the signal emerges from the sample with neither shape distortion nor energy loss, with a delay that is fully determined by the refractive index dispersion. Although absorption is not involved in the process, nearly all the signal energy is transferred either to the control field, in EIT, or to the off resonance atoms, in the hole-burning variant, during the signal transit. The signal recovers its initial energy at the medium output. Such a momentary pulse energy transfer to off-resonance atoms, without shape distortion, also exists in adiabatic following. [8]

Of course, the absorption coefficient and the refractive index are linked by the Kramers-Krönig relation, which follows from causality. At first glance, it might seem to be pointless to separate one from the other since both affect the atom-field dynamics. However, as is evidenced in the analysis of EIT signals, such an approach may prove productive.

In contrast to EIT, AFC is usually analyzed in terms of absorption rather than dispersion. The AFC scheme is based on the capture of the incoming pulse by an inhomogeneously broadened spectral profile, which has been shaped in the form of evenly spaced absorbing spikes, with no absorption in the interval between them. The stored radiation is progressively released through successive echoes. One may question the absorption oriented analysis. Indeed, when the spike width is much smaller than the comb spectral period, the imaginary part of susceptibility has a negligible effect on the atom-field dynamics. As in EIT, the interaction with the active medium is actually dominated by dispersion but, unlike EIT, the group velocity is not uniform over the pulse bandwidth, which results in a temporal shape distortion of the incoming signal. Specifically, the AFC output consists of the (partially) transmitted incident pulse, followed by a series of equally spaced echoes.

It was noticed quite early that nearly all the incident pulse energy is transmitted through the AFC filter, without absorption, in the narrow spike limit. [4, 9, 10] However, since analytic expressions for the successive echo contributions were not derived, one could not obtain the time profile of the energy retrieval. In the present work we remedy this situation, allowing us to determine the characteristic time of the radiation recovery. In addition, we extend the calculation to random spike media (RSM), consisting of randomly distributed absorbing spikes. For such a filter, the signal no longer consists of a number of echoes, although the average field energy is radiated on a time scale comparable to that of an AFC whose spike separation is equal to the average frequency spacing of the RSM.

The paper is organized as follows. In Section II, we provide a basic formalism for the problem, along the lines of Crisp. [11] To simplify matters, we adopt a model in Sec. III in which the incident pulse has a Gaussian temporal profile and the inhomogeneous frequency distribution of the atoms consists of a comb of Lorentzian spikes or "teeth." In Section IV, we obtain an analytic expression for the transmitted field amplitude for regularly spaced spikes (AFC). We then consider RSM in Sec. V, in which the spikes are placed at random positions in frequency space with some predetermined average spacing; analytic expressions are derived for the average transmitted field amplitude and intensity. In Sec. VI, we relate the characteristic time for energy transmission to the average group time delay of the pulse. The results are summarized in Sec. VII. In Appendix A, the results are generalized to allow for arbitrary pulse envelopes and arbitrary spike profiles, along with specific results for Gaussian and rectangular spike profiles. In Appendix B, derivations are provided for some of the asymptotic limits discussed in the main text.

II. BASIC FORMALISM

In general, when a radiation pulse is incident on an atomic medium, the resulting atom-field dynamics can be quite complicated, depending on the characteristics of the radiation pulse and the properties of the medium. We limit our discussion to a medium that has a broad inhomogeneous distribution of transition frequencies centered around frequency ω_0.

FIG. 1: Inhomogeneous frequency distribution $g(\delta)$. The spikes or teeth are separated by δ_0 and the spike width is γ_s. The dashed red curve is the frequency spectrum of the input pulse, whose temporal width is T_p. Not shown is the very broad inhomgeneous envelope of the frequency distribution, having width $\gamma_s \gg 1/T_p$.

From this distribution, an initial state is prepared which consists of an array of spikes, each having width γ_s. The atoms in the medium are modelled as two-level atoms, having upper level 2, lower level 1, and frequencies $\omega_{21}(\delta) = \omega_0 + \delta$. The values of δ are determined by the state preparation and are characterized by a frequency distribution $g(\delta)$. Two situations are envisioned. Either the spikes are equally spaced with frequency separation $\delta_0 \gg \gamma_s$ to form an AFC, or the spikes are centered at random frequencies having an average separation δ_0 (RSM - random spike media)). A radiation pulse, incident on the sample, is assumed to have a smooth envelope with temporal width T_p and a central frequency equal to ω_0. The excited state decay rate of the atoms, γ_2, is assumed to be much smaller than γ_s, and the pulse bandwidth δ_p is assumed to be much larger than δ_0; that is, the pulse bandwidth covers a large number of the spikes (although many of the expressions to be derived are valid for arbitrary ratios of δ_p/δ_0). The *overall* inhomogeneous width γ_w of the ensemble is much larger than the pulse bandwidth, $\delta_p = 1/T_p$. The pulse area is assumed to be much less than unity. The situation is summarized in Fig. 1.

The starting point of the calculation is the Maxwell-Bloch (MB) equations in the slowly varying amplitude and phase approximation for small area pulses. [11] The (stationary) atoms are confined to a cylindrical volume having length L. The cylinder axis is taken to be the $z-$axis and diffraction is neglected; that is, the incident and reradiated fields are assumed to be constant over the cross-sectional area of the medium. The electric field is of the form

$$E_T\left(z,t\right)=\frac{1}{2}E\left(z,t\right)e^{ikz-i\omega_0 t}+\text{c.c.}, \tag{1}$$

where $k=\omega_0/c=2\pi/\lambda$. We define a dimensionless slowly varying field amplitude by

$$F\left(z,t\right)=\sqrt{\frac{\epsilon_0 c}{2\mathcal{N}L\hbar\omega_0\gamma_2}}E\left(z,t\right), \tag{2}$$

where $E\left(z,t\right)$ is the field amplitude and $\mathcal{N}$ is the the atomic density, a dimensionless time by

$$\tau=t/T_p, \tag{3}$$

a dimensionless distance by

$$Z=z/L, \tag{4}$$

and a dimensionless frequency distribution by

$$G\left(\Delta\right)=\frac{1}{T_p}g\left(\frac{\Delta}{T_p}\right), \tag{5}$$

where $\Delta=\delta T_p$ is a dimensionless frequency.

In terms of these dimensionless units, the MB equations can be written as

$$\left(\frac{\partial}{\partial Z}+\frac{1}{\xi}\frac{\partial}{\partial\tau}\right)F\left(Z,\tau\right)=i\sqrt{C}\int\rho_{21}\left(Z,\tau,\Delta\right)d\Delta; \tag{6a}$$

$$\frac{\partial\rho_{21}\left(Z,\tau,\Delta\right)}{\partial\tau}=i\Gamma_2\sqrt{C}F\left(Z,\tau\right)G\left(\Delta\right)-\left(\frac{\Gamma_2}{2}+i\Delta\right)\rho_{21}\left(Z,\tau,\Delta\right), \tag{6b}$$

where

$$\xi=\frac{cT_p}{L} \tag{7}$$

is the spatial extent of the initial pulse divided by the length of the medium,

$$C=\frac{3}{8\pi}\mathcal{N}\lambda^2 L \tag{8}$$

is a cooperativity parameter, and

$$\Gamma_2=\gamma_2 T_p. \tag{9}$$

All frequencies have been expressed in units of T_p^{-1}. We have used a field interaction representation for the off-diagonal density matrix element,

$$\rho_{21}\left(Z,\tau,\Delta\right)=\tilde{\rho}_{21}\left(Z,\tau,\Delta\right)e^{ikLZ-i\omega_0 T_p\tau}, \tag{10}$$

and have dropped the tilde. The off-diagonal density matrix element $\rho_{21}(Z,\tau,\Delta)$ is that associated with an atom having transition frequency $\omega_{21}(\delta)=\omega_0+\delta$. Although not necessary, we shall assume that $\xi \gg 1$, typical of AFC experiments, which allows us to neglect the time derivative term in Eq. (6a).

The solution for $\rho_{21}(Z,\tau,\Delta)$ is

$$\begin{aligned}\rho_{21}(Z,\tau,\Delta) &= i\Gamma_2\sqrt{C}G(\Delta)\int_{-\infty}^{\tau} F(Z,\tau')\, e^{-\left(\frac{\Gamma_2}{2}+i\Delta\right)(\tau-\tau')}d\tau' \\ &= i\Gamma_2\sqrt{C}G(\Delta)\int_{0}^{\infty} F(Z,\tau-\tau')\, e^{-\left(\frac{\Gamma_2}{2}+i\Delta\right)\tau'}d\tau',\end{aligned} \tag{11}$$

which, when substituted into the field equation, yields

$$\frac{\partial F(Z,\tau)}{\partial Z} = -\Gamma_2 C\int d\Delta G(\Delta)\int_0^{\infty} F(Z,\tau-\tau')\, e^{-\left(\frac{\Gamma_2}{2}+i\Delta\right)\tau'}d\tau'. \tag{12}$$

This can be solved easily using Fourier transforms. [11] We set

$$\tilde{F}(Z,\Omega) = \int_{-\infty}^{\infty} e^{i\Omega\tau}F(Z,\tau)\,d\tau \tag{13}$$

($\Omega=\omega T_p$ is dimensionless) to arrive at

$$\frac{\partial \tilde{F}(Z,\Omega)}{\partial Z} = -\Gamma_2 C A(\Omega)\tilde{F}(Z,\Omega), \tag{14}$$

where

$$A(\Omega) = \int d\Delta \frac{G(\Delta)}{\frac{\Gamma_2}{2}+i(\Delta-\Omega)}. \tag{15}$$

Note that he electric susceptibility $\chi(\Omega)$ can be written in terms of $A(\Omega)$ as

$$\chi(\Omega) = i\frac{2\Gamma_2 C A(\Omega)}{kL}. \tag{16}$$

The solution for $\tilde{F}(Z,\Omega)$ is

$$\tilde{F}(Z,\Omega) = \tilde{F}(0,\Omega)\exp\left[-\Gamma_2 C A(\Omega) Z\right]. \tag{17}$$

Taking the inverse Fourier transform we find

$$F(Z,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\Omega\tau}\tilde{F}(0,\Omega)\,e^{-\Gamma_2 CZA(\Omega)}d\Omega. \tag{18}$$

The transmitted field intensity at the exit plane of the sample at time τ is proportional to a quantity $T(\tau)$ defined by

$$T(\tau) = |F(1,\tau)|^2, \tag{19}$$

the total energy that has exited the sample at time τ, normalized to the input field energy, is given by

$$W(\tau)=\frac{\int_{-\infty}^{\tau}\left|F\left(1,\tau\right)\right|^{2}d\tau'}{\int_{-\infty}^{\infty}\left|F\left(0,\tau\right)\right|^{2}d\tau}, \tag{20}$$

and the total energy that exits the sample, normalized to the input field energy, is calculated as

$$\begin{aligned} W=W\left(\infty\right)&=\frac{\int_{-\infty}^{\infty}\left|F\left(1,\tau\right)\right|^{2}d\tau}{\int_{-\infty}^{\infty}\left|F\left(0,\tau\right)\right|^{2}d\tau}\\ &=\frac{\int_{-\infty}^{\infty}\left|\tilde{F}\left(0,\Omega\right)\right|^{2}e^{-2\Gamma_{2}C\,\mathrm{Re}[A(\Omega)]}d\Omega}{\int_{-\infty}^{\infty}\left|\tilde{F}\left(0,\Omega\right)\right|^{2}d\Omega}. \end{aligned} \tag{21}$$

If $\mathrm{Re}\left[A(\Omega)\right]=0$ or, equivalently, if $\mathrm{Im}\left[\chi(\Omega)\right]=0$, then it follows from Eq. (21) that $W=1$ - there is no absorption.

