Classical simulation of quantum circuits using fewer Gaussian eliminations

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Improved Simulation of Quantum Circuits by Fewer Gaussian Eliminations

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A compelling way to quantify the separation between classical and quantum computing is to determine how many T gate magic states, t, a classical computer must simulate to calculate the probability of a universal quantum circuit's output. Unfortunately, efforts to determine the minimum number of stabilizer state inner products necessary to decompose T gate magic states (χ_t) have proven intractable past $t = 7$. By using a phase space formalism based on Wootters' discrete Weyl operator basis over a finite field, we develop a new algebraic approach to determining χ_t for single-Pauli measurements. This allows us to extend the bounds on χ_t to $t = 14$ for qutrits, effectively increasing the space searched by $> 10^{10^4}$. Our results show that by using such phase space methods it is possible to validate NISQ circuits of larger size than previously thought possible.

Strong quantum simulation is the task of calculating probabilities of arbitrary output strings of universal quantum circuits. This task is $\#P$-hard [1] and therefore any classical algorithm that improves on prior attempts can likely only lower the exponential coefficient in the cost of strong quantum simulation. However, such improvements are important for (i) simulating small-scale (NISQ [2]) quantum devices, and (ii) understanding the limits of classical simulation of quantum computers and the practical onset of quantum advantage [3].

One of the most compelling ways to determine when NISQ devices exhibit an advantage over current classical computers is to determine how many T gate magic states a classical computer much simulate to calculate the probability of the device's outcomes. Universal quantum computations can be written in terms of k-tensored T gate magic states $|T\rangle^{\otimes k}$ with $(n-k)$ computational states, that are then acted on by Clifford gates $\hat{U}_C$ and partially traced over to obtain a marginal over any qudit [4]:

$$P_k = \mathrm{Tr}\left[\hat{\Pi}\hat{\rho}\right], \tag{1}$$

where $\hat{\rho} = |\Psi\rangle\langle\Psi|$ for $|\Psi\rangle = \hat{U}_C |T\rangle^{\otimes k} |0\rangle^{\otimes(n-k)}$ and $\hat{\Pi}$ is a projector onto a single-qudit Pauli operator eigenstate. In this equation, the T gate magic state, which for qubits is $|T\rangle^{\otimes k} = \frac{1}{\sqrt{2}}(|0\rangle + e^{\pi i/4}|1\rangle)^{\otimes k}$, is a resource state that extends the Clifford classical subtheory to quantum universality in the limit of $k \to \infty$ [4]. We expect the cost of computing P_k to grow exponentially with k since this parameter dictates the degree of non-classicality of the circuit.

Stabilizer states $|\phi_i\rangle$ are eigenvalue $+1$ eigenstates of an Abelian subgroup of the Pauli group and they form an over-complete basis in Hilbert space. Since Cliffords simply permute stabilizer states, Eq. 1 can be rewritten in terms of only stabilizer states by simply expanding the T gate magic states in terms of their stabilizer decomposition: $|\Psi\rangle = \sum_{i=1}^{m_k} c_i |\phi_i\rangle$ for some $c_i \in \mathbb{C}$ (since $|0\rangle$ is a stabilizer state). Since Pauli projections take stabilizer states to stabilizer states, this expansion means Eq. 1 can be expressed as a linear combination of inner products of stabilizer states, $P_k = \sum_{i,j=1}^{m_k} c_i c_j^* \langle\phi_j|\phi_i'\rangle$, where $|\phi_i'\rangle = \hat{\Pi}|\phi_i\rangle$. The inner product of two n-qudit stabilizer states, $\langle\phi_i|\phi_j\rangle$, is governed by Gaussian elimination and therefore scales as $\mathcal{O}(n^3)$. Thus, calculating P_k scales as $\mathcal{O}(m_k^2 k^3)$. This scaling can be improved to $O(m_k k^3)$ by instead using an estimation technique that computes inner products between $|\phi_i\rangle$ and random stabilizer states from a uniform distribution [5], at the expense of adding dependence on relative error and probability of error. There has also been related work on the "approximate" stabilizer rank that exhibit similar scaling except in terms of a less demanding additive error [6–9].

Let χ_k be the *stabilizer rank* of the T gate magic state $|T\rangle^{\otimes k}$—the minimal number of states required in a stabilizer state decomposition of $|T\rangle^{\otimes k}$. Therefore, the smallest that m_k can be is χ_k and determining its scaling with k is crucial for understanding the optimal $\mathcal{O}(\chi_k k^3)$ cost of classically computing P_k. Although the T gate magic state is not only resource state possible, we focus on it in this study because it is postulated that its stabilizer rank χ_k grows slowest with k [10].

The property that the tensor product of two stabilizer states is a stabilizer state implies a *trivial tensor bound* on the stabilizer rank for all integer powers of a state: $\chi_t \le (\chi_k)^{t/k}$ where t is a multiple of k. However, it is possible that the actual stabilizer rank χ_t is strictly less than this trivial bound. If so, this implies that $|\Psi\rangle^{\otimes t'}$ has a more efficient stabilizer decomposition, for t' any multiple of t. Therefore, it is important to identify such reductions in rank over the trivial bound, a problem we tackle in this paper, as such reductions imply better asymptotic bounds.

Prior searches for these improved tensor bounds for the qubit T gate magic state have relied on numerical Monte Carlo search [10]. The results can be summarized in terms of four values: $\chi_1 = 2$, $\chi_2 = 2$, $\chi_3 = 3$, and $\chi_6 \le 7$. From these four data points and their tensor upper bounds, one can surmise that $\chi_4 \le (\chi_2)^2 = 4$, $\chi_5 \le \chi_3\chi_2 = 6$, and $\chi_7 \le (\chi_2)^2\chi_3 = 12$. All these bounds are conjectured to be tight and numerical searches support this claim [7]. The tensor bound implies the following upper bounds on the T gate stabilizer rank: $\chi_t \le (\chi_1)^t = 2^t$ for arbitrary t, $\chi_t \le (\chi_2)^{t/2} = 2^{0.5t}$ for even t, $\chi_t \le (\chi_3)^{t/3} = 2^{\sim 0.53t}$ for t a multiple of 3, and $\chi_t \le (\chi_6)^{t/6} = 2^{\sim 0.47t}$ for t a multiple of 6. These applications of the trivial tensor bound tell us about the asymptotic scaling of the strong simulation cost of P_k and it is clear that the last bound provides the most favorable such scaling.

To find a better asymptotic scaling requires reaching larger t. Unfortunately, the number of stabilizer states grows as $2^{(1/2+o(1))t^2}$ [11] and the stabilizer rank grows at least linearly with t, therefore any numerical search must contend with a prohibitive search space of size $> 2^{(1/2+o(1))t^3}$. Monte Carlo stops converging appreciably on current hardware at $t > 7$. Therefore, a non-numerical method is especially desirable.

In this direction, we previously showed that odd-prime-d dimensional qudit T gate magic states have the same stabilizer rank for $t=1$ and $t=2$ as has been found for qubits up to the exponential base factor—$2^{\alpha t} \leftrightarrow d^{\alpha t}$, i.e. $(\chi_1)^t = d^t$ and $(\chi_2)^t = d^{0.5t}$ [12]. In fact, we proved that stabilizer decompositions that achieve these stabilizer ranks for $t=1$ and $t=2$ have a one-to-one correspondence with the quadratic Gauss sums that decompose the T gate magic state's discrete Wigner function. Quadratic Gauss sums are the discrete analogue of Gaussian integrals: $\sum_{\boldsymbol{x}\in(\mathbb{Z}/d\mathbb{Z})^n} \exp[\frac{2\pi i}{d}(\boldsymbol{x}^T\boldsymbol{\mathcal{A}}\boldsymbol{x}+\boldsymbol{\beta}\cdot\boldsymbol{x})]$, where $\boldsymbol{\mathcal{A}} \in \mathbb{Z}^{n\times n}$ and $\boldsymbol{\beta}\in\mathbb{Z}^n$. Finding the minimum number of quadratic Gauss sums can be accomplished with an algebraic approach and so can be extended to higher numbers of qudits. Here we push this analysis further.

In the following, we will first introduce the Wigner-Weyl-Moyal (WWM) formalism that forms the basis of our approach in Section I and explain why its quadratic Gauss sums are operationally equivalent to stabilizer state inner products. We then sketch how to use the WWM formalism to algebraically determine the bound on the minimal number of necessary quadratic Gauss sums. This is followed by our main results for $t=3$, $t=6$ and $t=12$ in Section II. We then disucss the significance of the reduction we find in Section III and conclude in Section IV.

I. THE WWM FORMALISM

Instead of considering the magic state in terms of vectors in Hilbert space, we consider a kernel (or quasi-probability) representation; given a complete set of Hilbert-Schmidt orthogonal operators $\hat{R}(\boldsymbol{x})$, indexed by $\boldsymbol{x}\equiv(\boldsymbol{x}_p,\boldsymbol{x}_q)\in((\mathbb{Z}/d\mathbb{Z})^n)^2$, any operator $\hat{A}\in\mathcal{B}((\mathbb{C}^d)^n)$ can be represented as

$$\hat{A} = d^{-1}\sum_{\substack{\boldsymbol{x}\in\\(\mathbb{Z}/d\mathbb{Z})^{2n}}} \mathrm{Tr}(\hat{R}(\boldsymbol{x})\hat{A})\hat{R}(\boldsymbol{x}) \equiv \sum_{\boldsymbol{x}} A(\boldsymbol{x})\hat{R}(\boldsymbol{x}).$$

In particular, we consider the odd-prime d-dimensional Weyl operators introduced by Wootters [13–15],

$$\hat{R}(\boldsymbol{x}) = d^{-n}\sum_{\substack{\boldsymbol{y}_p,\boldsymbol{y}_q\in\\(\mathbb{Z}/d\mathbb{Z})^n}} e^{\frac{2\pi i}{d}(\boldsymbol{y}_p\cdot\boldsymbol{x}_q-\boldsymbol{y}_q\boldsymbol{x}_p-\frac{1}{2}\boldsymbol{y}_p\cdot\boldsymbol{y}_q)}\hat{Z}^{\boldsymbol{y}_p}\hat{X}^{\boldsymbol{y}_q},$$

where $\hat{X}$ and $\hat{Z}$ are d-dimensional generalized Pauli operators [16]. $\hat{R}(\boldsymbol{x})$ are Hermitian, self-inverse and unitary and so the coefficients $\rho(\boldsymbol{x})$ are real-valued. This representation is particularly simple for the Clifford subtheory: the Wigner function of states $A(\boldsymbol{x}) = \rho(\boldsymbol{x})$ are non-negative if and only if they are stabilizer states [17, 18] and the representation of Clifford gates $U_C(\boldsymbol{x})$ are symplectic positive maps that can be described as affine transformations: [17, 19]: $\boldsymbol{x}'\equiv(\boldsymbol{x}'_p,\boldsymbol{x}'_q)^T = \boldsymbol{\mathcal{M}}_C(\boldsymbol{x}_p,\boldsymbol{x}_q)^T+\boldsymbol{v}_C$.

