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Quantum state exchange is a quantum communication task in which two users exchange their respective quantum information in the asymptotic setting. In this work, we consider a one-shot version of the quantum state exchange task, in which the users hold a single copy of the initial state, and they exchange their parts of the initial state by means of entanglement-assisted local operations and classical communication. We first derive lower bounds on the least amount of entanglement required for carrying out this task, and provide conditions on the initial state such that the protocol succeeds with zero entanglement cost. Based on these results, we study how the users deal with their symmetric information in order to reduce the entanglement cost. Moreover, we show that it is possible for the users to gain extra shared entanglement after this task.

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I. INTRODUCTION

In quantum information theory, quantum state exchange [1, 2] is a quantum communication task in which two users, Alice and Bob, exchange their quantum information by means of local operations and classical communication (LOCC) assisted by shared entanglement. A main research aim in the study of the quantum state exchange is to evaluate the least amount of entanglement needed for the task, as in other quantum communication tasks, such as quantum state merging [3, 4] and quantum state redistribution [5, 6].

Most quantum communication tasks [3–8] including the quantum state exchange usually assume the *asymptotic scenario*, in which users can have an unbounded number of independent and identically distributed copies of an initial state, and they carry out their task with the copies. On the other hand, it is not easy in a realistic situation to prepare a sufficiently large number of state copies, and the amount of non-local resources available for the users is limited. To reflect these practical difficulties, quantum information research has focused more recently on the *one-shot scenario* [9–17].

Another reason for considering the one-shot scenario is that one-shot results can be applied to the asymptotic scenario. For example, in the original quantum state merging [3, 4], the authors devised a one-shot merging protocol in order to evaluate the minimum amount of entanglement needed for asymptotic merging. Since the optimal entanglement costs for the asymptotic state exchange tasks are unknown [1, 2], analysis of the one-shot scenario can be a good turning point in evaluating the entanglement cost.

In this work, we introduce and study the *one-shot quantum state exchange* (OSQSE) task. This is not only a useful quantum communication task, but can also have a potential application in quantum computation. Let us consider a specific situation as follows. Alice and Bob want to carry out the SWAP gate [18], which plays an important role in universal quantum computation [19]. The problem is that they cannot directly apply the SWAP gate, because they are far apart. If Alice and Bob are sharing prior entanglement, then the OSQSE can be a method to non-locally perform the SWAP gate, as both operationally provide the same result. Thus the OSQSE task can be useful for quantum computation.

This paper is organized as follows. In Sec. II, we formally define three different OSQSE protocols and their optimal entanglement costs. In Sec. III, we derive computable lower bounds on the latter, which in turn yield bounds for the asymptotic quantum state exchange [1, 2]. In addition, we provide two useful conditions to decide whether a given initial state enables OSQSE with zero entanglement cost in Sec. IV. In Sec. V, we present two examples which lead to properties of the OSQSE. In Sec. VI, we investigate under what conditions the optimal entanglement cost cannot be negative. We summarize our results and comment on some open problems in Sec. VII.

II. ONE-SHOT QUANTUM STATE EXCHANGE

Consider two users, Alice and Bob, holding parts A and B of the initial state $|\psi\rangle \equiv |\psi\rangle_{A_1B_1A_2B_2R}$ of systems $A = A_1A_2$ and $B = B_1B_2$, respectively, and R indicates the reference system on which neither Alice nor Bob can perform any operation. Their goal is either to exchange their parts A_1 and B_1 or to exchange their whole parts A and B.

Specifically, let ψ_{f_1} and $\psi_{f_{12}}$ be the final states of the task,

$$\begin{aligned}\psi_{f_1} &= (\mathbb{1}_{A_1\to A_1'} \otimes \mathbb{1}_{B_1\to B_1'} \otimes \mathbb{1}_{A_2B_2R})(\psi),\\ \psi_{f_{12}} &= (\mathbb{1}_{A\to A'} \otimes \mathbb{1}_{B\to B'} \otimes \mathbb{1}_{R})(\psi),\end{aligned} \quad (1)$$

where $\psi = |\psi\rangle\langle\psi|$, and B_1' and B' (A_1' and A') are Alice's

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(Bob's) systems whose dimensions are identical to those of systems B_1 and B (A_1 and A), respectively. Then three joint operations

$$\begin{aligned} \mathcal{E}^{1}_{\psi,K,L} &: A_1 E^{\text{in}}_{\text{A}} \otimes B_1 E^{\text{in}}_{\text{B}} \longrightarrow B'_1 E^{\text{out}}_{\text{A}} \otimes A'_1 E^{\text{out}}_{\text{B}}, \\ \mathcal{E}^{1|2}_{\psi,K,L} &: A E^{\text{in}}_{\text{A}} \otimes B E^{\text{in}}_{\text{B}} \longrightarrow B'_1 A_2 E^{\text{out}}_{\text{A}} \otimes A'_1 B_2 E^{\text{out}}_{\text{B}}, \\ \mathcal{E}^{12}_{\psi,K,L} &: A E^{\text{in}}_{\text{A}} \otimes B E^{\text{in}}_{\text{B}} \longrightarrow B' E^{\text{out}}_{\text{A}} \otimes A' E^{\text{out}}_{\text{B}}, \end{aligned} \tag{2}$$

are called the OSQSE protocols of $|\psi\rangle$, if they are performed by LOCC between Alice and Bob, and satisfy

$$\begin{aligned} \psi_{f_1} \otimes \Phi &= \left(\mathcal{E}^{1}_{\psi,K,L} \otimes \mathbb{1}_{A_2 B_2 R}\right)(\psi \otimes \Psi) \\ &= \left(\mathcal{E}^{1|2}_{\psi,K,L} \otimes \mathbb{1}_{R}\right)(\psi \otimes \Psi), \\ \psi_{f_{12}} \otimes \Phi &= \left(\mathcal{E}^{12}_{\psi,K,L} \otimes \mathbb{1}_{R}\right)(\psi \otimes \Psi), \end{aligned} \tag{3}$$

where Ψ and Φ are pure maximally entangled states with Schmidt rank K and L on systems $E^{\text{in}}_{\text{A}} E^{\text{in}}_{\text{B}}$ and $E^{\text{out}}_{\text{A}} E^{\text{out}}_{\text{B}}$, respectively. It is possible to generalize the above definitions by adding errors for approximation to Eq. (3), but it suffices to only consider error-free protocols to obtain our main results.

At this point, it is instructive to inform differences among the three protocols in Eq. (2) as follows: The first two protocols $\mathcal{E}^{1}_{\psi,K,L}$ and $\mathcal{E}^{1|2}_{\psi,K,L}$ indicate that only the parts A_1 and B_1 are exchanged, while the whole parts $A_1 A_2$ and $B_1 B_2$ are exchanged in the third protocol $\mathcal{E}^{12}_{\psi,K,L}$. In addition, the parts A_2 and B_2 can be used for exchanging A_1 and B_1 in the protocol $\mathcal{E}^{1|2}_{\psi,K,L}$, while A_2 and B_2 are untouched in the protocol $\mathcal{E}^{1}_{\psi,K,L}$. These protocols are described in Fig. 1.