Under normal AFC experimental conditions,

$$\Gamma_{2}\ll\Gamma_{s}\ll\Delta_{0}\ll1, \tag{22}$$

where $\Gamma_s=\gamma_s T_p$ is the (dimensionless) spike width and $\Delta_0=\delta_0 T_p$ is the (dimensionless) average spacing between spikes. As a consequences, we will always neglect terms of order Γ_2/Γ_s. Moreover, we will always assume that $\Gamma_s/\Delta_0\ll1$ and, in some cases, we will consider the asymptotic limit that $\Gamma_s/\Delta_0\sim0$, that is, the limit in which Γ_s/Δ_0 tends to zero. It should be noted that, for finite Γ_s, the normalized total transmitted energy is not equal to unity, even though $\Gamma_2/\Gamma_s\ll1$, since there is some net absorption resulting from the *inhomogeneous* frequency distribution associated with each spike. In these dimensionless units, $1/\Delta_0$ is the number of spikes contained in the bandwidth of the incident pulse.

III. GAUSSIAN INPUT PULSE AND LORENTZIAN FREQUENCY SPIKES

We now assume that the incident pulse has a Gaussian temporal profile,

$$F\left(0,\tau\right)=F_{0}e^{-\tau^{2}/2};\qquad\tilde{F}\left(0,\Omega\right)=\sqrt{2\pi}F_{0}e^{-\Omega^{2}/2}. \tag{23}$$

Moreover, to simplify the presentation in the main text, we assume that the inhomogeneous frequency distribution consists of a number of Lorentzian spikes superimposed on a very broad Gaussian distribution. That is, we take a frequency distribution

$$
\begin{aligned}
G_L(\Delta) &= \frac{\kappa}{\sqrt{\pi}\Gamma_w} e^{-\Delta^2/\Gamma_w^2} \sum_{n=1}^{\infty} \frac{1}{\pi} \frac{1}{1+\left(\frac{\Delta - D_n}{\Gamma_s}\right)^2} \\
&\approx \frac{\kappa}{\sqrt{\pi}\Gamma_w} \sum_{n=1}^{\infty} \frac{1}{\pi} \frac{1}{1+\left(\frac{\Delta - D_n}{\Gamma_s}\right)^2},
\end{aligned}
\tag{24}
$$

where $\Gamma_w = \gamma_w T_p \gg 1$ is the *total* inhomogeneous width (in units of T_p^{-1}), D_n is the dimensionless displacement of spike n from the central frequency of the inhomogeneous frequency distribution, and $\kappa < 1$ is a factor whose value depends on the manner in which the frequency distribution of the spikes was created in the preparation stage. The integral of $G_L(\Delta)$ over Δ is of order $\kappa\left(\Gamma_s/\Gamma_w\right)N_{st} \ll 1$, where N_{st} is the total number of spikes covered by the inhomogeneous frequency distribution, accounting for the fact that only a small percentage of the total inhomogeneous distribution is excited to the initial state. Although a Lorentzian spike profile results in more net absorption than a Gaussian profile having the same Γ_s owing to the long Lorentzian tails, in the limit that $\Gamma_s/\Delta_0 \sim 0$ which is of some interest in this paper, the transmission is independent of the spike profile. In Appendix A, the results are generalized to arbitrary pulse and spike profiles.

For a Gaussian incident pulse and Lorentzian spikes, Eq. (15) becomes

$$
A(\Omega) = \frac{\kappa\Gamma_s}{\sqrt{\pi}\Gamma_w} \sum_{n=1}^{\infty} \frac{1}{\Gamma_s - i\left(\Omega - D_n\right)}
\tag{25}
$$

and Eq. (18) reduces to

$$
F(Z,\tau) = \frac{F_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\Omega e^{-i\Omega\tau} e^{-\Omega^2/2} \exp\left[-\frac{\tilde{\alpha}_0\Gamma_s Z}{2} \sum_{n=1}^{\infty} \frac{1}{\Gamma_s - i\left(\Omega - D_n\right)}\right],
\tag{26}
$$

where $\tilde{\alpha}_0 = \alpha_0 L$ is the dimensionless *cw* absorption coefficient on resonance for a single spike given by

$$
\tilde{\alpha}_0 = \frac{2\Gamma_2 C\kappa}{\sqrt{\pi}\Gamma_w}.
\tag{27}
$$

Consistent with our approximations, we have neglected terms of order Γ_2/Γ_s in the final expression for $A(\Omega)$ [although for Lorentzian spikes they could be included simply by replacing Γ_s with $(\Gamma_2 + \Gamma_s)$].

Equation (26) can help us decide the relative importance of dispersion versus absorption in the narrow spike limit. As $\Gamma_s \sim 0$, we can use the identity [12]

$$
\lim_{\Gamma_s \to 0} \frac{1}{\Gamma_s - i\left(\Omega - D_n\right)} = \pi\delta\left(\Omega - D_n\right) + iP\left(\frac{1}{\Omega - D_n}\right)
\tag{28}
$$

where P is the Cauchy principal value, to rewrite Eq.(26) as

$$F(Z,\tau) \sim \frac{F_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\Omega e^{-i\Omega\tau} e^{-\Omega^2/2} \exp\left[-\frac{\tilde{\alpha}_0 \Gamma_s Z}{2} \sum_{n=1}^{\infty} \left[\pi\delta(\Omega - D_n) + iP\left(\frac{1}{\Omega - D_n}\right)\right]\right], \tag{29}$$

At $\Omega - D_n = 0$,

$$\exp\left[-\frac{\tilde{\alpha}_0 \Gamma_s Z}{2} \sum_{n=1}^{\infty} \pi\delta(\Omega - D_n)\right] \sim 0, \tag{30}$$

but these are sets of measure zero, implying that the real part of $A(\Omega)$ (or, equivalently, the imaginary part of the susceptibility), does not modify the expression for the field amplitude. As a consequence, the atom-field dynamics is determined entirely by the dispersive part of the susceptibility in the narrow spike limit. This will become even more apparent when we consider the specific cases of regularly and random spaced spikes.

IV. ATOMIC FREQUENCY COMBS (AFC)

For an AFC, $G_L(\Delta)$ can be written as

$$G_L(\Delta) = \frac{\kappa}{\sqrt{\pi}\Gamma_w} \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1 + \left(\frac{\Delta - n\Delta_0}{\Gamma_s}\right)^2}, \tag{31}$$

where Δ_0 is the (dimensionless) comb spacing. There are approximately $2\sqrt{2\ln 2}/\Delta_0$ spikes within the full width at half-maximum of $\tilde{F}(0,\Omega)$. For this choice of $G_L(\Delta)$, it is possible to express $A(\Omega)$ given in Eq. (15) as

$$A_L(\Omega) = \frac{\kappa\Gamma_s}{\sqrt{\pi}\Gamma_w} \int_{-\infty}^{\infty} d\Delta \mathcal{D}_{\Delta_0}(\Omega - \Delta) \frac{1}{\Gamma_s - i\Delta}, \tag{32}$$

where $\mathcal{D}_{\Delta_0}(\Delta)$ stands for the Dirac comb distribution function,

$$\mathcal{D}_{\Delta_0}(\Delta) = \sum_{n=-\infty}^{\infty} \delta(\Delta - n\Delta_0) = \frac{1}{\Delta_0} \sum_{n=-\infty}^{\infty} e^{2i\pi n\Delta/\Delta_0}. \tag{33}$$

Substituting the second form of $\mathcal{D}_{\Delta_0}(\Delta)$ into Eq. (32), we obtain

$$A_L(\Omega) = \frac{\sqrt{\pi}\kappa}{\Gamma_w} \frac{\Gamma_s}{\Delta_0} \left[1 + 2\sum_{n=1}^{\infty} e^{2\pi in\Omega/\Delta_0} e^{-2\pi n\Gamma_s/\Delta_0}\right]. \tag{34}$$

Writing $A_L(\Omega)$ in this form simplifies the calculation of the transmitted field. We shall see that transmitted signal at time $\tau_q = 2\pi q/\Delta_0$ depends only on those terms in the summation for which $n \leq q$. If the expression given in Eq. (31) is used for $G_L(\Delta)$ it would be necessary to include $n_{\max} \approx \Delta_0/\Gamma_s$ terms in the summation to calculate the transmitted fields. Note that, on integrating over Δ in Eq. (32), only positive n values contribute to Eq.(34). This is a reflection of causality, successively propagated from the atomic response, characterized by the frequency denominator $\left[\frac{\Gamma_2}{2} - i(\Omega - \Delta)\right]$, to the contributions from the spikes characterized by the frequency denominator $[\Gamma_s - i(\Omega - n\Delta_0)]$, and finally to $A_L(\Omega)$.

The field amplitude can then be written as

$$F_L(Z,\tau) = \frac{F_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\Omega e^{-i\Omega\tau} e^{-\Omega^2/2} K_L(\tilde{\alpha}_c Z, \Gamma_s/\Delta_0, \Omega/\Delta_0), \tag{35}$$

where the kernel $K_L(\tilde{\alpha}Z, \Gamma_s/\Delta_0, \Omega/\Delta_0)$ is defined as

$$K_L(\tilde{\alpha}_c Z, \Gamma_s/\Delta_0, \Omega/\Delta_0) = \exp\left[-\frac{\tilde{\alpha}_c Z}{2}\left(1 + 2\sum_{n=1}^{\infty} e^{\frac{2\pi i n}{\Delta_0}(\Omega + i\Gamma_s)}\right)\right] \tag{36}$$

and

$$\tilde{\alpha}_c = \tilde{\alpha}_0 \frac{\pi\Gamma_s}{\Delta_0} = \frac{2\sqrt{\pi} C \Gamma_2 \kappa}{\Gamma_w} \frac{\Gamma_s}{\Delta_0} \tag{37}$$

is an effective comb absorption coefficient.

The kernel K is a periodic function of Ω/Δ_0 and can be expanded in a Fourier series as

$$K_L(\tilde{\alpha}_c Z, \Gamma_s/\Delta_0, \Omega/\Delta_0) = \sum_{p=-\infty}^{\infty} c_{pL}(\tilde{\alpha}_c Z, \Gamma_s/\Delta_0) e^{2\pi i p \Omega/\Delta_0}, \tag{38}$$

where

$$c_{pL}(\tilde{\alpha}_c Z, y) = \int_{-1/2}^{1/2} dx K_L(\tilde{\alpha}_c Z, y, x) e^{-2\pi i p x}. \tag{39}$$

By carrying out the summation in Eq. (36) and combining the resulting equation with Eq. (39), we obtain

$$\begin{aligned} c_{pL}(\tilde{\alpha}_c Z, y) &= e^{-\tilde{\alpha}_c Z/2} \int_{-1/2}^{1/2} dx \exp\left[-\tilde{\alpha}_c Z \frac{e^{2\pi i(x+iy)}}{1 - e^{2\pi i(x+iy)}}\right] e^{-2\pi i p x} \\ &= e^{-\tilde{\alpha}_c Z/2} \int_{-1/2}^{1/2} dx \sum_{q=0}^{\infty} L_q^{-1}(\tilde{\alpha}_c Z) e^{2\pi i(x+iy)q} e^{-2\pi i p x} \\ &= \begin{cases} e^{-\tilde{\alpha}_c Z/2} e^{-2\pi p y} L_p^{-1}(\tilde{\alpha}_c Z) & p \geq 0 \\ 0 & p < 0 \end{cases}, \end{aligned} \tag{40}$$

where we have used the generating function for the generalized Laguerre polynomial $L_p^{-1}(x)$, namely

$$\exp\left[-\frac{xh}{1-h}\right] = \sum_{q=0}^{\infty} L_q^{-1}(x) h^q. \tag{41}$$

As a consequence, the field amplitude can be written as

$$\begin{aligned} F_L(Z,\tau) &= \sum_{p=0}^{\infty} F(0,\tau-2\pi p/\Delta_0)\, c_{pL}(\tilde{\alpha}Z,\Gamma_s/\Delta_0) \\ &= F_0 e^{-\tilde{\alpha}_c Z/2} \sum_{p=0}^{\infty} e^{-(\tau-2\pi p/\Delta_0)^2/2} e^{-2\pi p\Gamma_s/\Delta_0} L_p^{-1}(\tilde{\alpha}_c Z). \end{aligned} \tag{42}$$

The output field amplitude consists of the initial transmission plus a number of echoes, spaced at integral multiples of $2\pi/\Delta_0$. The field amplitude at the center of the nth echo is $F_0 e^{-\tilde{\alpha}_c Z/2} e^{-2\pi n\Gamma_s/\Delta_0} L_p^{-1}(\tilde{\alpha}_c)$.

