Criteria to detect genuine multipartite entanglement using spin measurements

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We derive conditions in the form of inequalities to detect the genuine N-partite entanglement of N systems. The inequalities are expressed in terms of variances of spin operators, and can be tested by local spin measurements performed on the individual systems. Violation of the inequalities is sufficient (but not necessary) to certify the multipartite entanglement, and occurs when a type of spin squeezing is created. The inequalities are similar to those derived for continuous-variable systems, but instead are based on the Heisenberg spin-uncertainty relation $\Delta J_x \Delta J_y \geq |\langle J_z \rangle|/2$. We also extend previous work to derive spin-variance inequalities that certify the full tripartite inseparability or genuine multi-partite entanglement among systems with fixed spin J, as in Greenberger-Horne-Zeilinger (GHZ) states and W states where $J = 1/2$. These inequalities are derived from the planar spin-uncertainty relation $(\Delta J_x)^2 + (\Delta J_y)^2 \geq C_J$ where C_J is a constant for each J. Finally, it is shown how the inequalities detect multipartite entanglement based on Stokes operators. We illustrate with experiments that create entanglement shared among separated atomic ensembles, polarization-entangled optical modes, and the clouds of atoms of an expanding spin-squeezed Bose-Einstein condensate. For each example, we give a criterion to certify the mutual entanglement.

I. INTRODUCTION

Genuine multipartite quantum entanglement is a resource required for many protocols in the field of quantum information and computation [1–9]. N systems are said to be genuinely N-partite entangled if the systems are mutually entangled in such a way that the entanglement cannot be constructed by mixing entangled states involving fewer than N parties [9–11]. Mathematically, a tripartite system is genuinely tripartite entangled if and only if the density operator characterizing the system cannot be represented in the biseparable form [9–12]

$$\rho_{BS} = P_1 \sum_R \eta_R^{(1)} \rho_1^R \rho_{23}^R + P_2 \sum_{R'} \eta_{R'}^{(2)} \rho_2^{R'} \rho_{13}^{R'}$$
$$+ P_3 \sum_{R'} \eta_{R''}^{(3)} \rho_3^{R''} \rho_{12}^{R''} , \tag{1}$$

where $\sum_{k=1}^3 P_k = 1$, $P_k \geq 0$, and $\sum_R \eta_R^{(k)} = 1$. ρ_k^R is an arbitrary density operator for the subsystem k, while ρ_{mn}^R is an arbitrary density operator for the subsystems m and n. The definition of genuine N-partite entanglement follows similarly.

Criteria to certify genuine N-partite entanglement for continuous variable (CV) systems have been derived by Shalm et al. [13] and Teh and Reid [14]. These criteria take the form of variance inequalities, similar to those derived for CV bipartite entanglement [15–17]. The work of Refs. [13, 14] extended earlier work by van Loock and Furusawa, who developed CV criteria for the related but different concept of full N-partite inseparability [18, 19] (see also Refs. [20, 21]). Although genuine N-partite entanglement implies full N-partite inseparability, the converse is not true, and full N-partite inseparability is therefore a weaker form of correlation. Nonetheless, for *pure* states, full N-partite inseparability is sufficient to imply genuine N-partite entanglement. Experiments have confirmed both full N-partite inseparability [19, 22–25] and genuine N-partite entanglement ($N \geq 3$) for CV

systems [13, 26–29]. Here, "continuous variable (CV)" refers to the use of measurements that have continuous-variable outcomes e.g. field quadrature phase amplitudes X and P, or position and momentum. The CV criteria are derived from the commutation relation $[X, P] = 2i$ and the associated uncertainty relations.

In this paper, we derive criteria for genuine N-partite entanglement that are useful for discrete variable systems involving spin degrees of freedom. In this case, measurements correspond to spin observables, and it is the spin commutation relation $[J_x, J_y] = iJ_z$ and associated spin-uncertainty relations that are relevant. The criteria we derive involve variances and apply to all physical systems, provided the measurements correspond to operators satisfying spin commutation relations. This approach extends to N systems that of Hofmann and Takeuchi [30] and Raymer et al. [31] who used spin-uncertainty relations to derive variance criteria for bipartite entanglement. The question of how to detect genuine N-partite entanglement has been studied previously but most work has been in the context of qubit (spin 1/2) systems [32–42] or systems of fixed dimension [43–48].

The development of criteria to certify the genuine multipartite entanglement of discrete systems, as in this paper, is motivated by the increasing number of experiments detecting entanglement with atoms. For example, bipartite entanglement has been created between atomic ensembles and separated atomic modes [49–51], and multi-partite entanglement has been created among the separated clouds of a Bose-Einstein condensate (BEC) [52]. It is sometimes possible to rewrite the spin commutation relation in a form that resembles the position-momentum commutation relation. This is often true where the spin observables are expressed as Schwinger operators, and justifies the use of CV entanglement criteria for the spin system in that case. For instance, Julsgaard et al. [49] characterize the entanglement in the collective spins between two atomic ensembles using CV criteria. However, as pointed out by Raymer et al. [31],

this is only valid in a restricted sense and will not give correct results in general. In other words, the complete spin commutation relation should be used in any derivation of criteria certifying the genuine multipartite entanglement of spin systems.

The program of characterizing entanglement in spin systems has been largely motivated by the observation that a spin-squeezed system exhibits quantum correlations among the spin particles. Sørensen et al. [53, 54] derived an N-partite entanglement criterion that implies the presence of an N-partite entangled state. Here, an N-partite entangled state is a state that cannot be expressed in the form

$$\rho_S = \sum_R P_R \rho_R^{(1)} \rho_R^{(2)} ... \rho_R^{(N)}, \tag{2}$$

where $\sum_R P_R = 1$. A host of criteria [55–59] were subsequently derived to certify the presence of N-partite entanglement in spin systems. However, these criteria only rule out the possibility of N-partite separable states of the form Eq. (2) and not the more general N-partite biseparable states of the form Eq. (1) (as extended to higher N) where all separable bipartitions (and mixtures of them) are considered. Hence they are not criteria for genuine N-partite entanglement, where the entanglement is mutually shared among all N parties. An exception are the spin-squeezing criteria of Sørensen and Mølmer (and others like it) which imply a genuine k-particle entanglement shared among k particles of an N-particle system ($k \leq N$) [60]. Such criteria differ from those derived in this paper, however, being based on collective spin measurements made on the composite system, rather than local measurements made on separated subsystems, and thus cannot directly test nonlocal models (as described in Ref. [61]).

The task of characterizing genuine multipartite entanglement in spin systems was carried out by Korbicz et al. [62, 63]. Korbicz and co-workers used the positivity of partial transpose (PPT) criterion or the Peres-Horodecki criterion [12, 64] as the starting point to derive entanglement criteria, and showed genuine tripartite entanglement for symmetric states. The PPT criterion, however, is less useful for N-partite separability when N is large [12]. In this paper, we derive criteria for genuine multipartite entanglement for spin systems by ruling out the possibility of the state in a biseparable form as in Eq. (1) using the uncertainty relations for spin operators.

The remainder of the paper is structured as follows. In Section II, we derive criteria for the genuine tripartite entanglement of three systems using spin measurements. The generalization to genuine N-partite entanglement is given in Section IV. These criteria are derived using methods similar to those developed by van Loock and Fursusawa [18], Teh and Reid [14] and Shalm et al. [13] for CV systems. In Section III, we extend variance criteria derived for Einstein-Podolsky-Rosen (EPR) steering by He and Reid [42], pointing out that these inequalities apply to certify genuine tripartite entanglement as well

as steering, which is a form of entanglement closely connected with the EPR paradox [61, 65, 66]. The criteria are derived using planar spin-uncertainty relations [67–71] and apply to subsystems with a fixed spin J. We show that the criteria may be used to detect the genuine tripartite entanglement of GHZ states, and the full tripartite inseparability of W states. Finally, in Section V, we explain how to generate genuinely-entangled spin systems based on Stokes operators. We then demonstrate using three examples the application of the criteria derived in Sections II and IV to certify the genuine N-partite entanglement.

II. CRITERIA FOR GENUINE TRIPARTITE ENTANGLEMENT

The criteria derived in this section involve variances of the sum of spin observables defined for each subsystem. These criteria only require the statistics of a set of observables and, in this sense, are state independent. In this work, all the caret symbols that denote the spin operators are dropped, unless specified otherwise, and we use the symbol $\Delta^2 x$ to denote the variance of x.

A. The sum inequalities

1. Sum of two variances

Consider the sum of $\Delta^2 u$ and $\Delta^2 v$ where

$$u = h_1 J_{x,1} + h_2 J_{x,2} + h_3 J_{x,3}$$
$$v = g_1 J_{y,1} + g_2 J_{y,2} + g_3 J_{y,3} \tag{3}$$

and h_k and g_k ($k = 1, 2, 3$) are real numbers. Here, $J_{x,k}, J_{y,k}, J_{z,k}$ are the spin operators for subsystem k, satisfying the commutation relation $[J_{x,k}, J_{y,k}] = iJ_{z,k}$. We derive the bound for $\Delta^2 u + \Delta^2 v$ such that the violation of the bound implies the genuine tripartite entanglement in the spin degree of freedom. This leads us to the following Criterion.

Criterion 1. Violation of the inequality

$$\Delta^2 u + \Delta^2 v \geq \min \{|g_1 h_1 \langle J_{z,1} \rangle| + |g_2 h_2 \langle J_{z,2} \rangle + g_3 h_3 \langle J_{z,3} \rangle|,$$
$$|g_2 h_2 \langle J_{z,2} \rangle| + |g_1 h_1 \langle J_{z,1} \rangle + g_3 h_3 \langle J_{z,3} \rangle|,$$
$$|g_3 h_3 \langle J_{z,3} \rangle| + |g_1 h_1 \langle J_{z,1} \rangle + g_2 h_2 \langle J_{z,2} \rangle|\} \tag{4}$$

is sufficient to confirm genuine tripartite entanglement.