Depending on the types of OSQSE protocols, we define three *optimal entanglement costs*

$$\begin{aligned} \mathbf{e}_{A_1 \leftrightarrow B_1}(\psi) &= \inf_{\mathcal{E}^{1}_{\psi,K,L}} (\log K - \log L), \\ \mathbf{e}^{A_2 B_2}_{A_1 \leftrightarrow B_1}(\psi) &= \inf_{\mathcal{E}^{1|2}_{\psi,K,L}} (\log K - \log L), \\ \mathbf{e}_{A \leftrightarrow B}(\psi) &= \inf_{\mathcal{E}^{12}_{\psi,K,L}} (\log K - \log L), \end{aligned} \tag{4}$$

where the quantity $\log K - \log L$ is called the *entanglement cost* of the OSQSE protocol, and the infimums are taken over all joint protocols $\mathcal{E}^{1}_{\psi,K,L}$, $\mathcal{E}^{1|2}_{\psi,K,L}$, and $\mathcal{E}^{12}_{\psi,K,L}$, respectively.

By the definitions of the optimal entanglement costs, we obtain the following proposition.

Proposition 1. *For any input state ψ, $\mathbf{e}_{A_1 \leftrightarrow B_1}(\psi) \geq \mathbf{e}^{A_2 B_2}_{A_1 \leftrightarrow B_1}(\psi)$.*

III. CONVERSE BOUNDS

A real number r is called a *converse bound* of the optimal entanglement cost if it is upper bounded by the entanglement cost of any OSQSE protocol. In this section, we firstly derive theoretical converse bounds of the three optimal entanglement costs and also provide computable converse bounds of them.

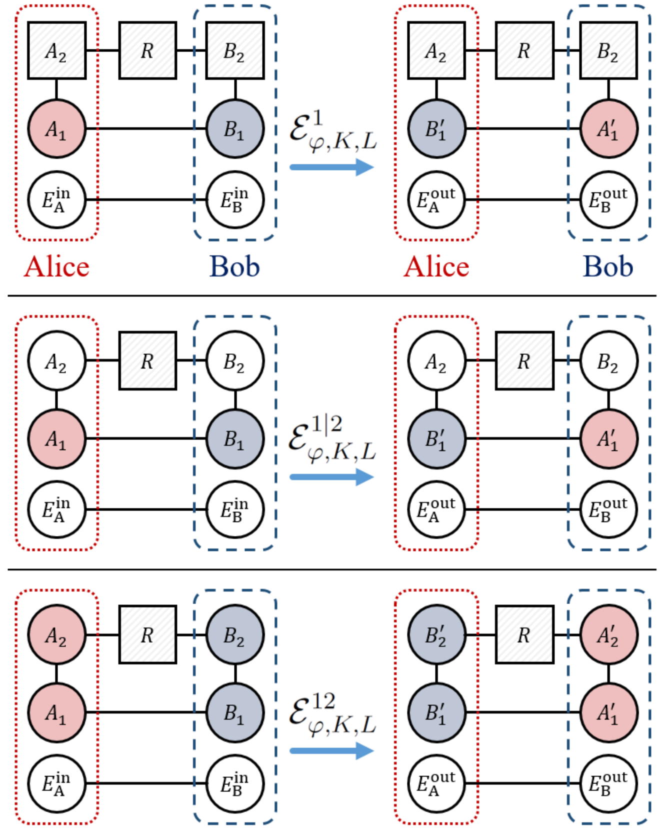

FIG. 1: Illustrations for three one-shot quantum state exchange protocols $\mathcal{E}^{1}_{\psi,K,L}$, $\mathcal{E}^{1|2}_{\psi,K,L}$, and $\mathcal{E}^{12}_{\psi,K,L}$: In each illustration, Alice and Bob can apply local operations to their parts represented by circles, while they cannot apply any local operations to those depicted by squares. Shaded circles indicate the systems which are exchanged from the OSQSE protocols.

Theorem 2. *Let F be an additive and Schur concave function such that $F(\sigma^M) = \log M$ for any M, where σ^M is the maximally mixed state with rank M. Let $\mathcal{N}$ be a quantum channel from R to R_{A}. Then for any initial state ψ,*

$$\mathbf{e}_{A_1 \leftrightarrow B_1}(\psi) \geq \max\{l_1(\psi), l_2(\psi)\}, \tag{5}$$

$$\mathbf{e}^{A_2 B_2}_{A_1 \leftrightarrow B_1}(\psi) \geq l_3(\psi), \tag{6}$$

$$\mathbf{e}_{A \leftrightarrow B}(\psi) \geq l_4(\psi), \tag{7}$$

where $l_i(\psi)$ are defined as

$$l_1(\psi) = \sup_{F,\mathcal{N}} \left| F(\mathcal{N}(\psi)_{B_1 R_{\text{A}}}) - F(\mathcal{N}(\psi)_{A_1 R_{\text{A}}}) \right|, \tag{8}$$

$$l_2(\psi) = \sup_{F,\mathcal{N}} \left| F(\mathcal{N}(\psi)_{B_1 A_2 R_{\text{A}}}) - F(\mathcal{N}(\psi)_{A R_{\text{A}}}) \right|, \tag{9}$$

$$l_3(\psi) = \sup_{F,\mathcal{N}} \left[F(\mathcal{N}(\psi)_{B_1 A_2 R_{\text{A}}}) - F(\mathcal{N}(\psi)_{A R_{\text{A}}}) \right], \tag{10}$$

$$l_4(\psi) = \sup_{F,\mathcal{N}} \left| F(\mathcal{N}(\psi)_{B R_{\text{A}}}) - F(\mathcal{N}(\psi)_{A R_{\text{A}}}) \right|. \tag{11}$$

Proof. As in the asymptotic scenario [1, 2], we consider a one-shot version of the R-assisted quantum state exchange task, in which the reference system R is divided into two systems R_{A} and R_{B}, and then Alice and Bob receive the divided parts R_{A}

and $R_{\rm B}$, respectively, so that the initial state $|\tilde{\psi}\rangle_{A_1B_1A_2B_2R_{\rm A}R_{\rm B}}$ is divided into Alice's parts $AR_{\rm A}$ and Bob's parts $BR_{\rm B}$. This can be realized by using a quantum channel $\mathcal{N}: R \longrightarrow R_{\rm A}$ and its complementary channel $\mathcal{N}^c: R \longrightarrow R_{\rm B}$ [20]. Let $\mathcal{R}^1_{\tilde{\psi},K,L}$, $\mathcal{R}^{1|2}_{\tilde{\psi},K,L}$, and $\mathcal{R}^{12}_{\tilde{\psi},K,L}$ be R-assisted OSQSE protocols of $\tilde{\psi}$,

$$\begin{aligned}\mathcal{R}^1_{\tilde{\psi},K,L} &: A_1E^{\rm in}_{\rm A}\otimes B_1E^{\rm in}_{\rm B} \longrightarrow B'_1E^{\rm out}_{\rm A}\otimes A'_1E^{\rm out}_{\rm B},\\ \mathcal{R}^{1|2}_{\tilde{\psi},K,L} &: AE^{\rm in}_{\rm A}\otimes BE^{\rm in}_{\rm B} \longrightarrow B'_1A_2E^{\rm out}_{\rm A}\otimes A'_1B_2E^{\rm out}_{\rm B},\\ \mathcal{R}^{12}_{\tilde{\psi},K,L} &: AE^{\rm in}_{\rm A}\otimes BE^{\rm in}_{\rm B} \longrightarrow B'E^{\rm out}_{\rm A}\otimes A'E^{\rm out}_{\rm B},\end{aligned} \tag{12}$$

