Dynamical quantum phase transitions in the random field Ising model

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Over the last few years it was pointed out that certain observables of time-evolving quantum systems may have singularities at certain moments in time, mimicking the singularities physical systems have when undergoing phase transitions. These were given the name of dynamical phase transitions. They were shown to exist in certain integrable (exactly solvable) quantum systems, and were seen numerically and experimentally in some models which were not integrable. The "universality classes" of such singularities were not yet convincingly established, however. We argue that random field Ising models feature singularities in time which may potentially be present in a wider variety of quantum systems, in particular in those which are many body localized, and describe these singularities in detail analytically.

Recently it has been proposed that some quantum systems evolving in time can have certain observables whose dependence on time is not analytic[1]. One such observable is the "return probability" related to the Loschmidt echo. Given an initial state $|\psi_0\rangle$ which is a ground state of a Hamiltonian H_0, one could define the following observable

$$Z(t) = \langle\psi_0| e^{-iHt} |\psi_0\rangle , \tag{1}$$

where $H \neq H_0$. It is closely related to the trace of the evolution operator $\operatorname{tr} e^{-iHt}$, and to the partition function of the system $\operatorname{tr} e^{-H/T}$. Indeed, expanding in the eigenstates of the Hamiltonian $|\psi_0\rangle = \sum_\alpha c_\alpha |\psi_\alpha\rangle$, such that $H|\psi_\alpha\rangle = E_\alpha |\psi_\alpha\rangle$, we find

$$Z(t) = \sum_\alpha e^{-iE_\alpha t} |c_\alpha|^2 . \tag{2}$$

For some choice of $|\psi_0\rangle$ the coefficients c_α may happen to be all equal to each other. Even if not, the sum in Eq. (2) closely resembles that in the definition of thermal partition functions (in fact, Eq. (2) can be shown[2,3] to be always equivalent to a partition function of some related system upon identifying $t = -i/T$). Partition functions are known to be non-analytic in T, signifying presence of phase transitions. It was proposed that $Z(t)$ may likewise be non-analytic in t. The physical significance of these singularities can be debated, but by now there is no doubt that they exist and can be measured[4,5]. The "universality classes" of these singularities have not yet been convincingly established. Most of the singularities discussed so far correspond to the discontinuities in the derivate $\partial \ln |Z|^2 / \partial t$, resembling first order transitions in statistical physics.

In this paper we demonstrate the existence of singularities in certain random models, of the new universality class $\partial \left[\ln |Z|^2\right] / \partial t \sim \ln |t - t_0|$. We argue that they should manifest themselves in a variety of random systems with Poisson level statistics, although probably not in the most generic many body localized systems.

We begin the discussion by reviewing the established facts, starting with the 1D classical Ising model[6]. Its Hamiltonian is given by

$$H = -J \sum_{n=1}^{N} \sigma^z_n \sigma^z_{m+1}, \tag{3}$$

with $\sigma^z_{N+1} \equiv \sigma^z_1$ (here and below, $\sigma^{x,y,z}$ denote Pauli matrices, while $\sigma = \pm 1$ will be the eigenvalues of σ^z). We could imagine the quench scenario where the system is initially in the state $|\psi_0\rangle$ which is the ground state of a different Hamiltonian

$$H_0 = -\gamma \sum_{n=1}^{N} \sigma^x_n. \tag{4}$$

In other words,

$$|\psi_0\rangle = \frac{1}{2^{N/2}} \prod_{n=1}^{N} \sum_{\sigma=\pm 1} |\sigma\rangle . \tag{5}$$

At a certain moment of time, the Hamiltonian suddenly changes (is quenched) to become H of Eq. (3). Then $Z(t)$ can be calculated in a straightforward way

$$Z(t) = \frac{1}{2^N} \sum_{\sigma=\pm 1} e^{itJ\sum_{n=1}^{N} \sigma_n \sigma_{n+1}} = \left(\cos(Jt)\right)^N + \left(i \sin(Jt)\right)^N . \tag{6}$$

In the large N limit $F = \ln\left(|Z|^2\right)/N$, the dynamical equivalent of free energy of the system, exhibits a non-analytic behavior

$$F = \begin{cases} \ln\left(\cos^2(tJ)\right), & Jt \in [-\pi/4, \pi/4] + \pi m, \\ \ln\left(\sin^2(tJ)\right), & Jt \in [\pi/4, 3\pi/4] + \pi m. \end{cases} \tag{7}$$

Here m is an arbitrary integer. Specifically, F has discontinuities in its derivative with respect to t at points in time $Jt_m = \pi/4 + \pi m/2$, corresponding to a sort of the "first order" singularities in $Z(t)$ of the 1D Ising model.