Proof. Firstly, we assume that the spin state is in a biseparable mixture state $\rho_{BS} = P_1 \sum \eta_R^{(1)} \rho_1^R \rho_{23}^R + P_2 \sum_{R'} \eta_{R'}^{(2)} \rho_{13}^{R'} \rho_2^{R'} + P_3 \sum_{R''} \eta_{R''}^{(3)} \rho_{12}^{R''} \rho_3^{R''}$ as in Eq. (1). This implies that the variance of an observable $\Delta^2 u$ is greater or equal to the sum of the variances of the observable of its component state $\Delta^2 u_R$, i.e. [30]

$$\Delta^2 u \geq \sum_R P_R \Delta^2 u_R. \tag{5}$$

The sum of $\Delta^2 u$ and $\Delta^2 v$ is then

$$\Delta^2 u + \Delta^2 v \geq P_1 \sum_R \eta_R^{(1)} \left[\Delta^2 u_R + \Delta^2 v_R \right]$$
$$+ P_2 \sum_{R'} \eta_{R'}^{(2)} \left[\Delta^2 u_{R'} + \Delta^2 v_{R'} \right]$$
$$+ P_3 \sum_{R''} \eta_{R''}^{(3)} \left[\Delta^2 u_{R''} + \Delta^2 v_{R''} \right] . \quad (6)$$

To proceed, we consider $\Delta^2 u_\zeta + \Delta^2 v_\zeta$ that corresponds to an arbitrary bipartition $\rho_k^\zeta \rho_{lm}^\zeta$:

$$\Delta^2 u_\zeta + \Delta^2 v_\zeta \geq |g_k h_k \langle J_{z,k} \rangle| + |g_l h_l \langle J_{z,l} \rangle + g_m h_m \langle J_{z,m} \rangle| . \quad (7)$$

The lower bound given in this inequality is derived in the Appendix 1, using the uncertainty relations for spin. We can always choose for the lower bound the smallest value of $\Delta^2 u_\zeta + \Delta^2 v_\zeta$ in Eq. (6). Hence, Eq. (6) becomes Eq. (4), where we use the fact that $\sum \eta_R^{(1)} = 1$ and $\sum P_k = 1$. In Eq. (4), the first term in the bracket {} is implied by the biseparable state $\rho_1 \rho_{23}$, the second term is implied by the biseparable state $\rho_2 \rho_{13}$, and the final term is implied by the biseparable state $\rho_3 \rho_{12}$. $\square$

The optimal values for g_k, h_k depend on the specific spin state. The criterion given by Eq. (4) is a general result that allows us to derive a host of other criteria. Examples of optimal choices for different types of spin states will be given in Section V.

2. Van Loock-Furusawa inequalities for spin

We can also derive the spin version of a set of inequalities derived by van Loock and Furusawa [18]. The quantities B_I, B_{II} and B_{III} are defined as

$$B_I \equiv \Delta^2 \left(J_{x,1} - J_{x,2} \right) + \Delta^2 \left(J_{y,1} + J_{y,2} + g_3 J_{y,3} \right)$$
$$B_{II} \equiv \Delta^2 \left(J_{x,2} - J_{x,3} \right) + \Delta^2 \left(g_1 J_{y,1} + J_{y,2} + J_{y,3} \right)$$
$$B_{III} \equiv \Delta^2 \left(J_{x,1} - J_{x,3} \right) + \Delta^2 \left(J_{y,1} + g_2 J_{y,2} + J_{y,3} \right) . \quad (8)$$

By choosing the coefficients g_k and h_k in Eq. (4), we obtain a set of inequalities satisfied by B_I, B_{II} and B_{III}. For example, the left side of the criterion in Eq. (4) is equal to B_I when $h_1 = 1$, $h_2 = -1$, $h_3 = 0$ and $g_1 = g_2 = 1$. The set of inequalities is given below:

$$B_I \geq (|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle|)$$
$$B_{II} \geq (|\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle|)$$
$$B_{III} \geq (|\langle J_{z,1} \rangle| + |\langle J_{z,3} \rangle|) . \quad (9)$$

We point out that $B_I \geq (|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle|)$ is implied by both the biseparable states $\rho_1 \rho_{23}$ and $\rho_2 \rho_{13}$, while $B_{II} \geq (|\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle|)$ is implied by the biseparable states $\rho_2 \rho_{13}$ and $\rho_3 \rho_{12}$. Finally, $B_{III} \geq (|\langle J_{z,1} \rangle| + |\langle J_{z,3} \rangle|)$ is satisfied by the biseparable states $\rho_1 \rho_{23}$ and $\rho_3 \rho_{12}$. Using

the inequalities in Eq. (9), we obtain a criterion that confirms genuine tripartite entanglement.

Criterion 2. Full tripartite inseparability is observed if any two of the inequalities (9) are violated. For a pure state, this is sufficient to imply genuine tripartite entanglement. For arbitrary states, genuine tripartite entanglement is observed if the inequality

$$B_I + B_{II} + B_{III} \geq |\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle| \quad (10)$$

is violated.

Proof. Full tripartite inseparability is observed if each one of the inequalities (9) is violated, because this certifies entanglement across all bipartitions. Following van Loock and Furusawa [18], in fact we see that tripartite inseparability is confirmed if any two inequalities are violated. This is so because: $B_I \geq |\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle|$ is implied by $\rho_1 \rho_{23}, \rho_2 \rho_{13}$; $B_{II} \geq |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle|$ is implied by $\rho_2 \rho_{13}, \rho_3 \rho_{12}$; and $B_{III} \geq |\langle J_{z,1} \rangle| + |\langle J_{z,3} \rangle|$ is implied by $\rho_1 \rho_{23}, \rho_3 \rho_{12}$. For pure states, the proof of full tripartite inseparability confirms genuine tripartite entanglement. Now we prove the second condition that applies to all states including mixed states. For brevity, we index the biseparable states $\rho_1 \rho_{23}$, $\rho_2 \rho_{13}$ and $\rho_3 \rho_{12}$ by $k = 1, 2, 3$, respectively. Let $B_{I,1}$ be the quantity B_I that is evaluated using the biseparable state $\rho_1 \rho_{23}$. Then,

$$B_I \geq \sum_k P_k B_{I,k}$$
$$\geq P_1 B_{I,1} + P_2 B_{I,2} \geq (P_1 + P_2) \left(|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| \right) .$$

Similarly, $B_{II} \geq (P_2 + P_3) \left(|\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle| \right)$ and $B_{III} \geq (P_1 + P_3) \left(|\langle J_{z,1} \rangle| + |\langle J_{z,3} \rangle| \right)$. In order to include all possible mixtures, we take the sum of B_I, B_{II} and B_{III}, and use the expansion in Eq. (1). The inequality they satisfy, derived below, provides a criterion for genuine tripartite entanglement:

$$B_I + B_{II} + B_{III} \geq (P_1 + P_2 + P_3)(|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle|)$$
$$+ P_1 |\langle J_{z,1} \rangle| + P_2 |\langle J_{z,2} \rangle| + P_3 |\langle J_{z,3} \rangle|$$

$$\geq (P_1 + P_2 + P_3)(|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle|)$$
$$= (|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle|) ,$$

where $\sum_k P_k = 1$. $\square$

The number of moment measurements in the criterion given by Eq. (10) can be reduced by using a criterion that only involves two of the three quantities B_I, B_{II} and B_{III}. Setting $g_1 = g_2 = g_3 = 1$, we see that the sum

$$B_I + B_{II} \geq |\langle J_{z,1} \rangle| + 2 |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle| \quad (11)$$

is satisfied by any mixture of all tripartite biseparable states. The violation of the criterion in Eq. (11) then implies genuine tripartite entanglement. This is also true for other combinations $B_I + B_{III} \geq 2 |\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle|$ and $B_{II} + B_{III} \geq |\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + 2 |\langle J_{z,3} \rangle|$.

B. The product inequalities

1. Product of two variances

Criteria involving products rather than sums can also be derived. Again, we consider the two quantities $\Delta^2 u = \Delta^2 (h_1 J_{x,1} + h_2 J_{x,2} + h_3 J_{x,3})$ and $\Delta^2 v = \Delta^2 (g_1 J_{y,1} + g_2 J_{y,2} + g_3 J_{y,3})$.

Criterion 3. Genuine tripartite entanglement is observed if the inequality

$$\Delta u \Delta v \geq \frac{1}{2} \min \left\{ |g_1 h_1 \langle J_{z,1} \rangle| + |g_2 h_2 \langle J_{z,2} \rangle + g_3 h_3 \langle J_{z,3} \rangle| \, , \right.$$
$$|g_2 h_2 \langle J_{z,2} \rangle| + |g_1 h_1 \langle J_{z,1} \rangle + g_3 h_3 \langle J_{z,3} \rangle| \, ,$$
$$\left. |g_3 h_3 \langle J_{z,3} \rangle| + |g_1 h_1 \langle J_{z,1} \rangle + g_2 h_2 \langle J_{z,2} \rangle| \right\} \quad (12)$$

is violated.

