

Solution of a Minimal Model for Many-Body Quantum Chaos

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We solve a minimal model for an ergodic phase in a spatially extended quantum many-body system. The model consists of a chain of sites with nearest-neighbor coupling under Floquet time evolution. Quantum states at each site span a q -dimensional Hilbert space, and time evolution for a pair of sites is generated by a $q^2 \times q^2$ random unitary matrix. The Floquet operator is specified by a quantum circuit of depth two, in which each site is coupled to its neighbor on one side during the first half of the evolution period and to its neighbor on the other side during the second half of the period. We show how dynamical behavior averaged over realizations of the random matrices can be evaluated using diagrammatic techniques and how this approach leads to exact expressions in the large- q limit. We give results for the spectral form factor, relaxation of local observables, bipartite entanglement growth, and operator spreading.

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I. INTRODUCTION

Random-matrix theory plays a central role in the understanding of chaotic quantum systems [1]. It is founded on the idea that for many systems of interest there is no privileged basis in Hilbert space. Some important phenomena in this field, however, arise only if there is spatial structure and a notion of locality. Diffusive transport in weakly disordered conductors is such an example for single-particle systems, and spreading of quantum information is a counterpart for many-body systems. It is natural to attempt to combine the simplifying features of random-matrix theory with extended spatial structure. For diffusive conductors, this goal is achieved in Wegner's n -orbital model [2] in which hopping between sites of a tight-binding system is governed by $n \times n$ random matrices, and disorder-averaged properties can be calculated exactly in the limit $n \rightarrow \infty$. Our aim in this paper is to establish a comparable simplification for spatially extended many-body systems.

Chaotic many-body quantum systems lie at the focus of efforts to understand the foundations of quantum statistical mechanics [3–6]. Generic features expected in the dynamics of such systems include rapid equilibration of local observables [5] and ballistic propagation of quantum

information [7], as well as ballistic growth of bipartite entanglement [8,9]. Conservation laws play a central part in dynamics, and one expects that systems with a given set of conservation laws will form a distinct class. Our focus in the following is on evolution arising from a time-dependent Hamiltonian: Without even energy as a conserved quantity, it constitutes a particularly simple example.

Random-matrix approaches offer natural routes to constructing models with minimal structure, and unitary quantum circuits provide an attractive way to formulate the evolution operator for time-dependent quantum systems. Unitary circuits that are random in both space and time have recently yielded valuable insights into chaotic quantum dynamics [10–13]. Here we initiate an analytic study of unitary circuits that are random in space but periodic in time.

We study a Floquet operator acting on a one-dimensional system consisting of q -state “spins” at each site. The Floquet operator is constructed from unitary matrices that couple adjacent sites. These $q^2 \times q^2$ matrices are drawn independently from the circular unitary ensemble (CUE), and we compute physical properties averaged over the ensemble. Our key simplification is to treat the large- q limit. We show that quantum dynamics in this system exhibits a range of features that are expected to be characteristic of ergodic many-body quantum systems: Correlators of local observables decay rapidly in time, and quantum information spreads ballistically, in the sense that the bipartite entanglement of an initial product state grows linearly in time, and the out-of-time-order correlator [14,15] (OTOC) shows the “butterfly” effect. Our approach also provides access to the spectral properties of the Floquet

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operator, and we show (on scales much larger than the level spacing) that its eigenvalue correlations are those of the CUE. At a technical level, our calculations are based on the application of diagrammatic techniques developed previously for single-particle problems in mesoscopic physics involving random matrices from the CUE [16].

Some features of the large- q limit are nongeneric: As for random unitary circuits [11,12] at large q , the distinction between the velocities associated with entanglement and operator spreading vanishes, and sublinear growth of entanglement expected at long times in one-dimensional disordered systems because of weak links [17] is absent.

The balance of this paper is organized as follows. In Sec. II, we define the model, observables, and results. In Sec. III, we develop the diagrammatic approach for taking the ensemble average of a given observable. In Sec. IV, we sketch the proof of the results presented in Sec. II using the tools developed in Sec. III. Lastly, in Sec. V, we summarize. Technical details are described in a series of Appendices.

II. MODEL, OBSERVABLES, AND RESULTS

We seek a minimal model for quantum chaos in a spatially extended many-body system with local interactions. We formulate the model directly in terms of the time-evolution operator rather than a Hamiltonian, and for simplicity we consider a Floquet problem. Taking a one-dimensional lattice, evolution over one period is separated into two half steps. Each even site is coupled to its neighbor on the left in the first half step and to its neighbor on the right in the second half step. This quantum circuit is illustrated in Fig. 1.