with the entanglement cost $\log K - \log L$ such that

$$\begin{aligned}\tilde{\psi}_{f_1}\otimes\Phi &= \left(\mathcal{R}^1_{\tilde{\psi},K,L}\otimes\mathbb{1}_{A_2B_2R_{\rm A}R_{\rm B}}\right)\left(\tilde{\psi}\otimes\Psi\right)\\ &= \left(\mathcal{R}^{1|2}_{\tilde{\psi},K,L}\otimes\mathbb{1}_{R_{\rm A}R_{\rm B}}\right)\left(\tilde{\psi}\otimes\Psi\right),\\ \tilde{\psi}_{f_{12}}\otimes\Phi &= \left(\mathcal{R}^{12}_{\tilde{\psi},K,L}\otimes\mathbb{1}_{R_{\rm A}R_{\rm B}}\right)\left(\tilde{\psi}\otimes\Psi\right),\end{aligned} \tag{13}$$

where

$$\begin{aligned}\tilde{\psi}_{f_1} &= (\mathbb{1}_{A_1\to A'_1}\otimes\mathbb{1}_{B_1\to B'_1}\otimes\mathbb{1}_{A_2B_2R_{\rm A}R_{\rm B}})(\tilde{\psi}),\\ \tilde{\psi}_{f_{12}} &= (\mathbb{1}_{A\to A'}\otimes\mathbb{1}_{B\to B'}\otimes\mathbb{1}_{R_{\rm A}R_{\rm B}})(\tilde{\psi}),\end{aligned} \tag{14}$$

and B'_1, B', A'_1, and A' are defined as in Eq. (1).

We first derive a converse bound of the optimal entanglement cost $\mathbf{e}_{A_1\leftrightarrow B_1}(\psi)$ as follows.

Note that the protocol $\mathcal{R}^1_{\tilde{\psi},K,L}$ is an LOCC protocol between Alice's part $AR_{\rm A}E^{\rm in}_{\rm A}$ and Bob's part $BR_{\rm B}E^{\rm in}_{\rm B}$. So, from the majorization condition for LOCC convertibility [21, 22], the state $\tilde{\rho}_{B'_1A_2R_{\rm A}}\otimes\sigma^L_{E^{\rm out}_{\rm A}}$ majorizes the state $\tilde{\rho}_{AR_{\rm A}}\otimes\sigma^K_{E^{\rm in}_{\rm A}}$, which can be more succinctly represented by using the notation $\prec$ as follows:

$$\tilde{\rho}_{AR_{\rm A}}\otimes\sigma^K_{E^{\rm in}_{\rm A}} \prec \tilde{\rho}_{B'_1A_2R_{\rm A}}\otimes\sigma^L_{E^{\rm out}_{\rm A}}. \tag{15}$$

Then, from the Schur concavity of the function F, the following inequality holds:

$$F(\tilde{\rho}_{B'_1A_2R_{\rm A}}\otimes\sigma^L_{E^{\rm out}_{\rm A}}) \le F(\tilde{\rho}_{AR_{\rm A}}\otimes\sigma^K_{E^{\rm in}_{\rm A}}). \tag{16}$$

Since $\tilde{\rho}_{B'_1A_2R_{\rm A}} = \tilde{\rho}_{B_1A_2R_{\rm A}}$ and F is additive, it follows that

$$\begin{aligned}\log K - \log L &\ge F(\tilde{\rho}_{B_1A_2R_{\rm A}}) - F(\tilde{\rho}_{AR_{\rm A}})\\ &= F(\mathcal{N}(\psi)_{B_1A_2R_{\rm A}}) - F(\mathcal{N}(\psi)_{AR_{\rm A}}).\end{aligned} \tag{17}$$

Let us now consider an R-assisted OSQSE protocol $\mathcal{R}'^1_{\tilde{\psi}_{f_1},K,L}$ exchanging B'_1 and A'_1 of the final state $\tilde{\psi}_{f_1}$, which is defined by exchanging Alice's role and Bob's role in the protocol $\mathcal{R}^1_{\tilde{\psi},K,L}$. That is, $\mathcal{R}'^1_{\tilde{\psi}_{f_1},K,L}$ is an LOCC protocol

$$\mathcal{R}'^1_{\tilde{\psi}_{f_1},K,L}: B'_1E^{\rm in}_{\rm A}\otimes A'_1E^{\rm in}_{\rm B} \longrightarrow A''_1E^{\rm out}_{\rm A}\otimes B''_1E^{\rm out}_{\rm B} \tag{18}$$

of the state $\tilde{\psi}_{f_1}$ satisfying

$$\left(\mathcal{R}'^1_{\tilde{\psi}_{f_1},K,L}\otimes\mathbb{1}_{A_2B_2R_{\rm A}R_{\rm B}}\right)\left(\tilde{\psi}_{f_1}\otimes\Psi\right) = \tilde{\psi}'_{f_1}\otimes\Phi, \tag{19}$$

where $\tilde{\psi}'_{f_1} = (\mathbb{1}_{A'_1\to A''_1}\otimes\mathbb{1}_{B'_1\to B''_1}\otimes\mathbb{1}_{A_2B_2R_{\rm A}R_{\rm B}})(\tilde{\psi}_{f_{12}})$ and A''_1 (B''_1) is Alice's (Bob's) system whose dimension equals A'_1 (B'_1). Then, by using the majorization condition for LOCC convertibility [21, 22] again, we have $\tilde{\rho}_{B'_1A_2R_{\rm A}}\otimes\sigma^K_{E^{\rm in}_{\rm A}} \prec \tilde{\rho}_{A''_1A_2R_{\rm A}}\otimes\sigma^L_{E^{\rm out}_{\rm A}}$, which implies that

$$\log K - \log L \ge F(\mathcal{N}(\psi)_{AR_{\rm A}}) - F(\mathcal{N}(\psi)_{B_1A_2R_{\rm A}}), \tag{20}$$

since $\tilde{\rho}_{B'_1A_2R_{\rm A}} = \tilde{\rho}_{B_1A_2R_{\rm A}}$, $\tilde{\rho}_{A''_1A_2R_{\rm A}} = \tilde{\rho}_{A_1A_2R_{\rm A}}$ and F is Schur concave and additive. Thus Eqs. (17) and (20) imply

$$\log K - \log L \ \ge\ \left|F(\mathcal{N}(\psi)_{B_1A_2R_{\rm A}}) - F(\mathcal{N}(\psi)_{AR_{\rm A}})\right|. \tag{21}$$

On the other hand, let us consider a situation that Alice and Bob want to exchange A_1 and B_1, by means of LOCC assisted by shared entanglement, when Alice and Bob hold $A_1R_{\rm A}$ and $B_1A_2B_2R_{\rm B}$ of $\tilde{\psi}$, respectively. In this case, we can apply the same technique used in obtaining Eq. (21) to Alice's part $A_1R_{\rm A}$ and Bob's part $B_1A_2B_2R_{\rm B}$ of $\tilde{\psi}$, and hence we have that

$$\log K - \log L \ge \left|F(\mathcal{N}(\psi)_{B_1R_{\rm A}}) - F(\mathcal{N}(\psi)_{A_1R_{\rm A}})\right|. \tag{22}$$

Since any protocol $\mathcal{E}^1_{\psi,K,L}$ is also an R-assisted OSQSE protocol $\mathcal{R}^1_{\tilde{\psi},K,L}$, the optimal entanglement cost $\mathbf{e}_{A_1\leftrightarrow B_1}(\psi)$ is lower bounded by $l_1(\psi)$ and $l_2(\psi)$, from Eqs. (21) and (22).