Proof. The product of two variances $\Delta^2 u$ and $\Delta^2 v$ satisfies the inequality

$$\Delta^2 u \Delta^2 v \geq \left[\sum_R P_R \Delta^2 u_R \right] \left[\sum_R P_R \Delta^2 v_R \right]$$
$$\geq \sum_R P_R \Delta^2 u_R \Delta^2 v_R \, , \quad (13)$$

where the Cauchy-Schwarz inequality is used. For an arbitrary bipartition $\rho_k^\zeta \rho_{lm}^\zeta$, $\Delta^2 u_\zeta \Delta^2 v_\zeta$ satisfies the inequality (see Appendix 2):

$$\Delta^2 u_\zeta \Delta^2 v_\zeta \geq \frac{1}{4} \left[|g_k h_k \langle J_{z,k} \rangle| + |g_l h_l \langle J_{z,l} \rangle + g_m h_m \langle J_{z,m} \rangle| \right]^2 . \quad (14)$$

Identical to the proof for Criterion 1, we can always choose the bipartition that gives us the smallest value of $\Delta u_\zeta \Delta v_\zeta$ in Eq. (13). Hence, Eq. (13) becomes (12). $\square$

2. Van Loock-Furusawa product inequalities

The product version of the van Loock-Furusawa inequalities can be obtained, using the criterion in Eq. (12). The quantities involved are S_I, S_{II}, and S_{III}, as defined below:

$$S_I \equiv \Delta (J_{x,1} - J_{x,2}) \Delta (J_{y,1} + J_{y,2} + g_3 J_{y,3})$$
$$S_{II} \equiv \Delta (J_{x,2} - J_{x,3}) \Delta (g_1 J_{y,1} + J_{y,2} + J_{y,3})$$
$$S_{III} \equiv \Delta (J_{x,1} - J_{x,3}) \Delta (J_{y,1} + g_2 J_{y,2} + J_{y,3}) . \quad (15)$$

By choosing the coefficients g_i and h_i in Eq. (12), we obtain a set of inequalities satisfied by S_I, S_{II} and S_{III}. For example, the left side of the criterion in Eq. (12) is equal to S_I when $h_1 = 1$, $h_2 = -1$, $h_3 = 0$ and $g_1 = g_2 = 1$. From Eq. (12) then, S_I, S_{II} and S_{III} satisfy the following inequalities:

$$S_I \geq \frac{1}{2} \left(|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| \right)$$
$$S_{II} \geq \frac{1}{2} \left(|\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle| \right)$$
$$S_{III} \geq \frac{1}{2} \left(|\langle J_{z,1} \rangle| + |\langle J_{z,3} \rangle| \right) . \quad (16)$$

Criterion 4. Full tripartite inseparability is observed if any two of the inequalities (16) are violated. Genuine tripartite entanglement is present if the following inequality is violated:

$$S_I + S_{II} + S_{III} \geq \frac{1}{2} \left(|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle| \right) . \quad (17)$$

Proof. The first result follows as for Criterion 2. Using the same notation as in the proof for Criterion 2, we index the biseparable states $\rho_1 \rho_{23}$, $\rho_2 \rho_{13}$ and $\rho_3 \rho_{12}$ by $k = 1, 2, 3$, respectively. Let $S_{I,1}$ be the quantity S_I that is evaluated using the biseparable state $\rho_1 \rho_{23}$. Then,

$$S_I \geq \sum_k P_k S_{I,k}$$
$$\geq P_1 S_{I,1} + P_2 S_{I,2} \geq \frac{1}{2} (P_1 + P_2) \left(|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| \right) .$$

Similarly, $S_{II} \geq (P_2 + P_3) \left(|\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle| \right) / 2$ and $S_{III} \geq (P_1 + P_3) \left(|\langle J_{z,1} \rangle| + |\langle J_{z,3} \rangle| \right) / 2$. In order to include all possible mixtures, we take the sum of S_I, S_{II} and S_{III}. The inequality they satisfy, derived below, provides a criterion for genuine tripartite entanglement:

$$S_I + S_{II} + S_{III} \geq \frac{(P_1 + P_2 + P_3)(|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle|)}{2}$$
$$+ \frac{1}{2} \left(P_1 |\langle J_{z,1} \rangle| + P_2 |\langle J_{z,2} \rangle| + P_3 |\langle J_{z,3} \rangle| \right)$$
$$\geq \frac{(P_1 + P_2 + P_3)(|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle|)}{2}$$
$$= \frac{1}{2} \left(|\langle J_{z,1} \rangle| + |\langle J_{z,2} \rangle| + |\langle J_{z,3} \rangle| \right) ,$$

where $\sum_k P_k = 1$. $\square$

III. INEQUALITIES INVOLVING PLANAR SPIN UNCERTAINTY RELATIONS

The inequalities in the previous two sections used the canonical spin uncertainty relations. For certain quantum states such as the multipartite spin GHZ state, the right side of these inequalities might be zero, giving the trivial relation that a sum or product of variances should be positive. Here, we consider the planar uncertainty relation, where the sum of uncertainties in two of the orthogonal spin observables has a lower bound that is a function of the spin value of the state. The planar

uncertainty relation was obtained for spin $J = 1/2$ [72] and $J = 1$ [30], and was later calculated for an arbitrary spin J by He et al. [67]. In that work, they minimized $\Delta^2 J_x + \Delta^2 J_y$ for a general quantum state written in the spin-z basis as

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{m=-J}^{J} R_m e^{-i\phi_m} |J,m\rangle, \qquad (18)$$

Here R_m, ϕ_m are real numbers characterizing the amplitude and phase of the basis state $|J,m\rangle$, while n is the normalization factor given by $n = \sum_{m=-J}^{J} R_m^2$. He et al. found the lower bound C_J $(C_J > 0)$ such that for a given J

$$\Delta^2 J_x + \Delta^2 J_y \geq C_J \qquad (19)$$

Also in that work [67], a criterion that verifies the N-partite inseparability was derived. Since the total N-partite separable state is a probabilistic sum of tensor product of N density operators, the planar uncertainty relation can be used. This is not the case for genuine multipartite entanglement where a biseparable state contains partitions that cannot be expressed as a product state of those particles/ modes in those partitions.

Nevertheless, the planar uncertainty relation can be used to detect genuine tripartite entanglement, if we use an inference variance method [15, 73].

Criterion 5. Consider the inequality given by

$$B_1 + B_2 + B_3 \geq C_J, \qquad (20)$$

where

$$B_1 = \Delta^2\left(J_{x,1} - O_{23}^{(1)}\right) + \Delta^2\left(J_{y,1} - P_{23}^{(1)}\right)$$
$$B_2 = \Delta^2\left(J_{x,2} - O_{13}^{(2)}\right) + \Delta^2\left(J_{y,2} - P_{13}^{(2)}\right)$$
$$B_3 = \Delta^2\left(J_{x,3} - O_{12}^{(3)}\right) + \Delta^2\left(J_{y,3} - P_{12}^{(3)}\right)$$

and $O_{lm}^{(k)}$, $P_{lm}^{(k)}$ are observables defining measurements that can be made on the combined subsystems that we denote by l and m. The violation of this inequality suffices to confirm genuine tripartite entanglement of the three systems denoted 1, 2 and 3. Full tripartite inseparability is observed if

$$B_k \geq C_J \qquad (21)$$

for each $k = 1, 2, 3$.

Proof. Consider $\Delta^2\left(J_{x,1} - O_{23}^{(1)}\right)$ and $\Delta^2\left(J_{y,1} - P_{23}^{(1)}\right)$ where $O_{23}^{(1)}$ and $P_{23}^{(1)}$ are operators for systems 2 and 3. We derive the following inequality that holds for an arbitrary pure state with a separable bipartition $\rho_1^\zeta \rho_{23}^\zeta$.

$$\begin{aligned} B_1 &= \Delta^2\left(J_{x,1} - O_{23}^{(1)}\right) + \Delta^2\left(J_{y,1} - P_{23}^{(1)}\right) \\ &\geq \Delta^2\left(J_{x,1}\right) + \Delta^2\left(J_{y,1}\right) \\ &\geq C_J \end{aligned} \qquad (22)$$

This holds also for all mixtures of separable bipartitions $\rho_1^\zeta \rho_{23}^\zeta$. Similarly, the inequalities

$$B_2 \geq \Delta^2\left(J_{x,2} - O_{13}^{(2)}\right) + \Delta^2\left(J_{y,2} - P_{13}^{(2)}\right) \geq C_J \quad (23)$$

and

$$B_3 \geq \Delta^2\left(J_{x,3} - O_{12}^{(3)}\right) + \Delta^2\left(J_{y,3} - P_{12}^{(3)}\right) \geq C_J \quad (24)$$

follow from the separable bipartitions $\rho_2^\zeta \rho_{13}^\zeta$ and $\rho_3^\zeta \rho_{12}^\zeta$ respectively. For a pure state, if all three inequalities are violated, we can conclude that the three systems are genuinely tripartite entangled. For a mixed state the conditions change. We require to falsify an arbitrary biseparable mixed state given by $\rho_{BS} = P_1 \sum_R \eta_R^{(1)} \rho_1^R \rho_{23}^R + P_2 \sum_{R'} \eta_{R'}^{(2)} \rho_2^{R'} \rho_{13}^{R'} + P_3 \sum_{R''} \eta_{R''}^{(3)} \rho_3^{R''} \rho_{12}^{R''}$, as defined by Eq. (1). We give a proof similar to those given for Criteria 2 and 4. For brevity, we index the biseparable states $\sum_R \eta_R^{(1)} \rho_1^R \rho_{23}^R$, $\sum_{R'} \eta_{R'}^{(2)} \rho_2^{R'} \rho_{13}^{R'}$ and $\sum_{R''} \eta_{R''}^{(3)} \rho_3^{R''} \rho_{12}^{R''}$ by $k = 1, 2, 3$, respectively. Thus, we denote $B_{1,1}$ to be the quantity B_1 that is evaluated using the biseparable state $\sum_R \eta_R^{(1)} \rho_1^R \rho_{23}^R$. Then, for the biseparable mixture,

$$\begin{aligned} B_1 &\geq \sum_k P_k B_{1,k} \\ &\geq P_1 B_{1,1} \geq P_1 C_J. \end{aligned}$$

Similarly, for a biseparable mixture, $B_2 \geq P_2 C_J$ and $B_3 \geq P_3 C_J$. In order to include all possible biseparable mixtures, we consider

$$B_1 + B_2 + B_3 \geq (P_1 + P_2 + P_3) C_J = C_J$$

using $\sum_k P_k = 1$. Thus, all biseparable mixtures are excluded when this inequality is violated. $\square$

This inequality has been derived in Ref. [42] in a similar context, to give a condition for genuine tripartite steering. Steering is a form of entanglement linked to the Einstein-Podolsky-Rosen paradox, and hence a steering criterion will also be a criterion for entanglement [65]. The entanglement criterion might be made stronger, if one can make use of uncertainty relations for the operators $O_{lm}^{(k)}$ and $P_{lm}^{(k)}$ once these are established for a given scenario.