If the echoes are well-separated in time, the total normalized transmitted energy is

$$\begin{aligned} W_L &= \frac{1}{\sqrt{\pi}\,|F_0|^2} \int_{-\infty}^{\infty} |F_L(1,\tau)|^2\, d\tau \\ &= e^{-\tilde{\alpha}_c} \sum_{p=0}^{\infty} e^{-4\pi p\Gamma_s/\Delta_0} \left[L_p^{-1}(\tilde{\alpha}_c)\right]^2. \end{aligned} \tag{43}$$

For $\Gamma_s = 0.001$, $\Delta_0 = 0.06$, and $\tilde{\alpha}_c = 2$, $W_L = 0.650$. Had we chosen Gaussian or rectangular spikes with the same width Γ_s, the total transmitted energy would be much closer to unity (see Appendix A), since those spike profiles do not have the long frequency tails that characterize a Lorentzian profile. In other words, Lorentzian spikes having the same full width at half maximum as either Gaussian or rectangular spikes result in more absorption.

1. $\Gamma_s/\Delta_0 \sim 0$

If $\Gamma_s/\Delta_0 \sim 0$, Eq. (42) reduces to

$$F(Z,\tau) = F_0 e^{-\tilde{\alpha}_c Z/2} \sum_{p=0}^{\infty} e^{-(\tau-2\pi p/\Delta_0)^2/2} L_p^{-1}(\tilde{\alpha}_c Z). \tag{44}$$

The corresponding ratio of the energy in the nth echo to the initial pulse energy (assuming the echoes are well-separated in time) is

$$W_n(\tilde{\alpha}_c) = e^{-\tilde{\alpha}_c} \left[L_n^{-1}(\tilde{\alpha}_c)\right]^2, \tag{45}$$

which also holds for for the initial transmitted field, $n = 0$. For example, with $\tilde{\alpha}_c = 2$, 0.135 of the incident pulse energy is transmitted initially by the medium, followed by 0.54 of the incident pulse energy in the first echo. That is, $0.54/0.87 = 0.62$ of the energy stored in the medium from the incident pulse is radiated in the first revival. This is clearly a collective emission, resulting from stimulated emission as the radiation propagates in the medium. With different choices of $\tilde{\alpha}_c$, you can maximize the intensity in the nth echo. From conservation of energy, we should expect that

$$\sum_{n=0}^{\infty} W_n\left(\tilde{\alpha}_c\right) = e^{-\tilde{\alpha}_c} \sum_{n=0}^{\infty} \left[L_n^{-1}(\tilde{\alpha}_c)\right]^2 = 1. \tag{46a}$$

A proof of this identity is given in the Appendix B.

The energy in successive echo pulses, $W_n\left(\tilde{\alpha}_c\right)$, converges slowly with n, since, for $n \gg 1$ and $n \gg \tilde{\alpha}_c$,

$$W_n\left(\tilde{\alpha}_c\right) = e^{-\tilde{\alpha}_c} \left[L_n^{-1}(\tilde{\alpha}_c)\right]^2 \sim \frac{\sqrt{\tilde{\alpha}_c}}{2n^{3/2}\pi} \left[1 - \sin\left(4\sqrt{n\tilde{\alpha}_c}\right)\right]. \tag{47}$$

For $\tilde{\alpha}_c$ of order unity, Eq. (47) is a remarkably good approximation for $n \geq 3$. If we set $n = \Delta_0 \tau / 2\pi$, we see that the normalized energy in the nth echo falls off as $\tau^{-3/2}$. Moreover the total normalized transmitted energy at time τ_n for $n \gg 3$ and $\tilde{\alpha}_c$ of order unity is given approximately by

$$\begin{aligned} W\left(\tau_n = \frac{2\pi n}{\Delta_0}\right) &= \sum_{p=0}^{n} W_n\left(\tilde{\alpha}_c\right) \\ &\approx 1 - \frac{1}{\pi}\sqrt{\frac{\tilde{\alpha}_c}{n}} = 1 - \sqrt{\frac{2\tilde{\alpha}_c}{\pi \Delta_0 \tau_n}}. \end{aligned} \tag{48}$$

In other words, the characteristic time for energy transmission is of order $(2\tilde{\alpha}_c/\pi\Delta_0)$. From Eq. (37), it then follows that the characteristic time is proportional to the cooperativity C, reminiscent of subradiant decay from an atomic ensemble. [14]

2. *Finite* Γ_s/Δ_0

Of course, Eq. (48) holds only in the limit $\Gamma_s/\Delta_0 \sim 0$. For a finite ratio of Γ_s/Δ_0, there are two time scales for the fall-off with n of the radiated echo signals. One is given by Eq. (48), with a time scale on the order of several $\tau_0 \approx 2\pi/\Delta_0$, for $\tilde{\alpha}_c$ of order unity The second time scale is determined by the condition that $n\Gamma_s/\Delta_0 \approx 1$, giving a time scale of order $\tau_s \approx 2\pi/\Gamma_s$, effectively determined by the spike width, as in "normal" free induction decay. If $\Gamma_s/\Delta_0 \ll 1$, as has been assumed, the signal falls off on a time scale τ_0 for $\tilde{\alpha}_c$ of order unity. On the other hand, for finite Γ_s/Δ_0, the total normalized transmitted energy is no longer equal to unity - an energy of order Γ_s/Δ_0 is stored in the medium for times $\Gamma_2^{-1} \ll \tau \ll \Gamma_s^{-1}$. That is, there is some net absorption of the incident energy by the atoms that would ultimately be radiated as incoherent spontaneous emission for times $\Gamma_2\tau = \gamma_2 t > 1$. For example, in Appendix B, it is shown that, for $4\pi\Gamma_s/\Delta_0 \ll 1$ and $\tilde{\alpha}_c$ of order unity,

$$W_L \approx 1 - 2\sqrt{\frac{\tilde{\alpha}_c\Gamma_s}{\Delta_0}}. \tag{49}$$

The corresponding results for Gaussian and rectangular spikes are given in Appendix A.

3. Absorption versus Dispersion

Before leaving this section, we would like to return to the relative contributions of absorption and dispersion to the signal in the narrow spike limit. By carrying out the summation in Eq. (36), it follows that

$$\begin{aligned} K_L\left(\tilde{\alpha}_c Z, y, x\right) &= \exp\left[-\frac{\tilde{\alpha}_c Z}{2}\left(\frac{1+e^{2\pi i(x+iy)}}{1-e^{2\pi i(x+iy)}}\right)\right] \\ &= e^{-i\frac{\tilde{\alpha}_c Z}{2}\cot[\pi(x+iy)]}, \end{aligned} \tag{50}$$

which, together with Eq. (35), implies that

$$F_L\left(Z,\tau\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\Omega\tau}e^{-\Omega^2/2}e^{-i\frac{\tilde{\alpha}_c Z}{2}\cot[\pi(\Omega+i\Gamma_s)/\Delta_0]}d\Omega. \tag{51}$$

Equation (51) provides explicit proof that, in the limit $\Gamma_s/\Delta_0 \sim 0$, it is only the real part of the susceptibility that contributes to the field dynamics.

V. RANDOM SPIKE MEDIA (RSM)

We now examine how the situation changes if we replace the AFC by a medium whose $G\left(\Delta\right)$ is characterized by frequency spikes placed at random positions, having an average spacing equal to Δ_0. The initial transmission is about the same as for a regularly spaced comb, as is the total transmitted energy. Where the results differ is that the average field intensity following the initial pulse no longer consists of a sequence of echoes - instead there is a slowly decreasing average intensity that falls off in a time of order of several $2\pi/\Delta_0$.

The field amplitude is given by Eq. (26), namely

$$F_{rL}\left(Z,\tau\right) = \frac{F_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\Omega e^{-i\Omega\tau}e^{-\Omega^2/2}\exp\left[-\frac{\tilde{\alpha}_0\Gamma_s Z}{2}\sum_{n=1}^{N_s}\frac{1}{\Gamma_s - i\left(\Omega - D_n\right)}\right], \tag{52}$$

For a specific history, this expression can be evaluated numerically with the D_n chosen at random in the interval $(-N_s\Delta_0/2, N_s\Delta_0/2)$, where N_s is the total number of spikes. To average over all histories of random spike positions in frequency space, we recognize that, on average, each factor in the product is the same; in other words,

$$\langle F_{rL}(Z,\tau)\rangle = \frac{F_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\Omega\tau}e^{-\Omega^2/2}\left\langle \exp\left[-\frac{\tilde{\alpha}_0\Gamma_s Z}{2}\frac{1}{\Gamma_s - i(\Omega - D_n)}\right]\right\rangle^{N_s} \tag{53}$$

where the average of any function $v(D_n)$ is defined as

$$\langle v(D_n)\rangle = \frac{1}{N_s\Delta_0}\int_{-N_s\Delta_0/2}^{N_s\Delta_0/2} v(D_n)\, dD_n. \tag{54}$$

Let us set

$$\psi_L = \left\langle \exp\left[-\frac{\tilde{\alpha}_0\Gamma_s Z}{2}\frac{1}{\Gamma_s - i(\Omega - D_n)}\right]\right\rangle \tag{55}$$

and expand the exponential to obtain

$$\begin{aligned}\psi_L &= \frac{1}{N_s\Delta_0}\int_{-N_s\Delta_0/2}^{N_s\Delta_0/2} dD_n \sum_{q=0}^{\infty}\left[-\frac{\tilde{\alpha}_0\Gamma_s Z}{2}\frac{1}{\Gamma_s - i(\Omega - D_n)}\right]^q \frac{1}{q!} \\ &\approx 1 + \lim_{N_s\to\infty}\frac{1}{N_s\Delta_0}\sum_{q=1}^{\infty}\int_{-\infty}^{\infty} dy\left[-\frac{\tilde{\alpha}_0\Gamma_s Z}{2}\frac{1}{\Gamma_s + iy}\right]^q \frac{1}{q!},\end{aligned} \tag{56}$$

where we have allowed the number of spikes to approach infinity. Note that the result no longer depends on Ω, implying that

$$\langle F_{rL}(Z,\tau)\rangle = F(0,\tau)\lim_{N_s\to\infty}\psi_L^{N_s}. \tag{57}$$

Using the fact that, for $q \geq 1$,

$$\int_{-\infty}^{\infty} dy\left[\frac{1}{\Gamma_s + iy}\right]^q = \pi\delta_{q,1}, \tag{58a}$$

we obtain

$$\lim_{N_s\to\infty}\psi_L^{N_s} = \lim_{N_s\to\infty}\left(1 - \frac{\pi\tilde{\alpha}_0\Gamma_s Z}{2N_s\Delta_0}\right)^{N_s} = e^{-\tilde{\alpha}_c Z/2} \tag{59}$$

and

$$\langle F_r(Z,\tau)\rangle = F(0,\tau)\, e^{-\tilde{\alpha}_c Z/2} = F_0 e^{-\tilde{\alpha}_c Z/2}e^{-\tau^2/2}, \tag{60}$$

where $\tilde{\alpha}_c$, as defined by Eq. (37), is the analogous value of $\tilde{\alpha}_c$ for a regularly spaced comb with spacing Δ_0. The average transmitted field amplitude has the same temporal profile as the input field amplitude and vanishes once the input field has traversed the sample. This is a general result, valid for any spike profile (see Appendix A), which is why the L subscript has been suppressed in Eq. (60). Although the temporal profile of the average transmitted field is identical to that of the input field, the same cannot be said of the average transmitted field intensity.