To write the evolution operator explicitly, let $U_{2n-1,2n}$ denote the unitary matrix that couples sites $2n-1$ and $2n$ in the first half step, and let $U_{2n,2n+1}$ be the matrix that couples sites $2n$ and $2n+1$ in the second half step. Each $U_{i,i+1}$ is independently distributed with the Haar measure, and we calculate physical properties as an average over this ensemble denoted by $\langle \dots \rangle$. For a system of L sites, the full Hilbert space has dimension q^L . Taking L even for definiteness, the first half step is represented by an evolution operator acting in this space with the form

$$W_1 = U_{1,2} \otimes U_{3,4} \otimes \dots \otimes U_{L-1,L}. \quad (1)$$

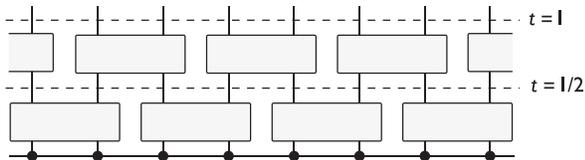


FIG. 1. Illustration of the model for Floquet time evolution studied in this paper. Space and time are represented by the horizontal and vertical directions. Lattice sites are indicated by filled dots, and the coupling of pairs of sites under time evolution is shown with rectangles.

The evolution operator for the second half step is similarly

$$W_2 = \mathbb{1}_q \otimes U_{2,3} \otimes U_{4,5} \dots U_{L-2,L-1} \otimes \mathbb{1}_q, \quad (2)$$

where $\mathbb{1}_q$ denotes the $q \times q$ unit matrix. The Floquet operator describing evolution over a single complete period is

$$W = W_2 W_1. \quad (3)$$

We denote the evolution operator for an integer number t of periods (the t th power of W) by $W(t)$.

To demonstrate that this model has the features expected in a chaotic quantum many-body system, we examine a range of physical properties, as detailed below. Our results for entanglement spreading and for the OTOC are the same as those for random unitary circuits with the same structure as our model if one takes the large- q limit in previous results [11,12]. In this context, one of our significant conclusions is that random unitary circuits do indeed share important physical features with deterministic systems. At the same time, our model opens up discussion of the spectral properties of the evolution operator. This has no equivalent for random unitary circuits since they lack any such fixed operator.

A. Spectral form factor

Our results for the spectral form factor show that the Floquet operator for the model has exactly the same eigenvalue correlations in the large- q limit as an ensemble of Haar-distributed unitary matrices. This insensitivity to the spatial structure of the system is a striking emergent feature.

The spectral form factor $K(t)$ is the Fourier transform of the two-point correlation function of the eigenvalue density. Denote the eigenphases of W by $\{\theta_m\}$ for $m = 1, \dots, q^L$. Then,

$$K(t) \equiv \langle \text{Tr}[W(t)] \text{Tr}[W^\dagger(t)] \rangle = \sum_{m,n} \langle e^{i(\theta_m - \theta_n)t} \rangle. \quad (4)$$

For $N \times N$ matrices from the CUE, one has [1]

$$K(t) = \begin{cases} N^2 & t = 0, \\ |t| & 0 < |t| \leq N, \\ N & N \leq |t|. \end{cases} \quad (5)$$

The behavior of $K(t)$ on scales $|t| \ll N$ reflects level correlations at separations much larger than the mean spacing, and the linear dependence of $K(t)$ on $|t|$ in this regime is a consequence of Coulombic suppression of long wavelength fluctuations in the eigenvalue density. Conversely, the form of $K(t)$ for $|t| \sim N$ encodes spectral correlations on the scale of the level spacing.

We obtain exactly the CUE form $K(t) = |t|$ for $t \neq 0$ in the large- q limit at fixed L and t . We stress that this result is a consequence of coupling between all sites, and it is not simply a reflection of the properties of individual random matrices $U_{i,i+1}$. To illustrate the point, consider an alternative model in which the couplings $U_{2n,2n+1}$ on even bonds are replaced with unit matrices $\mathbb{1}_{q^2}$. For this toy system of $L/2$ -independent pairs of sites, one has much larger fluctuations in eigenvalue density, with $K(t) = |t|^{L/2}$ for $0 < |t| \leq q^2$.

Returning to the original model, it is interesting to speculate on behavior in regimes other than the one we are able to analyze exactly. At large but fixed q and fixed t , it is natural to expect $K(t)$ to increase with L if L is sufficiently large, since distant parts of the system should be only weakly correlated; the results for this regime will be presented elsewhere [18]. In the opposite limit, fixing q and L and varying t , the form of $K(t)$ at $|t| \sim q^L$ probes correlations on the scale of the level spacing. We expect correlations on this scale to be of CUE form for all q and L , provided only that $q^L \gg 1$. They should therefore show a transition from $K(t) = |t|$ for $|t| \lesssim q^L$ to $K(t) = q^L$ for $|t| \geq q^L$. The transition to constant $K(t)$ at large t is, however, well known to be inaccessible in a perturbation expansion in inverse powers of q [19,20].