Similarly, we obtain that $l_3(\psi)$ and $l_4(\psi)$ are converse bounds of the optimal entanglement costs $\mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}$ and $\mathbf{e}_{A\leftrightarrow B}$, respectively, by applying the above technique to the protocols $\mathcal{R}^{1|2}_{\tilde{\psi},K,L}$ and $\mathcal{R}^{12}_{\tilde{\psi},K,L}$. □

In Theorem 2, if R is directly sent to either Alice or Bob without splitting, and we restrict the function F to the quantum Rényi entropy $S_\alpha(\varrho)$ of order α [23] for a quantum state ϱ, then we obtain the following computable converse bounds.

Corollary 3. *For any input state* ψ,

$$\begin{aligned}\mathbf{e}_{A_1\leftrightarrow B_1}(\psi) &\ge \max_{\alpha\in[0,\infty]}\max\{f^{(1)}_\psi(\alpha), |f^{(2)}_\psi(\alpha)|, |f^{(3)}_\psi(\alpha)|\}, &(23)\\ \mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi) &\ge l^{\rm c}_{\rm new}(\psi) \equiv \max_{\alpha\in[0,\infty]}\max\{f^{(2)}_\psi(\alpha), f^{(3)}_\psi(\alpha)\}, &(24)\\ \mathbf{e}_{A\leftrightarrow B}(\psi) &\ge \max_{\alpha\in[0,\infty]} f^{(4)}_\psi(\alpha), &(25)\end{aligned}$$

where $f^{(i)}_\psi(\alpha)$ *are functions of* $|\psi\rangle$ *and* α *defined by*

$$\begin{aligned}f^{(1)}_\psi(\alpha) &= \max\{|S_\alpha(\rho_{A_1}) - S_\alpha(\rho_{B_1})|, |S_\alpha(\rho_{AB_2}) - S_\alpha(\rho_{A_2B})|\},\\ f^{(2)}_\psi(\alpha) &= S_\alpha(\rho_{A_1B_2}) - S_\alpha(\rho_B),\\ f^{(3)}_\psi(\alpha) &= S_\alpha(\rho_{B_1A_2}) - S_\alpha(\rho_A),\\ f^{(4)}_\psi(\alpha) &= |S_\alpha(\rho_A) - S_\alpha(\rho_B)|.\end{aligned} \tag{26}$$

Remark that the converse bounds in Corollary 3 can be easily computed by means of analytical or numerical methods, since the functions $f^{(i)}_\psi(\alpha)$ are one-variable and differentiable on $(0,\infty)$. In addition, we can know that if $l^{\rm c}_{\rm new}(\psi) < 0$, then $\mathbf{e}_{A_1\leftrightarrow B_1}(\psi) > 0$, by observing the bounds in Corollary 3.

Proof of Corollary 3. It suffices to show that, for each i, there exists a number $\alpha_0^{(i)} \in [0,\infty]$ such that $\max_{\alpha\in[0,\infty]} f_\psi^{(i)}(\alpha) = f_\psi^{(i)}(\alpha_0^{(i)})$. Note that the function $f_\psi^{(i)}(\alpha)$ is continuous on the compact set $[0,1]$. So the extreme value theorem implies that there exists a number $\alpha_1^{(i)} \in [0,1]$ such that $f_\psi^{(i)}(\alpha_1^{(i)}) \geq f_\psi^{(i)}(\alpha)$ for all $\alpha \in [0,1]$. Let us consider the function $g^{(i)}(x)$ on the interval $[0,1]$ defined as

$$g^{(i)}(x) = \begin{cases} f_\psi^{(i)}(\infty) & \text{if } x = 0 \\ f_\psi^{(i)}(\frac{1}{x}) & \text{otherwise,} \end{cases} \tag{27}$$

then $g^{(i)}(x)$ is continuous on $[0,1]$. By using the extreme value theorem again, there exists a number $x_0^{(i)} \in [0,1]$ such that $g^{(i)}(x_0^{(i)}) \geq g^{(i)}(x)$ for all $x \in [0,1]$. It follows that there exists a number $\alpha_2^{(i)} \in [1,\infty]$ such that $f_\psi^{(i)}(\alpha_2^{(i)}) \geq f_\psi^{(i)}(\alpha)$ for all $\alpha \in [1,\infty]$. By setting $\alpha_0^{(i)} = \max\left\{\alpha_1^{(i)}, \alpha_2^{(i)}\right\}$, we obtain that

$$\max_{\alpha\in[0,\infty]} f_\psi^{(i)}(\alpha) = f_\psi^{(i)}(\alpha_0^{(i)}) \geq f_\psi^{(i)}(\alpha), \tag{28}$$

for all $\alpha \in [0,\infty]$. Similarly, we know that, for each i, there exists $\beta^{(i)} \in [0,\infty]$ such that $\max_{\alpha\in[0,\infty]} |f_\psi^{(i)}(\alpha)| = |f_\psi^{(i)}(\beta^{(i)})|$. □

We also remark that in Theorem 2, if F is chosen as the von Neumann entropy [20], then the converse bound l_3 recovers a theoretical converse bound in Ref. [2]. In addition, a computable converse bound therein is just $l_{\rm old}^{\rm c}(\psi) = \max\{f_\psi^{(2)}(1), f_\psi^{(3)}(1)\}$ in Corollary 3. By virtue of the additivity of F, it is clear that l_3 and $l_{\rm new}^{\rm c}$ are also converse bounds of the optimal entanglement cost for the asymptotic quantum state exchange task. Hence, our converse bounds improve the existing bounds in Ref. [2]. For example, if the initial state $|\psi_1\rangle \equiv |\psi_1\rangle_{A_1B_1A_2B_2R}$ has the specific form

$$|\psi_1\rangle = \tfrac{1}{5}|00000\rangle + \sqrt{\tfrac{3}{50}}|00010\rangle + \tfrac{3}{5}|01001\rangle + \sqrt{\tfrac{27}{50}}|11100\rangle, \tag{29}$$

then we can find a value $\alpha_0 \in [0,\infty]$ such that

$$l_{\rm new}^{\rm c}(\psi_1) = \max\left\{f_{\psi_1}^{(2)}(\alpha_0), f_{\psi_1}^{(3)}(\alpha_0)\right\} > l_{\rm old}^{\rm c}(\psi_1) \tag{30}$$

as depicted in Fig. 2. This example shows that our bound $l_{\rm new}^{\rm c}(\psi)$ is tighter than the existing bound $l_{\rm old}^{\rm c}(\psi)$.

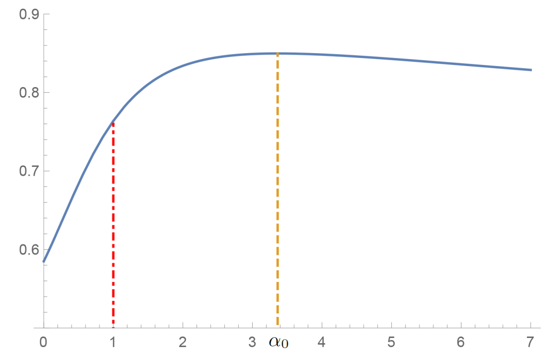

FIG. 2: The graph of the function $\max\{f_{\psi_1}^{(2)}(\alpha), f_{\psi_1}^{(3)}(\alpha)\}$ for a specific initial state $|\psi_1\rangle$ in Eq. (29). The maximum of the function is attained at the point α_0 ($\approx$ 3.362), leading to an improved converse bound compared to that in Ref. [2]. In the graph, α_0 is represented as the yellow dashed line, and 1 is represented as the red dashed dotted line.