The transmitted intensity at time τ, defined by Eqs. (19) and (26), is

$$\begin{aligned} T_{rL}(\tau) = {} & \frac{|F_0|^2}{2\pi} \int_{-\infty}^{\infty} d\Omega' \int_{-\infty}^{\infty} d\Omega e^{-\left(\Omega^2+\Omega'^2\right)/2} \\ & \times e^{-i(\Omega-\Omega')\tau} \exp\left[-\frac{\tilde{\alpha}_0 \Gamma_s}{2} \sum_{n=1}^{\infty} \frac{[2\Gamma_s + i(\Omega'-\Omega)]}{[\Gamma_s - i(\Omega - D_n)]\,[\Gamma_s + i(\Omega' - \Omega_n)]}\right]. \end{aligned} \tag{61}$$

Setting

$$\bar{\Omega} = \Omega' - \Omega, \tag{62}$$

we can average Eq. (61) over all the D_n's and write it as

$$\langle T_{rL}(\tau)\rangle = \frac{|F_0|^2}{2\pi} \lim_{N_s\to\infty} \int_{-\infty}^{\infty} d\bar{\Omega} \int_{-\infty}^{\infty} d\Omega\, e^{-\left[\Omega^2+\left(\Omega+\bar{\Omega}\right)^2\right]/2} e^{i\bar{\Omega}\tau} \left[\psi_{L1}(\bar{\Omega},\Omega)\right]^{N_s}, \tag{63}$$

where

$$\psi_{L1}(\bar{\Omega},\Omega) = \frac{1}{N_s\Delta_0} \int_{-N_s\Delta_0/2}^{N_s\Delta_0/2} dD_n \exp\left[-\tilde{\alpha}_c \Delta_0 H_L(\bar{\Omega},\Omega,D_n)\right] \tag{64}$$

and

$$H_L(\bar{\Omega},\Omega,D_n) = \frac{1}{2\pi} \frac{\left(2\Gamma_s + i\bar{\Omega}\right)}{[\Gamma_s - i(\Omega - D_n)]\left[\Gamma_s + i(\bar{\Omega} + \Omega - D_n)\right]}. \tag{65}$$

In the case of the field amplitude, it was possible to expand the exponential and carry out the integrations over D_n since all but the first two terms in the expansion vanished. The situation is not so simple here since all terms in the the expansion contribute; however, for the Lorentzian spike profile that we have chosen, it is still possible to analytically evaluate all the integrals. To see this, we write Eq. (64) as

$$\psi_{L1}(\bar{\Omega},\Omega) = 1 + \lim_{N_s\to\infty} \frac{1}{N_s\Delta_0} \sum_{q=1}^{\infty} \int_{-\infty}^{\infty} dD_n \frac{\left[-\tilde{\alpha}_c\Delta_0 H_L(\bar{\Omega},\Omega,D_n)\right]^q}{q!}. \tag{66}$$

We can now change the integration variable from D_n to $D_n = (D_n - \Omega)$ so that the integrand no longer depends on Ω. We then carry out the integration over Ω in Eq. (63) to arrive at

$$\langle T_{rL}(\tau)\rangle = \frac{|F_0|^2}{2\sqrt{\pi}} \lim_{N_s\to\infty} \int_{-\infty}^{\infty} d\bar{\Omega}\, e^{-\bar{\Omega}^2/4} e^{i\bar{\Omega}\tau} \left[\tilde{\psi}_{L1}(\bar{\Omega})\right]^{N_s}, \tag{67}$$

where

$$\tilde{\psi}_{L1}(\bar{\Omega}) = 1 + \lim_{N_s \to \infty} \frac{1}{N_s \Delta_0} \sum_{q=1}^{\infty} \int_{-\infty}^{\infty} d\bar{D}_n \frac{(-\tilde{\alpha}_c \Delta_0)^q}{q!} \times \left[\frac{1}{2\pi} \frac{\left(2\Gamma_s + i\bar{\Omega}\right)}{\left(\Gamma_s + i\bar{D}_n\right)\left[\Gamma_s + i(\bar{\Omega} - \bar{D}_n)\right]} \right]^q . \tag{68}$$

The integral of each term in the sum can be calculated analytically, leading to

$$\psi_{L1}(\Omega) = 1 + \lim_{N_s \to \infty} \frac{2\pi i\left(\Omega - 2i\Gamma_s\right)}{N_s \Delta_0} \sum_{q=1}^{\infty} \frac{[i\beta(\Omega)/2]^q}{q!} \binom{2q-2}{q-1} \tag{69}$$

where

$$\beta(\Omega) = \frac{\tilde{\alpha}_c \Delta_0}{\pi\left(\Omega - 2i\Gamma_s\right)}. \tag{70}$$

Using the fact that

$$\sum_{q=1}^{\infty} [i\beta(\Omega)/2]^q \frac{1}{q!} \binom{2q-2}{q-1} = [i\beta(\Omega)/2]\, e^{i\beta(\Omega)} \left\{ I_0\left[i\beta(\Omega)\right] - I_1\left[i\beta(\Omega)\right] \right\}, \tag{71}$$

where I_n is a modified Bessel function of the first kind, and Eqs. (64) and (69), we obtain the average transmitted field intensity at time τ as

$$\langle T_{rL}(\tau) \rangle = \frac{|F_0|^2}{2\sqrt{\pi}} \int_{-\infty}^{\infty} d\Omega e^{-\Omega^2/4} e^{i\Omega\tau} e^{-\tilde{\alpha}_c e^{i\beta(\Omega)} \{I_0[i\beta(\Omega)] - I_1[i\beta(\Omega)]\}}. \tag{72}$$

In Fig. 2, we plot $\sqrt{\langle T_r(\tau) \rangle}/F_0$ for $\tilde{\alpha}_c = 2$, $\Gamma_s = 0$,and $\Delta_0 = 0.06$, along with the corresponding results for a regularly spaced comb (for $\Gamma_s = 0$, Eq. (72) is valid for an arbitrary spike profile so the L subscript is dropped). The transmitted intensity initially follows the input pulse profile, but then develops a long tail.

The normalized averaged transmitted energy at time τ,

$$\langle W_{rL}(\tau) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\tau} d\tau' \int_{-\infty}^{\infty} d\Omega e^{-\Omega^2/4} e^{i\Omega\tau'} e^{-\tilde{\alpha}_c e^{i\beta(\Omega)} \{I_0[i\beta(\Omega)] - I_1[i\beta(\Omega)]\}}, \tag{73}$$

is plotted as a function of τ for both a regularly spaced comb and a RSM, for the same parameters as those in Fig. 3. It can be seen from the figure that the RSM transmitted energy is roughly the same as for an AFC for long times. This is consistent with the fact, derived in Appendix B, that, for $\Gamma_s/\Delta_0 \sim 0$, the asymptotic form of $\langle W_{rL}(\tau) \rangle$ is identical to that of the corresponding AFC [Eq. (48)].

Finally, the total average transmitted energy is given by

$$\begin{aligned}\langle W_{rL} \rangle = \langle W_{rL}(\infty) \rangle &= e^{-\tilde{\alpha}_c e^{i\beta(0)}\{I_0[i\beta(0)]-I_1[i\beta(0)]\}} \\ &= e^{-\tilde{\alpha}_c e^{-\sigma}[I_0(\sigma)+I_1(\sigma)]},\end{aligned} \tag{74}$$

where

$$\sigma = -i\beta(0) = \frac{\tilde{\alpha}_c \Delta_0}{2\pi\Gamma_s}. \tag{75}$$

For $\Gamma_s = 0.001$, $\Delta_0 = 0.06$, and $\tilde{\alpha}_c = 2$, $\langle W_{rL} \rangle = 0.696$, which is slightly larger than the corresponding value $W_L = 0.650$ for a regularly spaced comb (the corresponding difference between the AFC and RSM results for Gaussian rectangular spikes having the same width Γ_s is much smaller, since the spikes in those cases have less overlap). In the limit that $\sigma \gg 1$, it follows from the asymptotic form of the Bessel functions that

$$\langle W_{rL} \rangle \approx \exp\left(-2\sqrt{\frac{\tilde{\alpha}_c \Gamma_s}{\Delta_0}}\right). \tag{76}$$

If, in addition, $\tilde{\alpha}_c \Gamma_s/\Delta_0 \ll 1$,

$$\langle W_{rL} \rangle \approx 1 - 2\sqrt{\frac{\tilde{\alpha}_c \Gamma_s}{\Delta_0}} \tag{77}$$

and we recover the same result as that given in Eq. (49) for an AFC. In the limit that $\Gamma_s/\Delta_0 \sim 0$, $\langle W_{rL} \rangle \sim 1$ - there is no absorption. We present in Table I a summary of the results.

VI. CONNECTION BETWEEN ENERGY TRANSMISSION TIME AND GROUP DELAY

The delayed transit of a light pulse in an EIT medium [1, 2] or in a medium having a spectral hole burned in its inhomogeneous frequency distribution [6, 7] is conveniently analyzed in terms of the pulse's group velocity v_g. Such an approach is appropriate because v_g is uniform over the incoming pulse spectrum, resulting in pulse propagation without distortion. As already pointed out, the incoming pulses undergo strong distortion in other dispersion-dominated processes such as AFC. The question then remains as to whether or not the concept of group velocity is of any use in explaining pulse propagation in AFC. As

FIG. 2: Square root of the averaged transmitted intensity $\sqrt{\langle T_r(\tau)\rangle}/F_0$ (red, solid curve) for a RSM as a function of τ for $\tilde{\alpha}_c = 2$, $\Gamma_s = 0$, and an average spike spacing $\Delta_0 = 0.06$. The blue, dashed curve is the corresponding AFC result.