B. Dynamics

We next examine the physical properties chosen to reveal both the dynamics of local degrees of freedom and the spreading of quantum information.

1. Local relaxation

A characteristic feature of chaotic dynamics is that local observables relax rapidly towards equilibrium. To probe relaxation in our model, we introduce an operator O_x representing an observable at site x and obeying $O_x^\dagger = O_x$, $\text{Tr}O_x = 0$, and $O_x^2 = \mathbb{1}_q$. Let

$$O(x) = \mathbb{1}_q \otimes \dots \otimes \mathbb{1}_q \otimes O_x \otimes \dots \otimes \mathbb{1}_q \quad (6)$$

and $O(x, t) = W^\dagger(t)O(x)W(t)$. The statistical average for a system with density matrix ρ is $[\dots]_{\text{av}} = \text{Tr}[\rho \dots]$. In our Floquet model, the infinite-temperature density matrix $\rho = q^{-L} \mathbb{1}_{q^L}$ is the natural choice, and we denote the normalized many-body trace by $\text{tr} \equiv q^{-L} \text{Tr}$.

One anticipates that $[O(x, t)O(x)]_{\text{av}}$ will relax to the value $[O(x)]_{\text{av}}^2$ [which is zero with this form for $O(x)$] on a microscopic timescale of order the Floquet period. At large q , we in fact find complete relaxation within a single period, obtaining for $q \rightarrow \infty$ the result

$$\langle \text{tr}[O(x, t)O(x)] \rangle = \delta_{t,0}. \quad (7)$$

We discuss the leading corrections to this result for finite q and small t in Sec. IV B.

2. Entanglement spreading

We probe entanglement spreading via the time dependence of the ensemble-averaged bipartite entanglement purity (and, more generally, ensemble averages of exponentials of Rényi entropies) for an initial state $|\psi\rangle$ that is a direct product over sites. Specifically, let A be the left half of the system (with site labels $1 \leq n \leq L/2$) and B the right half, and consider the reduced density matrix

$$\rho_A(t) = \text{Tr}_B[W(t)|\psi\rangle\langle\psi|W^\dagger(t)]. \quad (8)$$

The Rényi entropies $S_\alpha(t)$ are given by

$$e^{(1-\alpha)S_\alpha(t)} = \text{Tr}_A[\rho_A(t)]^\alpha, \quad (9)$$

and the entanglement purity is $\mathcal{P}(t) = e^{-S_2(t)}$. We compute $\langle e^{(1-\alpha)S_\alpha(t)} \rangle$ for $\alpha = 2$ and $\alpha = 3$. The purity shows an exponential decay in time until it falls to a value $\mathcal{P}(t) \sim q^{-L/2}$ typical of random states in the many-body Hilbert space. In detail, we obtain at large q ,

$$\langle e^{(1-\alpha)S_\alpha(t)} \rangle \sim \begin{cases} f_\alpha(t)q^{-2(\alpha-1)t} & t \leq L/4, \\ K_\alpha q^{-(\alpha-1)L/2} & t > L/4. \end{cases} \quad (10)$$

The function $f_\alpha(t)$ grows exponentially with t , and in particular, we find $f_2(t) = 4^t$ and $f_3(t) \simeq [(4 + 3\sqrt{2})/2]^{2t}$ for $t \gg 1$. At times $t > L/4$, the Rényi entropies saturate, and we conjecture that $K_\alpha = \text{Cat}(\alpha)$, where Cat is the α th Catalan number. The latter result is proven for α up to 5 (see Appendix F). This result is consistent with the generalized Page formula for bipartite entanglement in random states [21,22] and with the results from the random unitary circuit [23].

An interpretation of these results is that the reduced density matrix for the pure initial state spreads at time t over the Hilbert space spanned by basis states at $2t$ sites. As an illustration, suppose that $\rho_A(t)$ has q^{2t} nonzero eigenvalues that are all equal. Then, $\text{Tr}_A \rho_A(t) = q^{-2(\alpha-1)t}$, which is consistent with the leading-order behavior of Eq. (10) for $t \leq L/4$. This result demonstrates that the entanglement spreads ballistically with a velocity at large q that is the same as the naive light-cone velocity $v = 2$ introduced below in Fig. 2. The values of $f_2(t)$ and $f_3(t)$ give information on the distribution of nonzero eigenvalues of $\rho_A(t)$, and it is noteworthy that these two quantities are distinct.