IV. CONDITIONS FOR ZERO ENTANGLEMENT COST

We now present conditions for OSQSE at zero entanglement cost.

By the converse bounds in Corollary 3, it is obvious that if there exist Alice's and Bob's local isometries performing the OSQSE task, then the optimal entanglement cost is zero. We first characterize this type of strategy. Let (X,Y) be a pair of two systems, which can be either (A_1,B_1) or (A,B), and consider a spectral decomposition of the reduced state ρ_{XY} for $|\psi\rangle$, $\rho_{XY} = \sum_{i=1}^N \lambda_i |\xi_i\rangle\langle\xi_i|_{XY}$, where $\lambda_i > 0$ with $\sum_{i=1}^N \lambda_i = 1$. For each i, we define the matrix $\Omega_{XY}^{(i)}(\psi)$ as

$$\Omega_{XY}^{(i)}(\psi) = \sum_{j,k} \left(\langle j|_X \otimes \langle k|_Y\right)|\xi_i\rangle_{XY} |j\rangle\langle k|, \tag{31}$$

where $\{|j\rangle\}$ and $\{|k\rangle\}$ indicate the computational bases on Alice's and Bob's systems, respectively. Then we obtain the following sufficient condition.

Theorem 4. *Let (X,Y) be either (A_1,B_1) or (A,B). If there exist isometries U and V such that, for each i,*

$$\left(\Omega_{XY}^{(i)}(\psi)\right)^t = U\Omega_{XY}^{(i)}(\psi)V, \tag{32}$$

where W^t is the transpose of the matrix W, then $\mathbf{e}_{X\leftrightarrow Y}(\psi) = 0$.

Here, the isometries U and V indicate Alice's and Bob's local operations exchanging the parts X and Y without shared entanglement.

Proof of Theorem 4. For $X = A$ and $Y = B$, we consider the Schmidt decomposition, $|\psi\rangle_{ABR} = \sum_{i=1}^N \sqrt{\lambda_i}|\xi_i\rangle_{AB} \otimes |\iota_i\rangle_R$, where $\lambda_i > 0$ with $\sum_{i=1}^N \lambda_i = 1$. For the computational bases $\{|j\rangle\}$ and $\{|k\rangle\}$ on the systems A and B, respectively, we have

$$|\psi\rangle_{ABR} = \sum_{i=1}^N \sqrt{\lambda_i} \sum_{j,k} [\Omega_{AB}^{(i)}(\psi)]_{jk} |j\rangle_A \otimes |k\rangle_B \otimes |\iota_i\rangle_R\,, \tag{33}$$

where $[\Omega_{AB}^{(i)}(\psi)]_{jk} = (\langle j|_A \otimes \langle k|_B)|\xi_i\rangle_{AB}$. If the parts A and B are perfectly exchanged, then Alice and Bob hold the final state

$$|\psi\rangle_{BAR} = \sum_{i=1}^N \sqrt{\lambda_i} \sum_{j,k} [\Omega_{AB}^{(i)}(\psi)]_{kj} |j\rangle_B \otimes |k\rangle_A \otimes |\iota_i\rangle_R\,. \tag{34}$$

Assume that there exist isometries U and V such that

$$\left(\Omega_{AB}^{(i)}(\psi)\right)^t = U\Omega_{AB}^{(i)}(\psi)V \tag{35}$$

for each i. Then we have, for each i,

$$[\Omega_{AB}^{(i)}(\psi)]_{kj} = \sum_{l,m}[\Omega_{AB}^{(i)}(\psi)]_{lm} \langle j| U |l\rangle \langle k| V^t |m\rangle, \quad (36)$$

which implies that

$$\begin{aligned}|\psi\rangle_{BAR} &= \sum_{i=1}^{N} \sqrt{\lambda_i} \sum_{l,m}[\Omega_{AB}^{(i)}(\psi)]_{lm} \sum_j |j\rangle \langle j| U |l\rangle \\ &\quad \otimes \sum_k |k\rangle \langle k| V^t |m\rangle \otimes |\iota_i\rangle_R \\ &= \sum_{i=1}^{N} \sqrt{\lambda_i} \sum_{l,m}[\Omega_{AB}^{(i)}(\psi)]_{lm} U |l\rangle \otimes V^t |m\rangle \otimes |\iota_i\rangle_R \\ &= \left(U \otimes V^t \otimes I_R\right) |\psi\rangle_{ABR}. \quad (37)\end{aligned}$$

Hence, $\mathbf{e}_{A\leftrightarrow B}(\psi) = 0$. Similarly, for $X = A_1$ and $Y = B_1$, we show that $\mathbf{e}_{A_1\leftrightarrow B_1}(\psi) = 0$ by using isometries U' and V' such that for each i, $\left(\Omega_{A_1B_1}^{(i)}(\psi)\right)^t = U'\Omega_{A_1B_1}^{(i)}(\psi)V'$. □

From the converse bounds in Corollary 3, observe that if the spectrum of Alice's state is different from that of Bob's state, then the optimal entanglement cost cannot be zero. Based on this observation, we obtain the following theorem, whose proof can be found in Appendix A.

Theorem 5. *Let (X, Y) be either (A_1, B_1) or (A, B). If $\mathbf{e}_{X\leftrightarrow Y}(\psi) = 0$, then there exists an isometry $U_{X\to Y}$ such that $\rho_Y = U_{X\to Y}\rho_X(U_{X\to Y})^\dagger$.*

We remark that the converse of Theorem 5 is not true in general. Let us consider the following simple initial state

$$|\psi_2\rangle_{A_1B_1A_2B_2} = \tfrac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle), \quad (38)$$

then, from Corollary 3, we know that $\mathbf{e}_{A_1\leftrightarrow B_1}(\psi_2) \geq |f_{\psi_2}^{(3)}(\alpha)| = 2$ for any α. In addition, Alice and Bob can exchange A_1 and B_1, by using quantum teleportation [24]. In this case, the entanglement cost is two ebits. Thus we obtain that $\mathbf{e}_{A_1\leftrightarrow B_1}(\psi_2) = 2$. However, the state $|\psi_2\rangle$ satisfies the necessary condition in Theorem 5, since its reduced states ρ_{A_1} and ρ_{B_1} are identical.

V. EXAMPLES

In this section, we present two examples, which show properties of the OSQSE task.

A. Symmetric information

For the initial state $|\psi\rangle$, let us consider a scenario in which Alice and Bob exchange their whole information A and B. Assume that their parts A_2 and B_2 are symmetric, while the remaining parts A_1 and B_1 are not symmetric, i.e., the initial state $|\psi\rangle$ satisfies $(\mathrm{SWAP}_{A_1\leftrightarrow B_1})(\psi) \neq \psi$ and $(\mathrm{SWAP}_{A_2\leftrightarrow B_2})(\psi) = \psi$, where $\mathrm{SWAP}_{X\leftrightarrow Y}$ is the operation swapping quantum states in systems X and Y.