	AFC	RSM
$F_L(Z,\tau)$	$F_0 e^{-\tilde{\alpha}_c Z/2} \sum_{p=0}^{\infty} e^{-(\tau - 2\pi p/\Delta_0)^2/2} \times e^{-2\pi p \Gamma_s/\Delta_0} L_p^{-1}(\tilde{\alpha}_c Z)$	$F_0 e^{-\tilde{\alpha}_c Z/2} e^{-\tau^2/2}$
$W_L\left(\tau_n = \frac{2\pi n}{\Delta_0}\right)$, $\langle W_{rL}(\tau)\rangle$	$e^{-\tilde{\alpha}_c} \sum_{p=0}^{n} e^{-4\pi p \Gamma_s/\Delta_0} \times \left[L_p^{-1}(\tilde{\alpha}_c)\right]^2$	$\frac{1}{2\pi}\int_{-\infty}^{\tau} d\tau' \int_{-\infty}^{\infty} d\Omega e^{-\Omega^2/4} e^{i\Omega\tau'} \times e^{-\tilde{\alpha}_c e^{i\beta(\Omega)}\{I_0[i\beta(\Omega)] - I_1[i\beta(\Omega)]\}}$
W_L, $\langle W_{rL}\rangle$	$e^{-\tilde{\alpha}_c} \sum_{p=0}^{\infty} e^{-4\pi p \Gamma_s/\Delta_0} \left[L_p^{-1}(\tilde{\alpha}_c)\right]^2$	$e^{-\tilde{\alpha}_c e^{-\left(\frac{\tilde{\alpha}_c\Delta_0}{\pi\Gamma_s}\right)}} \times \left\{ I_0\left(\frac{\tilde{\alpha}_c\Delta_0}{\pi\Gamma_s}\right) + I_1\left(\frac{\tilde{\alpha}_c\Delta_0}{\pi\Gamma_s}\right)\right\}$
$\Gamma_s/\Delta_0 \sim 0$; $W(\tau)$, $\langle W_r(\tau)\rangle$	$1 - \sqrt{\frac{2\tilde{\alpha}_c}{\pi\Delta_0\tau}}$; $\tau \gg \Delta_0^{-1}$	
$4\pi\Gamma_s/\Delta_0 \ll 1$; W_L, $\langle W_{Lr}\rangle$	$1 - 2\sqrt{\frac{\tilde{\alpha}_c\Gamma_s}{\Delta_0}}$	

TABLE I: Summary of results for Lorentzian spikes and a Gaussian input pulse. The asymptotic results are valid for $\tilde{\alpha}_c$ of order unity. In the limit that $\Gamma_s/\Delta_0 \sim 0$, the results are valid for arbitrary spike profiles. The quantity $\beta(\Omega)$ is defined by Eq. (70).

FIG. 3: Transmitted AFC energy (blue, dashed curve) and average RSM transmitted energy (red, solid curve) for $\tilde{\alpha}_c = 2$, $\Gamma_s = 0$,and $\Delta_0 = 0.06$. The transmitted energy is normalized to the input field energy.

we shall see, the characteristic time for energy transmission in an AFC is related to the average group velocity of the pulse.

An analytical expression of the AFC field amplitude has been obtained in Sec. IV as a summation of the contributions from all the frequency spikes in the inhomogeneous frequency distribution. From Eqs. (18) and (51), it follows that, in the limit $\Gamma_s/\Delta_0 \sim 0$, the Fourier transform of the field amplitude is given by

$$\tilde{F}\left(Z,\Omega\right) = \tilde{F}\left(0,\Omega\right)e^{iK(\Omega)Z}, \tag{78}$$

where

$$K(\Omega) = -\frac{\tilde{\alpha}_c}{2}\cot\left(\pi\Omega/\Delta_0\right) \tag{79}$$

is a dimensionless propagation constant, $K(\Omega) = k(\Omega)L$. The dimensionless group velocity, $V_g(\Omega) = v_g T_p/L$, as defined by

$$V_g(\Omega) = \left(\frac{dK}{d\Omega}\right)^{-1} = \frac{2\Delta_0}{\pi\tilde{\alpha}_c}\sin^2\left(\pi\Omega/\Delta_0\right), \tag{80}$$

is a periodic function of Ω.

At the centers, $\Omega_{c,n} = (n + 1/2)\,\Delta_0$, of the intervals between adjacent spikes, V_g is maximum, while it equals zero at the spike positions. Averaging over one period, we find $\bar{V}_g = \Delta_0/\pi\tilde{\alpha}_c$, with relative standard deviation $1/\sqrt{2}$. At the the exit plane of the medium, $Z = 1$, the corresponding (dimensionless) average time delay is defined by

$$\bar{\tau}_g = \frac{t_g}{T_p} = \frac{1}{\bar{V}_g} = \frac{\pi\tilde{\alpha}_c}{\Delta_0}. \tag{81}$$

It turns out that $\bar{\tau}_g$ provides a rough estimate of the time at which the echo having the most energy is radiated. The echo having the maximum intensity is centered at $\tau = r\bar{\tau}_g$, where r equals unity for $\tilde{\alpha}_c = 2$, and decreases with increasing $\tilde{\alpha}_c$, asymptotically approaching a value $r \approx 0.5$. It is noteworthy that, for $\tilde{\alpha}_c = 2$, the first echo contains the most energy and is centered at $\tau = \bar{\tau}_g$.

Given the strong distortion of the incoming pulse temporal shape by the AFC filter, it might be better to associate the average time delay with the normalized, integrated retrieved energy $W(\tau)$ defined by Eq. (20) rather than the transmitted field itself. For an AFC, the normalized transmitted energy following the nth echo is.

$$W\left(\tilde{\alpha}_c, \tau_n = \frac{2\pi n}{\Delta_0}\right) = e^{-\tilde{\alpha}_c}\sum_{q=0}^{n}\left[L_q^{-1}(\tilde{\alpha}_c)\right]^2. \tag{82a}$$

Of course, $\bar{\tau}_g = 0$ if $\tilde{\alpha}_c = 0$. Rather remarkably, for $n = \tilde{\alpha}_c$, $W(\tilde{\alpha}_c, \tau_n) \approx 2/3$ (if $\tilde{\alpha}_c$ is not integral, that is, if $\tilde{\alpha}_c = n_0 + s$, where $0 \leq s \leq 1$, $[(1-s)\,W(\tilde{\alpha}_c, \tau_{n_0}) + sW(\tilde{\alpha}_c, \tau_{n_0+1})] \approx 2/3$). In other words, approximately 2/3 of the incident pulse energy is transmitted in a time of order $2\bar{\tau}_g$. This result is consistent with Eq. (48), derived for large values of n,

$$W\left(\tilde{\alpha}_c, \tau_n = \frac{2\pi n}{\Delta_0}\right) \approx 1 - \frac{1}{\pi}\sqrt{\frac{\tilde{\alpha}_c}{n}} = 1 - \frac{\sqrt{2}}{\pi}\sqrt{\frac{\bar{\tau}_g}{\tau_n}}, \tag{83}$$

which predicts that 2/3 of the energy is transmitted for $\tau_n = 2\bar{\tau}_g\,(3/\pi)^2$.

These results suggest that both the emission time of the maximum echo, as well as the characteristic time for a large fraction of the incident pulse energy to be transmitted is linked to the average group delay. It is more difficult to define an average group delay for RSM. However, the temporal evolution of the AFC retrieved energy is very similar to that of RSM, as illustrated in Fig. 3.

VII. CONCLUSIONS

We have studied the transmission properties of regularly and randomly spaced atomic frequency arrays. Throughout the discussion, we have emphasized that the transmission originates from a continuous exchange of energy between the atoms and the field. In the limit of narrow comb teeth, there is negligible absorption of the field by the medium, even though the field energy may be temporarily stored in the medium. In other words, it is optical dispersion rather than absorption that determines the atom-field dynamics. For a regularly spaced comb, the output consists of the (partially) transmitted incident pulse, followed by a succession of regularly spaced echoes. For a RSM, the output intensity has a continuous rather than echo-like dependence. The time scale for total transmission depends on both the spacing of the absorption teeth and the ratio of the tooth width to the tooth spacing. In the limit that the spike width is much lass than the spike separation, the time of emission for an AFC has been related to the group delay of the medium.

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VIII. APPENDIX A - GENERALIZED RESULTS

In this appendix, we generalize the results to allow for arbitrary pulse and spike profiles. The resulting expressions are then evaluated for a Gaussian input pulse and both Gaussian and rectangular spike profiles.

A. AFC

We replace the frequency distribution given by Eq. (24) with

$$\begin{aligned} G\left(\Delta\right) &= \frac{\kappa}{\sqrt{\pi}\Gamma_w} e^{-\Delta^2/\Gamma_w^2} \sum_{n=-\infty}^{\infty} h\left(\frac{\Delta - n\Delta_0}{\Gamma_s}\right) \\ &\approx \frac{\kappa}{\sqrt{\pi}\Gamma_w} \sum_{n=-\infty}^{\infty} h\left(\frac{\Delta - n\Delta_0}{\Gamma_s}\right), \end{aligned} \tag{84}$$

where the function $h(x)$ is an arbitrary function having a width of order unity normalized such that

$$\int_{-\infty}^{\infty} h\left(\frac{\Delta}{\Gamma_s}\right) d\Delta = \Gamma_s \int_{-\infty}^{\infty} h(x)\, dx = \Gamma_s. \tag{85}$$

Then, following the same procedure as in Sec. IV, we find

$$A(\Omega) = \frac{\pi\kappa}{\sqrt{\pi}\Gamma_w}\frac{\Gamma_s}{\Delta_0}\left[1 + 2\sum_{n=1}^{\infty} e^{2\pi in\Omega/\Delta_0} b_n\left(\Gamma_s/\Delta_0\right)\right] \tag{86}$$

with

$$b_n(y) = \int_{-\infty}^{\infty} dx\, h(x)\, e^{-2\pi inxy}. \tag{87}$$

In the asymptotic limit that $\Gamma_s/\Delta_0 \sim 0$, $b_n\left(\Gamma_s/\Delta_0\right) \sim 1$, proving that the results are independent of the shape of the spikes in this limit.

Setting

$$\tilde{\alpha}_c = \frac{2\sqrt{\pi}C\Gamma_2\kappa}{\Gamma_w}\frac{\Gamma_s}{\Delta_0} = \frac{\tilde{\alpha}_0}{h(0)}\frac{\Gamma_s}{\Delta_0}, \tag{88}$$

where

$$\tilde{\alpha}_0 = \frac{2\sqrt{\pi}\Gamma_2 C\kappa}{\Gamma_w} h(0) \tag{89}$$

is the dimensionless *cw* absorption coefficient on resonance for a single spike, we obtain the field amplitude as

$$F(Z,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\Omega\tau}\tilde{F}(0,\Omega)\, K\left(\tilde{\alpha}_c Z, \Gamma_s/\Delta_0, \Omega/\Delta_0\right) d\Omega, \tag{90}$$

with

$$K\left(\tilde{\alpha}_c Z, \Gamma_s/\Delta_0, \Omega/\Delta_0\right) = e^{-\tilde{\alpha}_c Z/2}\exp\left[-\tilde{\alpha}_c Z\sum_{n=1}^{\infty} b_n\left(\Gamma_s/\Delta_0\right) e^{2\pi in\Omega/\Delta_0}\right] \tag{91}$$

The quantity $\tilde{\alpha}_c$ is an effective comb absorption coefficient.