It is straightforward to see that the inequality is violated for the GHZ state [74], defined as

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle\right) \qquad (25)$$

where $|\uparrow\uparrow\uparrow\rangle$ $(|\downarrow\downarrow\downarrow\rangle)$ is the state with z-spins up (down) for all subsystems $k = 1, 2, 3$. This is because, as is well-known for the GHZ state, the z-spin, x-spin and the y-spin of any of the three subsystems can be inferred by joint measurements made on the other two subsystems. This result is clear for inferring the value of $J_{z,k}$. The inequality (20) applies for all spin pairs, and if we replace $J_{y,i}$ with $J_{z,i}$, it is clear that by taking $P_{lm}^{(k)} = J_z^{(l)}$,

one can achieve $\Delta^2\left(J_{z,k} - P_{lm}^{(k)}\right) = 0$ for each k. For inferring $J_{x,k}$, it is also clear, since the GHZ state is an eigenstate of $J_{x,1}J_{x,2}J_{x,3}$ with eigenvalue -1. Thus, $O_{lm}^{(k)}$ is the measurement given as follows: Measure the spin J_x of each of the other subsystems l and m, and assign the value of the measurement by multiplying the spins values together. If the product is $+1$, then the outcome of $O_{lm}^{(k)}$ is -1. If the product is -1, then the outcome of $O_{lm}^{(k)}$ is $+1$. In this way, we see that $\Delta^2\left(J_{x,k} - O_{lm}^{(k)}\right) = 0$, for each $k = 1, 2, 3$ with $l \neq m \neq k$. Hence, the inequality (20) is violated, giving a simple method to detect the genuine tripartite entanglement of GHZ states (or approximate GHZ states) in an experiment.

We may ask whether the inequality is also violated for the W state [75] given by

$$|W\rangle = \frac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle).\qquad(26)$$

Here we will use the criterion expressed in Pauli spins, so that $B_i = \Delta^2\left(\sigma_{z,i} - O_{jk}^{(1)}\right) + \Delta^2\left(\sigma_{x,i} - P_{jk}^{(1)}\right)$ where $i \neq j \neq k$. The conditions then utilize $C_J = 1$ since $J = 1/2$ [72]. The spin σ_z of system 1 can be inferred by measuring the spin product of 2 and 3. We find that $\Delta^2(\sigma_{z,1} - O_{23}^{(1)}) = 0$. Now consider that the spins σ_x of systems 2 and 3 are simultaneously measured. We consider the measurement $P_{23}^{(1)}$ to have an outcome of 1 if both spins are measured as $+1$; an outcome -1 if the spins are measured as -1; and zero otherwise. Simple calculation tells us that $\Delta^2(\sigma_{x,1} - P_{23}^{(1)}) = \frac{1}{2}$. By symmetry of the W state, this result holds for all permutations of the subsystems. Thus we see that we are able to confirm entanglement across each bipartition, since the condition (22) for Pauli spins reduces to $B_1 \geq 1$. Since we find $B_1 = B_2 = B_3 = \frac{1}{2}$, the condition for tripartite inseparability is satisfied. If in an experiment we are able to verify a pure state, then this implies genuine tripartite entanglement. We note the above condition for mixed states, $B_1 + B_2 + B_3 < 1$ is not satisfied. The W state (26) *is* genuinely tripartite entangled. That the condition is not satisfied merely reflects that the criteria we derive are sufficient, but not necessary, to certify genuine tripartite entanglement.

Svetlichny derived conditions to detect the genuine tri-partite entanglement of three spin $1/2$ systems in the form of Bell inequalities. Further criteria for the certification of the genuine tripartite entanglement of GHZ, W and cluster states have been derived in Refs. [20, 35, 63]. The method given above is not necessarily advantageous over these earlier methods. It can be readily extended (by applying uncertainty relation (19)) however to conditions for higher J.

IV. CRITERIA FOR GENUINE N-PARTITE ENTANGLEMENT

The method used in Section II to derive criteria for genuine tripartite entanglement can be extended to N-partite systems. The complication arises in that the set of possible bipartitions scales as $\left(2^{N-1} - 1\right)$, and every bipartition has to be taken into account in the derivation of these criteria that certify genuine N-partite entanglement.

Here, we generalize the criterion in Eq. (4) for N-partite spin systems.

Criterion 6. We denote each bipartition by $S_r - S_s$, where S_r and S_s are two sets of modes in the partitions in a specific bipartition. Then, the violation of the inequality

$$\Delta^2 u + \Delta^2 v \geq \min\{S_B\},\qquad(27)$$

implies genuine N-partite entanglement, where S_B is $\left(\left|\sum_{k_r=1}^m h_{k_r} g_{k_r}\langle J_{z,k_r}\rangle\right| + \left|\sum_{k_s=1}^n h_{k_s} g_{k_s}\langle J_{z,k_s}\rangle\right|\right)$. The proof for this inequality follows from the proof for the inequality in Eq. (4).

Criterion 7. Similarly, the corresponding product inequality is given by

$$\Delta u \Delta v \geq \frac{1}{2}\min\{S_B\}.\qquad(28)$$

A. Criteria for genuine four-partite entanglement

1. Sum and product inequalities

Criterion 8. For $N = 4$, there will be $2^{4-1} - 1 = 7$ bipartitions. They are, using the $S_r - S_s$ notation, $1-234$, $2-134$, $3-124$, $4-123$, $12-34$, $13-24$ and $14-23$. The sum inequality in Eq. (27) is then

$$\begin{aligned}
\Delta^2 u + \Delta^2 v \geq \min\{&|g_1 h_1 \langle J_{z,1}\rangle| + |g_2 h_2\langle J_{z,2}\rangle + g_3 h_3\langle J_{z,3}\rangle + g_4 h_4\langle J_{z,4}\rangle|,\\
&|g_2 h_2\langle J_{z,2}\rangle| + |g_1 h_1\langle J_{z,1}\rangle + g_3 h_3\langle J_{z,3}\rangle + g_4 h_4\langle J_{z,4}\rangle|,\\
&|g_3 h_3\langle J_{z,3}\rangle| + |g_1 h_1\langle J_{z,1}\rangle + g_2 h_2\langle J_{z,2}\rangle + g_4 h_4\langle J_{z,4}\rangle|,\\
&|g_4 h_4\langle J_{z,4}\rangle| + |g_1 h_1\langle J_{z,1}\rangle + g_2 h_2\langle J_{z,2}\rangle + g_3 h_3\langle J_{z,3}\rangle|,\\
&|g_1 h_1\langle J_{z,1}\rangle + g_2 h_2\langle J_{z,2}\rangle| + |g_3 h_3\langle J_{z,3}\rangle + g_4 h_4\langle J_{z,4}\rangle|,\\
&|g_1 h_1\langle J_{z,1}\rangle + g_3 h_3\langle J_{z,3}\rangle| + |g_2 h_2\langle J_{z,2}\rangle + g_4 h_4\langle J_{z,4}\rangle|,\\
&|g_1 h_1\langle J_{z,1}\rangle + g_4 h_4\langle J_{z,4}\rangle| + |g_2 h_2\langle J_{z,2}\rangle + g_3 h_3\langle J_{z,3}\rangle|\} \equiv \min\{S_{B,4}\}.
\end{aligned}\qquad(29)$$

Criterion 9. Similarly, the product inequality for genuine four-partite entanglement is given by

$$\Delta u \Delta v \geq \frac{1}{2}\min\{S_{B,4}\} ,\qquad (30)$$

where $S_{B,4}$ is defined in Eq. (29). The violation of inequality in Eq. (29) or Eq. (30) implies the presence of genuine four-partite entanglement.

2. Criteria involving van Loock-Furusawa inequalities

Van Loock and Furusawa [18] derived a set of six inequalities to rule out four-partite inseparability. We can derive similar inequalities to certify genuine four-partite entanglement. The six spin inequalities are given by

$$
\begin{aligned}
B_I &\equiv \Delta^2\left(J_{x,1}-J_{x,2}\right)+\Delta^2\left(J_{y,1}+J_{y,2}+g_3 J_{y,3}+g_4 J_{y,4}\right) \geq \left(|\langle J_{z,1}\rangle|+|\langle J_{z,2}\rangle|\right) ,\\
B_{II} &\equiv \Delta^2\left(J_{x,2}-J_{x,3}\right)+\Delta^2\left(g_1 J_{y,1}+J_{y,2}+J_{y,3}+g_4 J_{y,4}\right) \geq \left(|\langle J_{z,2}\rangle|+|\langle J_{z,3}\rangle|\right) ,\\
B_{III} &\equiv \Delta^2\left(J_{x,1}-J_{x,3}\right)+\Delta^2\left(J_{y,1}+g_2 J_{y,2}+J_{y,3}+g_4 J_{y,4}\right) \geq \left(|\langle J_{z,1}\rangle|+|\langle J_{z,3}\rangle|\right) ,\\
B_{IV} &\equiv \Delta^2\left(J_{x,3}-J_{x,4}\right)+\Delta^2\left(g_1 J_{y,1}+g_2 J_{y,2}+J_{y,3}+J_{y,4}\right) \geq \left(|\langle J_{z,3}\rangle|+|\langle J_{z,4}\rangle|\right) ,\\
B_V &\equiv \Delta^2\left(J_{x,2}-J_{x,4}\right)+\Delta^2\left(g_1 J_{y,1}+J_{y,2}+g_3 J_{y,3}+J_{y,4}\right) \geq \left(|\langle J_{z,2}\rangle|+|\langle J_{z,4}\rangle|\right) ,\\
B_{VI} &\equiv \Delta^2\left(J_{x,1}-J_{x,4}\right)+\Delta^2\left(J_{y,1}+g_2 J_{y,2}+g_3 J_{y,3}+J_{y,4}\right) \geq \left(|\langle J_{z,1}\rangle|+|\langle J_{z,4}\rangle|\right) .
\end{aligned}
\qquad (31)
$$

Criterion 10. The violation of any three of the above inequalities implies that the four-partite system is not in any biseparable states, and thus signifies four-partite inseparability (refer Ref. [18] for the proof). Genuine four-partite entanglement is verified if the inequality

$$\sum_{J=1}^{6} B_J \geq |\langle J_{z,1}\rangle|+|\langle J_{z,2}\rangle|+|\langle J_{z,3}\rangle|+|\langle J_{z,4}\rangle| \qquad (32)$$

is violated. These criteria are sufficient but not necessary conditions for four-partite inseparability, or genuine four-partite entanglement.