In the WWM formalism, Eq. 1 becomes

$$P_k = \sum_{\boldsymbol{x}\in D}\left[\prod_{i=1}^{k}\rho_T(\boldsymbol{x}_i)\prod_{j=k+1}^{n}\delta(x_{q_j})\right], \tag{2}$$

for

$$D = \left\{\boldsymbol{x}\,\middle|\,\left(\boldsymbol{\mathcal{M}}_C^{-1}\boldsymbol{x}+\boldsymbol{v}\right)_{n+1} \bmod d^h = 0\right\} \tag{3}$$

for some $h\in\mathbb{Z}^+$ and $(\boldsymbol{x})_i$ is the ith element of $\boldsymbol{x}$. The Clifford sequence $\hat{U}_C$ changes the restriction of the domain of the sum from $x_{n+1}\equiv x_{q_1}=0$ to D.

We showed previously [12] that

$$\rho_{T^{\otimes k}}(\boldsymbol{x}) = \prod_{i=1}^{k}\rho_T(\boldsymbol{x}_i) = \sum_{\boldsymbol{y}_q\in(\mathbb{Z}/d\mathbb{Z})^k} e^{\frac{2\pi i}{d^h}P(\boldsymbol{y}_q,\boldsymbol{x})}, \tag{4}$$

for P a polynomial in $\boldsymbol{y}_q$ and $\boldsymbol{x}$ over $\mathbb{Z}$, for p-odd-prime qudits. We refer to $\boldsymbol{y}_q$ as intermediate variables in order to distinguish them from $\boldsymbol{x}\equiv(\boldsymbol{x}_p,\boldsymbol{x}_q)$, which are the final variables at which the Wigner function is evaluated.

For instance, we found that the Wigner function of the two-qutrit tensored T gate magic state [20, 21], $|T\rangle^{\otimes 2} = (|0\rangle + e^{\frac{2\pi i}{9}}|1\rangle + e^{-\frac{2\pi i}{9}}|2\rangle)^{\otimes 2}$, can be transformed from initial to final variables of $\rho_{T^{\otimes 2}}$ with $C^2_{1,2}$ to obtain [12],

$$\frac{1}{3^2}\sum_{y_{q_1}\in\mathbb{Z}/3\mathbb{Z}}\exp\left\{\frac{2\pi i}{3^2}\left[8x_{q_1}^3+7y_{q_1}^3\right]\right\}\mathcal{A}_2(y_{q_1},\boldsymbol{x}). \tag{5}$$

where

$$\mathcal{A}_2(y_{q_1},\boldsymbol{x}) = \sum_{y_{q_2}\in\mathbb{Z}/3\mathbb{Z}} e^{\frac{2\pi i}{3}P(y_{q_1},y_{q_2},\boldsymbol{x})},$$

for $P(y_{q_1},y_{q_2},\boldsymbol{x})$ a polynomial over $\mathbb{Z}$ that is quadratic in y_{q_2}, where

$$\begin{aligned}
&\mathcal{A}_2(y_{q_1},\boldsymbol{x}) \\
&= \sum_{y_{q_2}\in\mathbb{Z}/3\mathbb{Z}} e^{\frac{2\pi i}{3^2}\left[3x_{q_1}^2x_{q_2}+6x_{q_1}x_{q_2}^2+6x_{q_1}^2y_{q_1}+6x_{q_1}x_{q_2}y_{q_1}\right]} \\
&\times e^{\frac{2\pi i}{3^2}\left[6x_{q_2}^2y_{q_1}+6x_{q_1}y_{q_1}^2+3x_{q_2}y_{q_1}^2+3x_{q_1}^2y_{q_2}+3x_{q_1}x_{q_2}y_{q_2}\right]} \\
&\times e^{\frac{2\pi i}{3^2}\left[6x_{q_1}y_{q_1}y_{q_2}+3x_{q_2}y_{q_1}y_{q_2}+6y_{q_1}^2y_{q_2}+6x_{q_1}y_{q_2}^2+3y_{q_1}y_{q_2}^2\right]} \\
&\times e^{\frac{2\pi i}{3^2}\left[x_{p_1}(6y_{q_1}+3x_{q_1})+x_{p_2}(6y_{q_2}+3x_{q_2})\right]}.
\end{aligned} \tag{6}$$

Eq. 5 is a Wigner function that is a linear combination of three terms indexed by y_{q_1}, each of which is a

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
qubit:														
χ_k	2	2	3	4	6	7	12	*inaccessible to Monte Carlo*						
$\chi_k^{t/k}$	2^t	$2^{0.5t}$	$2^{\sim 0.528t}$			$2^{\sim 0.468t}$								
qutrit:														
χ_k	3	3	≤ 8	*inaccessible to Monte Carlo*										
ξ_k	3	3	8	9	24	24	≤ 72	72	≤ 216	216	≤ 486	486	≤ 1458	1458
$\xi_k^{t/k}$	3^t	$3^{0.5t}$	$3^{\sim 0.631t}$			$3^{\sim 0.482t}$					$3^{\sim 0.512t}$	$3^{\sim 0.469t}$		

TABLE I. Upper bound of qubit and qutrit T gate magic state stabilizer ranks χ_k are tabulated and compared to qutrit quadratic Gauss sum ranks ξ_k along with their tensor upper bounds ($\chi_k^{t/k}$ and $\xi_k^{t/k}$ respectively). The reductions in the qubit scaling for $k = 1$, $k = 2$, $k = 3$ and $k = 6$ are observed for qutrits as well. Moreover, a further reduction is observed for qutrits for $k = 12$, a result beyond the reach of Monte Carlo numerical search. $(\chi_k)^{t/k}$ is only listed at the k values at which there is a reduction over the trivial tensor bound.

quadratic Gauss sum over y_{q_2}. Quadratic Gauss sums require $\mathcal{O}(k^3)$ computations to evaluate for k qudits; their value depends on the determinant of their covariance matrix and so they are governed by the cost of Gaussian elimination of a matrix of size $k \times k$ with entries in $\mathbb{Z}/d\mathbb{Z}$ [12]. Importantly, stabilizer state inner products require Gaussian elimination of a matrix with the same properties [11]. WWM quadratic Gauss sums are thus operationally equivalent to Hilbert space stabilizer state inner products.

As a result, we proceed to determine the cost of evaluating Eq. 2 in terms of the number of quadratic Gauss sums in its sum. We previously showed that to find this number it is sufficient to just determine the number of quadratic Gauss sums necessary to evaluate $\rho_{T^{\otimes k}}(\boldsymbol{x})$ for fixed $\boldsymbol{x}$ [12][22]. We define the minimum of this number ξ_k, and call it the *quadratic Gauss sum rank*. More precisely, the overall cost of evaluating Eq. 2 scales as $\mathcal{O}(\xi_k)$ for $k \leq 2$ [12], and empirically continues to do so for $k > 2$. Since the product of two Wigner functions on separate qudits is also a Wigner function, ξ_k satisfies the same trivial tensor bound property as χ_k: $\xi_t \leq (\xi_k)^{t/k}$ for t a multiple of k. Therefore, calculating Eq. 2 using the WWM formalism scales as $\mathcal{O}(\xi_k k^3)$, similarly to how using stabilizer state inner products scales as $\mathcal{O}(\chi_k k^3)$.

As described in [12], the prime-d exponential sum, for arguments in $\mathbb{Z}$, is invariant under some linear transformation $\boldsymbol{\mathcal{M}}$:

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{y}_q \in \\ (\mathbb{Z}/d\mathbb{Z})^m}} \exp \frac{2\pi i}{d} P(\boldsymbol{\mathcal{M}} \boldsymbol{y}_q, \boldsymbol{x}_p, \boldsymbol{x}_q) \\
&= \sum_{\substack{\boldsymbol{\mathcal{M}}^{-1}\boldsymbol{y}_q \in \\ (\mathbb{Z}/d\mathbb{Z})^m}} \exp \frac{2\pi i}{d} P(\boldsymbol{y}_q, \boldsymbol{x}_p, \boldsymbol{x}_q) \\
&= \sum_{\substack{\boldsymbol{y}_q \in \\ (\mathbb{Z}/d\mathbb{Z})^m}} \exp \frac{2\pi i}{d} P(\boldsymbol{y}_q, \boldsymbol{x}_p, \boldsymbol{x}_q).
\end{aligned} \tag{7}$$

This is because a linear transformation over the domain of a field merely permutes the order of the sum.

It further follows that, given a marginal trace over a single degree of freedom, a linear transformation $\boldsymbol{\mathcal{M}}$ merely changes the degree of freedom that is traced over:

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{y}_q \in \\ (\mathbb{Z}/d\mathbb{Z})^m}} \sum_{\substack{\boldsymbol{x} \in \\ (\mathbb{Z}/d\mathbb{Z})^{2m} \\ \boldsymbol{x}_{m+1}=0}} e^{\frac{2\pi i}{d} P(\boldsymbol{y}_q, \boldsymbol{\mathcal{M}}\boldsymbol{x})} \\
&= \sum_{\substack{\boldsymbol{y}_q \in \\ (\mathbb{Z}/d\mathbb{Z})^m}} \sum_{\substack{\boldsymbol{\mathcal{M}}^{-1}\boldsymbol{x} \in \\ (\mathbb{Z}/d\mathbb{Z})^{2m} \\ (\boldsymbol{\mathcal{M}}^{-1}\boldsymbol{x})_{m+1}=0}} e^{\frac{2\pi i}{d} P(\boldsymbol{y}_q, \boldsymbol{x})}. \\
&= \sum_{\substack{\boldsymbol{y}_q \in \\ (\mathbb{Z}/d\mathbb{Z})^m}} \sum_{\substack{\boldsymbol{x} \in \\ (\mathbb{Z}/d\mathbb{Z})^{2m}}} e^{\frac{2\pi i}{d} P(\boldsymbol{y}_q, (\boldsymbol{x}_p, \boldsymbol{x}_q))} \delta((\boldsymbol{\mathcal{M}}^{-1}\boldsymbol{x})_{m+1}),
\end{aligned} \tag{8}$$

where again, the final simplification results from recognizing that the linear transformation only permutes the order of the sum.

These identities generalize to displaced linear transformations (affine transformations) [12]. The latter capture Clifford transformations, which form a symplectic subgroup.

Eq. 2 is of the form of Eq. 8. We investigate the minimal number of terms in its sum after any Clifford transformation $\hat{U}_C$ in Eq. 2. Therefore, due to Eq. 7 and Eq. 8, it follows that we are free to additionally transform Eq. 2's $\boldsymbol{y}_q$ and $\boldsymbol{x}$ variables by a linear tranformation corresponding to Clifford transformations without affecting the worst-case analysis of the minimum number of sums.