One should note that the thermal equivalent of Eq. (3) gives

$$Z(T) = \sum_{\sigma=\pm 1} e^{\frac{J}{T}\sum_{n=1}^{N} \sigma_n \sigma_{n+1}} =$$

$$\left(2\cosh\left(J/T\right)\right)^N + \left(2\sinh(J/T)\right)^N. \quad (8)$$

This quantity is analytic in t at large N, because $|\cosh(J/T)| > |\sinh(J/T)|$, signifying the absence of conventional thermal phase transitions in a one dimensional Ising model (or in any other one dimensional system).

Quite remarkably, $Z(t)$ defined for more general quantum systems also appear to have singularities at certain moments in time. For example, given a 1D transverse field Ising model[1]

$$H = -J\sum_{n+1}^{N} \sigma_n^z \sigma_{n+1}^z - \gamma \sum_{n=1}^{N} \sigma_n^x, \quad (9)$$

choosing H_0 to be H with some choice of the parameters J, h, with $|\psi_0\rangle$ its ground state, and evolving this state with the Hamiltonian H with some other values of these parameters, $Z(t)$ can be shows to have singularities at certain moments of time t_m as long as the initial and final values of J and γ belong to two different phases of the 1D transverse field Ising model. The example of the 1D classical Ising model then becomes a particular case of this with the initial $J=0$ and the final $\gamma = 0$.

Furthermore, given that 1D transverse field Ising model is equivalent to free fermions by Jordan-Winger transformation, this construction was generalized to other free fermion systems[7] with their $Z(t)$ shown to have similar singularities at certain times t_m under the right parameter quench. All such solvable systems however are examples of exactly solvable (integrable) models.

One can wonder if the singularities of $Z(t)$ in the examples above survive if the system considered is not integrable. Indeed, the moments in time t_m where the singularities occur are related to level spacing J^{-1} in the example worked out above. A generic quantum system with N degrees of freedom (such as N spin-1/2's) will have level spacing of the order of e^{-N}, leading to singularities, if any, occurring at enormous times $t \sim e^N$, becoming unobservable for large systems $N \to \infty$. Nevertheless, nonintegrable generalizations of (9) were studied numerically and found to still have the singularities in time[8–11]. Furthermore, these singularities were measured in an experiment[4] for 1D transverse field Ising model with nonlocal interactions in space, which is also not integrable.

As an example of a very generic non-integrable (chaotic) quantum model we could consider random matrix theory[12] (RMT). The quantity of interest to us is nothing but the spectral form factor of RMT

$$|Z_{\rm RMT}(t)|^2 = \sum_{\alpha\beta} e^{it(E_\alpha - E_\beta)}, \quad (10)$$

where E_α are energy levels of a random matrix. The derivatives of the RMT form factors over t are known to have a discontinuity at a critical value[12] of t of the order of level spacing, just as argued above. One should note however that usually one studies the average form factor, while we are interested in the typical form factor which could be represented as the average of the logarithm of this quantity. The form factor is known not to be self-averaging in RMT[13], and its typical structure is not completely understood. Recent studies also looked at this and related quantities in quantum chaotic models such as the SYK model[14].

Instead of a most generic quantum system with the Wigner-Dyson level statistics, let us consider a system with a Poissonian level statistics. Those models routinely appear in the context of many body localization[15]. The simplest such model is 1D the random field classical Ising model, with the Hamiltonian

$$H = -J\sum_{n=1}^{N} \sigma_n^z \sigma_{n+1}^z - \sum_{n=1}^{N} h_n \sigma_n^z. \quad (11)$$

Here h_n are random independent variables distributed uniformly on the interval $h \in [-J, J]$ (precise form of the probability distribution, as well as disorder strength, as we argue below, turns out not to be important). The thermodynamics of this model was extensively studied in the past and was arguably found to be unremarkable[16]. Here we are not interested in its thermodynamics, however. We again envision putting the system in the ground state Eq. (5) of the Hamiltonian Eq. (4), and then quenching it to Eq. (11). The Loschmidt echo is again proportional to the imaginary temperature partition function of Eq. (11), to give

$$Z = \frac{1}{2^N} \sum_{\sigma=\pm1} e^{itJ\sum_{n=1}^{N}\sigma_n\sigma_{n+1} + it\sum_{n=1}^{N} h_n\sigma_n}. \quad (12)$$