When the kernel is expanded as in Eq. (38),we find the expansion coefficients

$$c_p\left(\tilde{\alpha}_c Z, \Gamma_s/\Delta_0\right) = \begin{cases} e^{-\tilde{\alpha}_c Z/2} Q_p\left(\tilde{\alpha}_c Z, \Gamma_s/\Delta_0\right) & p \geq 0 \\ 0 & p < 0 \end{cases}, \tag{92}$$

and

$$F(Z,\tau) = F_0 e^{-\tilde{\alpha}_c Z/2}\sum_{p=0}^{\infty} e^{-\left(\tau - 2\pi p/\Delta_0\right)^2/2} Q_p\left(\tilde{\alpha}_c Z, \Gamma_s/\Delta_0\right), \tag{93}$$

with

$$Q_p(x,y) = \int_{-1/2}^{1/2} d\sigma \exp\left[-x\sum_{n=1}^{\infty} b_n(y)\, e^{2\pi in\sigma}\right] e^{-2\pi ip\sigma}. \tag{94}$$

The exponential can be expanded as

$$\exp\left[-x\sum_{n=1}^{\infty}b_n\left(y\right)e^{2\pi in\sigma}\right]=\prod_{n=1}^{\infty}\sum_{q=0}^{\infty}\frac{\left(-x\right)^q}{q!}b_n^q\left(\Gamma_s/\Delta_0\right)e^{2\pi inq\sigma}$$

$$=\left[1-xb_1\left(y\right)e^{2\pi i\sigma}+\frac{\left[xb_1\left(y\right)\right]^2}{2!}e^{4\pi i\sigma}-\frac{\left[xb_1\left(y\right)\right]^3}{3!}e^{6\pi i\sigma}+\frac{\left[xb_1\left(y\right)\right]^4}{4!}e^{8\pi i\sigma}-\frac{\left[xb_1\left(y\right)\right]^5}{5!}e^{10\pi i\sigma}\right]$$

$$\times\left[1-xb_2\left(y\right)e^{4\pi i\sigma}+\frac{\left[xb_2\left(y\right)\right]^2}{2!}e^{8\pi i\sigma}\right]\left[1-xb_3\left(y\right)e^{6\pi i\sigma}\right]\left[1-xb_4\left(y\right)e^{8\pi i\sigma}\right]\left[1-xb_5\left(y\right)e^{10\pi i\sigma}\right],\tag{95}$$

where we have written out a few terms. The integral in Eq. (94) is now calculated easily using

$$\int_{-1/2}^{1/2}d\sigma\exp e^{2\pi in\sigma}e^{-2\pi ip\sigma}=\delta_{n,p},\tag{96}$$

where $\delta_{n,p}$ is a Kronecker delta. In this manner, we obtain

$$Q_0\left(x,y\right)=1;\tag{97a}$$

$$Q_1\left(x,y\right)=-b_1x;\tag{97b}$$

$$Q_2\left(x,y\right)=-b_2x+b_1^2\frac{x^2}{2};\tag{97c}$$

$$Q_3\left(x,y\right)=-b_3x+\left(b_2b_1\right)x^2-b_1^3\frac{x^3}{3!};\tag{97d}$$

$$Q_4\left(x,y\right)=-b_4x+\left(b_3b_1+\frac{b_2^2}{2!}\right)x^2-\frac{b_2b_1^2}{2!}x^3+b_1^4\frac{x^4}{4!};\tag{97e}$$

$$Q_5\left(x,y\right)=-b_5x+\left(b_4b_1+b_3b_2\right)x^2-\left(\frac{b_3b_1^2}{2!}+\frac{b_1b_2^2}{2!}\right)x^3+b_2b_1^3\frac{x^4}{3!}-b_1^5\frac{x^5}{5!};\tag{97f}$$

$$Q_6\left(x,y\right)=-b_6x+\left(b_5b_1+b_4b_2+\frac{b_3^2}{2!}\right)x^2-\left(\frac{b_4b_1^2}{2!}+\frac{b_2^3}{3!}+b_1b_2b_3\right)x^3$$
$$+\left(\frac{b_3b_1^3}{3!}+\frac{b_1^2}{2!}\frac{b_2^2}{2!}\right)x^4-b_2\frac{b_1^4}{4!}x^5+b_1^6\frac{x^6}{6!},\tag{97g}$$

where $b_n\equiv b_n(y)$. No further simplification is possible, as was the case for Lorentzian spikes. In the limit that $\Gamma_s/\Delta_0\sim 0$, all the b_n's can be set equal to unity and we recover

$$c_p\left(\tilde{\alpha}_cZ,0\right)\approx c_{pL}\left(\tilde{\alpha}_cZ,0\right)=e^{-\tilde{\alpha}_cZ/2}L_p^{-1}\left(\tilde{\alpha}_cZ\right),\tag{98}$$

found previously for narrow Lorentzian spikes.

IX. RSM

The distribution $G(\Delta)$ is still given by Eq. (24), with the sum from $n = -\infty$ to ∞ replaced by a sum from $n = 1$ to $n = N_s$ and with $n\Delta_0$ replaced by D_n, where D_n is a frequency chosen at random in the interval $(-N_s\Delta_0/2, N_s\Delta_0/2)$, with N_s allowed to approach infinity. The field amplitude is then given by

$$F_r(Z,\tau) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\Omega\tau}\tilde{F}(\Omega,0)\exp\left[-\frac{\tilde{\alpha}_0 Z}{2\pi h(0)}\sum_{n=1}^{N_s}\int_{-\infty}^{\infty}\frac{h\left(\frac{\Delta-D_n}{\Gamma_s}\right)}{\frac{\Gamma_2}{2}-i(\Omega-\Delta)}d\Delta\right], \tag{99}$$

which can be rewritten as

$$F_r(Z,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\Omega\tau}\tilde{F}(\Omega,0)\prod_{n=1}^{N_s}\exp\left[-\frac{\tilde{\alpha}_0 Z\Gamma_s}{2\pi h(0)}\int_{-\infty}^{\infty}\frac{h\left(\frac{\Delta-D_n}{\Gamma_s}\right)}{\frac{\Gamma_2}{2}-i(\Omega-\Delta)}d\Delta\right]. \tag{100}$$

This expression can be evaluated numerically with the D_n chosen at random in the interval $(-N_s\Delta_0/2, N_s\Delta_0/2)$

We now want to average this expression over random spike positions in frequency space. On average, each factor in the product is the same; in other words,

$$\langle F_r(Z,\tau)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\Omega\tau}\tilde{F}(\Omega,0)\left\langle\exp\left[-\frac{\tilde{\alpha}_0 Z}{2\pi h(0)}\int_{-\infty}^{\infty}\frac{h\left(\frac{\Delta-D_n}{\Gamma_s}\right)}{\frac{\Gamma_2}{2}-i(\Omega-\Delta)}d\Delta\right]\right\rangle^{N_s}. \tag{101}$$

Let us define

$$\psi = \left\langle\exp\left[-\frac{\tilde{\alpha}_0 Z}{2\pi h(0)}\int_{-\infty}^{\infty}\frac{h\left(\frac{\Delta-D_n}{\Gamma_s}\right)}{\frac{\Gamma_2}{2}-i(\Omega-\Delta)}d\Delta\right]\right\rangle \tag{102}$$

and expand the exponential to obtain

$$\begin{aligned}
\psi &= \frac{1}{N_s\Delta_0}\int_{-N_s\Delta_0/2}^{N_s\Delta_0/2} dD_n \sum_{q=0}^{\infty}\left[-\frac{\tilde{\alpha}_0 Z}{2\pi h(0)}\int_{-\infty}^{\infty}\frac{h\left(\frac{\Delta-D_n}{\Gamma_s}\right)}{\frac{\Gamma_2}{2}-i(\Omega-\Delta)}d\Delta\right]^q \frac{1}{q!} \\
&\approx 1 + \lim_{N_s\to\infty}\frac{1}{N_s\Delta_0}\sum_{q=1}^{\infty}\int_{-\infty}^{\infty} dD_n\left[-\frac{\tilde{\alpha}_0 Z}{2\pi h(0)}\int_{-\infty}^{\infty}\frac{h\left(\frac{x}{\Gamma_s}\right)}{\frac{\Gamma_2}{2}-i(\Omega-x-D_n)}dx\right]^q\frac{1}{q!} \\
&= 1 + \lim_{N_s\to\infty}\frac{1}{N_s\Delta_0}\sum_{q=1}^{\infty}\int_{-\infty}^{\infty} dy\left[-\frac{\tilde{\alpha}_0 Z}{2\pi h(0)}\int_{-\infty}^{\infty}\frac{h\left(\frac{x}{\Gamma_s}\right)}{\frac{\Gamma_2}{2}+i(y+x)}dx\right]^q\frac{1}{q!},
\end{aligned} \tag{103}$$

where we have used the fact that the number of spikes is taken to approach infinity. Note that the result no longer depends on Ω, implying that

$$F_r(Z,\tau) = F(0,\tau)\lim_{N_s\to\infty}\psi^{N_s}. \tag{104}$$

Using the fact that

$$\int_{-\infty}^{\infty} dy \left[\int_{-\infty}^{\infty} \frac{h\left(\frac{x}{\Gamma_s}\right)}{\frac{\Gamma_2}{2} + i\left(y+x\right)} dx \right]^q = \pi \Gamma_s \delta_{q,1}, \tag{105a}$$

and letting N_s approach infinity, we obtain

$$F_r\left(Z,\tau\right) = F\left(0,\tau\right) \exp\left[-\frac{\tilde{\alpha}_0}{2h\left(0\right)} \frac{\Gamma_s}{\Delta_0} Z \right] = F\left(0,\tau\right) e^{-\tilde{\alpha}_c Z/2}, \tag{106}$$

where

$$\tilde{\alpha}_c = \frac{\tilde{\alpha}_0}{h\left(0\right)} \frac{\Gamma_s}{\Delta_0} \tag{107}$$

is the analogous value of $\tilde{\alpha}$ for a regularly spaced comb with spacing Δ_0. Regardless of the spike profile, the average transmitted field has the same temporal profile as the input field and vanishes once the input field has traversed the sample.

The transmitted intensity at time τ, defined by Eq. (19), is

$$\begin{aligned} T_r\left(\tau\right) = &\frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\Omega' \int_{-\infty}^{\infty} d\Omega \tilde{F}\left(\Omega,0\right) \left[\tilde{F}\left(\Omega',0\right) \right]^* \\ &\times e^{-i(\Omega-\Omega')\tau} \exp\left[-\frac{\tilde{\alpha}_0}{2\pi h(0)} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{h\left(\frac{\Delta - D_n}{\Gamma_s}\right)\left[\Gamma_2 + i(\Omega' - \Omega)\right]}{\left[\frac{\Gamma_2}{2} - i(\Omega - \Delta)\right]\left[\frac{\Gamma_2}{2} + i(\Omega' - \Delta)\right]} d\Delta \right]. \end{aligned} \tag{108}$$

Setting

$$\bar{\Omega} = \Omega' - \Omega, \tag{109}$$

$$y = \Delta - \Omega, \tag{110}$$

and defining

$$B(\bar{\Omega}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\Omega \tilde{F}\left(\Omega,0\right) \left[\tilde{F}\left(\bar{\Omega} + \Omega, 0\right) \right]^*, \tag{111}$$

we can average Eq. (61) and write it as

$$\left\langle T_r\left(\tau\right) \right\rangle = 2\pi \lim_{\Gamma_2 \to 0} \int_{-\infty}^{\infty} d\Omega\, B(\Omega) e^{i\Omega\tau} \left[\psi_1(\Omega)\right]^{N_s}, \tag{112}$$

where

$$\psi_1(\Omega) = \frac{1}{N_s \Delta_0} \lim_{\Gamma_2 \to 0} \int_{-N_s\Delta_0/2}^{N_s\Delta_0/2} dD_n \exp\left[-\frac{\tilde{\alpha}_0}{2\pi h(0)} \int_{-\infty}^{\infty} \frac{\left(\Gamma_2 + i\Omega\right) h\left(\frac{y - \Omega - D_n}{\Gamma_s}\right)}{\left[\frac{\Gamma_2}{2} + iy\right]\left[\frac{\Gamma_2}{2} - i(y - \Omega)\right]} dy \right]. \tag{113}$$

For convergence, we need to retain the Γ_2's in the denominator.