The growth of entanglement found here is similar to that obtained for integrable systems using conformal field theories [24]. In integrable systems, this behavior is associated with the presence of quasiparticles that travel ballistically; a quite different physical picture is required for ergodic systems.

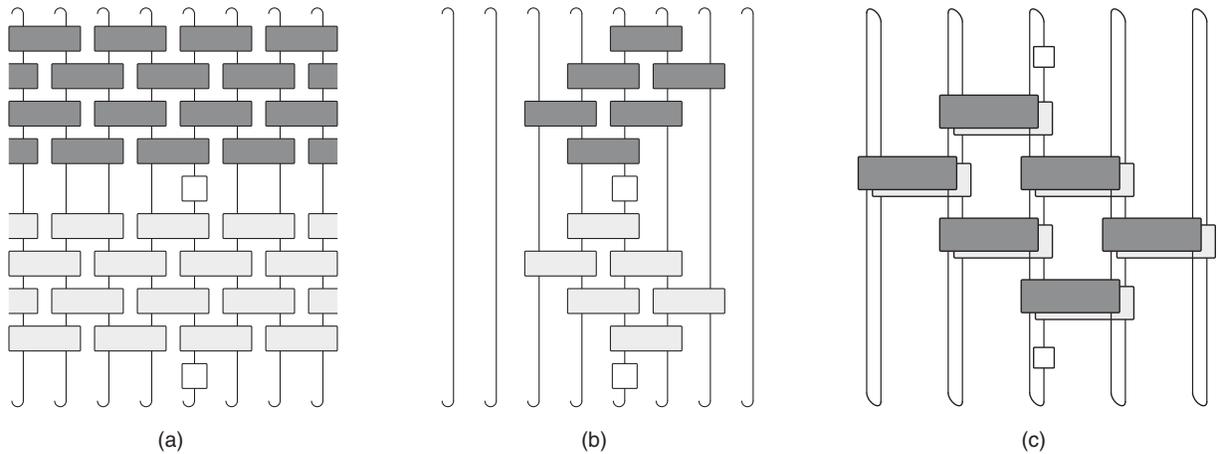


FIG. 2. Illustration of $\text{Tr}[O(t)O(0)]$ for $t = 2$. U 's and U^\dagger 's are represented by rectangles, respectively, shaded in light and dark gray. The local operator O_x is represented by a square. The curled lines at the top and bottom edge indicate closed loops which denote the trace operation. (a) Initial expression. (b) After cancellation of pairs of U with U^\dagger wherever possible. The boundaries of the regions within which U 's and U^\dagger 's remain form light cones with a velocity $v = 2$ set by the construction of the model. (c) Illustration of $\text{Tr}[O(t)O(0)]$ for $t = 2$ formed by folding Fig. 2 (right) so that timelines run upward for W and downward for W^\dagger .

3. Out-of-time-order correlator

The spreading with time of local operators is characterized by the behavior of the (ensemble-averaged) commutator

$$\mathcal{C}(x-y, t) = \frac{1}{2} \langle \text{tr}[[O(x, t), O(y)]^2] \rangle, \quad (11)$$

which measures how a local perturbation at site x affects measurements at a later time at the site y . With our normalization for local operators, one has

$$\mathcal{C}(x-y, t) = 1 - \langle \text{tr}[O(x, t)O(y)O(x, t)O(y)] \rangle, \quad (12)$$

and the second term on the right-hand side is the OTOC. For short times t and large separations $x-y$, the operators $O(x, t)$ and $O(y)$ commute and $\mathcal{C}(x-y, t)$ is zero, while for large times, the OTOC is small and $\mathcal{C}(x-y, t)$ approaches unity. We obtain at large q a sharp light-cone behavior

$$\mathcal{C}(x-y, t) = \begin{cases} 0 & |t| < |x-y|/2, \\ 1 & |t| \geq |x-y|/2. \end{cases} \quad (13)$$

Hence, operator spreading occurs with a butterfly velocity that, like the entanglement velocity, is equal to the naive light-cone velocity $v = 2$ in the large- q limit. The results on random unitary circuits [11,12] suggest that all three velocities should be distinct for q finite. From the same comparison, we also expect at finite q that the step function of Eq. (13) will broaden.

III. ENSEMBLE AVERAGING

We now set out in several steps the general approach that we use to obtain these results.