$|Z|^2$, when averaged over random fields, is time independent at large enough time. Indeed,

$$\left\langle |Z|^2 \right\rangle = \frac{1}{2^{2N}} \sum_{\substack{\sigma=\pm1\\ \mu=\pm1}} \int_{-J}^{J} \prod_{n=1}^{N} \left[\frac{dh_n}{2J} \times \right. \\ \left. e^{itJ(\sigma_n\sigma_{n+1} - \mu_n\mu_{n+1}) + ith_n(\sigma_n - \mu_n)} \right]. \quad (13)$$

When $tJ \gg 1$, the integrals over h_n give Kronecker deltas $\delta_{\sigma_n\mu_n}$, resulting in the time-independent $Z(t)$, $\left\langle |Z(t)|^2 \right\rangle = 2^{-N}$. Its Fourier transform with respect to t reflects the fact that the energy levels of Eq. (11) are not correlated (compare with Eq. (10)), indicative of the Poisson statistics of the levels.

However, the typical values of $F = \ln\left(|Z|^2\right)/N$ display a totally different and far more interesting behavior. Fig, (1) shows this quantity plotted numerically for $N = 5000$ spins, for a single realization of random h_n. We see that unlike the average spectral form-factor this function displays criticalities which superficially look similar to the disorder-free case, but as we now verify are qualitatively different from it.

To understand the nature of these singularities we observe (and justify later) that at large enough times it is sufficient to think of th_n as being uniformly distributed

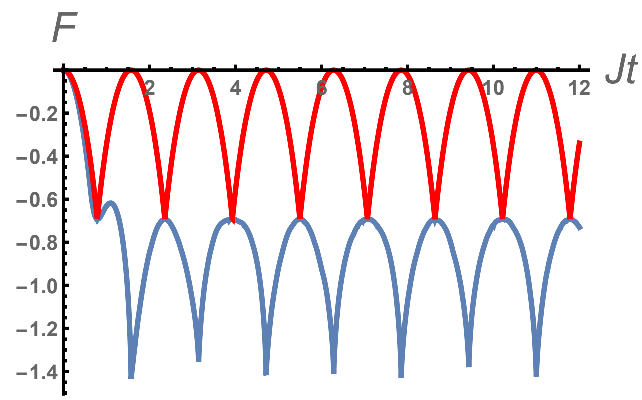

FIG. 1: The typical value of $F = \ln\left(|Z|^2\right)/N$ from Eq. (12) computed for $N = 5000$ spins and random $h_n \in [-J, J]$ plotted as a function of Jt (lower curve). The upper curve represents the disorder free 1D Ising model Eq. (6) shown for comparison.

on the interval $th_n \in [-\pi, \pi]$. To confirm this, we plot the result of numerically evaluating

$$Z = \frac{1}{2^N} \sum_{\sigma=\pm 1} e^{itJ \sum_{n=1}^{N} \sigma_n \sigma_{n+1} + i \sum_{n=1}^{N} h_n \sigma_n}. \quad (14)$$

where h_n are now randomly distributed on the interval $h_n \in [-\pi, \pi]$ in Fig. (2). The resulting curve coincides with Eq. (12) for large enough t.

From Fig. (1 the singularities seem to occur at times $t_m = \pi m/(2J)$. In fact, these are special times where the term proportional to J in Eq. (12) can be set to zero without changing the value of $|Z|^2$. Choosing t_m a multiple of $2\pi/J$ for simplicity (the algebra is similar if slightly different for other values of t_m) we find

$$Z = \frac{1}{2^N} \sum_{\sigma=\pm 1} e^{i \sum_{n=1}^{N} h_n \sigma_n} = \prod_{n=1}^{N} \cos\left(h_n\right), \quad (15)$$

where from now on we take h_n above to be uniformly distributed on the unit circle $h_n \in [-\pi, \pi]$ and no longer multiply them by t. We expect F to be a self averaging