Proof. For brevity, we index the biseparable states $\rho_1\rho_{234}$, $\rho_2\rho_{134}$, $\rho_3\rho_{124}$, $\rho_4\rho_{123}$, $\rho_{12}\rho_{34}$, $\rho_{13}\rho_{24}$ and $\rho_{14}\rho_{23}$ by $k=1,2,...,7$, respectively. Let $B_{I,1}$ be the quantity B_I that is evaluated using the biseparable state $\rho_1\rho_{234}$. Then,

$$
\begin{aligned}
B_I &\geq \sum_k P_k B_{I,k}\\
&\geq P_1 B_{I,1}+P_2 B_{I,2}+P_6 B_{I,6}+P_7 B_{I,7}\\
&\geq \left(P_1+P_2+P_6+P_7\right)\left(|\langle J_{z,1}\rangle|+|\langle J_{z,2}\rangle|\right) .
\end{aligned}
\qquad (33)
$$

Similarly, $B_{II} \geq \left(P_2+P_3+P_5+P_6\right)\left(|\langle J_{z,2}\rangle|+|\langle J_{z,3}\rangle|\right)$, $B_{III} \geq \left(P_1+P_3+P_5+P_7\right)\left(|\langle J_{z,1}\rangle|+|\langle J_{z,3}\rangle|\right)$, $B_{IV} \geq \left(P_3+P_4+P_6+P_7\right)\left(|\langle J_{z,3}\rangle|+|\langle J_{z,4}\rangle|\right)$, $B_V \geq \left(P_2+P_4+P_5+P_7\right)\left(|\langle J_{z,2}\rangle|+|\langle J_{z,4}\rangle|\right)$ and $B_{VI} \geq \left(P_1+P_4+P_5+P_6\right)\left(|\langle J_{z,1}\rangle|+|\langle J_{z,4}\rangle|\right)$. In order to include all possible mixtures, we take the sum of B_I, B_{II}, B_{III}, B_{IV}, B_V and B_{VI}. The inequality they satisfy, derived below, provides a criterion for genuine four-partite entanglement. The violation of the following inequality implies genuine four-partite entanglement:

$$
\begin{aligned}
\sum_{J=1}^{6} B_J &\geq \left(P_1+P_2+P_3+P_4+P_5+P_6+P_7\right)\left(|\langle J_{z,1}\rangle|\right.\\
&\quad +|\langle J_{z,2}\rangle|+|\langle J_{z,3}\rangle|+\left.|\langle J_{z,4}\rangle|\right)\\
&\quad +\left(2P_1+P_5+P_6+P_7\right)|\langle J_{z,1}\rangle|\\
&\quad +\left(2P_2+P_5+P_6+P_7\right)|\langle J_{z,2}\rangle|\\
&\quad +\left(2P_3+P_5+P_6+P_7\right)|\langle J_{z,3}\rangle|\\
&\quad +\left(2P_4+P_5+P_6+P_7\right)|\langle J_{z,4}\rangle|\\
&\geq \left(P_1+P_2+P_3+P_4+P_5+P_6+P_7\right)\left(|\langle J_{z,1}\rangle|\right.\\
&\quad +|\langle J_{z,2}\rangle|+|\langle J_{z,3}\rangle|+\left.|\langle J_{z,4}\rangle|\right)\\
&= |\langle J_{z,1}\rangle|+|\langle J_{z,2}\rangle|+|\langle J_{z,3}\rangle|+|\langle J_{z,4}\rangle| ,
\end{aligned}
\qquad (34)
$$

where $\sum_k P_k = 1$. $\square$

V. APPLICATIONS

We now show how one may create N-partite entangled states satisfying the criteria derived in Sections II and IV of this paper. In Section V A, we outline optical experiments involving polarization entanglement, where the measured observables at each site are the Stokes operators for two polarization modes. We then consider, in Section V B, experiments that entangle spatially-separated atomic ensembles. In Section V C, we analyze recent experiments that generate entanglement between spatially-separated clouds of atoms formed from a spin-squeezed Bose-Einstein condensate. Here, for each separated subsystem, the measured observable is a Schwinger operator involving two internal atomic levels. The Schwinger and Stokes operators satisfy the same
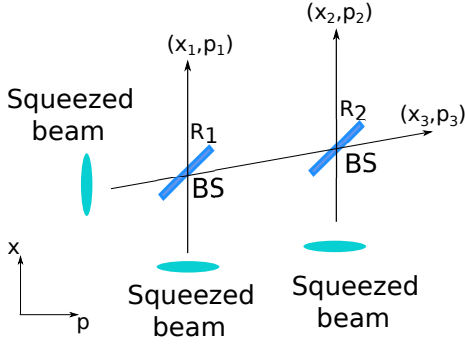
Figure 1. Generation of the tripartite-entangled CV GHZ state. The configuration uses three squeezed-vacuum inputs and two beam splitters (BS) with reflectivities $R_1 = 1/3$ and $R_2 = 1/2$. The x_i and p_i are the two orthogonal quadrature-phase amplitudes of the spatially separated optical modes i ($i = 1, 2, 3$).
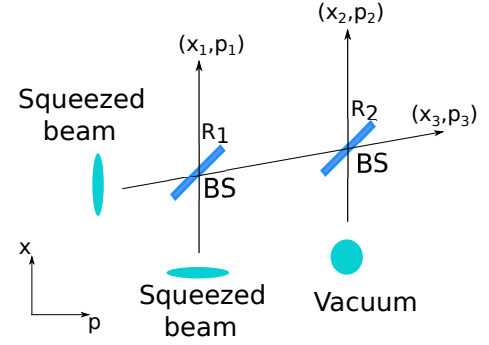

Figure 2. Generation of the tripartite entangled EPR-type state. The configuration uses two squeezed-vacuum inputs, a coherent-vacuum input, and two beam splitters BS with reflectivities $R_1 = R_2 = 1/2$. The x_i and p_i are the two orthogonal quadrature-phase amplitudes of the spatially separated optical modes i ($i = 1, 2, 3$).

commutation relation as spin operators, and hence all the criteria derived in Sections II-IV are applicable.

A. Polarization entanglement

The polarization of a quantum state can be characterized by the Stokes operators defined as [76]

$$
\begin{aligned}
S_x &= a_H^\dagger a_H - a_V^\dagger a_V \\
S_y &= a_H^\dagger a_V e^{i\theta} + a_V^\dagger a_H e^{-i\theta} \\
S_z &= i a_V^\dagger a_H e^{-i\theta} - i a_H^\dagger a_V e^{i\theta} ,
\end{aligned}
\tag{35}
$$

where a_H and a_V are the annihilation operators of the horizontal and vertical polarization modes respectively, and θ is the phase difference between these polarization modes. In the work of Bowen et al. [76], bipartite polarization entanglement was created by first generating CV bipartite entanglement in the quadrature degree of freedom, and then transferring the entanglement into the polarization degree of freedom.

This scheme can be extended to generate genuine tripartite polarization entanglement. Genuine CV tripartite entanglement in the quadratures is first created in an optical setup involving squeezed vacuums and beam splitters, as shown in Figs. 1 and 2. The three entangled modes from the outputs of these beam splitters are horizontally polarized. Each of these modes is subsequently mixed with a bright coherent beam with vertical polarization using a polarizing beam splitter. At each site $i = 1, 2, 3$ prior to mixing, one can define pairs of orthogonally polarized modes (with annihilation operators a_{Hi}, $a_{V,i}$). The choice of polariser angle determines which Stokes observable is measured, after a number difference is taken. The final readout is given as a difference current. After the mixing, the genuine CV entanglement has been transformed into genuine tripartite polarization entanglement, as illustrated in Figure 3.
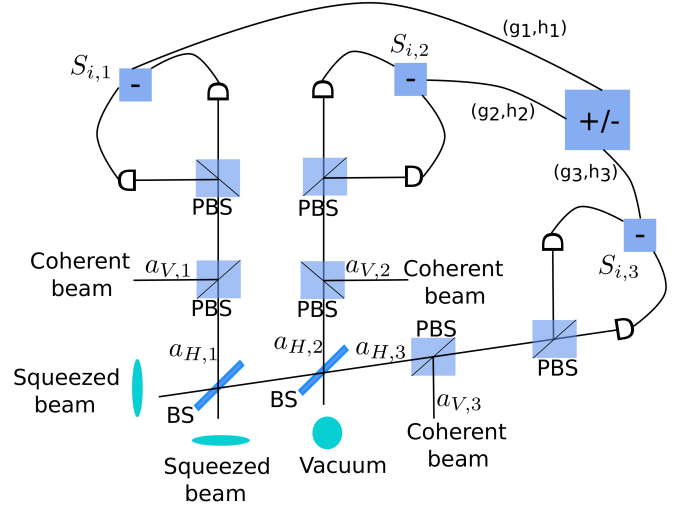
Figure 3. The experimental setup to generate genuine tripartite polarization entanglement from genuine tripartite CV entanglement. In this schematic diagram, an EPR-type genuine tripartite-entanglement is generated as shown in Figure 2. The outputs are mixed with coherent fields, as described in the text. The $S_{i,k}$ denotes the polarisation $S_{x,k}$, $S_{y,k}$ or $S_{z,k}$ for the site k ($k = 1, 2, 3$). The (g_k, h_k) are the gains used in the criteria, and are introduced in the final currents. By using a third squeezed input state at the second beam splitter instead of the vacuum input, the CV GHZ genuine tripartite entanglement (refer Figure 1) can be transformed into an equivalent genuine tripartite polarization entanglement. Alternatively, by using only one squeezed input, one may transfer the genuine tripartite entanglement depicted in Figure 4.