First, it is useful to extend the notation of Fig. 1 in various ways so as to represent pictorially the quantities defined in Sec. II. An example for $\text{Tr}[O(x, t)O(x, 0)]$ is shown in Fig. 2. The vertical timelines of sites join rectangles representing factors of $U_{i,i+1}$ and indicate matrix multiplication. Repeated copies of W denote multiple time steps; W^\dagger is shown as a differently shaded version of W , and local operators appear as squares. The matrix trace is shown by joining the timelines of sites to form closed loops. Finally, product states in the site basis (which do not appear in this example) are shown using circles at the ends of the timelines.

A straightforward simplification in many instances is that some factors of U and U^\dagger cancel, as illustrated in Fig. 2(b), and the naive light-cone velocity $v = 2$ clearly emerges.

As a second step, it is helpful to fold the pictorial representations so that while the timelines in W run upward, those in W^\dagger run downward, as shown in Fig. 2(c).

These folded pictures provide a direct depiction of physical quantities but are cumbersome. A simpler representation is possible if we focus on the time evolution of a single site. To this end, and in anticipation of the disorder average, we switch to an alternative notation for $U_{i,i+1}$ and $U_{i,i+1}^\dagger$ in which individual sites appear separately, as shown in Fig. 3.

With these preliminaries in place, we can set out a diagrammatic representation for ensemble averages. It is a many-body extension of the one introduced for averages over the CUE by Brouwer and Beenakker [16]. It can be applied to an arbitrary observable $f(W)$, but to be definite, we illustrate it for $K(t)$ with $t = 2$ and $L = 2$ so that $W(2) = [U_{1,2}]^2$. In general, three steps are involved.

- (i) The observable $f(W)$ is represented as a collection of single-site diagrams using the notation introduced

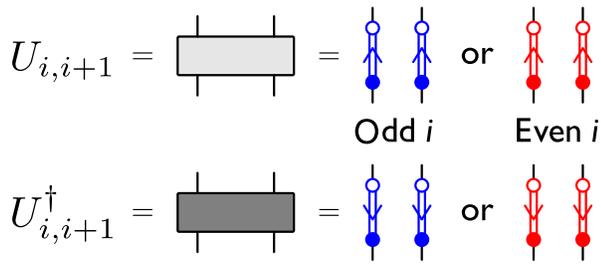


FIG. 3. Adaptations to diagrammatic notation made in order to show time evolution of an individual site. The shaded rectangles representing $U_{i,i+1}$ and $U_{i,i+1}^\dagger$ are replaced by pairs of double lines, with an arrow directed from the column label to the row label of the matrix. To record the distinction between odd and even i , the double lines are blue in the first case and red in the second. This color convention allows us to represent single-site diagrams without ambiguities (see, e.g., Fig. 7).

in Fig. 3. For the example of $K(t=2)$, one starts from the many-body diagram shown in Fig. 4 (right) and obtains the single-site representation shown in Fig. 5 (left).

- (ii) The ensemble average $\langle f(W) \rangle$ is computed by generating a collection of single-site “contracted” diagrams $G = \{G_i\}_{i=1}^L$ as follows. On each site, filled U dots are connected to filled U^\dagger dots of the same color on the same site with a dashed line (a contraction) in all possible ways and likewise for the empty dots.

Since U 's and U^\dagger 's act on neighboring pairs of sites, these contractions must be matched: For even i , the choice of contractions between blue dots must be the same in the diagrams G_{i-1} and G_i and similar for the red dots and the diagrams G_i, G_{i+1} . We refer to this matching requirement as the *bond constraint*.

- (iii) Each contracted site diagram G_i gives rise to an algebraic expression $\mathcal{A}(G_i)$ obtained as the product of two factors

$$\mathcal{A}(G_i) = \mathcal{A}_T(G_i)\mathcal{A}_U(G_i). \tag{14}$$

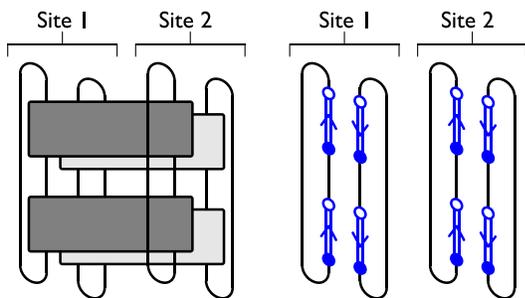


FIG. 4. Alternative diagrammatic representations of $\text{Tr}[W(t=2)]\text{Tr}[W^\dagger(t=2)]$, which gives $K(t=2)$ after ensemble averaging using the notation of the left and right sides of Fig. 3.