In the OSQSE, the proper use of the symmetric parts A_2 and B_2 can more efficiently reduce the entanglement cost compared to exchanging only A_1 and B_1 without using A_2 and B_2. To be specific, there exists an initial state $|\psi\rangle$ such that the parts A_2 and B_2 are symmetric and $\mathbf{e}_{A\leftrightarrow B}(\psi) = 0$ while the rest parts A_1 and B_1 are not symmetric. Consider the specific initial state

$$|\phi_1\rangle_{A_1B_1A_2B_2R} = \tfrac{1}{\sqrt{2}}(|00000\rangle + |01111\rangle), \quad (39)$$

where A_2 and B_2 are symmetric but A_1 and B_1 are not. Since $\Omega_{AB}^{(1)}(\phi_1) = |00\rangle\langle 00|$ and $\Omega_{AB}^{(2)}(\phi_1) = |01\rangle\langle 11|$, we can show that $\Omega_{AB}^{(1)}(\phi_1)$ and $\Omega_{AB}^{(2)}(\phi_1)$ satisfy the condition in Theorem 4, by setting

$$U = V = |00\rangle\langle 00| + |01\rangle\langle 11| + |10\rangle\langle 10| + |11\rangle\langle 01|. \quad (40)$$

Thus we obtain that $\mathbf{e}_{A\leftrightarrow B}(\phi_1) = 0$, which means that A and B can be exchanged by means of LOCC without consuming any non-local resource.

The above example also shows that the use of the symmetric parts A_2 and B_2 can reduce the entanglement cost for exchanging A_1 and B_1. From the converse bound in Corollary 3, we obtain $\mathbf{e}_{A_1\leftrightarrow B_1}(\phi_1) \geq f_{\phi_1}^{(1)}(\alpha) = 1$ for any α. Using quantum teleportation [24], B_1 can be sent from Bob to Alice by consuming an ebit, and Bob can prepare the part A_1. This implies that $\mathbf{e}_{A_1\leftrightarrow B_1}(\phi_1) = 1$. Observe that the isometry U (V) in Eq. (40) represents Alice's (Bob's) local operation $\mathrm{CNOT_A}$ ($\mathrm{CNOT_B}$) whose target and controlled systems are A_1 (B_1) and A_2 (B_2), respectively. This implies that Alice and Bob can exchange A_1 and B_1 by using local operations. It follows that $0 \geq \mathbf{e}_{A_1\leftrightarrow B_1}^{A_2B_2}(\phi_1)$. In fact, $\mathbf{e}_{A_1\leftrightarrow B_1}^{A_2B_2}(\phi_1) = 0$ from Corollary 3. Therefore, we obtain $\mathbf{e}_{A_1\leftrightarrow B_1}(\phi_1) > \mathbf{e}_{A_1\leftrightarrow B_1}^{A_2B_2}(\phi_1)$.

When A_2 and B_2 are symmetric, we can show the following relation between the optimal entanglement costs by definition.

Proposition 6. $\mathbf{e}_{A\leftrightarrow B}(\psi) = \mathbf{e}_{A_1\leftrightarrow B_1}^{A_2B_2}(\psi)$, *if the parts A_2 and B_2 of $|\psi\rangle$ are symmetric.*

From Proposition 6, we can see that, when Alice and Bob exchange systems A and B of $|\psi\rangle$ with symmetric parts A_2 and B_2, they can achieve the optimal entanglement cost by exchanging only A_1 and B_1, making the most of this symmetry.

B. Negative entanglement cost

As in the asymptotic quantum state exchange task [1, 2], there exist initial states to show that the entanglement cost of the OSQSE task can be negative. Assume that Alice and Bob exchange the parts A_1 and B_1 of the initial state

$$|\phi_2\rangle_{A_1B_1A_2B_2} = \tfrac{1}{2}\textstyle\sum_{i,j=0}^{1} |i\rangle_{A_1} |j\rangle_{B_1} |j\rangle_{A_2} |i\rangle_{B_2}, \quad (41)$$

where $|\phi_2\rangle$ consists of two ebits $|e\rangle_{A_1B_2}$ and $|e\rangle_{B_1A_2}$. To exchange A_1 and B_1, both Alice and Bob prepare an ebit, respectively, and they locally implement entanglement swapping [25] by performing two Bell measurements on A_2, B_2,

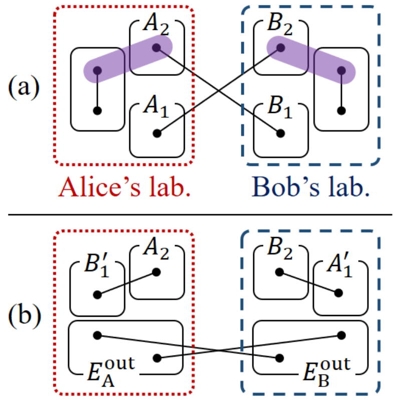

FIG. 3: Illustration of the one-shot quantum state exchange protocol of $|\phi_2\rangle$ in Eq. (41). (a) In order to exchange A_1 and B_1, Alice and Bob locally prepare an ebit each, and they apply Bell measurements to the shaded areas. (b) By performing local operations corresponding to the measurement outcomes, the parts A_1 and B_1 can be exchanged. At the same time, Alice and Bob can share two ebits.

and the parts of the ebits, as described in Fig. 3. Then they can exchange A_1 and B_1, and can share two ebits at the same time. In fact, we have $\mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\phi_2) = -2$ from Corollary 3. This means that the entanglement cost can be negative.

We note that, in Ref. [2], the negativity of the entanglement cost has been theoretically shown by using the merge-and-merge strategy, which is not optimal in general. On the other hand, our example in Eq. (41) elucidates the OSQSE strategy, in which Alice and Bob can exactly achieve the negative optimal entanglement cost.

Moreover, this example tells us that it is worth using Alice's and Bob's parts A_2 and B_2 in order to reduce the entanglement cost. Assume that Alice and Bob do not apply any local operations on A_2 and B_2, then they can exchange A_1 and B_1 by using quantum teleportation [24] twice. From the converse bound in Corollary 3, $\mathbf{e}_{A_1\leftrightarrow B_1}(\phi_2) \geq 2$, and so we obtain that $\mathbf{e}_{A_1\leftrightarrow B_1}(\phi_2) = 2$ and the optimal OSQSE protocol for ϕ_2 is just two quantum teleportation protocols for A_1 and B_1. This means that it is not always possible for Alice and Bob to reduce the amounts of entanglement and classical communication, even though they know the information about the initial state. On the other hand, in this case, if Alice and Bob use their parts A_2 and B_2, then the entanglement cost can be reduced as follows:

$$\mathbf{e}_{A_1\leftrightarrow B_1}(\phi_2) = 2 > -2 = \mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\phi_2). \tag{42}$$

VI. NON-NEGATIVITY CONDITIONS FOR ENTANGLEMENT COST

From Proposition 6, we can know that if A_2 and B_2 are symmetric, then $\mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi)$ cannot be negative, contrary to the example in Sec. V B. One may ask the question: Is there any condition that implies the non-negativity of the optimal entanglement cost $\mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}$? To answer this question, we present the following inequalities.