Here, we find that products of the Clifford controlled-not gate are sufficient for our purposes.

The choice of a sequence of controlled-not transformations is not unique and only serves to make any reduction in the number of quadratic Gauss sums to be more easily recognized in an algebraic analysis. Generally, we choose a transformation so that half of the qudit degrees of freedom are "cubic" variables and the other half are "quadratic" variables—for fixed cubic variables, the remaining variables form a quadratic Gauss sum. This allows the "cubic" variables to function as indices that label the quadratic Gauss sums and determine their covariance and linear coefficients.

Further controlled-not transformations are made to reduce the number of quadratic Gauss sums by transforming the coefficients of some cubic indexing variables so that they only depend on other cubic indexing variables. This allows for repetitions or simplifications to become apparent without relying on quadratic Gauss sum identities. This allows only the indexing variables to be in the arguments of any additional Kronecker delta functions that reduce the number of quadratic Gauss sums that must be included in the full sum.

We employ this approach to operationally define the cost of evaluating P_k in Eq. 2. In particular, this paper examines the odd-prime d-dimensional qudit ξ_k for $k > 2$. We focus on the smallest such qudit: the qutrit ($d = 3$).

II. RESULTS

In much the same way that stabilizer decompositions of a state are generally non-unique, decompositions of a Wigner function in terms of quadratic Gauss sums are also generally non-unique. This freedom is due to the invariance of the discrete sum in Eq. 4 under linear transformations of its variables $\boldsymbol{y}_q$ as these lie on a finite odd-prime field $\mathbb{Z}/p\mathbb{Z}$. As we showed in the previous section, this is true for both the intermediate variables $\boldsymbol{y}_q$ and the final phase space variables $\boldsymbol{x}$ when the full trace is taken in Eq. 2, despite the additional restriction in the domain. We use this freedom to algebraically lower the number of quadratic Gauss sums that are obtained from the trivial tensor bound for the Wigner function of higher tensor powers of the T gate magic state. We find that products of linear transformations corresponding to the Clifford controlled-not $C_{i,j}$ gate between qudit i and j, $\boldsymbol{\mathcal{M}}_{C_{i,j}} : (x_{p_i}, x_{p_j}, x_{q_i}, x_{q_j}) \to (x_{p_i}, x_{p_j} - x_{p_i} \mod d, x_{q_i} + x_{q_j} \mod d, x_{q_j})$, are sufficient for this purpose.

The trivial tensor bound indicates that $\xi_3 \leq 9$. After transformation by a $C^2_{1,2}$, $C^2_{1,3}$ and $C_{2,3}$ (the overall transformation we call C_3), we find that the three-qutrit T gate magic state can be written as:

$$\begin{aligned} &\rho_{T^{\otimes 3}}(\boldsymbol{\mathcal{M}}_{C_3}\boldsymbol{x}) \qquad (9)\\ &= \sum_{\substack{y_{q_1}, y_{q_2} \\ \in \mathbb{Z}/3^2\mathbb{Z}}} \exp\left[\frac{2\pi i}{9}\left(7y^3_{q_1} + 8x^3_{q_1}\right)\right] \mathcal{A}_3(y_{q_1}, y_{q_2}, \boldsymbol{x}) \\ &\qquad \times \left[\delta(\neg(y_{q_1} - x_{q_1})) + \delta(y_{q_1} - x_{q_1})\delta(\Delta)\right], \end{aligned}$$

where $\delta(\neg\alpha) = \delta((\alpha^{p-1} - 1)^{p-1})$ is logical negation. Logical negation of an argument α for prime p is simply $\alpha^{p-1} \mod p$: if $x \neq 0$ then $x^{p-1} \mod p = 0$ and if $x = 0$ then $(x-1)^{p-1} \mod p = 1$. $\mathcal{A}_3$ is a quadratic Gauss sum and Δ is a the linear coefficient of this sum (see Appendix A for their explicit form).

Eq. 9 is a linear combination of nine quadratic Gauss sums indexed by y_{q_1} and y_{q_2}. However, the additional Kronecker delta functions explicitly express the contrapositive of the condition that these quadratic Gauss sums are zero at $y_{q_1} = x_{q_1}$ and $\Delta \in \{1, 2\}$. Moreover, the delta function terms are disjoint; given any $(\boldsymbol{x}_p, \boldsymbol{x}_q)$ and $\boldsymbol{y}_q$, only one term can be non-zero. However, given $\boldsymbol{x}$, all the terms are zero for at least one value of $\boldsymbol{y}_q$ in the sum, thereby reducing the number of quadratic Gauss sums.

Hence, the Wigner function of three tensored qutrit magic states can be expressed in terms of only $\xi_3 = 8$ non-zero quadratic Gauss sums. Extrapolating to higher t using the tensor bound, this result shows that $(\xi_3)^{t/3} = 3^{\frac{\log 8}{3 \log 3}t} = 3^{\sim 0.63t}$ quadratic Gauss sums can represent t magic states, for t a multiple of 3.

We also find numerical evidence that $\chi_3 = 8$ from running the same Monte Carlo search algorithm as [10] but adapted for qutrits; we perform a random walk on the set of χ stabilizer states and try to maximize the projection between the linear subspace they span and the k-tensored T gate magic state ϕ: $F = ||\Pi\phi||$, where Π is the projector onto the linear subspace spanned by the stabilizer states. At each step, one of the stabilizer states ϕ_i is randomly selected and a random Pauli operator is applied to it: $\phi_i \to (I - P)(I - \omega P)\phi_i$, for $\omega = \exp\frac{2\pi i}{3}$. The new (renormalized) stabilizer state is accepted if it increases F's value. It is rejected if $(I - P)(I - \omega P)\phi_i = 0$. Otherwise, it is accepted with probability $\exp\left[-\beta(F - F')\right]$, where F and F' are the values of the projection before and after the step, respectively. The walk is stopped when $F = 1$, its maximum. We begin with a small β and "anneal" to a large final value.

This approach produces results that are possibly not converged for the 3-tensored qutrit T gate magic state at lower numbers of stabilizer states, since the search space is very large (comparable to $t = 7$ for qubits). The results are illustrated in Figure 1 and show a stabilizer decomposition upper bound of 8. This further establishes evidence that $\chi_k = \xi_k$ for $k = 1$, 2, and 3.

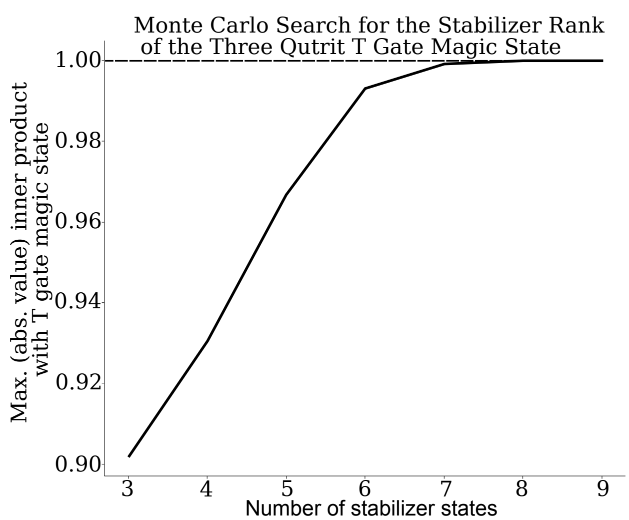

FIG. 1. Three qutrit Monte Carlo stabilizer rank search.

The six-qutrit case undergoes a further reduction due to two sets of quadratic Gauss sums evaluating to the same value (see Appendix B). These sets are indexed by the cubic intermediate variable y_{q_3}, for y_{q_1} fixed. In the worst case over Clifford gates $\hat{U}_C$ in Eq. 1, only this last reduction occurs and so the Wigner function consists of $\xi_6 = 24(= 3^3 - 3)$ quadratic Gauss sums. This leads to a trivial tensor bound of $\xi_t \leq (\xi_6)^{t/6} = 3^{\sim 0.482t}$ for t a multiple of 6.

Lastly, the twelve-qutrit case possesses two sets of quadratic Gauss sums that evaluate to the same value, in the worst case over Clifford gates $\hat{U}_C$, and are indexed by two values of the intermediate cubic variable y_{q_6}. However, compared to the six-qutrit case, this condition holds for more indexing variables than would be proportionally expected: $y_{q_1}, \ldots, y_{q_4}$. Therefore, in the worst-case this Wigner function consists of $\xi_{12} = 3^4 \times 6 = 486$ quadratic Gauss sums. This leads to a trivial tensor bound of $\xi_t \leq (\xi_{12})^{t/12} = 3^{\sim 0.469t}$, for t a multiple of 12.

Results up to $t = 14$, including the bounds discussed above, are tabulated in Table I.

The trivial tensor bounds set the cost of classical strong simulation of P_k, and we can compare this cost to that of existing simulation methods. In Figure 2, we compare these bounds to the cost of a Monte Carlo numerical method [23] based on qutrit Wigner function sampling [24]. Note that the direct evaluation of P_k using the WWM formalism is an explicit algorithm (see Table 1 in [12]) that provably saturates the bounds shown in Figure 2 for $k \leq 2$, and empirically for $k > 2$. We find that the new bounds provide an exponential improvement over existing methods.

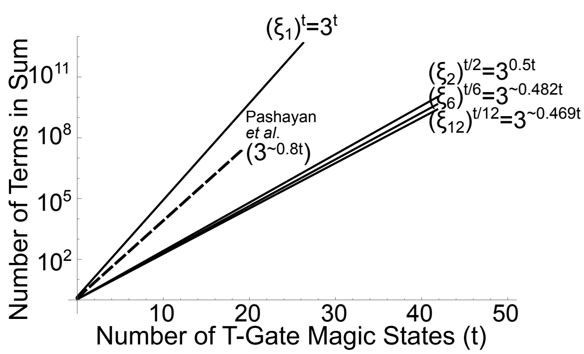

FIG. 2. Logarithm (base 10) of the worst-case number of terms required to evaluate P_k for qutrits in Eq. 2 for a Monte Carlo method based on Wigner negativity [23] (dashed curve) compared to the qutrit trivial tensor bound from $(\xi_1)^t$, $(\xi_2)^{t/2}$, $(\xi_6)^{t/6}$ and $(\xi_{12})^{t/12}$ (solid curves).

III. DISCUSSION

We have found that reductions in ξ_t over the trivial tensor bound also exist for $t = 3$ and $t = 6$ as they do in χ_t for qubits. Unlike the numerical approach, we are able to push far past $t = 7$ and extend our search to $t = 14$. We found that the upper bound cannot be improved over the trivial tensor bound until $t = 12$ where the new rank produces an improved scaling bound of $< 3^{\sim 0.469t}$ for qutrits (for t a multiple of 12).