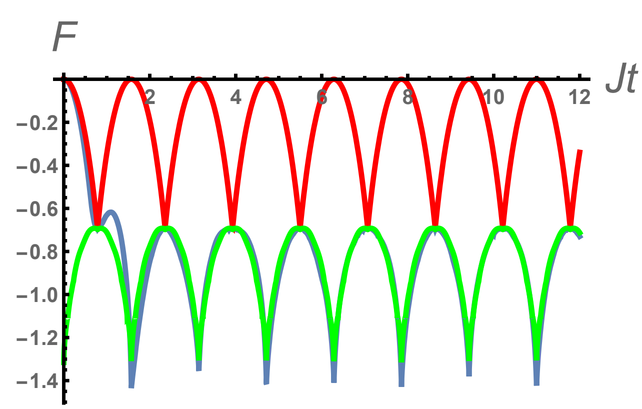

FIG. 2: Same as Fig. (1) but with the result of Eq. (14) also shown.

quantity, based on its analogy with free energy of the Gibbs ensemble. Averaging it over random h_n gives

$$F = \frac{1}{N} \int_{-\pi}^{\pi} \prod_{n=1}^{N} \left[\frac{dh_n}{2\pi}\right] \sum_{n=1}^{n} \ln \cos^2(h_n) = -\ln 4. \quad (16)$$

This is consistent with the observed values of F at $t_m = \pi m/(2J)$ as seen in Fig. (1).

Nearby these values of t, we substitute $t = t_m + J^{-1}\epsilon$ and expand Z in powers of ϵ up to terms of the order ϵ^2. This gives

$$Z = \frac{\cos^N(\epsilon)}{2^N} \sum_{\sigma=\pm 1} \prod_{n=1}^{N} \left(1 + i \tan(\epsilon) \sigma_n \sigma_{n+1}\right) e^{i h_n \sigma_n} \approx \prod_{n=1}^{n} \cos(h_n) \left(1 - i\epsilon \sum_{n=1}^{N} \tan(h_n)\tan(h_{n+1}) - \epsilon^2 \sum_{n-m\geq 2} \tan(h_n)\tan(h_{n+1})\tan(h_m)\tan(h_{m+1}) + \epsilon^2 \sum_{n=1}^{N} \tan(h_n)\tan(h_{n+2})\right). \quad (17)$$

This can be used to average F over random h_n, accomplished by integrating it over $dh_n/(2\pi)$ over the interval $[-\pi, \pi]$ for each h_n. A convenient change of variables $\tan(h_n) = x_n$ brings the relevant expression to the form

$$F \approx -\ln 4 + \frac{1}{\pi^N} \int_{-\infty}^{\infty} \prod_{n=1}^{N} \frac{dx_n}{1 + x_n^2} \times \ln\left(1 + \epsilon^2 \sum_{n=1}^{N} \left(x_n^2 x_{n+1}^2 + 2x_n x_{n+2} + 2x_n x_{n+1}^2 x_{n+2}\right)\right). \quad (18)$$

So far everything appears to be analytic in ϵ. However, the integral over x_n makes the result nonanalytic. Indeed, expanding the logarithm in ϵ^2 under the sign of integral we can easily see that the resulting integral is divergent, indicating that the result should be larger than ϵ^2. Note that infinite x corresponds to h in the vicinity of $\pm\pi/2$, thus we predict that $Jt \geq \pi/2$ for the singularities to appear in Eq. (12).

The result of evaluation of Eq. (18) for small ϵ gives[17]

$$F \approx -\ln 4 + \frac{4}{\pi} \epsilon \ln \frac{1}{\epsilon}. \quad (19)$$

This result is valid for $t = \pi m/(2J) + \epsilon$ for $\epsilon \ll 1/J$.

To verify this we plot F as a function of $\log \epsilon$ for small ϵ, shown in Fig. (3). The result is consistent with Eq. (19).

Thus despite appearing qualitatively similar in Fig. (1), the singularities of the random field 1D Ising equation are much sharper than those for the nonrandom 1D Ising model, with the first derivative of F diverging logarithmically as t approaches any of the singularities.

A natural question is whether these singularities survive the addition of quantum terms to the Hamiltonian.

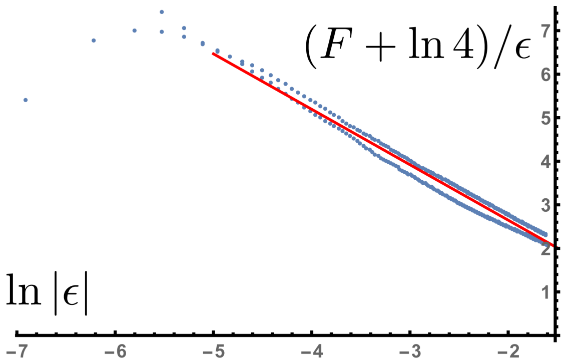

FIG. 3: Vicinity of singularity: $(F + \ln 4)/\epsilon$ is plotted as a function of $\ln|\epsilon|$ for Eq. (14) at $N = 5000$. The straight line has the slope of $-4/\pi$. Two sets of data correspond to the two signs of ϵ.