Proposition 7.

$$\begin{aligned}\mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi) + \mathbf{e}^{A_2B_2}_{B_1'\leftrightarrow A_1'}(\psi_{f_1}) &\geq 0,\\ \mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi) + \mathbf{e}^{B_1'A_1'}_{A_2\leftrightarrow B_2}(\psi_{f_1}) &\geq \mathbf{e}_{A\leftrightarrow B}(\psi),\end{aligned} \tag{43}$$

where $\mathbf{e}^{A_2B_2}_{B_1'\leftrightarrow A_1'}(\psi_{f_1})$ is the optimal entanglement cost for exchanging B_1' and A_1' when using A_2 and B_2, and $\mathbf{e}^{B_1'A_1'}_{A_2\leftrightarrow B_2}(\psi_{f_1})$ is the optimal entanglement cost for exchanging A_2 and B_2 when using B_1' and A_1'.

In Proposition 7, the first inequality comes from the fact that Alice and Bob cannot increase the amount of entanglement between them by means of LOCC [26], while the second one is straightforward from the definitions of the optimal entanglement costs. From Proposition 7, we can see that if $\mathbf{e}^{A_2B_2}_{B_1'\leftrightarrow A_1'}(\psi_{f_1})$ or $\mathbf{e}^{B_1'A_1'}_{A_2\leftrightarrow B_2}(\psi_{f_1})$ is non-positive, then $\mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi)$ cannot be negative. Moreover, if the condition $\mathbf{e}^{B_1'A_1'}_{A_2\leftrightarrow B_2}(\psi_{f_1}) \leq \mathbf{e}_{A\leftrightarrow B}(\psi)$ holds, then Proposition 7 implies $\mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi) \geq 0$.

In particular, let us assume that A_1 and B_1 are symmetric. Then it is obvious that $0 \geq \mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi)$, from Proposition 1. If $0 > \mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi)$ then it follows from Proposition 7 that $\mathbf{e}^{A_2B_2}_{B_1'\leftrightarrow A_1'}(\psi_{f_1}) > 0$. However, since B_1' and A_1' are also symmetric, Proposition 1 implies $\mathbf{e}^{A_2B_2}_{B_1'\leftrightarrow A_1'}(\psi_{f_1}) \leq 0$, which leads to a contradiction. Therefore, we obtain the following corollary.

Corollary 8. *If the parts A_1 and B_1 of $|\psi\rangle$ are symmetric, then we have $\mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi) = 0$.*

This tells us that if A_1 and B_1 are symmetric, Alice and Bob cannot increase the amount of shared entanglement after the OSQSE task, even if they make use of the parts A_2 and B_2.

VII. CONCLUSION

In this work, we have introduced a one-shot version of the original quantum state exchange task, formally defining the OSQSE task and its optimal entanglement costs. We have derived converse bounds on the optimal entanglement costs, and have presented conditions on the initial state to achieve zero entanglement cost. As a related open problem, we can ask the following question: If $\mathbf{e}_{A\leftrightarrow B}(\psi) = 0$, then is it possible to exchange the parts A and B, without classical communication and entanglement, that is, are there local operations L_{A} and L_{B} such that $\psi_{f_{12}} = (L_{\mathrm{A}} \otimes L_{\mathrm{B}})(\psi)$?

We have also provided two interesting properties of the OSQSE, by presenting specific examples. One of the properties tells us that it is worth using the symmetric parts in order to optimally perform the OSQSE. The other shows that the entanglement cost of the OSQSE can be negative. Moreover, we have found the conditions for non-negative optimal entanglement costs. By observing the aforementioned examples, we can provide another interesting open problem: If $\mathbf{e}^{A_2B_2}_{A_1\leftrightarrow B_1}(\psi) \leq 0$, do there exist Alice's and Bob's local operations L_{A}' and L_{B}' such that $\psi_{f_1} \otimes \Phi = \left(L_{\mathrm{A}}' \otimes L_{\mathrm{B}}'\right)(\psi)$?

A further open problem is whether the catalytic use of entanglement [27–29] can reduce the optimal entanglement cost for the OSQSE. To be more specific, for the initial state $|\psi\rangle$, do there exist a bipartite entangled state $|\psi_{\mathrm{c}}\rangle_{A_3B_3}$ shared by Alice and Bob and an OSQSE protocol $C_{K,L} : AA_3E_{\mathrm{A}}^{\mathrm{in}} \otimes BB_3E_{\mathrm{B}}^{\mathrm{in}} \longrightarrow B'A_3E_{\mathrm{A}}^{\mathrm{out}} \otimes A'B_3E_{\mathrm{B}}^{\mathrm{out}}$ such that $\psi_{f_{12}} \otimes \psi_{\mathrm{c}} \otimes \Phi = (C_{K,L} \otimes \mathbb{1}_R)(\psi \otimes \psi_{\mathrm{c}} \otimes \Psi)$ and $\log K - \log L < \mathbf{e}_{A\leftrightarrow B}(\psi)$?

Theoretically, the OSQSE is a powerful two-user quantum communication task, which includes quantum teleportation [24] and quantum state merging [3, 4] as special cases. Practically, this task can be a fundamental building block for applications involving multiple users, such as distributed quantum computation [30, 31] and quantum network [32–35].

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Appendix A: Proof of Theorem 5

We use the following lemma in order to prove Theorem 5.

Lemma 9. *Let Z and W be any discrete random variables on alphabets $\mathcal{Z}$ and $\mathcal{W}$ with $|\mathcal{Z}| = N$ and $|\mathcal{W}| = M$. Let $\{p_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^M$ be probability distributions for Z and W, respectively. If the following equality holds for all $\alpha \in [0, \infty]$,*

$$H_\alpha(Z) = H_\alpha(W), \tag{A1}$$

where $H_\alpha(\cdot)$ is the Rényi entropy of classical random variables, then $|\mathcal{Z}| = |\mathcal{W}|$ and there exists a permutation $\sigma \in S_N$ such that $p_i = q_{\sigma(i)}$ for all $i \in [N]$, where S_N is the set of all permutations on $[N] = \{1, \dots, N\}$.

Note that, for each $\alpha \in [0, \infty]$, $H_\alpha(Z) = \lim_{x\to\alpha} H_x(Z)$ and $S_\alpha(\rho_A) = \lim_{x\to\alpha} S_x(\rho_A)$.

Proof of Lemma 9. Suppose that $H_\alpha(Z) = H_\alpha(W)$ for all $\alpha \in [0, \infty]$. Since $H_0(Z) = H_0(W)$, it holds that $|\mathcal{Z}| = |\mathcal{W}|$. For convenience, we assume that any probability distribution $\{r_i\}_{i=1}^N$ satisfies $r_1 \geq r_i$ for all $i \in [N]$.

We now prove the statement by using mathematical induction on N.

(i) If $N = 2$, then $H_\infty(Z) = H_\infty(W)$ implies $p_1 = q_1$ and so $p_2 = 1 - p_1 = 1 - q_1 = q_2$. Thus the statement is true.