Using the same argument as in [11], and the fact that the n-qutrit stabilizer group has size 3^n [25], the number of pure stabilizer states on n qutrits is

$$3^n \prod_{k=0}^{n-1} (7^n/2^n - 2^k) / \prod_{k=0}^{n-1} (3^n - 2^k) = 3^{(1/2+o(1))n^2}.$$

Despite the fact that this space grows faster than the $2^{(1/2+o(1))n^2}$ n-qubit stabilizer subspace [11], with the algebraic approach presented here we are able to bound well past $t = 7$ to $t = 14$, an increase in the stabilizer subspace of $> 10^{10^4}$ if the newly discovered upper bound $\xi_{14} \leq 1458$ is tight.

Examining the results in Table I, a deviation from the relationship $2^{\alpha t} \leftrightarrow 3^{\alpha t}$ can be observed for the tensor upper bounds $(\chi_k)^{t/k}$ and $(\xi_k)^{t/k}$ for $k > 2$. A similar deviation was found for the *approximate* stabilizer rank of qutrits [26]. This is due to the conversion issue that occurs from the exponential factor α being a real number while χ_k and ξ_k are constrained to be integers.

Finally, the techniques used in this work can be extended to $d = 2$ despite the fact that the WWM formalism is not real-valued for qubits. This was accomplished in recent work [27] and produced bounds on the classical simulation cost of qubit circuits that similarly further lowered the stabilizer rank for the qubit T gate magic state from tensor products for $k > 7$.

IV. CONCLUSION

In this study we found that the cost of classical strong simulation of universal quantum circuits with qutrit T gate magic states using the WWM formalism, which produces a linear combination of terms that are cost-equivalent to stabilizer decompositions, exhibits novel reductions for $t = 1$, 2, 3 and 6 qutrits, in agreement with the qubit case. Moreover, as this is an algebraic method that is more tractable than numerical search for stabilizer rank by Monte Carlo methods, we are able to derive simulation cost bounds up to $t = 14$ qutrit magic states and find an improvement to the trivial tensor bound from the 12-qutrit T gate magic state. Numerical implementation of this method may allow for increasing this search to even larger t values.

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Appendix A: Three Qutrit Magic State

We act on the initial state with $C_{1,2}^2$, $C_{1,3}^2$, and $C_{2,3}$ (the overall transformation we call C_3), which transforms $y_{q_1} \to y_{q_1} - y_{q_2}$, $y_{q_1} \to y_{q_1} - y_{q_3}$, and $y_{q_2} \to y_{q_2} + y_{q_3}$, respectively. We also act on the final phase space variables with the same operators.This produces:

$$\rho(\boldsymbol{\mathcal{M}}_{C_3}\boldsymbol{x}) = \sum_{\substack{y_{q_1},y_{q_2}\\ \in\mathbb{Z}/3^2\mathbb{Z}}} \exp\left[\frac{2\pi i}{9}\left(7y_{q_1}^3 + 8x_{q_1}^3\right)\right] \mathcal{A}_3(y_{q_1}, y_{q_2}, y_{q_3}, \boldsymbol{x})$$

$$
\begin{aligned}
&\mathcal{A}_3(y_{q_1}, y_{q_2}, y_{q_3}, x_{p_1}, x_{p_2}, x_{p_3}, x_{q_1}, x_{q_2}, x_{q_3}) \\
&= \sum_{\substack{y_{q_3} \\ \in \mathbb{Z}/3^2\mathbb{Z}}} \exp\Bigg\{ \frac{2\pi i}{3} \Bigg[-y_{q_1}^2 y_{q_2} + y_{q_1} y_{q_2}^2 - y_{q_2} x_{p_2} - y_{q_1}^2 x_{q_1} - y_{q_1} y_{q_2} x_{q_1} - y_{q_2}^2 x_{q_1} + x_{p_1} x_{q_1} \\
&\quad + y_{q_3}^2 (y_{q_1} - x_{q_1}) + y_{q_1}^2 x_{q_2} + y_{q_1} y_{q_2} x_{q_2} + x_{p_2} x_{q_2} + x_{q_1}^2 x_{q_2} - x_{q_1} x_{q_2}^2 + y_{q_2} x_{q_1} (x_{q_1} + x_{q_2}) \\
&\quad - y_{q_1} (x_{p_1} + x_{q_1}^2 + x_{q_1} x_{q_2} + x_{q_2}^2) - y_{q_1}^2 x_{q_3} - y_{q_1} y_{q_2} x_{q_3} + y_{q_2}^2 x_{q_3} + x_{p_3} x_{q_3} - x_{q_1}^2 x_{q_3} \\
&\quad - x_{q_1} x_{q_2} x_{q_3} + x_{q_2}^2 x_{q_3} - y_{q_2} (x_{q_1} + x_{q_2}) x_{q_3} + y_{q_1} (x_{q_1} - x_{q_2}) x_{q_3} - y_{q_1} x_{q_3}^2 - x_{q_1} x_{q_3}^2 - x_{q_3} + \Delta y_{q_3} \Bigg] \Bigg\}
\end{aligned} \tag{A1}
$$

and

$$
\begin{aligned}
\Delta = {} & y_{q_1}^2 + y_{q_1} y_{q_2} - y_{q_2}^2 - x_{p_3} - x_{q_1}^2 - x_{q_1} x_{q_2} + x_{q_2}^2 \\
& - y_{q_2} (x_{q_1} + x_{q_2}) + x_{q_1} x_{q_3} + y_{q_1} (x_{q_1} - x_{q_2} + x_{q_3}) + 1.
\end{aligned} \tag{A2}
$$

Appendix B: Six Qutrit Magic State

After transforming the intermediate and final phase space variables by $C_{1,2}^2$, $C_{3,4}^2$, $C_{5,6}^2$, $C_{3,5}^2$, $C_{1,3}^2$, $C_{4,3}^2$, $C_{6,5}^2$, $C_{3,5}^2$, $C_{6,3}^2$, $C_{5,2}^2$, $C_{2,4}^2$, $C_{6,3}^2$, $C_{4,5}$, $C_{4,3}^2$, $C_{3,2}^2$, $C_{6,2}$, $C_{5,3}^2$, $C_{6,3}^2$, $C_{2,1}$, $C_{1,5}$, $C_{3,5}$, $C_{5,3}^2$, $C_{3,5}$, and $C_{1,3}$ (the overall transformation we call C_6), we find:

$$
\begin{aligned}
&\rho(\boldsymbol{\mathcal{M}}_{C_6}(x_{p_1}, x_{p_2}, x_{p_3}, x_{p_4}, x_{p_5}, x_{p_6}, x_{q_1}, x_{q_2}, x_{q_3}, x_{q_4}, x_{q_5}, x_{q_6})) \\
&= \sum_{\substack{y_{q_1}, y_{q_3}, y_{q_4} \\ \in \mathbb{Z}/3^2\mathbb{Z}}} \exp\left[\frac{2\pi i}{9}\left(4y_{q_1}^3 + 2x_{q_1}^3\right)\right] \exp\left[\frac{2\pi i}{3}\Gamma_6(y_{q_1}, y_{q_3}, y_{q_4}, \boldsymbol{x}_p, \boldsymbol{x}_q)\right] \mathcal{A}_6(\boldsymbol{y}_q, \boldsymbol{x}_p, \boldsymbol{x}_q)
\end{aligned} \tag{B1}
$$

where

$$
\begin{aligned}
&\Gamma_6(y_{q_1}, y_{q_3}, y_{q_4}, \boldsymbol{x}_p, \boldsymbol{x}_q) \\
&\equiv \Gamma_6(y_{q_1}, y_{q_3}, y_{q_4}, x_{p_1}, x_{p_2}, x_{p_3}, x_{p_4}, x_{p_5}, x_{p_6}, x_{q_1}, x_{q_2}, x_{q_3}, x_{q_4}, x_{q_5}, x_{q_6}) \\
&= 2y_{q_3}^2 y_{q_4} + y_{q_4}^3 + 2y_{q_3} x_{p_3} + 2y_{q_4} x_{p_4} + 2y_{q_4}^2 x_{q_1} + x_{p_1} x_{q_1} + 2y_{q_4} x_{q_1}^2 + 2y_{q_3} y_{q_4} x_{q_3} + x_{p_3} x_{q_3} + y_{q_4} x_{q_3}^2 \\
&\quad + (x_{p_4} + x_{q_1}(y_{q_4} + 2x_{q_1}) + (y_{q_3} + x_{q_3})^2) x_{q_4} + 2x_{q_1} x_{q_4}^2 + 2x_{q_4}^3 + y_{q_1}^2 (y_{q_4} + 2(x_{q_1} + x_{q_4})) \\
&\quad + y_{q_1} (y_{q_4}^2 + 2(x_{p_1} + x_{q_1}^2) + x_{q_1} x_{q_4} + 2x_{q_4}^2 + y_{q_4}(x_{q_1} + x_{q_4})),
\end{aligned} \tag{B2}
$$

$$
\begin{aligned}
&\mathcal{A}_6(\boldsymbol{y}_q, \boldsymbol{x}_p, \boldsymbol{x}_q) \\
&= \sum_{\substack{y_{q_2}, y_{q_5}, y_{q_6} \\ \in \mathbb{Z}/3^2\mathbb{Z}}} \exp\Bigg\{\frac{2\pi i}{3}\Bigg[x_{p_2} x_{q_2} + x_{q_2}^2 (2y_{q_1} + 2y_{q_4} + 2x_{q_1} + 2x_{q_4}) + \Sigma_{y_{q_5}} y_{q_5}^2 \\
&\qquad + \Sigma_{y_{q_6}} y_{q_6}^2 + \Sigma_{y_{q_2}} y_{q_2}^2 + \Delta_{y_{q_2}} y_{q_2} + (y_{q_3} + 2y_{q_4} + x_{q_3} + 2x_{q_4}) x_{q_5}^2 + \Delta_{2q_5} y_{q_5} \\
&\qquad + x_{p_5} x_{q_5} + x_{p_6} x_{q_6} + 2(y_{q_3} + y_{q_4} + x_{q_3} + x_{q_4}) x_{q_6}^2 + \Sigma_{y_{q_6}} y_{q_6}\Bigg]\Bigg\},
\end{aligned} \tag{B3}
$$

for

$$\Sigma_{y_{q_2}} = y_{q_1} + y_{q_4} + 2(x_{q_1} + x_{q_4}), \tag{B4}$$
$$\Sigma_{y_{q_5}} = 2y_{q_3} + y_{q_4} + x_{q_3} + 2x_{q_4}, \tag{B5}$$
$$\Sigma_{y_{q_6}} = y_{q_3} + y_{q_4} + 2x_{q_3} + 2x_{q_4}, \tag{B6}$$
$$\Delta_{y_{q_2}} = 2x_{p_2} + x_{q_2}(y_{q_1} + y_{q_4} + x_{q_1} + x_{q_4}), \tag{B7}$$
$$\Delta_{y_{q_5}} = 2x_{p_5} + (2y_{q_3} + y_{q_4} + 2x_{q_3} + x_{q_4}) x_{q_5}, \tag{B8}$$

and

$$\Delta_{y_{q_6}} = 2x_{p_6} + (y_{q_3} + y_{q_4} + x_{q_3} + x_{q_4})x_{q_6}. \tag{B9}$$

In Eq. B1, the intermediate variables y_{q_2}, y_{q_5}, and y_{q_6} are quadratic while y_{q_1}, y_{q_3}, and y_{q_4} are cubic. y_{q_1} is the only intermediate variable that lies in the full 9-cycle and with respect to the intermediate variables it only has cross-terms with the cubic ones. Hence, y_{q_1} indexes the quadratic sums over the intermediate quadratic variables in terms of 3-cocycles $\{0, 3, 6\}$, $\{1, 4, 7\}$, and $\{2, 5, 8\}$. The three cubic variables each take three non-periodic values which leads to $3^3 = 9 \times 3$ quadratic Gauss sums (that are 3−dimensional). However, we can reduce this number by noticing some properties.