To verify the tripartite polarization entanglement, we consider the sum inequality

$$
\begin{aligned}
&\Delta^2 \left[S_{y,1} + h \left(S_{y,2} + S_{y,3} \right) \right] + \Delta^2 \left[S_{z,1} + g \left(S_{z,2} + S_{z,3} \right) \right] \\
&\geq 2 \min \{ \alpha_v^2 + 2 |gh| \alpha_v^2, |gh| \alpha_v^2 + \alpha_v^2 |1 + gh| \} ,
\end{aligned}
\tag{36}
$$

where α_v is the coherent amplitude of the vertically po-

larized coherent beam. The variances are

$$\Delta^2\left[S_{y,1}+h\left(S_{y,2}+S_{y,3}\right)\right]=\alpha_v^2\Delta^2\left[P_{H,1}+h(P_{H,2}+P_{H,3})\right]$$
$$\Delta^2\left[S_{z,1}+g\left(S_{z,2}+S_{z,3}\right)\right]=\alpha_v^2\Delta^2\left[X_{H,1}+g(X_{H,2}+X_{H,3})\right].\tag{37}$$

Here, $S_{x,k}$, $S_{y,k}$ and $S_{z,k}$ are the Stokes operators defined in (35) for each mode pair at site k. $X_{H,k}$ and $P_{H,k}$ are the X and P quadratures for beam k. The h and g are gain factors defined in the criteria, where we take $h_1=1$, $h_2=h_3=h$ and $g_1=1$, $g_2=g_3=g$. Note that the commutation relations satisfied by these Stokes operators are $[S_x,S_y]=2iS_z$, which differ from the spin commutation relations by a factor of 2. As a result, the sum and product inequalities below have an extra factor of 2 compared to the sum and product inequalities in Eqs. (4) and (12) respectively. With these variances, the sum inequality Eq. (4) and the product inequality Eq. (12) are respectively transformed into a continuous-variable genuine tripartite entanglement sum and product criterion, according to

$$\frac{\Delta^2\left[S_{y,1}+h\left(S_{y,2}+S_{y,3}\right)\right]+\Delta^2\left[S_{z,1}+g\left(S_{z,2}+S_{z,3}\right)\right]}{2\alpha_v^2\min\{1+2\left|gh\right|,\left|gh\right|+\left|1+gh\right|\}}$$

$$=\frac{\Delta^2[X_{H,1}+g\left(X_{H,2}+X_{H,3}\right)]+\Delta^2[P_{H,1}+h\left(P_{H,2}+P_{H,3}\right)]}{2\min\{1+2\left|gh\right|,\left|gh\right|+\left|1+gh\right|\}}$$

$$\geq 1\tag{38}$$

and

$$\frac{\Delta\left[S_{y,1}+h\left(S_{y,2}+S_{y,3}\right)\right]\Delta\left[S_{z,1}+g\left(S_{z,2}+S_{z,3}\right)\right]}{\min\{\alpha_v^2+2\left|gh\right|\alpha_v^2,\left|gh\right|\alpha_v^2+\alpha_v^2\left|1+gh\right|\}}$$

$$=\frac{\Delta\left[X_{H,1}+g\left(X_{H,2}+X_{H,3}\right)\right]\Delta\left[P_{H,1}+h\left(P_{H,2}+P_{H,3}\right)\right]}{\min\{1+2\left|gh\right|,\left|gh\right|+\left|1+gh\right|\}}$$

$$\geq 1\,.\tag{39}$$

Hence, any CV genuine tripartite quadrature entanglement then implies genuine tripartite polarization entanglement.

There are two types of states that show genuine tripartite entanglement in the quadratures. These are the CV GHZ and CV EPR-type states, defined in Refs. [18] and [14], and illustrated in Figs. 1 and 2 respectively. It has been shown previously that these two states violate both the quadrature sum inequality in Eq. (38) and the product inequality in Eq. (39) with specific values for the gains, $g_1=h_1=1$ and $g_{i>1}=g$, $h_{i>1}=h$ [14]. The gains g,h are chosen such that the variance sum and product are minima, and are given in Table I. With these gain values, as shown in Ref. [14], Criteria 1 and 3 are always violated for any nonzero squeezing of the squeezed vacuum inputs, implying the presence of genuine tripartite entanglement. The inequalities of Criteria 2 and 4 are also useful in showing genuine tripartite

r	CV GHZ		CV EPR	
	g	h	g	h
0	0	0	0	0
0.25	0.36	-0.27	0.33	-0.33
0.50	0.68	-0.40	0.54	-0.54
0.75	0.86	-0.46	0.64	-0.64
1.00	0.95	-0.49	0.68	-0.68
1.50	0.99	-0.50	0.70	-0.70
2.00	1.00	-0.50	0.70	-0.70

Table I. Values of the gains g and h that minimize the variance sum and product in Criteria 1 and 3.
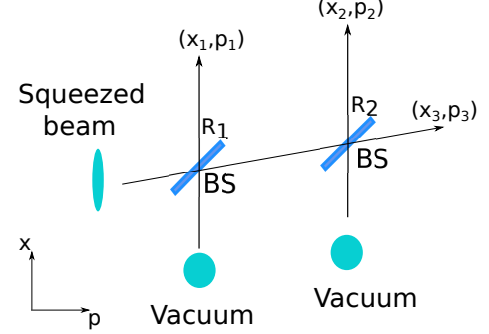
Figure 4. Generation of tripartite entanglement using a squeezed vacuum beam with squeezing along the P (or X) quadrature. All other beam splitter ports have vacuum inputs. The reflectivity for the first beam splitter is $R_1=1/3$ and $R_2=1/2$ for the second beam splitter. A calculation of the genuine tripartite entanglement generated from this configuration is given in Appendix 3.

entanglement. The optimal gains for these inequalities can be found in Ref. [14].

Genuine tripartite entanglement is also created using a third configuration involving only one squeezed input, shown in Fig. 4. Normally, two squeezed vacuum inputs are combined across a beam splitter to create strong EPR-correlations between the output modes [19, 73]. It is also possible to create EPR-entangled modes, using only one squeezed vacuum input [15]. While the EPR correlations are weaker, the entanglement is sufficiently strong that a subsequent beam-splitter interaction with a non-squeezed vacuum input can create genuine tripartite entanglement. A summary of this calculation is given in the Appendix 3, where we show how the entanglement that is generated can be detected by Criterion 5 of Ref. [14] with the gains $h=-1/2$ and $g=1/2$. This tripartite entanglement is not sufficiently strong to generate tripartite EPR-steering correlations, but can be transformed into genuine tripartite polarization-entanglement using the configuration of Fig. 3. The spin sum-inequality given by Criterion 1 is then useful to detect the genuine tripartite entanglement.

entanglement that can be detected using a similar spin criterion [81–83].

The experiment of Fadel et al. observed bipartite entanglement and EPR steering, but did not investigate tripartite entanglement. It is likely however that one could detect a genuine tripartite entanglement for clouds generated by further splitting the BEC. This would seem possible, given the result obtained in the Appendix 3 and depicted in Fig. 4, where tripartite entanglement is generated using only one squeezed input, followed by a sequence of splitting of the modes using beam splitter interactions. This works, because entangled modes can be created from a beam splitter with only one squeezed vacuum input [15]. The tripartite entanglement created in the three modes of Fig. 4 can be detected using the Criterion 5 of Ref. [14] with the gains $h = -1/2$ and $g = 1/2$. If one considers transforming into an equivalent tripartite entanglement in the Schwinger operators, then the suitable criterion would be Criterion 3 in Eq. (12) with the gains $h = -1/2$ and $g = 1/2$.

A realization of a beam splitter interaction for the BEC can be obtained in several ways. An analogy of optical beam splitters with the splitting of a condensate (which is envisaged to be a realization of the final beam splitter of Fig. 4) is explained in Ref [84]. The splitting into two modes is described by the interaction Hamiltonian

$$H_{I+} = e^{i\phi} a_+^\dagger a_{+0} + e^{-i\phi} a_+ a_{+0}^\dagger \,, \qquad (45)$$

where a_+, a_{+0} are the annihilation operators for modes labelled A_+ and $A_{+,0}$ respectively, and ϕ is the phase difference between these two modes. The transformation is equivalent to the beam splitter relations

$$a_{+,\text{out}} = a_+ \cos\tau - i e^{i\phi} a_{+0} \sin\tau$$
$$a_{+0,\text{out}} = a_{+0} \cos\tau - i e^{-i\phi} a_+ \sin\tau \,, \qquad (46)$$

where τ is the interaction time and $a_{+,\text{out}} = a_+(\tau)$, $a_{+0,\text{out}} = a_{+0}(\tau)$. One can adjust the effective transmission to reflection ratio by adjusting the interaction time between the two modes.