We define

$$H(\Omega, D_n) = \frac{1}{2\pi\Gamma_s} \lim_{\Gamma_2\to 0} \int_{-\infty}^{\infty} \frac{(\Gamma_2 + i\Omega)\, h\left(\frac{y-\Omega-D_n}{\Gamma_s}\right)}{\left[\frac{\Gamma_2}{2} + iy\right]\left[\frac{\Gamma_2}{2} - i(y-\Omega)\right]} dy, \tag{114}$$

such that

$$\psi_1(\Omega) = \frac{1}{N_s\Delta_0} \lim_{\Gamma_2\to 0} \int_{-N_s\Delta_0/2}^{N_s\Delta_0/2} dD_n \exp\left[-\tilde{\alpha}_c \Delta_0 H(\Omega, D_n)\right]. \tag{115}$$

For a given spike profile h, it is rather a straightforward, but computationally challenging, to obtain $\langle T_r(\tau)\rangle$ using a value $N_s\Delta_0 \gg 1$, so as to cover the entire bandwidth of the incident pulse (which is equal to unity in these dimensionless units). Note that, in the asymptotic limit that $\Gamma_s \sim 0$, $h\left[(y-\Omega-D_n)/\Gamma_s\right] \sim \Gamma_s \delta(y-\Omega-D_n)$ and $H(\Omega, D_n)$ becomes independent of spike shape.

A. Specific Examples - Gaussian Input Pulse and Gaussian and Rectangular Spike.

We would like to present some results for a Gaussian input pulse

$$F(0,\tau) = F_0 e^{-\tau^2/2}; \qquad \tilde{F}(0,\Omega) = \sqrt{2\pi} F_0 e^{-\Omega^2/2}, \tag{116}$$

and both Gaussian and rectangular spike profiles,

$$h_G(\Delta/\Gamma_s) = \frac{1}{\sqrt{\pi}} e^{-\Delta^2/\Gamma_s}; \tag{117a}$$

$$h_R(\Delta/\Gamma_s) = \Theta\left(1 - \frac{|\Delta|}{\Gamma_s/2}\right). \tag{117b}$$

For a regularly spaced comb of Gaussian or rectangular spikes,

$$\begin{aligned}
2C\Gamma_2 A_G(\Omega) &= \tilde{\alpha}_{0G} \sum_{n=-\infty}^{\infty} w\left(\frac{\Omega - n\Delta_0}{\Gamma_s}\right) \\
&= \tilde{\alpha}_c \left(1 + 2\sum_{n=1}^{\infty} e^{2\pi i n\Omega/\Delta_0} e^{-n^2\pi^2\Gamma_s^2/\Delta_0^2}\right) \\
&= \tilde{\alpha}_c \left(\mathrm{ET}\left(3, \frac{\pi\Omega}{\Delta_0}, e^{-\pi^2\Gamma_s^2/\Delta_0^2}\right) + 2i\sum_{n=1}^{\infty} \sin\left(\frac{2\pi n\Omega}{\Delta_0}\right) e^{-n^2\pi^2\Gamma_s^2/\Delta_0^2}\right);
\end{aligned} \tag{118a}$$

$$\begin{aligned}
2C\Gamma_2 A_R(\Omega) &= \tilde{\alpha}_{0R} \sum_{n=-\infty}^{\infty} \Theta\left(1 - \frac{|\Omega - n\Delta_0|}{\Gamma_s/2}\right) + \frac{i}{\pi} \ln\left|\frac{\frac{\Omega-n\Delta_0}{\Gamma_s/2} + 1}{\frac{\Omega-n\Delta_0}{\Gamma_s/2} - 1}\right| \\
&= \tilde{\alpha}_c \left(1 + 2\sum_{n=1}^{\infty} e^{2\pi i n\Omega/\Delta_0} \frac{\sin\left(\frac{n\pi\Gamma_s}{\Delta_0}\right)}{\frac{n\pi\Gamma_s}{\Delta_0}}\right) \\
&= \tilde{\alpha}_c \left(1 - \frac{i}{\pi}\frac{\Delta_0}{\Gamma_s}\left[\ln\left(1 - e^{i(2\Omega-\Gamma_s)/\Delta_0}\right) - \ln\left(1 - e^{i(2\Omega+\Gamma_s)/\Delta_0}\right)\right]\right),
\end{aligned} \tag{118b}$$

where ET is a Jacobi theta function (EllipticTheta in Mathematica),

$$w(x) = e^{-x^2}\left[1 + \Phi(ix)\right], \tag{119}$$

$\Phi(x)$ is the error function,

$$\tilde{\alpha}_{0G} = \frac{2C\Gamma_2\kappa}{\Gamma_w}; \tag{120a}$$

$$\tilde{\alpha}_{0R} = \frac{2C\Gamma_2\sqrt{\pi}\kappa}{\Gamma_w}, \tag{120b}$$

and

$$\tilde{\alpha}_c = \tilde{\alpha}_{0G}\sqrt{\pi}\frac{\Gamma_s}{\Delta_0} = \tilde{\alpha}_{0R}\frac{\Gamma_s}{\Delta_0} = \frac{2\sqrt{\pi}C\Gamma_2\kappa}{\Gamma_w}\frac{\Gamma_s}{\Delta_0}. \tag{121a}$$

For these pulses,

$$b_{nG}\left(\frac{\Gamma_s}{\Delta_0}\right) = e^{-n^2\pi^2\Gamma_s^2/\Delta_0^2}; \tag{122a}$$

$$b_{nR}\left(\frac{\Gamma_s}{\Delta_0}\right) = \frac{\sin\left(\frac{n\pi\Gamma_s}{\Delta_0}\right)}{\frac{n\pi\Gamma_s}{\Delta_0}}. \tag{122b}$$

For an AFC the amplitude of the first several echoes is calculated easily using Eqs. (93), (92), and (122). For RSM

$$2C\Gamma_2A_{Gr}(\Omega) = \tilde{\alpha}_{0G}\sum_{n=1}^{\infty} w\left(\frac{\Omega - D_n}{\Gamma_s}\right) \tag{123a}$$

$$2C\Gamma_2A_{Rr}(\Omega) = \tilde{\alpha}_{0R}\sum_{n=1}^{\infty}\Theta\left(1 - \frac{|\Omega - D_n|}{\Gamma_s/2}\right) + \frac{i}{\pi}\ln\left|\frac{\frac{\Omega-D_n}{\Gamma_s/2}+1}{\frac{\Omega-D_n}{\Gamma_s/2}-1}\right| \tag{123b}$$

The output field amplitude is given by

$$F(Z,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\Omega\tau}e^{-\Omega^2/2}e^{-\Gamma_2CZA(\Omega)}d\Omega. \tag{124}$$

The real part of $A(\Omega)$ is equal to zero except for a range of frequencies of order Γ_s centered at frequencies $\Omega_n = n\Delta_0$ or $\Omega_n = D_n$, while $\mathrm{Im}\left[A(\Omega)\right]$ consists of dispersion like profiles centered at these values. In calculating the field amplitude, therefore, it is a very good approximation to neglect the contributions from the real part of $A(\Omega)$ since they are limited to a narrow frequency range and do not affect the integral significantly. It is the dispersive part of the susceptibility that determines the field evolution. For example, in the limit that $\Gamma_s/\Delta_0 \sim 0$, it follows from Eqs. (118) and (124) that the field amplitude for an AFC becomes

$$F(Z,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\Omega\tau}e^{-\Omega^2/2}e^{-\frac{\tilde{\alpha}_c}{2}\cot(\pi\Omega/\Delta_0)}d\Omega, \tag{125}$$

consistent with Eq. (51).

For finite Γ_s/Δ_0, the exact field expression for an AFC with Gaussian spikes is

$$F_G(Z,\tau) \approx \frac{F_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\Omega\tau} e^{-\Omega^2/2} \exp\left[-\frac{\tilde{\alpha}_{cG} Z}{2}\left(1 + 2\sum_{n=1}^{\infty} e^{2\pi i n\Omega/\Delta_0} e^{-n^2\pi^2\Gamma_s^2/\Delta_0^2}\right)\right], \quad (126)$$

whereas the approximate expression, neglecting the real part of the susceptibility, is

$$F_G^{ap}(Z,\tau) = \frac{F_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\Omega\tau} e^{-\Omega^2/2} \exp\left[-i\tilde{\alpha}_{cG} Z \sum_{n=1}^{\infty} e^{-n^2\pi^2\gamma_0^2/\Delta_0^2} \sin\left(\frac{2\pi n\Omega}{\Delta_0}\right) d\Omega\right]. \quad (127)$$

For $\Delta_0 = 0.06$, and $\tilde{\alpha}_c = 1$, $F_G(1,\tau)/F_0$ (red, solid curves) and $F_G^{ap}(1,\tau)/F_0$ (blue, dashed curves) for the third echo are plotted in Fig. 4 for $\Gamma_s = 0.001$ and Fig. 5 for $\Gamma_s = 0.0005$, respectively. As can be seen the two expressions differ very little and would overlap entirely in the limit that $\Gamma_s/\Delta_0 \sim 0$. Although Eqs. (126) and (127) are equal in the limit that $\Gamma_s/\Delta_0 \ll 1$, computationally, in the case of an AFC, it is easier to use the *exact* expression for the field amplitude. That is, to calculate the field amplitude for the qth echo using the exact expression, it is necessary only to keep terms in the summation in Eq. (126) up to $n_{\max} = q$, whereas, it is necessary to keep terms in the summation in Eq. (126) up to $n_{\max} = \Delta_0/\Gamma_s \gg 1$ at all times. The reason for this is that only terms varying as $\exp(2\pi i p\Omega/\Delta_0)$ with $p \geq 0$ appear in the expansion of the exponential Eq. (126), whereas, if we expand the exponential in Eq. (127), both positive and negative values of p will enter. In calculating the approximate amplitude we make an (absolute) error in the amplitude of each amplitude of order Γ_s/Δ_0, resulting from the fact that we have neglected the extinction of the integrand in small intervals having width $\sqrt{\pi}\Gamma_s$ centered about $\Omega = 2\pi n/\Delta_0$. For $\Gamma_s/\Delta_0 \approx 0.017$, this deviation of the approximate solution from the exact one can be seen in Fig. 4.

FIG. 4: Exact transmitted field amplitude $F_G(1,\tau)/F_0$ (red, solid curves) and approximate (including only dispersion) transmitted field amplitude $F_G^{ap}(1,\tau)/F_0$ (blue, dashed curves) for the third echo are plotted for $\tilde{\alpha}_c = 1$ $\Delta_0 = 0.06$, and $\Gamma_s = 0.001$.

FIG. 5: Same as Fig. 4, except that $\Gamma_s = 0.0005$.