If the linear coefficient of y_{q_5} or y_{q_6} is non-zero anywhere, it is non-zero for at least three values of (y_{q_3}, y_{q_4}) independent of y_{q_1}. If the linear coefficient of y_{q_2} is non-zero anywhere, it is non-zero for at least three values of (y_{q_1}, y_{q_4}) independent of y_{q_3}. Otherwise, $y_{q_1} = x_{q_1}$, $y_{q_3} \neq x_{q_3}$ gives you a set of quadratic Gauss sums (indexed by y_{q_4} and y_{q_3}) that must add up to a real number because y_{q_3}'s quadratic coefficient is w.r.t. x_{q_3} and so is equal for $y_{q_3} - x_{q_3} \neq 0$ and so only its linear coefficient differs (linearly) meaning that any imaginary parts must cancel out when running through all values of its quadratic coefficient: $(y_{q_4} - x_{q_4})$. Hence, $y_{q_3} \neq x_{q_3}$ index two sets of quadratic Gauss sums indexed by y_{q_4} that each sum up to the same total. Thus, it is sufficient to sum up one set and multiply by two. This takes six quadratic Gauss sums and replaces them with three.

This can be summarized by the following equation:

$$\begin{aligned}
&\rho_{T^{\otimes 6}}(\boldsymbol{\mathcal{M}}_{C_6}\boldsymbol{x}) \\
&= \sum_{\substack{y_{q_1}, y_{q_3}, y_{q_4} \\ \in \mathbb{Z}/3^2\mathbb{Z}}} e^{\frac{2\pi i}{9}\left(4y_{q_1}^3 + 2x_{q_1}^3\right)} e^{\frac{2\pi i}{3}\Gamma_6(y_{q_1}, y_{q_3}, y_{q_4}, \boldsymbol{x}_p, \boldsymbol{x}_q)} \\
&\qquad \times \mathcal{A}_6(\boldsymbol{y}_q, \boldsymbol{x}_p, \boldsymbol{x}_q) \\
&\quad \times \Bigg[\delta((\Sigma_{y_{q_2}} \wedge \Delta_{y_{q_2}}) \vee (\Sigma_{y_{q_5}} \wedge \Delta_{y_{q_5}}) \vee (\Sigma_{y_{q_6}} \wedge \Delta_{y_{q_6}})) \\
&\qquad + [\delta(y_{q_3} - x_{q_3}) + 2\delta(y_{q_3} - x_{q_3} + 1)] \\
&\qquad\qquad \times \delta(\neg(\Sigma_{y_{q_2}} \vee \Sigma_{y_{q_5}} \vee \Sigma_{y_{q_6}}))\delta(y_{q_1} - x_{q_1}) \\
&\qquad + \delta(\neg(\Sigma_{y_{q_2}} \vee \Sigma_{y_{q_5}} \vee \Sigma_{y_{q_6}}))\delta(\neg(y_{q_1} - x_{q_1})) \Bigg],
\end{aligned} \tag{B10}$$

where

$$\begin{aligned}
\delta(\alpha \vee \beta) &= \delta(\alpha)\delta(\neg\beta) + \delta(\neg\alpha)\delta(\beta) \\
&\quad + \delta(\alpha)\delta(\beta), \text{ (logical inclusive disjunction)}
\end{aligned} \tag{B11}$$

and

$$\delta(\alpha \wedge \beta) = \delta(\alpha)\delta(\beta), \text{ (logical conjunction)} \tag{B12}$$

and the arguments for the delta functions are taken mod p odd-prime.

In Eq. B10, the first term includes all cases where the quadratic coefficients of y_{q_2}, y_{q_5}, and y_{q_6} are zero along with their respective linear coefficients, as these produce plane waves that do not evaluate to zero. The second term includes all cases where all these quadratic coefficients are non-zero and $y_{q_1} = x_{q_1}$ when the two sets of quadratic Gauss sums indexed by $y_{q_3} \neq x_{q_3}$ sum up to the same value. The third term includes the remaining terms when the quadratic coefficients of y_{q_2}, y_{q_5}, and y_{q_6} are non-zero and $y_{q_1} \neq x_{q_1}$.

As before for three-qutrit T gate magic state, these three terms are disjoint–only one term is non-zero given $(\boldsymbol{x}_p, \boldsymbol{x}_q)$ and $\boldsymbol{y}_q$. However, from the discussion earlier, there are fewer terms here than for the unfettered sum over $\boldsymbol{y}_q$; the number of quadatic Gauss sums is reduced by three.

The same sort of analysis can be found for the corresponding final phase space variables.

Appendix C: Twelve Qutrit Magic State

After transforming the intermediate and final phase space variables by $C^2_{12,11}$, $C^2_{10,9}$, $C^2_{8,7}$, $C^2_{6,5}$, $C^2_{4,3}$, $C^2_{2,1}$, $C^2_{10,12}$ $C^2_{8,10}$, $C^2_{6,8}$, $C^2_{4,6}$, $C^2_{2,4}$, $C^2_{12,9}$, $C_{9,12}$, $C^2_{9,6}$, $C_{6,9}$, $C^2_{10,7}$, $C_{7,10}$, $C^2_{3,5}$, $C^2_{7,5}$, $C_{5,7}$, $C^2_{3,1}$, $C_{1,3}$, $C^2_{5,3}$, $C_{3,5}$, $C^2_{8,5}$, $C_{5,8}$, $C_{9,7}$,

$C^2_{1,3}$, $C_{5,3}$, $C_{4,5}$, $C_{6,3}$, $C^2_{5,3}$, $C_{3,5}$, $C_{5,6}$, $C^2_{6,1}$, $C_{11,6}$, $C^2_{6,1}$, $C_{11,1}$, $C_{11,2}$, $C_{11,3}$, $C^2_{11,4}$, $C_{7,11}$, $C_{7,5}$, $C^2_{11,7}$, and $C_{7,11}$, (the overall transformation we call C_{12}), we find:

$$\begin{aligned}&\rho(\boldsymbol{\mathcal{M}}_{C_{12}}(x_{p_1},\ldots,x_{p_{12}},x_{q_1},\ldots,x_{q_{12}}))\\ &=\sum_{\substack{y_{q_1},\ldots,y_{q_6}\\ \in\mathbb{Z}/3^2\mathbb{Z}}}\exp\left[\frac{2\pi i}{9}\left(7y^3_{q_2}+8x^3_{q_2}\right)\right]\exp\left[\frac{2\pi i}{3}\Gamma_{12}(y_{q_1},\ldots,y_{q_6},\boldsymbol{x}_p,\boldsymbol{x}_q)\right]\mathcal{A}_{12}(\boldsymbol{y}_q,\boldsymbol{x}_p,\boldsymbol{x}_q),\end{aligned}\tag{C1}$$