(ii) Suppose that the statement is true for $N = k - 1$. Let Z and W be discrete random variables on alphabets $\mathcal{Z}$ and $\mathcal{W}$ with $|\mathcal{Z}| = |\mathcal{W}| = k$. Let $\{p_i\}_{i=1}^k$ and $\{q_i\}_{i=1}^k$ be probability distributions for Z and W, respectively. Since $H_\infty(Z) = H_\infty(W)$, $p_1 = q_1$. By setting $p_i' = \frac{p_{i+1}}{1-p_1}$ and $q_i' = \frac{q_{i+1}}{1-p_1}$ for each $i \in [k-1]$, we can construct random variables Z' and W' on alphabets $\mathcal{Z}'$ and $\mathcal{W}'$ whose probability distributions are $\{p_i'\}_{i=1}^{k-1}$ and $\{q_i'\}_{i=1}^{k-1}$, respectively. Obviously, $|\mathcal{Z}'| = |\mathcal{W}'| = k - 1$, and so $H_0(Z') = H_0(W')$. Observe that for $\alpha \in (0, 1) \cup (1, \infty)$,

$$\begin{aligned}
& H_\alpha(Z) = H_\alpha(W) \\
\Longrightarrow\ & \frac{1}{1-\alpha} \log\left(\sum_{i=1}^k p_i^\alpha\right) = \frac{1}{1-\alpha} \log\left(\sum_{i=1}^k q_i^\alpha\right) \\
\Longrightarrow\ & \sum_{i=2}^k p_i^\alpha = \sum_{i=2}^k q_i^\alpha \\
\Longrightarrow\ & \sum_{i=1}^{k-1} \left(\frac{p_{i+1}}{1-p_1}\right)^\alpha = \sum_{i=1}^{k-1} \left(\frac{q_{i+1}}{1-p_1}\right)^\alpha \\
\Longrightarrow\ & \frac{1}{1-\alpha} \log\left(\sum_{i=1}^{k-1} (p_i')^\alpha\right) = \frac{1}{1-\alpha} \log\left(\sum_{i=1}^{k-1} (q_i')^\alpha\right) \\
\Longrightarrow\ & H_\alpha(Z') = H_\alpha(W').
\end{aligned} \tag{A2}$$

In addition, if $\alpha = 1$, then

$$\begin{aligned}
& H_1(Z) = H_1(W) \\
\Longrightarrow\ & \sum_{i=1}^k p_i \log \frac{1}{p_i} = \sum_{i=1}^k q_i \log \frac{1}{q_i} \\
\Longrightarrow\ & \sum_{i=2}^k p_i \log \frac{1}{p_i} = \sum_{i=2}^k q_i \log \frac{1}{q_i} \\
\Longrightarrow\ & (1 - p_1) \log (1 - p_1) + \sum_{i=2}^k p_i \log \frac{1}{p_i} \\
& = (1 - p_1) \log (1 - p_1) + \sum_{i=2}^k q_i \log \frac{1}{q_i} \\
\Longrightarrow\ & \sum_{i=2}^k \frac{p_i}{1-p_1} \log \frac{1-p_1}{p_i} = \sum_{i=2}^k \frac{q_i}{1-p_1} \log \frac{1-p_1}{\tau_i} \\
\Longrightarrow\ & \sum_{i=1}^{k-1} p_i' \log \frac{1}{p_i'} = \sum_{i=1}^{k-1} q_i' \log \frac{1}{q_i'} \\
\Longrightarrow\ & H_1(Z') = H_1(W').
\end{aligned} \tag{A3}$$

Finally, we have

$$\begin{aligned}
H_\infty(Z') - H_\infty(W') &= \lim_{\alpha\to\infty} H_\alpha(Z') - \lim_{\alpha\to\infty} H_\alpha(W') \\
&= \lim_{\alpha\to\infty} (H_\alpha(Z') - H_\alpha(W')) = 0.
\end{aligned} \tag{A4}$$

It follows that $H_\alpha(Z') = H_\alpha(W')$ for all $\alpha \in [0, \infty]$. By the induction hypothesis, there exists a permutation $\sigma' \in S_{k-1}$ such that $p_i' = q_{\sigma'(i)}'$ for all $i \in [k - 1]$. Define $\sigma(1) = 1$ and $\sigma(i) = \sigma'(i - 1)$ with $i \neq 1$. Then $\sigma \in S_k$ and $p_i = q_{\sigma(i)}$ for all $i \in [k]$. Therefore, the statement is true for $N = k$. □

In fact, we can prove Lemma 9 by assuming a weaker condition as follows. Let S be a subset of $[0, \infty]$ including 0, the extended real number ∞, and a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} s_n = \infty$. Then we can show that if $H_\alpha(Z) = H_\alpha(W)$ holds for all $\alpha \in S$, then Z and W have the same probability distribution.

The contrapositive of the following lemma proves Theorem 5.

Lemma 10 (Sufficient conditions on the initial state $|\psi\rangle$ with $\mathbf{e}_{X\leftrightarrow Y}(\psi) > 0$)**.** *Let (X, Y) be the pair of two systems, which can be either (A_1, B_1) or (A, B). Let $\{\lambda_i\}_{i=1}^N$ and $\{\tau_i\}_{i=1}^M$ be non-zero eigenvalues for the reduced states ρ_X and ρ_Y of $|\psi\rangle$, respectively, which satisfy $\lambda_1 \geq \ldots \geq \lambda_N$, $\tau_1 \geq \ldots \geq \tau_M$, and $\sum_{i=1}^N \lambda_i = \sum_{i=1}^M \tau_i = 1$. Then $\mathbf{e}_{X\leftrightarrow Y} > 0$, if one of the following conditions holds:*

(i) $N \neq M$.

(ii) $N = M$ and $\lambda_{i'} \neq \tau_{i'}$ for some $i' \in [N] = \{1, \ldots, N\}$.

Proof. (i) If $N \neq M$, then $\mathrm{rank}(\rho_X) \neq \mathrm{rank}(\rho_Y)$, which means

$$\mathbf{e}_{X\leftrightarrow Y}(\psi) \geq |S_0(\rho_X) - S_0(\rho_Y)| > 0, \tag{A5}$$

by the converse bounds in Corollary 3.

(ii) Suppose that $|\psi\rangle$ satisfies $N = M$ and $\lambda_{i'} \neq \tau_{i'}$ for some $i' \in [N]$. Let Z and W be discrete random variables on alphabets $\mathcal{Z}$ and $\mathcal{W}$ with $|\mathcal{Z}| = |\mathcal{W}| = N$, whose probability distributions are $\{\lambda_i\}_{i=1}^N$ and $\{\tau_i\}_{i=1}^N$, respectively. Let us consider the set

$$A = \{i \in [N] | \lambda_i \neq \tau_i\}, \tag{A6}$$

then A is a non-empty subset of $[N]$, since $i' \in A$. So we can choose the largest element in A, say j. Then $\lambda_j \neq \tau_j$ and $\lambda_i = \tau_i$ for all $i > j$ by the definition of the set A. If $\lambda_j > \tau_j$ (or $\lambda_j < \tau_j$) than $\lambda_i > \tau_j$ (or $\lambda_j < \tau_i$) for all $i \in [j]$. Thus $\lambda_i \neq \tau_j$ (or $\lambda_j \neq \tau_i$) for all $i \in [j]$, which shows that for each $\sigma \in S_j$, there exists $i \in [j]$ such that $\lambda_i \neq \tau_{\sigma(i)}$. It follows that for each $\sigma \in S_N$, there exists $i \in [N]$ such that $\lambda_i \neq \tau_{\sigma(i)}$. From the contrapositive of Lemma 9, there exists $\alpha' \in [0, \infty]$ such that $H_{\alpha'}(X) \neq H_{\alpha'}(Y)$. Therefore, from the converse bounds in Corollary 3, we obtain

$$\mathbf{e}_{X\leftrightarrow Y}(\psi) \geq |S_{\alpha'}(\rho_X) - S_{\alpha'}(\rho_Y)| > 0. \tag{A7}$$

□

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