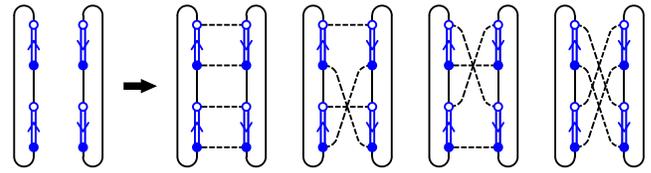


FIG. 5. Single-site diagrams associated with $\text{Tr}[W(t=2)]\text{Tr}[W^\dagger(t=2)]$ (left) and its ensemble average $K(t=2)$ (right).

These factors are associated with loops of two kinds called T loops and U loops in Ref. [16]. The T loops simply record pairings of matrix labels and give rise to powers of q in the contribution of a diagram. The U loops distinguish different contributions from the average of factors of U and U^\dagger . Examples of these two types of loops are illustrated in Fig. 6.

A T loop is a closed sequence of alternating single and dashed lines. It carries an index corresponding to one of the q basis states at a site. We generate the contribution associated with each T loop by summing over its index. This summation leads to a factor of q for a T loop of single lines if it does not pass through any operator insertion or of $\text{Tr}(O_1 O_2 \dots)$ if the T loop passes through the operators $\{O_1, O_2, \dots\}$. The overall factor $\mathcal{A}_T(G_i)$ is obtained as the product of the individual factors coming from each T loop.

A U loop is a closed sequence of alternating double and dashed lines. The length c of a U loop is defined as half of the number of double lines it contains. Let $R_i = \{c_k^{(r)}\}_{k=1}^{r_i}$ and $B_i = \{c_k^{(b)}\}_{k=1}^{b_i}$ be the sets of lengths of red and blue U loops in G_i . Then, from the theory of CUE averages [16],

$$\mathcal{A}_U(G_i) = (V_{R_i} V_{B_i})^{1/2}, \tag{15}$$

where the explicit form of the coefficients $V_{R_i} \equiv V_{c_1, \dots, c_{r_i}}$ and $V_{B_i} \equiv V_{c_1, \dots, c_{b_i}}$ (also known as the Weingarten functions) is given in Appendix A. Note that the exponent 1/2 in Eq. (15) arises because we distribute the contribution arising from unitary operators equally over the two sites on which the operators act.

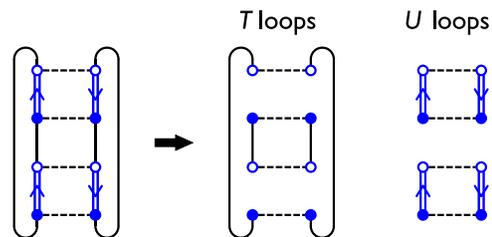


FIG. 6. The T and U loops (right) associated with a contracted diagram for $\text{Tr}[W(t=2)]\text{Tr}[W^\dagger(t=2)]$ (left in this figure and top middle in Fig. 5). The associated algebraic factors are $\mathcal{A}_T(G_i) = q^2$ and $\mathcal{A}_U(G_i) = (V_{1,1})^{1/2}$.

The final value of the ensemble average reads

$$\langle f(W) \rangle = \sum_G \prod_i \mathcal{A}(G_i), \quad (16)$$

where the sum runs over all contracted diagrams G satisfying the bond constraint. For the example of $\langle K(t=2) \rangle$, all possible single-site contracted diagrams are drawn in Fig. 5 right. Explicit evaluation of the algebraic expressions for this case leads to

$$\langle K(t=2) \rangle = (q^4 V_{1,1} + q^2 V_2 + q^2 V_2 + q^4 V_{1,1}) \approx 2. \quad (17)$$

The factors of q within the brackets are contributions from the T loops. We take the large- q limit on the far right using $V_{1,1} = (q^4 - 1)^{-1}$ and $V_2 = -[q^2(q^4 - 1)]^{-1}$. At large q , the first and fourth diagrams are leading order, while the second and third are subleading.

The above procedure is exact for any q , but the sum in Eq. (16) is problematic in general, as it may involve an extremely large number of terms. For example, a total of $(t!)^{2(L-1)}$ diagrams contribute to $K(t)$. It is therefore useful to establish which terms dominate in the large- q limit. For large q , one has $\mathcal{A}_U(G_i) \sim q^{u_i - n_i}$, where $u_i = r_i + b_i$ is the total number of U loops, and $n_i = \sum_k c_k$ is the number of contractions on the i th site. Let τ_i be the number of T loops in G_i that contribute a factor q . Then we have the large- q expansion

$$\prod_i \mathcal{A}(G_i) \sim \prod_{i \in \text{site}} q^{\tau_i + u_i - n_i} \equiv \mathcal{O}(G), \quad (18)$$

where we introduce the order $\mathcal{O}(G)$ and omit a proportionality constant independent of q . In Appendix B, we discuss ways of enumerating the leading-order diagrams. As a general rule, since the total number of contractions $\sum_i n_i$ is fixed, we should retain in Eq. (16) only the diagrams that maximize the total number of T and U loops.