where

$$\begin{aligned}&\Gamma_{12}(y_{q_1},\ldots,y_{q_6},\boldsymbol{x}_p,\boldsymbol{x}_q)=\\
&2y^3_{q_3}+2y^2_{q_3}y_{q_4}+y_{q_3}y^2_{q_4}+2y^2_{q_3}y_{q_5}+2y_{q_3}y_{q_4}y_{q_5}+2y_{q_3}y^2_{q_5}+2y^2_{q_3}y_{q_6}+2y_{q_3}y_{q_4}y_{q_6}\\
&+y_{q_3}x_{p_{11}}+y_{q_5}x_{p_{11}}+y_{q_6}x_{p_{11}}+y_{q_5}x_{p_3}+y_{q_6}x_{p_3}+y_{q_3}x_{p_4}+2y_{q_4}x_{p_4}+y_{q_5}x_{p_4}+y_{q_6}x_{p_4}+2y_{q_3}x_{p_5}\\
&+y_{q_5}x_{p_5}+2y_{q_6}x_{p_5}+2y_{q_6}x_{p_6}+2y_{q_3}y_{q_4}x_{q_1}+2y_{q_3}y_{q_5}x_{q_1}+2y_{q_5}y_{q_6}x_{q_1}+x_{p_1}x_{q_1}+2x_{p_{11}}x_{q_1}\\
&+2x_{p_3}x_{q_1}+2x_{p_4}x_{q_1}+x_{p_5}x_{q_1}+y^2_{q_3}x_{q_2}+y_{q_3}y_{q_4}x_{q_2}+2y^2_{q_4}x_{q_2}+2y_{q_3}y_{q_5}x_{q_2}+2y_{q_4}y_{q_6}x_{q_2}+2y^2_{q_6}x_{q_2}\\
&+x_{p_2}x_{q_2}+y_{q_3}x_{q_1}x_{q_2}+y_{q_3}x^2_{q_2}+y_{q_4}x^2_{q_2}+2y_{q_6}x^2_{q_2}+2y_{q_3}y_{q_4}x_{q_3}+2y^2_{q_4}x_{q_3}+2y_{q_3}y_{q_5}x_{q_3}+y_{q_4}y_{q_5}x_{q_3}\\
&+y^2_{q_5}x_{q_3}+2y_{q_3}y_{q_6}x_{q_3}+y_{q_4}y_{q_6}x_{q_3}+2x_{p_{11}}x_{q_3}+2x_{p_4}x_{q_3}+x_{p_5}x_{q_3}+2y_{q_4}x_{q_1}x_{q_3}+2y_{q_5}x_{q_1}x_{q_3}\\
&+2y_{q_3}x_{q_2}x_{q_3}+y_{q_4}x_{q_2}x_{q_3}+2y_{q_5}x_{q_2}x_{q_3}+x_{q_1}x_{q_2}x_{q_3}+x^2_{q_2}x_{q_3}+y_{q_4}x^2_{q_3}+y_{q_5}x^2_{q_3}+y_{q_6}x^2_{q_3}+x_{q_2}x^2_{q_3}+x^3_{q_3}\\
&+y^2_{q_3}x_{q_4}+y_{q_3}y_{q_4}x_{q_4}+y_{q_3}y_{q_5}x_{q_4}+y_{q_3}y_{q_6}x_{q_4}+x_{p_4}x_{q_4}+2y_{q_3}x_{q_1}x_{q_4}+y_{q_3}x_{q_2}x_{q_4}+y_{q_4}x_{q_2}x_{q_4}+2y_{q_6}x_{q_2}x_{q_4}\\
&+x^2_{q_2}x_{q_4}+2y_{q_3}x_{q_3}x_{q_4}+y_{q_4}x_{q_3}x_{q_4}+y_{q_5}x_{q_3}x_{q_4}+y_{q_6}x_{q_3}x_{q_4}+2x_{q_1}x_{q_3}x_{q_4}+x_{q_2}x_{q_3}x_{q_4}+x^2_{q_3}x_{q_4}+2y_{q_3}x^2_{q_4}+2x_{q_2}x^2_{q_4}\\
&+2x_{q_3}x^2_{q_4}+y^2_{q_3}x_{q_5}+2y_{q_3}y_{q_4}x_{q_5}+y^2_{q_4}x_{q_5}+2y_{q_4}y_{q_5}x_{q_5}+2y^2_{q_5}x_{q_5}+y_{q_3}y_{q_6}x_{q_5}+2y_{q_4}y_{q_6}x_{q_5}+2x_{p_3}x_{q_5}\\
&+x_{p_5}x_{q_5}+2y_{q_3}x_{q_1}x_{q_5}+y_{q_4}x_{q_1}x_{q_5}+y_{q_5}x_{q_1}x_{q_5}+2y_{q_6}x_{q_1}x_{q_5}+2y_{q_4}x_{q_2}x_{q_5}+y_{q_5}x_{q_2}x_{q_5}+2x_{q_1}x_{q_2}x_{q_5}\\
&+2x^2_{q_2}x_{q_5}+2y_{q_3}x_{q_3}x_{q_5}+2y_{q_4}x_{q_3}x_{q_5}+y_{q_6}x_{q_3}x_{q_5}+2x_{q_1}x_{q_3}x_{q_5}+x^2_{q_3}x_{q_5}+2y_{q_3}x_{q_4}x_{q_5}+2y_{q_4}x_{q_4}x_{q_5}\\
&+2y_{q_5}x_{q_4}x_{q_5}+2y_{q_6}x_{q_4}x_{q_5}+x_{q_1}x_{q_4}x_{q_5}+2x_{q_2}x_{q_4}x_{q_5}+2x_{q_3}x_{q_4}x_{q_5}+x^2_{q_4}x_{q_5}+2y_{q_3}x^2_{q_5}+2y_{q_5}x^2_{q_5}\\
&+y_{q_6}x^2_{q_5}+x_{q_1}x^2_{q_5}+2x_{q_2}x^2_{q_5}+2x_{q_3}x^2_{q_5}+2x^3_{q_5}+(y^2_{q_3}+2x_{p_{11}}+2x_{p_3}+2x_{p_4}+x_{p_5}+x_{p_6}+2y_{q_5}x_{q_1}\\
&+x_{q_2}(y_{q_6}+2x_{q_2})+x^2_{q_3}+(2x_{q_2}+x_{q_3})x_{q_4}+y_{q_3}(y_{q_4}+2x_{q_3}+x_{q_4}+x_{q_5})+x_{q_5}(2x_{q_1}+x_{q_3}+2x_{q_4}+x_{q_5})\\
&+y_{q_4}(2x_{q_2}+x_{q_3}+2x_{q_5}))x_{q_6}+2x_{q_2}x^2_{q_6}+y_{q_1}(y_{q_5}y_{q_6}+2x_{p_1}+x_{p_{11}}+x_{p_3}+x_{p_4}+2x_{p_5}+2y_{q_4}x_{q_3}+2y_{q_5}x_{q_3}+x_{q_2}x_{q_3}\\
&+2x_{q_3}x_{q_4}+y_{q_4}x_{q_5}+y_{q_5}x_{q_5}+2y_{q_6}x_{q_5}+2x_{q_2}x_{q_5}+2x_{q_3}x_{q_5}+x_{q_4}x_{q_5}+x^2_{q_5}+y_{q_2}(2y_{q_3}+x_{q_3}+2x_{q_5})\\
&+y_{q_3}(y_{q_4}+y_{q_5}+x_{q_2}+2(x_{q_4}+x_{q_5}))+2(y_{q_5}+x_{q_5})x_{q_6})+y^2_{q_2}(2y_{q_3}+2y_{q_4}+y_{q_6}+2x_{q_2}+x_{q_3}+x_{q_4}+2(x_{q_5}+x_{q_6}))\\
&+y_{q_2}(2y^2_{q_3}+y^2_{q_4}+y^2_{q_6}+2x_{p_2}+y_{q_6}x_{q_2}+2x^2_{q_2}+2y_{q_5}x_{q_3}+x_{q_1}x_{q_3}+2x_{q_2}x_{q_3}+x^2_{q_3}+2y_{q_6}x_{q_4}+2x_{q_2}x_{q_4}\\
&+x_{q_3}x_{q_4}+2x^2_{q_4}+y_{q_3}(2y_{q_4}+y_{q_5}+x_{q_1}+2(x_{q_2}+x_{q_3})+x_{q_4})+y_{q_5}x_{q_5}+2x_{q_1}x_{q_5}+x_{q_2}x_{q_5}+2x_{q_4}x_{q_5}+2x^2_{q_5}+\\
&(y_{q_6}+x_{q_2}+2x_{q_4})x_{q_6}+2x^2_{q_6}+y_{q_4}(y_{q_6}+2x_{q_2}+x_{q_3}+x_{q_4}+2(x_{q_5}+x_{q_6})))\end{aligned}\tag{C2}$$