We thus consider two separated clouds A and B that show spin entanglement with respect to the difference operators $g_z S_{z,A} + S_{z,B}$ and $g_y S_{y,A} + S_{y,B}$ so that the criterion of Eq. (43) is satisfied. These two clouds are analogous to the entangled outputs after the first beam splitter BS of the configuration shown in Fig. 4. Each cloud is identified with Schwinger spin observables. For example, $S_{z,A}$ and $S_{y,A}$ are measurements that can be made on cloud A, where $S_{z,A} = \frac{1}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$ and a_+, a_- correspond to the two atomic levels. To generate the tripartite entanglement, the system A is transformed according to a beam splitter interaction (splitting) modeled as Eq. (45). Since the splitting is insensitive to the internal spin degrees of freedom, there is a similar independent interaction for a_-. The outputs of $a_{\pm 1}$ and $a_{\pm 2}$ are then spatially separated, so that three separate clouds are created, labelled $A1$, $A2$ and B, these being
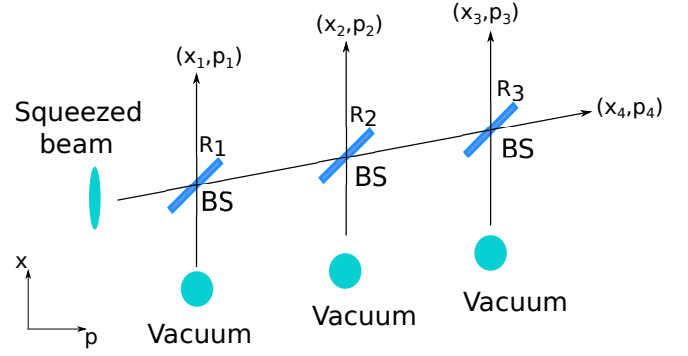

Figure 5. Generation of four-partite entanglement using a squeezed beam along the P (or X) quadrature. All other beam splitter ports have vacuum inputs. The reflectivity for the first beam splitter $R_1 = 1/4$, $R_2 = 1/3$ for the second beam splitter and $R_3 = 1/2$ for the third beam splitter.

analogous to the three outputs of the configuration of Fig. 4. The final Schwinger operators at A_1 and A_2 are defined by the $a_{\pm 1}$ at A_1, and the $a_{\pm 2}$ at A_2. The different Schwinger components can be measured using Rabi rotations or equivalent [50, 78]. The calculation carried out in Appendix 5 predicts a tripartite entanglement between the three clouds that could be detected by Criteria 1 and 3. Using Eqs. (55) and (56) in Appendix 5, the inequality of Criterion 3 is then

$$\Delta[g_z(S_{z,A1}+S_{z,A2})+S_{z,B}]\Delta[g_y(S_{y,A1}+S_{y.A2})+S_{y,B}]$$
$$\geq \frac{1}{2}\min\{|g_z g_y| |\langle S_{x,A1}\rangle + \langle S_{x,A2}\rangle| + |\langle S_{x,B}\rangle| \,,$$
$$|(g_z g_y \langle S_{x,A1}\rangle + \langle S_{x,B}\rangle)| + |g_z g_y| |\langle S_{x,A2}\rangle|\}\,. \qquad (47)$$

The violation of this inequality implies genuine tripartite entanglement. We show in Appendix 5 that, assuming the number of atoms is large, $S_{z,A1} + S_{z,A2} \approx S_{z,A}$, $S_{y,A1} + S_{y,A2} \approx S_{y,A}$, and $S_{x,A1} + S_{x,A2} \approx S_{x,A}$. The criterion for genuine tripartite entanglement will therefore be satisfied if there is sufficient entanglement as measured by the bipartite criterion given in Eq. (43). Assuming $S_{x,A}$ and $S_{x,B}$ correspond to the Bloch vectors, with the directions of axes being chosen to ensure $\langle S_{x,A}\rangle$ and $\langle S_{x,B}\rangle$ are positive, we see that the beam splitter transformation (refer Appendix 5) ensures the signs of $S_{x,A1}$ and $S_{x,A2}$ are also positive. The right-side of the inequality is then either precisely that given by the right-side of Eq. (43) (if $g_z g_y > 0$), or is less than this value (if $g_z g_y < 0$).

We note from the results reported in Refs. [14, 18, 24] that we can generate N-partite entangled states ($N > 3$) by successive use of beam splitters with vacuum inputs, once an initial entangled state is created from two squeezed inputs or some other means. This has been implemented for a BEC by Ref. [52] (for $N = 5$). We show in the Appendix 4 that we can also create genuinely $4-$partite entangled states from a single squeezed input (refer Fig. 5), followed by multiple beam splitter combinations and vacuum inputs (with no squeezing). This

may provide an avenue (using successive splittings) for the generation of multi-partite entanglement in experiments such as Ref. [50].

VI. CONCLUSION

In summary, we have derived several different criteria for certifying genuine N-partite entanglement using spin measurements. The criteria are inequalities expressed in terms of variances of spin observables measured at each of the N sites.

In Sections II and IV, we derive criteria based on the standard spin uncertainty relation, involving $|\langle J_z \rangle|$. These criteria are valid for any systems, provided at each site the outcomes are reported faithfully, as results of accurately calibrated quantum measurements [9, 85] . We present in Section V three examples of application of these criteria. In these examples, entanglement is created that can be detected using Stokes or Schwinger operators defined at each site. These observables arise naturally in atomic ensembles, where the creation and detection of multi-partite entanglement is important for testing the quantum mechanics of massive systems. The criteria we develop may be useful for this purpose. In particular we specifically propose how to extend the experiments of Julsgaard et al. [49] and Fadel et al. [50], to generate three or more genuinely-entangled spatially-separated ensembles of atoms. The experiment of Kunkel et al. [52] succeeded in generating genuine 5-partite entanglement.

Where Stokes operators are defined for atomic systems, it is possible to introduce a normalization with respect to total atom number. This concept was introduced by He et al. [81, 86] and Żukowski et al. [87–90]. These authors show how the detection of entanglement and nonlocality can be enhanced using such a normalization. It is likely that the criteria derived in Sections II and IV may also be further improved using this technique.

In Section III, we have outlined criteria derived from the planar spin uncertainty relation $\Delta^2 J_x + \Delta^2 J_y \geq C_J$ valid for a system of fixed spin J. This is useful for states where $\langle J_z \rangle = 0$, such as the GHZ states. Such criteria were developed previously for genuine tripartite steering. Although genuine tripartite steering implies genuine tripartite entanglement, we have extended the results of the earlier work by giving details of the application of these criteria to certify the genuine tripartite entanglement and the full tripartite inseparability of the GHZ and W states respectively. While other methods exist to detect the genuine tripartite entanglement of these states (for example [33, 35, 63]), the criteria we present in Section III are readily extended to higher spin J.

APPENDIX

1. Lower bound of the sum inequality for an arbitrary bipartition

We derive the inequality in Eq. (7) for an arbitrary pure state bipartition $\rho_k^\zeta \rho_{lm}^\zeta$.

$$
\begin{aligned}
&\Delta^2 u_\zeta + \Delta^2 v_\zeta \\
&= \Delta^2 \left(h_k J_{x,k} \right) + \Delta^2 \left(h_l J_{x,l} + h_m J_{x,m} \right) \\
&+ \Delta^2 \left(g_k J_{y,k} \right) + \Delta^2 \left(g_l J_{y,l} + g_m J_{y,m} \right) \\
&\geq \left| g_k h_k \left[J_{x,k}, J_{y,k} \right] \right| + \left| g_l h_l \left[J_{x,l}, J_{y,l} \right] + g_m h_m \left[J_{x,m}, J_{y,m} \right] \right| \\
&= \left| g_k h_k \langle J_{z,k} \rangle \right| + \left| g_l h_l \langle J_{z,l} \rangle + g_m h_m \langle J_{z,m} \rangle \right| . \quad (48)
\end{aligned}
$$

Here, the uncertainty relation $\Delta^2 \left(h J_x \right) + \Delta^2 \left(g J_y \right) \geq \langle \left| g h \left[J_x, J_y \right] \right| \rangle$ is used to obtain the first inequality in Eq. (48). The spin commutation relation $\left[J_x, J_y \right] = i J_z$ is used in the last line.

2. Lower bound of the product inequality for an arbitrary bipartition

We derive the inequality in Eq. (14) for an arbitrary bipartition $\rho_k^\zeta \rho_{lm}^\zeta$.

$$\Delta^2 u_\zeta \Delta^2 v_\zeta = \left[\Delta^2\left(h_k J_{x,k}\right) + \Delta^2\left(h_l J_{x,l} + h_m J_{x,m}\right)\right]\left[\Delta^2\left(g_k J_{y,k}\right) + \Delta^2\left(g_l J_{y,l} + g_m J_{y,m}\right)\right]$$
$$= \Delta^2\left(h_k J_{x,k}\right)\Delta^2\left(g_k J_{y,k}\right) + \Delta^2\left(h_l J_{x,l} + h_m J_{x,m}\right)\Delta^2\left(g_l J_{y,l} + g_m J_{y,m}\right)$$
$$+ \Delta^2\left(h_l J_{x,l} + h_m J_{x,m}\right)\Delta^2\left(g_k J_{y,k}\right) + \Delta^2\left(h_k J_{x,k}\right)\Delta^2\left(g_l J_{y,l} + g_m J_{y,m}\right)$$
$$\geq \Delta^2\left(h_k J_{x,k}\right)\Delta^2\left(g_k J_{y,k}\right) + \Delta^2\left(h_l J_{x,l} + h_m J_{x,m}\right)\Delta^2\left(g_l J_{y,l} + g_m J_{y,m}\right)$$
$$+ 2\Delta\left(h_l J_{x,l} + h_m J_{x,m}\right)\Delta\left(g_k J_{y,k}\right)\Delta\left(h_k J_{x,k}\right)\Delta\left(g_l J_{y,l} + g_m J_{y,m}\right)$$
$$= \left[\Delta\left(h_k J_{x,k}\right)\Delta\left(g_k J_{y,k}\right) + \Delta\left(h_l J_{x,l} + h_m J_{x,m}\right)\Delta\left(g_l J_{y,l} + g_m J_{y,m}\right)\right]^2$$
$$\geq \left[\frac{|g_k h_k \langle J_{z,k}\rangle|}{2} + \frac{|g_l h_l \langle J_{z,l}\rangle + g_m h_m \langle J_{z,m}\rangle|}{2}\right]^2 . \tag{49}$$

In going from the second equality to the first inequality, the inequality for two real numbers x and y, $x^2 + y^2 \geq 2xy$, is used. The uncertainty relation in the final line is $\Delta\left(h J_x\right)\Delta\left(g J_y\right) \geq \langle|gh\left[J_x, J_y\right]|\rangle/2$.