A natural approximation at large q is to treat the elements of U as independent Gaussian random variables so that a standard Wick theorem applies. This approximation corresponds to including all diagrams where all U loops are of unit length and omitting all others. We refer to this set as the Gaussian diagrams. As an example, in Fig. 5, the first and fourth diagrams on the right-hand side are Gaussian. For these diagrams, Eq. (18) holds with proportionality constant 1; therefore, counting the overall number of U and T loops is sufficient to obtain the leading contribution of the diagram.

In this paper, the leading contributions to all quantities calculated (except for the autocorrelation function of a local observable for $t > 0$) are Gaussian. A procedure that goes beyond the Gaussian approximation is necessary, first, for the proof of this statement, and second, for future discussion of the subleading contributions. Since the number

of diagrams contributing to each of the quantities we consider is finite at fixed L and t , our results are exact in the limit $q \rightarrow \infty$ with L, t fixed.

IV. DIAGRAMMATIC EVALUATION OF RESULTS

We now show how this diagrammatic approach can be used to generate the results given in Sec. II. We sketch the main ideas here, deferring formal proofs to the Appendices.

A. Spectral form factor

Consider the spectral form factor $K(t)$ introduced in Eq. (5). Applying the procedure described in Sec. III (and generalizing the example given for $L = 2$ and $t = 2$ in Fig. 5), we obtain for $L > 2$ a many-body diagram consisting of single-site diagrams as shown in Fig. 7(a). Here, each time step in $W(t)$ contributes with two unitaries, leading to $4t$ dots divided into four types: blue or red, and empty or filled.

Next we consider ensemble averaging this diagram, making all possible contractions. On a single site, one can easily check that one of the diagrams with the maximum number of loops is, e.g., the site diagram in Fig. 7(b). Since this diagram is Gaussian and $\tau_i = u_i = 2t$ and $n_i = 4t$, its contribution for large q is simply 1. There are t -equivalent diagrams of this kind obtained by cyclically shifting the right dots with respect to the left ones, as shown in Figs. 7(c) and 7(d). Additionally, once one of these configurations is chosen on the site i , the bond constraint forces all other sites to be in the same configuration in order to maximize the number of loops. All other diagrams are smaller by powers of q for $q \rightarrow \infty$. Consequently, we get the

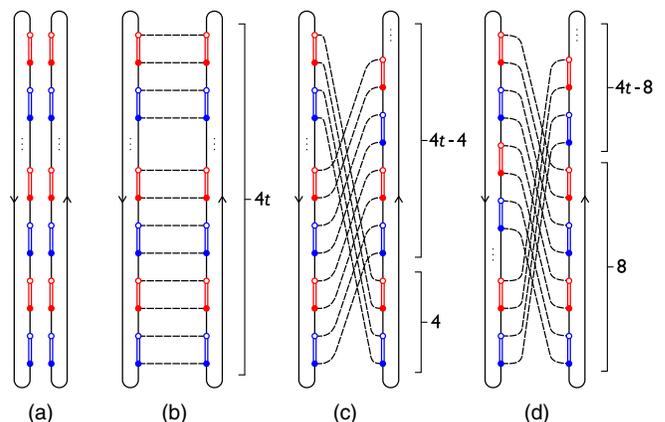


FIG. 7. (a) The site diagram associated with $\text{Tr}[W(t)]\text{Tr}[W^\dagger(t)]$. It is color coded as follows: For an odd site i , the blue dots and double lines are contributions from $U_{i,i+1}$ or $U_{i,i+1}^\dagger$ and the red ones are from $U_{i-1,i}$ or $U_{i-1,i}^\dagger$. For an even site, this coding is reversed. (b)–(d) Three examples from a total of t leading-order diagrams for $K(t)$ at $t > 0$.

result $K(t) = |t|$. A proof of this statement is given in Appendix C.

B. Relaxation of local observables

Contributions to the autocorrelation function $\text{tr}[O(x, t)O(x)]$ are generated by contractions of the site diagrams shown in Figs. 8(a) and 8(b). The leading contribution G_i at a site $i \neq x$ is from a contraction of the form shown in Fig. 8(c). However, if this contraction is made at every site $i \neq x$, then because of the bond constraint, it also applies at site x and yields $\mathcal{A}(G_x) \propto \text{Tr}O_x = 0$ in Fig. 8(d). An example of an alternative contraction, for which $\mathcal{A}(G_x) \neq 0$, is shown in Fig. 8(e). With such a choice at $i = x$, the bond constraint imposes contractions at nearby sites $i \neq x$ that are subleading in q .