$$
\begin{aligned}
&\mathcal{A}_{12}(\boldsymbol{y}_q, \boldsymbol{x}_p, \boldsymbol{x}_q) \qquad\qquad \text{(C3)}\\
&= \sum_{\substack{y_{q_7},\dots,y_{q_{12}} \\ \in \mathbb{Z}/3^2\mathbb{Z}}} \exp\Bigg\{ \frac{2\pi i}{3} \Bigg[\Sigma_{y_{q_{12}}} y_{q_{12}}^2 + x_{q_{12}}^2 (2y_{q_3} + 2y_{q_5} + 2x_{q_3}) + \Sigma_{y_{q_{10}}} y_{q_{10}}^2 + x_{q_{10}}^2 (2y_{q_3} + 2x_{q_3} + x_{q_5}) + \Sigma_{y_{q_8}} y_{q_8}^2 \\
&\quad + \Sigma_{y_{q_9}} y_{q_9}^2 + \Sigma_{y_{q_7}}(y_{q_6}) y_{q_7}^2 + \Sigma_{y_{q_{11}}}(y_{q_6}) y_{q_{11}}^2 + x_{q_{11}}^2 (2y_{q_3} + 2y_{q_5} + 2y_{q_6} + 2x_{q_3} + 2x_{q_6}) \\
&\quad + x_{q_{12}} (y_{q_1} y_{q_3} + y_{q_1} y_{q_5} + y_{q_3} y_{q_6} + y_{q_5} y_{q_6} + x_{p_{12}} + y_{q_3} x_{q_1} + y_{q_5} x_{q_1} + y_{q_1} x_{q_3} \\
&\quad + y_{q_6} x_{q_3} + x_{q_1} x_{q_3} + y_{q_3} x_{q_6} + y_{q_5} x_{q_6} + x_{q_3} x_{q_6}) + \Delta_{y_{q_{12}}} y_{q_{12}} \\
&\quad + x_{q_{11}} (2y_{q_1} y_{q_3} + 2y_{q_1} y_{q_5} + y_{q_3} y_{q_5} + y_{q_5}^2 + 2y_{q_1} y_{q_6} + y_{q_5} y_{q_6} + x_{p_{11}} + 2y_{q_3} x_{q_1} \\
&\quad + 2y_{q_5} x_{q_1} + 2y_{q_6} x_{q_1} + 2y_{q_1} x_{q_3} + y_{q_5} x_{q_3} + 2x_{q_1} x_{q_3} + y_{q_3} x_{q_5} + y_{q_5} x_{q_5} + y_{q_6} x_{q_5} + x_{q_3} x_{q_5} \\
&\quad + 2y_{q_1} x_{q_6} + y_{q_5} x_{q_6} + 2x_{q_1} x_{q_6} + x_{q_5} x_{q_6}) + x_{q_{10}} (y_{q_1} y_{q_3} + 2y_{q_3}^2 + y_{q_3} y_{q_5} + y_{q_3} y_{q_6} \\
&\quad + x_{p_{10}} + y_{q_3} x_{q_1} + y_{q_1} x_{q_3} + y_{q_3} x_{q_3} + y_{q_5} x_{q_3} + y_{q_6} x_{q_3} + x_{q_1} x_{q_3} + 2x_{q_3}^2 + 2y_{q_1} x_{q_5} + 2y_{q_5} x_{q_5} \\
&\quad + 2y_{q_6} x_{q_5} + 2x_{q_1} x_{q_5} + x_{q_5}^2 + y_{q_3} x_{q_6} + x_{q_3} x_{q_6} + 2x_{q_5} x_{q_6}) + \Delta_{y_{q_{10}}} y_{q_{10}} + \Delta_{y_{q_{11}}} y_{q_{11}} \\
&\quad + (2y_{q_1}^2 + y_{q_1} y_{q_2} + 2y_{q_2} y_{q_3} + y_{q_3}^2 + 2y_{q_1} y_{q_4} + y_{q_3} y_{q_4} + 2y_{q_1} y_{q_5} + 2y_{q_2} y_{q_5} \\
&\quad + y_{q_4} y_{q_5} + 2y_{q_5}^2 + 2y_{q_1} y_{q_6} + y_{q_2} y_{q_6} + 2y_{q_3} y_{q_6} + 2y_{q_4} y_{q_6} + y_{q_5} y_{q_6} + x_{p_7} + y_{q_1} x_{q_1} \\
&\quad + y_{q_2} x_{q_1} + 2y_{q_4} x_{q_1} + 2y_{q_5} x_{q_1} + 2y_{q_6} x_{q_1} + 2x_{q_1}^2 + y_{q_1} x_{q_2} + 2y_{q_3} x_{q_2} + 2y_{q_5} x_{q_2} + y_{q_6} x_{q_2} \\
&\quad + x_{q_1} x_{q_2} + 2y_{q_2} x_{q_3} + 2y_{q_3} x_{q_3} + y_{q_4} x_{q_3} + 2y_{q_6} x_{q_3} + 2x_{q_2} x_{q_3} + x_{q_3}^2 + 2y_{q_1} x_{q_4} + y_{q_3} x_{q_4} \\
&\quad + y_{q_5} x_{q_4} + 2y_{q_6} x_{q_4} + 2x_{q_1} x_{q_4} + x_{q_3} x_{q_4} + 2y_{q_1} x_{q_5} + y_{q_3} x_{q_5} + y_{q_5} x_{q_5} + 2y_{q_6} x_{q_5} + 2x_{q_1} x_{q_5} \\
&\quad + x_{q_3} x_{q_5} + 2y_{q_1} x_{q_6} + y_{q_2} x_{q_6} + 2y_{q_3} x_{q_6} + 2y_{q_4} x_{q_6} + y_{q_5} x_{q_6} + 2x_{q_1} x_{q_6} + x_{q_2} x_{q_6} + 2x_{q_3} x_{q_6} \\
&\quad + 2x_{q_4} x_{q_6} + 2x_{q_5} x_{q_6}) x_{q_7} + (y_{q_1} + 2y_{q_3} + 2y_{q_5} + y_{q_6} + x_{q_1} + 2x_{q_3} + x_{q_6}) x_{q_7}^2 + \Delta_{y_{q_7}} y_{q_7} \\
&\quad + (2y_{q_1} y_{q_2} + y_{q_2}^2 + 2y_{q_1} y_{q_3} + 2y_{q_2} y_{q_3} + y_{q_3}^2 + y_{q_1} y_{q_4} + y_{q_2} y_{q_4} + y_{q_3} y_{q_4} \\
&\quad + y_{q_4}^2 + y_{q_2} y_{q_5} + y_{q_3} y_{q_5} + 2y_{q_4} y_{q_5} + y_{q_2} y_{q_6} + y_{q_3} y_{q_6} + 2y_{q_4} y_{q_6} + x_{p_8} + 2y_{q_2} x_{q_1} \\
&\quad + 2y_{q_3} x_{q_1} + y_{q_4} x_{q_1} + 2y_{q_1} x_{q_2} + 2y_{q_2} x_{q_2} + 2y_{q_3} x_{q_2} + y_{q_4} x_{q_2} + y_{q_5} x_{q_2} + y_{q_6} x_{q_2} + 2x_{q_1} x_{q_2} \\
&\quad + x_{q_2}^2 + 2y_{q_1} x_{q_3} + 2y_{q_2} x_{q_3} + 2y_{q_3} x_{q_3} + y_{q_4} x_{q_3} + y_{q_5} x_{q_3} + y_{q_6} x_{q_3} + 2x_{q_1} x_{q_3} + 2x_{q_2} x_{q_3} \\
&\quad + x_{q_3}^2 + y_{q_1} x_{q_4} + y_{q_2} x_{q_4} + y_{q_3} x_{q_4} + 2y_{q_4} x_{q_4} + 2y_{q_5} x_{q_4} + 2y_{q_6} x_{q_4} + x_{q_1} x_{q_4} + x_{q_2} x_{q_4} \\
&\quad + x_{q_3} x_{q_4} + x_{q_4}^2 + y_{q_1} x_{q_5} + 2y_{q_2} x_{q_5} + 2y_{q_3} x_{q_5} + y_{q_4} x_{q_5} + 2y_{q_5} x_{q_5} + 2y_{q_6} x_{q_5} + x_{q_1} x_{q_5} \\
&\quad + 2x_{q_2} x_{q_5} + 2x_{q_3} x_{q_5} + x_{q_4} x_{q_5} + y_{q_2} x_{q_6} + y_{q_3} x_{q_6} + 2y_{q_4} x_{q_6} + x_{q_2} x_{q_6} + x_{q_3} x_{q_6} + 2x_{q_4} x_{q_6} \\
&\quad + 2x_{q_5} x_{q_6}) x_{q_8} + (2y_{q_2} + 2y_{q_3} + y_{q_4} + 2x_{q_2} + 2x_{q_3} + x_{q_4} + x_{q_5}) x_{q_8}^2 + \Delta_{y_{q_8}} y_{q_8} + (2y_{q_1}^2 \\
&\quad + 2y_{q_1} y_{q_3} + y_{q_3} y_{q_4} + y_{q_4}^2 + y_{q_1} y_{q_6} + y_{q_3} y_{q_6} + y_{q_4} y_{q_6} + x_{p_9} + y_{q_1} x_{q_1} + 2y_{q_3} x_{q_1} \\
&\quad + y_{q_6} x_{q_1} + 2x_{q_1}^2 + 2y_{q_1} x_{q_3} + y_{q_4} x_{q_3} + y_{q_6} x_{q_3} + 2x_{q_1} x_{q_3} + y_{q_3} x_{q_4} + 2y_{q_4} x_{q_4} + y_{q_6} x_{q_4} \\
&\quad + x_{q_3} x_{q_4} + x_{q_4}^2 + y_{q_1} x_{q_5} + 2y_{q_4} x_{q_5} + 2y_{q_6} x_{q_5} + x_{q_1} x_{q_5} + 2x_{q_4} x_{q_5} + y_{q_1} x_{q_6} + y_{q_3} x_{q_6} \\
&\quad + y_{q_4} x_{q_6} + x_{q_1} x_{q_6} + x_{q_3} x_{q_6} + x_{q_4} x_{q_6} + 2x_{q_5} x_{q_6}) x_{q_9} + (2y_{q_1} + 2y_{q_3} + 2y_{q_4} + 2x_{q_1} + 2x_{q_3} \\
&\quad + 2x_{q_4} + x_{q_5}) x_{q_9}^2 + \Delta_{y_{q_9}} y_{q_9} \Bigg] \Bigg\}.
\end{aligned}
$$

where

$$\Sigma_{y_{q_7}}(y_{q_6}) = (2y_{q_1} + y_{q_3} + y_{q_5} + 2y_{q_6} + x_{q_1} + 2x_{q_3} + x_{q_6}) \tag{C4}$$
$$\Sigma_{y_{q_8}} = (y_{q_2} + y_{q_3} + 2y_{q_4} + 2x_{q_2} + 2x_{q_3} + x_{q_4} + x_{q_5}) \tag{C5}$$
$$\Sigma_{y_{q_9}} = (y_{q_1} + y_{q_3} + y_{q_4} + 2x_{q_1} + 2x_{q_3} + 2x_{q_4} + x_{q_5}) \tag{C6}$$
$$\Sigma_{y_{q_{10}}} = (y_{q_3} + 2x_{q_3} + x_{q_5}) \tag{C7}$$
$$\Sigma_{y_{q_{11}}}(y_{q_6}) = (y_{q_3} + y_{q_5} + y_{q_6} + 2x_{q_3} + 2x_{q_6}) \tag{C8}$$
$$\Sigma_{y_{q_{12}}} = (y_{q_3} + y_{q_5} + 2x_{q_3}) \tag{C9}$$
$$\begin{aligned}\Delta_{y_{q_7}} = (&y_{q_1}^2 + 2y_{q_1}y_{q_2} + y_{q_2}y_{q_3} + 2y_{q_3}^2 + y_{q_1}y_{q_4} + 2y_{q_3}y_{q_4} + y_{q_1}y_{q_5} + y_{q_2}y_{q_5} + 2y_{q_4}y_{q_5} + y_{q_5}^2\\ &+y_{q_1}y_{q_6} + 2y_{q_2}y_{q_6} + y_{q_3}y_{q_6} + y_{q_4}y_{q_6} + 2y_{q_5}y_{q_6} + 2x_{p_7} + y_{q_1}x_{q_1} + y_{q_2}x_{q_1} + 2y_{q_4}x_{q_1} + 2y_{q_5}x_{q_1}\\ &+2y_{q_6}x_{q_1} + 2x_{q_1}^2 + y_{q_1}x_{q_2} + 2y_{q_3}x_{q_2} + 2y_{q_5}x_{q_2} + y_{q_6}x_{q_2} + x_{q_1}x_{q_2} + 2y_{q_2}x_{q_3} + 2y_{q_3}x_{q_3} + y_{q_4}x_{q_3}\\ &+2y_{q_6}x_{q_3} + 2x_{q_2}x_{q_3} + x_{q_3}^2 + 2y_{q_1}x_{q_4} + y_{q_3}x_{q_4} + y_{q_5}x_{q_4} + 2y_{q_6}x_{q_4} + 2x_{q_1}x_{q_4} + x_{q_3}x_{q_4} + 2y_{q_1}x_{q_5}\\ &+y_{q_3}x_{q_5} + y_{q_5}x_{q_5} + 2y_{q_6}x_{q_5} + 2x_{q_1}x_{q_5} + x_{q_3}x_{q_5} + 2y_{q_1}x_{q_6} + y_{q_2}x_{q_6} + 2y_{q_3}x_{q_6} + 2y_{q_4}x_{q_6} + y_{q_5}x_{q_6}\\ &+2x_{q_1}x_{q_6} + x_{q_2}x_{q_6} + 2x_{q_3}x_{q_6} + 2x_{q_4}x_{q_6} + 2x_{q_5}x_{q_6} + (2y_{q_1} + y_{q_3} + y_{q_5} + 2y_{q_6} + 2x_{q_1} + x_{q_3} + 2x_{q_6})x_{q_7})\end{aligned} \tag{C10}$$
$$\begin{aligned}\Delta_{y_{q_8}} = (&y_{q_1}y_{q_2} + 2y_{q_2}^2 + y_{q_1}y_{q_3} + y_{q_2}y_{q_3} + 2y_{q_3}^2 + 2y_{q_1}y_{q_4} + 2y_{q_2}y_{q_4} + 2y_{q_3}y_{q_4} + 2y_{q_4}^2\\ &+2y_{q_2}y_{q_5} + 2y_{q_3}y_{q_5} + y_{q_4}y_{q_5} + 2y_{q_2}y_{q_6} + 2y_{q_3}y_{q_6} + y_{q_4}y_{q_6} + 2x_{p_8} + 2y_{q_2}x_{q_1} + 2y_{q_3}x_{q_1}\\ &+y_{q_4}x_{q_1} + 2y_{q_1}x_{q_2} + 2y_{q_2}x_{q_2} + 2y_{q_3}x_{q_2} + y_{q_4}x_{q_2} + y_{q_5}x_{q_2} + y_{q_6}x_{q_2} + 2x_{q_1}x_{q_2} + x_{q_2}^2 + 2y_{q_1}x_{q_3}\\ &+2y_{q_2}x_{q_3} + 2y_{q_3}x_{q_3} + y_{q_4}x_{q_3} + y_{q_5}x_{q_3} + y_{q_6}x_{q_3} + 2x_{q_1}x_{q_3} + 2x_{q_2}x_{q_3} + x_{q_3}^2 + y_{q_1}x_{q_4} + y_{q_2}x_{q_4} + y_{q_3}x_{q_4}\\ &+2y_{q_4}x_{q_4} + 2y_{q_5}x_{q_4} + 2y_{q_6}x_{q_4} + x_{q_1}x_{q_4} + x_{q_2}x_{q_4} + x_{q_3}x_{q_4} + x_{q_4}^2 + y_{q_1}x_{q_5} + 2y_{q_2}x_{q_5} + 2y_{q_3}x_{q_5} + y_{q_4}x_{q_5}\\ &+2y_{q_5}x_{q_5} + 2y_{q_6}x_{q_5} + x_{q_1}x_{q_5} + 2x_{q_2}x_{q_5} + 2x_{q_3}x_{q_5} + x_{q_4}x_{q_5} + y_{q_2}x_{q_6} + y_{q_3}x_{q_6} + 2y_{q_4}x_{q_6} + x_{q_2}x_{q_6} + x_{q_3}x_{q_6}\\ &+2x_{q_4}x_{q_6} + 2x_{q_5}x_{q_6} + (y_{q_2} + y_{q_3} + 2y_{q_4} + x_{q_2} + x_{q_3} + 2x_{q_4} + 2x_{q_5})x_{q_8})\end{aligned} \tag{C11}$$
$$\begin{aligned}\Delta_{y_{q_9}} = (&y_{q_1}^2 + y_{q_1}y_{q_3} + 2y_{q_3}y_{q_4} + 2y_{q_4}^2 + 2y_{q_1}y_{q_6} + 2y_{q_3}y_{q_6} + 2y_{q_4}y_{q_6} + 2x_{p_9} + y_{q_1}x_{q_1} + 2y_{q_3}x_{q_1}\\ &+y_{q_6}x_{q_1} + 2x_{q_1}^2 + 2y_{q_1}x_{q_3} + y_{q_4}x_{q_3} + y_{q_6}x_{q_3} + 2x_{q_1}x_{q_3} + y_{q_3}x_{q_4} + 2y_{q_4}x_{q_4} + y_{q_6}x_{q_4} + x_{q_3}x_{q_4} + x_{q_4}^2\\ &+y_{q_1}x_{q_5} + 2y_{q_4}x_{q_5} + 2y_{q_6}x_{q_5} + x_{q_1}x_{q_5} + 2x_{q_4}x_{q_5} + y_{q_1}x_{q_6} + y_{q_3}x_{q_6} + y_{q_4}x_{q_6} + x_{q_1}x_{q_6} + x_{q_3}x_{q_6} + x_{q_4}x_{q_6}\\ &+2x_{q_5}x_{q_6} + (y_{q_1} + y_{q_3} + y_{q_4} + x_{q_1} + x_{q_3} + x_{q_4} + 2x_{q_5})x_{q_9})\end{aligned} \tag{C12}$$
$$\begin{aligned}\Delta_{y_{q_{10}}} = (&2y_{q_1}y_{q_3} + y_{q_3}^2 + 2y_{q_3}y_{q_5} + 2y_{q_3}y_{q_6} + 2x_{p_{10}} + y_{q_3}x_{q_1} + y_{q_1}x_{q_3} + y_{q_3}x_{q_3} + y_{q_5}x_{q_3} + y_{q_6}x_{q_3}\\ &+x_{q_1}x_{q_3} + 2x_{q_3}^2 + 2y_{q_1}x_{q_5} + 2y_{q_5}x_{q_5} + 2y_{q_6}x_{q_5} + 2x_{q_1}x_{q_5} + x_{q_5}^2 + x_{q_{10}}(y_{q_3} + x_{q_3} + 2x_{q_5}) + y_{q_3}x_{q_6}\\ &+x_{q_3}x_{q_6} + 2x_{q_5}x_{q_6})\end{aligned} \tag{C13}$$
$$\begin{aligned}\Delta_{y_{q_{11}}} = (&y_{q_1}y_{q_3} + y_{q_1}y_{q_5} + 2y_{q_3}y_{q_5} + 2y_{q_5}^2 + y_{q_1}y_{q_6} + 2y_{q_5}y_{q_6} + 2x_{p_{11}} + 2y_{q_3}x_{q_1} + 2y_{q_5}x_{q_1} + 2y_{q_6}x_{q_1}\\ &+2y_{q_1}x_{q_3} + y_{q_5}x_{q_3} + 2x_{q_1}x_{q_3} + y_{q_3}x_{q_5} + y_{q_5}x_{q_5} + y_{q_6}x_{q_5} + x_{q_3}x_{q_5} + 2y_{q_1}x_{q_6} + y_{q_5}x_{q_6} + 2x_{q_1}x_{q_6} + x_{q_5}x_{q_6}\\ &+x_{q_{11}}(y_{q_3} + y_{q_5} + y_{q_6} + x_{q_3} + x_{q_6}))\end{aligned} \tag{C14}$$