$\Delta^2\left[P_{a,\text{out}} + 1/2\left(P_{b,\text{out}} + P_{c,\text{out}}\right)\right] \geq 2$, as shown in Criterion 5 of Ref. [14] with the gains $h = -1/2$ and $g = 1/2$. This inequality is violated and hence the final output state in Fig. 4 is genuinely tripartite entangled. We can also consider the input to be squeezed along X, in which case the gains g and h will have opposite sign.

3. Generating genuine tripartite entangled states using 3 beam splitters and one squeezed input

Here we consider the configuration of Fig. 4. The output mode operators a_{out}, b_{out} and c_{out} are

$$a_{\text{out}} = \frac{1}{\sqrt{3}}a_{\text{in}} + \sqrt{\frac{2}{3}}b_{\text{in}}$$
$$b_{\text{out}} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{3}}a_{\text{in}} - \frac{1}{\sqrt{3}}b_{\text{in}}\right) + \frac{1}{\sqrt{2}}c_{\text{in}}$$
$$c_{\text{out}} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{3}}a_{\text{in}} - \frac{1}{\sqrt{3}}b_{\text{in}}\right) - \frac{1}{\sqrt{2}}c_{\text{in}} . \tag{50}$$

Now, we consider $X_{a,\text{out}} - 1/2\left(X_{b,\text{out}} + X_{c,\text{out}}\right)$ and $P_{a,\text{out}} + 1/2\left(P_{b,\text{out}} + P_{c,\text{out}}\right)$. Their variances are then

$$\Delta^2\left[X_{a,\text{out}} - \frac{\left(X_{b,\text{out}} + X_{c,\text{out}}\right)}{2}\right] = \frac{3}{2}\Delta^2 X_{b,\text{in}} = \frac{3}{2}$$
$$\Delta^2\left[P_{a,\text{out}} + \frac{\left(P_{b,\text{out}} + P_{c,\text{out}}\right)}{2}\right] = \frac{2}{3}\Delta^2 P_{a,\text{in}} + \frac{1}{6}\Delta^2 P_{b,\text{in}}$$
$$= \frac{2}{3}e^{-2r} + \frac{1}{6} \tag{51}$$

and their sum is

$$\Delta^2\left[X_{a,\text{out}} - \frac{\left(X_{b,\text{out}} + X_{c,\text{out}}\right)}{2}\right] + \Delta^2\left[P_{a,\text{out}} + \frac{\left(P_{b,\text{out}} + P_{c,\text{out}}\right)}{2}\right]$$
$$= \frac{10}{6} + \frac{2}{3}e^{-2r}, \tag{52}$$

giving a minimum of $10/6 = 1.6667$ for large squeezing parameter r. The sum inequality for those variances is $\Delta^2\left[X_{a,\text{out}} - 1/2\left(X_{b,\text{out}} + X_{c,\text{out}}\right)\right] +$

4. Generating genuine four-partite entangled states using 4 beam splitters and one squeezed input

Here we consider the configuration of Fig. 5. The output mode operators a_{out}, b_{out}, c_{out} and d_{out} are

$$a_{\text{out}} = \frac{1}{\sqrt{4}}a_{\text{in}} + \sqrt{\frac{3}{4}}b_{\text{in}}$$
$$b_{\text{out}} = \frac{1}{\sqrt{3}}\left(\sqrt{\frac{3}{4}}a_{\text{in}} - \frac{1}{\sqrt{4}}b_{\text{in}}\right) + \sqrt{\frac{2}{3}}c_{\text{in}}$$
$$c_{\text{out}} = \frac{1}{\sqrt{3}}\left(\sqrt{\frac{3}{4}}a_{\text{in}} - \frac{1}{\sqrt{4}}b_{\text{in}}\right) - \frac{1}{\sqrt{6}}c_{\text{in}} + \frac{1}{\sqrt{2}}d_{\text{in}}$$
$$d_{\text{out}} = \frac{1}{\sqrt{3}}\left(\sqrt{\frac{3}{4}}a_{\text{in}} - \frac{1}{\sqrt{4}}b_{\text{in}}\right) - \frac{1}{\sqrt{6}}c_{\text{in}} - \frac{1}{\sqrt{2}}d_{\text{in}} . \tag{53}$$

Now, we consider $X_{a,\text{out}} - 1/3\left(X_{b,\text{out}} + X_{c,\text{out}} + X_{d,\text{out}}\right)$ and $P_{a,\text{out}} + 1/3\left(P_{b,\text{out}} + P_{c,\text{out}} + P_{d,\text{out}}\right)$. Their variances are then

$$\Delta^2\left[X_{a,\text{out}} - \frac{\left(X_{b,\text{out}} + X_{c,\text{out}} + X_{d,\text{out}}\right)}{3}\right] = \frac{4}{3}\Delta^2 X_{b,\text{in}} = \frac{4}{3}$$
$$\Delta^2\left[P_{a,\text{out}} + \frac{\left(P_{b,\text{out}} + P_{c,\text{out}} + P_{d,\text{out}}\right)}{3}\right] = \Delta^2 P_{a,\text{in}} + \frac{1}{3}\Delta^2 P_{b,\text{in}}$$
$$= e^{-2r} + \frac{1}{3} \tag{54}$$

and their sum is $5/3 + e^{-2r}$, giving a minimum of $5/3 = 1.6667$ for large squeezing parameter r. The sum inequality for those variances is $\Delta^2\left[X_{a,\text{out}} - 1/3\left(X_{b,\text{out}} + X_{c,\text{out}} + X_{d,\text{out}}\right)\right] + \Delta^2\left[P_{a,\text{out}} + 1/3\left(P_{b,\text{out}} + P_{c,\text{out}} + P_{d,\text{out}}\right)\right] \geq 16/9$, as

shown in Criterion 8 of Ref. [14] for $N = 4$ and with the gains $h = -1/3$ and $g = 1/3$. This inequality is violated and hence the final output state in Fig. 5 is genuinely four-partite entangled. We note we can also consider the input to be squeezed along X, in which case the gains g and h will have opposite sign.

5. Beam splitter operation as a model for splitting BEC clouds

We define the mode operators $a_+ = (a_{+1} - ia_{+2})/\sqrt{2}$ and $a_- = (a_{-1} - ia_{-2})/\sqrt{2}$, and their corresponding auxiliary mode operators $a_{\mathrm{vac}+} = (a_{+1} - ia_{+2})/\sqrt{2}$ and $a_{\mathrm{vac}-} = (a_{+1} - ia_{+2})/\sqrt{2}$. This allows us to model the splitting of a BEC cloud with the beam splitter operations where the mode operators $a_+, a_{\mathrm{vac}+}$ are the inputs of a beam splitter. Since the different spin species does not interact, the mode operators $a_-, a_{\mathrm{vac}-}$ are also the inputs of a beam splitter and are split independently of the other spin species. With these mode operators, the Schwinger spin operators after splitting are

$$S_{z,A1} = \frac{1}{2}\left(a_{+1}^\dagger a_{+1} - a_{-1}^\dagger a_{-1}\right)$$
$$= \frac{1}{4}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) + F\left(a_{\mathrm{vac}+}, a_{\mathrm{vac}-}\right) \quad (55)$$
$$S_{z,A2} = \frac{1}{2}\left(a_{+2}^\dagger a_{+2} - a_{-2}^\dagger a_{-2}\right)$$
$$= \frac{1}{4}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) + G\left(a_{\mathrm{vac}+}, a_{\mathrm{vac}-}\right). \quad (56)$$

Here we take the orientation of x, y, z so that S_z corresponds to the number difference. $S_{z,A1}$ and $S_{z,A2}$ are the Schwinger spin operators along the z-axis for clouds $A1$ and $A2$ respectively, F and G are terms containing $a_{\mathrm{vac}+}, a_{\mathrm{vac}+}^\dagger, a_{\mathrm{vac}-}, a_{\mathrm{vac}-}^\dagger$. Similar Schwinger spin operators along the x and y-axes have the same expressions as Eqs. (55) and (56) but the spin up and down are relative to their respective axis. From Eqs. (55) and (56), we see that $S_{z,A1} + S_{z,A2} = S_{z,A} + F + G \approx S_{z,A}$. Here we assume the terms F and G involving the incoming unoccupied modes can be neglected in the calculation of the variances, relative to the leading terms which come from the incoming modes with a high occupation (the number of atoms being assumed large). Using a similar argument, we consider $S_\theta = \frac{1}{2}\left(a_+^\dagger a_- e^{i\theta} + a_-^\dagger a_+ e^{-i\theta}\right)$.

$$S_{\theta,A1} = \frac{1}{2}\left(a_{+1}^\dagger a_{-1} e^{i\theta} + a_{-1}^\dagger a_{+1} e^{-i\theta}\right)$$
$$= \frac{1}{4}\left(a_+^\dagger a_- e^{i\theta} + a_-^\dagger a_+ e^{-i\theta}\right) + F(a_{\mathrm{vac}+}, a_{\mathrm{vac}-})$$

$$S_{\theta,A2} = \frac{1}{2}\left(a_{+2}^\dagger a_{-2} e^{i\theta} + a_{-2}^\dagger a_{+2} e^{-i\theta}\right)$$
$$= \frac{1}{4}\left(a_+^\dagger a_- e^{i\theta} + a_-^\dagger a_+ e^{-i\theta}\right) + G(a_{\mathrm{vac}+}, a_{\mathrm{vac}-})$$

So, for large numbers of atoms, $S_{y,A1} + S_{y,A2} = S_{y,A} + F + G \approx S_{y,A}$ and similarly, $S_{x,A1} + S_{x,A2} \approx S_{x,A}$.

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