The total transmitted energy, normalized to the total energy of the pulse, is

$$\begin{aligned}
W_G &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\Omega^2} \exp\left[-\tilde{\alpha}_{0G}\sum_{n=-\infty}^{\infty}\exp\left[-\left(\frac{\Omega-n\Delta_0}{\Gamma_s}\right)^2\right]\right] d\Omega \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\Omega^2} \exp\left[-\tilde{\alpha}_{cG}\left(1+2\sum_{n=1}^{\infty}\cos\left(2\pi n\Omega/\Delta_0\right)e^{-n^2\pi^2\Gamma_s^2/\Delta_0^2}\right)\right] d\Omega; \qquad (128a) \\
W_{Gr} &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\Omega^2} \exp\left[-\tilde{\alpha}_{0G}\sum_{n=1}^{\infty}\exp\left[-\left(\frac{\Omega-D_n}{\Gamma_s}\right)^2\right]\right] d\Omega; \qquad (128b) \\
W_R &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\Omega^2} \exp\left[-\tilde{\alpha}_{0R}\sum_{n=-\infty}^{\infty}\Theta\left(1-\frac{|\Omega-n\Delta_0|}{\Gamma_s/2}\right)\right] d\Omega; \qquad (128c) \\
W_{Rr} &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\Omega^2} \exp\left[-\tilde{\alpha}_{0R}\sum_{n=1}^{\infty}\Theta\left(1-\frac{|\Omega-D_n|}{\Gamma_s/2}\right)\right]. \qquad (128d)
\end{aligned}$$

The total, normalized transmitted field energy is equal to unity, with a small correction of order Γ_s/Δ_0 for $\tilde{\alpha}_c$ of order unity. To a good approximation, for $\Gamma_s/\Delta_0 \ll 1$,

$$W_G \approx \langle W_{Gr}\rangle \approx 1 - 2\frac{\Gamma_s}{\Delta_0}\sqrt{\ln\left(\frac{\tilde{\alpha}_{0G}}{\ln 2}\right)} = 1 - 2\frac{\Gamma_s}{\Delta_0}\sqrt{\ln\left(\frac{\Delta_0\tilde{\alpha}_c}{\Gamma_s\sqrt{\pi}\ln 2}\right)}; \qquad (129a)$$

$$W_R \approx \langle W_{Rr}\rangle \approx 1 - \frac{\Gamma_s}{\Delta_0}, \qquad (129b)$$

where the average is for random spikes having average separation Δ_0. For $\Gamma_s = 0.001$, $\Delta_0 = 0.06$, and $\tilde{\alpha}_c = 2$, $W_G = 0.928$ and $W_R = 0.983$, much closer to unity than the value $W_L = 0.650$ found for Lorentzian spikes.

X. APPENDIX B - ASYMPTOTIC RESULTS

In this appendix, we derive the asymptotic results for the total transmitted energy W_L in the limit that $\Gamma_s/\Delta_0 \ll 1$ and for the average transmitted energy at time τ, $\langle W_{rL}(\tau)\rangle$ in the limit that $\Delta_0\tau \gg 1$ and $\Gamma_s/\Delta_0 \sim 0$.

A. W_L in the limit that $\Gamma_s/\Delta_0 \ll 1$ for an AFC

We start from Eq. (43),

$$W_L = e^{-\tilde{\alpha}_c}\sum_{q=0}^{\infty} e^{-bq}\left[L_p^{-1}(\tilde{\alpha}_c)\right]^2, \qquad (130)$$

with $b = 4\pi\Gamma_s/\Delta_0$ and define

$$y\left(x\right) = e^{-x}\sum_{q=0}^{\infty}\left[L_q^{-1}(x)\right]^2 e^{-bq}.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= -y + 2e^{-x}\sum_{q=0}^{\infty}L_q^{-1}(x)\frac{d\left[L_q^{-1}(x)\right]}{dx}e^{-bq} \\ &= -y - 2e^{-x}\sum_{q=1}^{\infty}L_q^{-1}(x)L_{q-1}^{0}(x)e^{-bq}.\end{aligned} \tag{131}$$

We can use the identity

$$L_q^{-1}(x)L_{q-1}^{0}(x) = \frac{\left[L_q^{-1}(x)+L_{q-1}^{0}(x)\right]^2 - \left[L_q^{-1}(x)\right]^2 - \left[L_{q-1}^{0}(x)\right]^2}{2} \tag{132}$$

along with the recursion relation

$$L_q^{k}(x) + L_{q-1}^{k+1}(x) = L_q^{k+1}(x) \tag{133}$$

to arrive at

$$\begin{aligned}2\sum_{q=1}^{\infty}L_q^{-1}(x)L_{q-1}^{0}(x)e^{-bq} &= \sum_{q=1}^{\infty}\left(\left[L_q^{0}(x)\right]^2 - \left[L_q^{-1}(x)\right]^2 - \left[L_{q-1}^{0}(x)\right]^2\right)e^{-bq} \\ &= -\sum_{q=0}^{\infty}\left[L_q^{-1}(x)\right]^2 e^{-bq} + \left(1-e^{-b}\right)\sum_{q=0}^{\infty}\left[L_q^{0}(x)\right]^2 e^{-bq},\end{aligned} \tag{134}$$

where we have used the fact that $L_0^0(x) = L_0^{-1}(x) = 1$. Combining Eqs. (131) and (134), we find

$$\frac{dy}{dx} = -\left(1-e^{-b}\right)e^{-x}\sum_{q=0}^{\infty}\left[L_q^{0}(x)\right]^2 e^{-bq}. \tag{135}$$

Since $y\left(0\right) = 1$,

$$y\left(x\right) = 1 - \left(1-e^{-b}\right)\int_0^x dx'\, y^{(0)}\left(x'\right), \tag{136}$$

where

$$y^{(0)}\left(x\right) = e^{-x}\sum_{q=0}^{\infty}\left[L_q(x)\right]^2 e^{-qb} \tag{137}$$

and $L_q(x) \equiv L_q^0(x)$. We break up the sum as

$$y^{(0)}\left(x\right) = e^{-x}\sum_{q=0}^{n-1}\left[L_q(x)\right]^2 e^{-qb} + e^{-x}\sum_{n}^{\infty}\left[L_q(x)\right]^2 e^{-qb} \tag{138}$$

and take n sufficiently large to allow us to replace $[L_q(x)]^2$ by its asymptotic expansion for large q,

$$e^{-x}\left[L_q(x)\right]^2 \sim \frac{1}{2\pi\sqrt{qx}}\left[1+\sin\left(4\sqrt{qx}\right)\right]. \tag{139}$$

We then replace the second sum in Eq. (138) by an integral and calculate

$$g(n,b) = \int_n^\infty \frac{e^{-qb}}{2\pi\sqrt{qx}}\left[1+\sin\left(4\sqrt{qx}\right)\right]dq, \tag{140}$$

an integral that can be evaluated in terms of error functions. In the limit that $b \ll 1$,

$$g(n,b,x) \approx \frac{1}{2\pi\sqrt{x}}\left(\sqrt{\frac{\pi}{b}}+\frac{\cos\left(4\sqrt{nx}\right)}{\sqrt{x}}-2\sqrt{n}\right). \tag{141}$$

If $\sqrt{bn} \ll 1$ and $x \gtrsim 1$, $g(n,b,x) \approx \sqrt{1/\left(4\pi xb\right)}$ and

$$y^{(0)}\left(x\right) \approx e^{-x}\sum_{q=0}^{n}\left[L_q(x)\right]^2 e^{-qb}+\frac{1}{\sqrt{4\pi xb}} \tag{142}$$

Moreover, for $\sqrt{bn} \ll 1$ and $x \gtrsim 1$, the summation term is much less than the second term and

$$y^{(0)}\left(x\right) \approx \frac{1}{\sqrt{4\pi xb}}. \tag{143}$$

Substituting this into Eq. (136), we arrive at

$$y(x) \approx 1-\left(1-e^{-b}\right)\sqrt{\frac{x}{\pi b}} \approx 1-\sqrt{\frac{bx}{\pi}} \tag{144}$$

or

$$W_L = y(\tilde{\alpha}_c) \approx 1-2\sqrt{\tilde{\alpha}_c\Gamma_s/\Delta_0}. \tag{145}$$

B. $\langle W_{rL}\left(\tau\right)\rangle$ for $\Delta_0\tau \gg 1$ in the limit that $\Gamma_s/\Delta_0 \sim 0$ for RSM

We start from Eq. (73),

$$\begin{aligned}\langle W_{rL}\left(\tau\right)\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\tau}d\tau'\int_{-\infty}^{\infty}d\Omega e^{-\Omega^2/4}e^{i\Omega\tau'}e^{-\tilde{\alpha}_c e^{i\beta(\Omega)}\{I_0[i\beta(\Omega)]-I_1[i\beta(\Omega)]\}}\\ &= \langle W_{rL}\rangle-\frac{1}{2\pi}\int_{\tau}^{\infty}d\tau'\int_{-\infty}^{\infty}d\Omega e^{-\Omega^2/4}e^{i\Omega\tau'}e^{-\tilde{\alpha}_c e^{i\beta(\Omega)}\{I_0[i\beta(\Omega)]-I_1[i\beta(\Omega)]\}},\end{aligned} \tag{146}$$

where $\langle W_{rL}\rangle = \langle W_{rL}\left(\infty\right)\rangle$ is the average total transmitted energy and

$$\beta(\Omega) = \frac{\tilde{\alpha}_c\Delta_0}{\pi\left(\Omega-2i\Gamma_s\right)}. \tag{147}$$

For $\Delta_0\tau \gg 1$, the oscillating factor $e^{i\Omega\tau'}$ in the second line of Eq. (146) insures that only values

$$\Omega \ll \Delta_0 \ll 1 \tag{148}$$

contribute to the integral. As a consequence, $|\beta(\Omega)| \gg 1$ and the Bessel functions can be replaced by their asymptotic values

$$e^{i\beta(\Omega)}\left\{I_0\left[i\beta(\Omega)\right] - I_1\left[i\beta(\Omega)\right]\right\} \sim \sqrt{\frac{2i}{\pi\beta(\Omega)}} = \sqrt{\frac{2}{\tilde{\alpha}_c\Delta_0}}\sqrt{2\Gamma_s + i\Omega}. \tag{149}$$

Substituting this result into Eq. (146), we obtain

$$\langle W_{rL}(\tau)\rangle \approx \langle W_{rL}\rangle - \frac{1}{2\pi}\int_\tau^\infty d\tau' \int_{-\infty}^\infty d\Omega \exp\left[i\Omega\tau' - \sqrt{\frac{2\tilde{\alpha}_c}{\Delta_0}}\sqrt{2\Gamma_s + i\Omega}\right], \tag{150}$$

where we have dropped the $e^{-\Omega^2/4}$ factor in Eq. (146) since $\Omega \ll 1$.

To make further progress, we now take the asymptotic limit $\Gamma_s/\Delta_0 \sim 0$ for which $\langle W_{rL}\rangle \sim 1$ and

$$\langle W_{rL}(\tau)\rangle \sim 1 - \frac{1}{2\pi}\int_\tau^\infty d\tau' \int_{-\infty}^\infty d\Omega \exp\left[i\Omega\tau' - \sqrt{\frac{2i\tilde{\alpha}_c\Omega}{\Delta_0}}\right]. \tag{151}$$

The integral over Ω is tabulated,

$$\int_{-\infty}^\infty d\Omega \exp\left[i\Omega\tau' - \sqrt{\frac{2i\tilde{\alpha}_c\Omega}{\Delta_0}}\right] = \frac{1}{(\tau')^{3/2}}\sqrt{\frac{2\pi\tilde{\alpha}_c}{\Delta_0}}e^{-\tilde{\alpha}_c/(2\Delta_0\tau')}. \tag{152}$$

Since it has been assumed that $\Delta_0\tau \gg 1$. it follows from Eqs. (150) and (152) that, for $\tilde{\alpha}_c$ of order unity,

$$\langle W_{rL}(\tau)\rangle \sim 1 - \sqrt{\frac{\tilde{\alpha}_c}{2\pi\Delta_0}}\int_\tau^\infty d\tau' \frac{1}{(\tau')^{3/2}} = 1 - \sqrt{\frac{2\tilde{\alpha}_c}{\pi\Delta_0\tau}}. \tag{153}$$

As we have proved already, in the asymptotic limit $\Gamma_s/\Delta_0 \sim 0$, all results are independent of spike shape, implying that Eq. (153) is valid for arbitrary spike profiles.

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