and

$$\begin{aligned}\Delta_{y_{q_{12}}} = (&2y_{q_1}y_{q_3} + 2y_{q_1}y_{q_5} + 2y_{q_3}y_{q_6} + 2y_{q_5}y_{q_6} + 2x_{p_{12}} + y_{q_3}x_{q_1} + y_{q_5}x_{q_1} + y_{q_1}x_{q_3} + y_{q_6}x_{q_3} + x_{q_1}x_{q_3}\\ &+x_{q_{12}}(y_{q_3} + y_{q_5} + x_{q_3}) + y_{q_3}x_{q_6} + y_{q_5}x_{q_6} + x_{q_3}x_{q_6}).\end{aligned} \tag{C15}$$

In Eq. C1, the intermediate variables y_{q_7}, ..., $y_{q_{12}}$ are quadratic while y_{q_1}, ..., y_{q_6} are cubic. y_{q_2} is the only intermediate coordinate that lies in the full 9-cycle and with respect to the intermediate variables it only has cross-terms with the cubic ones. Hence, y_{q_2} indexes the quadratic sums over the intermediate quadratic variables in terms of 3-cocycles $\{0,3,6\}$, $\{1,4,7\}$, and $\{2,5,8\}$. The three cubic variables each take three non-periodic values which leads to 3^6 quadratic Gauss sums (that are 3^6-dimensional). However, we can reduce this number by noticing some properties.

Given y_{q_1}, y_{q_2}, y_{q_3} and y_{q_4}, if the quadratic coefficients of the quadratic variables is zero, then the number of quadratic Gauss sums reduces from 3^2 (indexed by y_{q_5} and y_{q_6}) to at most 3. This reduces the sum to 3^5 quadratic Gauss sums.

Otherwise, if the quadratic coefficients of the quadratic variables are not zero, then the Wigner function exhibits another property. The coefficient of the $y_{q_6}^2$ term is only dependent on y_{q_2} of the cubic indexing intermediate variables. The coefficients of the y_{q_6} term is dependent on the $y_{q_7}^2$ and $y_{q_{11}}^2$ quadratic variables. Given a fixed y_{q_1}, y_{q_2}, y_{q_3}, and y_{q_4} that does not set any of the quadratic coefficients of the quadratic variables to zero, it follows that for two values of y_{q_6} (and say $y_{q_5} = 0$), the quadratic coefficients of $y_{q_7}^2$ and $y_{q_{11}}^2$ are switched. Since these are separable quadratic coefficients, it follows that the Gauss sum at these two values of y_{q_6} has the same magnitude for all y_{q_5}.

The Wigner function is real so the imaginary parts of these quadratic Gauss sums of equal magnitude must cancel out. Their complex conjugates lie across y_{q_1}, y_{q_2}, y_{q_3} or y_{q_4}, but not y_{q_6}. There are only three options for the phase for every y_{q_2} (corresponding to the three co-cycles) for the pairs of equal magnitude quadratic Gauss sums indexed by y_{q_6}. For $y_{q_2} = 0$, one of the options is for no imaginary part and so it follows that the quadratic Gauss sums in that sector that have equal magnitude must be real, or have the same imaginary part. For the two other sectors ($y_{q_2} \neq 0$), the same behavior holds since the only term that displaces a phase from being real is the sole $y_{q_2}^3$ term, which is independent of y_{q_5} and y_{q_6}.

This latter case produces $3^4 \times 6 = 486$ quadratic Gauss sums, which is the worst case.

This can be summarized by the following equation:

$$\begin{aligned}
&\rho(\boldsymbol{\mathcal{M}}_{C_{12}}(x_{p_1}, \ldots, x_{p_{12}}, x_{q_1}, \ldots, x_{q_{12}})) \\
&= \sum_{\substack{y_{q_1}, \ldots, y_{q_6} \\ \in \mathbb{Z}/3^2\mathbb{Z}}} \exp\left[\frac{2\pi i}{9}\left(7y_{q_2}^3 + 8x_{q_2}^3\right)\right] \exp\left[\frac{2\pi i}{3}\Gamma_{12}(y_{q_1}, \ldots, y_{q_6}, \boldsymbol{x}_p, \boldsymbol{x}_q)\right] \mathcal{A}_{12}(\boldsymbol{y}_q, \boldsymbol{x}_p, \boldsymbol{x}_q) \\
&\quad \times \Bigg\{ \delta(\vee_{i=7}^{12}(\Sigma_{y_{q_i}} \wedge \Delta_{y_{q_i}})) + \delta(\neg(\vee_{i=7}^{12}(\Sigma_{y_{q_i}} \wedge \Delta_{y_{q_i}}))) \Big[\delta(\Sigma_{y_{q_7}}(0) - \Sigma_{y_{q_{11}}}(1)) \big[2\delta(y_{q_6}) + \delta(y_{q_6} - 2)\big] \\
&\qquad + \delta(\Sigma_{y_{q_7}}(1) - \Sigma_{y_{q_{11}}}(2)) \big[2\delta(y_{q_6} - 1) + \delta(y_{q_6})\big] + \delta(\Sigma_{y_{q_7}}(0) - \Sigma_{y_{q_{11}}}(2)) \big[2\delta(y_{q_6}) + \delta(y_{q_6} - 1)\big] \Big] \Bigg\}
\end{aligned} \tag{C16}$$

The first term includes all cases where the quadratic coefficients of $y_{q_7}, \ldots, y_{q_{12}}$ are zero along with their respective linear coefficients, as these produce plane waves that do not evaluate to zero. The second term includes all cases where all these quadratic coefficients are non-zero and y_{q_5} indexes the same quadratic Gauss sums for two values of y_{q_6} for some values of the the cubic variables.

The same simplification can be made for the final phase space variables $(\boldsymbol{x}_p, \boldsymbol{x}_q)$. In the worst-case, this produces $\xi_{12} = 3^4 \times 6 = 486$ quadratic Gauss sums, which produces a tensor upper bound of $\xi_{12}^{t/12} = 3^{\sim 0.469t}$.

Numerical examination of this Wigner function seems to indicate that this number can be lowered even further.