For example, we could consider quenching Eq. (4) to the Hamiltonian

$$H = -J \sum_{n=1}^{N} \sigma_n^z \sigma_{n+1}^z - \sum_{n=1}^{N} h_n \sigma_n^z - \gamma \sum_{n=1}^{N} \sigma_n^x, \qquad (20)$$

with h_n random as before. This model is not integrable and no good analytic methods exist to study its behavior. It is believed to have no quantum phase transitions at zero temperature[18] and to be many-body localized[19]. As is well appreciated now, this implies that there exist a number of operators, called l-bits, in terms of which the Hamiltonian can be effectively diagonalized[20],

$$H = -\sum_{n=1}^{N} J_n^{(1)} \tau_n^z - \sum_{n=1}^{N} J_n^{(2)} \tau_n^z \tau_{n+1}^z + \dots \qquad (21)$$

where dots denote terms with a higher number of interacting spins, and $J_n^{(1)}$, $J_n^{(2)}$ all random. Such models however, where spin-spin interactions are now random, wash out the singularities studied above, as is clear from Fig. (4). Thus a generic quantum random many-body localized model with energy levels obeying Poisson statistics would not have singularities of the type discussed here.

Therefore, the program to look for many body localized systems which have singularities in their Loschmidt echo $Z(t)$ consists of looking for those models which, when expressed in terms of l-bits, map into random field-type models Eq. (11) as opposed to the random bond models Eq. (21), as well as classifying all l-bit Hamiltonians with singularities, going beyond just Eq. (11). This is possible to do; the result of the investigation along these lines will be reported elsewhere.

Nevertheless even in the absence of explicit examples of quantum many-body localized Hamiltonians with singular Loschmidt echo, the behavior of the Loschmidt echo found here can be argued to be fairly generic. From Eq. (17) leading to Eq. (18) it should be clear that a larger variety of classical models beyond 1D random field

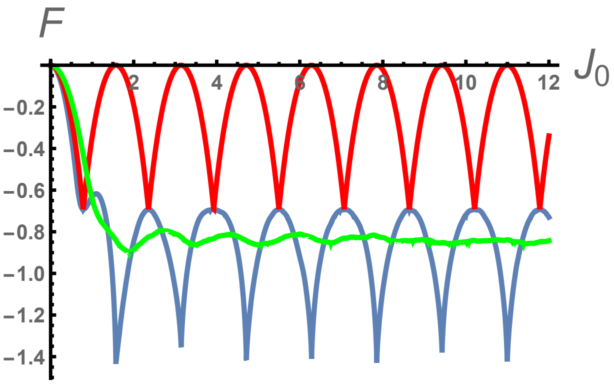

FIG. 4: Same as Fig. (1) but with additional line added representing F for bonds J randomly and uniformly distributed on the interval $[-J_0, J_0]$ in addition to h_n uniform and random on the same interval.

Ising model should exhibit similar behavior. For example, consider classical 2D or 3D Ising models which are quenched from the initial paramagnetic state similar to Eq. (5), and whose Loschmidt echo is simply their partition function computed at imaginary temperature. When expanded in powers of ϵ, random field averaging of F leads to an integral broadly similar to Eq. (18) which should also produce the singularity $\epsilon \ln(1/\epsilon)$.

Even more generally, one could observe that a large number of quantum systems can be thought of as consisting of fermionic "quasiparticles" with the energy spectrum ϵ_α and quasiparticle occupation numbers $n_\alpha = 0, 1$. The energy of such a system is

$$E = \sum_\alpha \epsilon_\alpha n_\alpha + J \sum_{\alpha\beta} n_\alpha n_\beta, \qquad (22)$$

where J can be thought of as being quasiparticle type independent. Fermi liquids could be examples of such systems. Such systems have Poisson level statistics, as opposed to other "more generic" quantum system whose levels obey Wigner-Dyson statistics. It should be clear from the preceding discussion that all such models should have Loschmidt echo having the singularities of the type described here, as long as J does not itself depend on α and β in some random fashion. This construction gives a rather generic realization of the models considered here.

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