We also evaluate the leading nonzero contributions to an autocorrelation function for large q at small values of t . We find

$$\langle [O(x, t)O(x)]_{\text{av}} \rangle = \begin{cases} 1 & \text{for } t = 0, \\ 0 & \text{for } t = 1, \\ q^{-7} & \text{for } t = 2, \\ 16q^{-11} & \text{for } t = 3. \end{cases} \quad (19)$$

These results are interesting for two reasons. First, the equivalent quantity for time evolution with a random unitary circuit is identically zero at all $t \neq 0$. Hence, the results expose a difference between our Floquet model and random unitary circuits. The finite relaxation rate at finite q

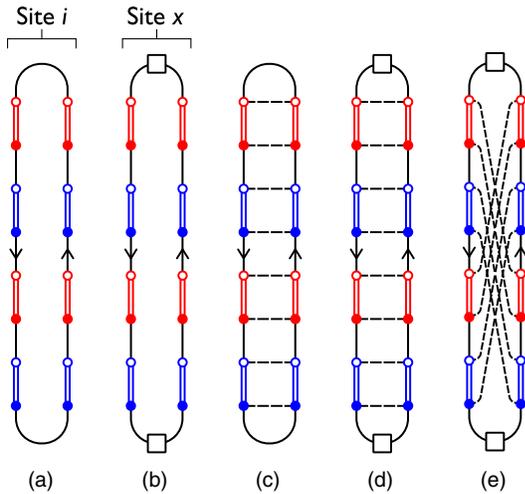


FIG. 8. (a),(b) Site diagrams associated with $[O(x, t=2)O(x)]_{\text{av}}$ at sites $i \neq x$ and sites $i = x$, respectively. Color coding is as in Fig. 7. (c) The leading contraction of the site diagram for $x \neq i$. (d),(e) Two alternative contractions of the site diagram for $[O(x, t=2)O(x)]_{\text{av}}$ at site x . (d) gives a contribution $\mathcal{A}(G_x) \propto \text{Tr}O_x = 0$. (e) has $\mathcal{A}(G_x) \neq 0$ but forces similar contractions on adjacent sites via the bond constraint, and these contributions vanish as $q \rightarrow \infty$.

in the Floquet model is consistent with expected generic behavior, whereas complete relaxation for any $t \neq 0$ is likely to be a special feature of random unitary circuits. Second, and quite separate, the dominant contributions arise from non-Gaussian diagrams: For example, at $t = 3$ the largest Gaussian term is $2q^{-9}$, but it is canceled by non-Gaussian contributions.

C. Purity and Rényi entropies

We now show that the large- q calculation of the purity can be reduced to counting DW configurations with positive weights, in a problem analogous to one from classical statistical mechanics. Related ideas apply (albeit more elaborately) to the evaluation of averages of the exponential of the Rényi entropy $S_\alpha(t)$ for general positive integer α , and we discuss the case $\alpha = 3$ in Appendix F. A domain structure similar to what we derive here also appears in recent treatments of random unitary circuits [10,11], and it is striking to find it as an emergent property of our Floquet model.

The main steps in our procedure are summarized in Fig. 9. We wish to average $\text{Tr}_A[\rho_A(t)^\alpha]$ for $\alpha = 2$. Using the conventions of Fig. 2, Eq. (9) has the pictorial representation shown in Fig. 9(a). Since each $\rho_A(t) = W^\dagger(t)|\psi\rangle\langle\psi|W(t)$, we have four sectors containing $W(t)$, $W^\dagger(t)$, $W(t)$, and $W^\dagger(t)$, respectively. It is convenient to fold this diagram, as we discuss in Sec. III and Fig. 2. This procedure leads to a folded representation containing four layers, as shown in Fig. 9(b) top. The folded site diagrams in regions A and B are shown in Fig. 9(b) bottom left. Note, in particular, how the timelines of the sites in the regions A and B connect differently at the top of these diagrams because of the structure of traces in Eq. (9). After averaging, it turns out that the leading contributions come from contractions of the U 's and U^\dagger 's that are brought to lie in a stack on top of each other by the folding operation. We call a contraction of this type *local*. Such a stack of unitaries is indicated with a large blue box in Fig. 9(b) top. In the evaluation of the purity (where two U 's and two U^\dagger 's are involved), there are two possible local contractions [Fig. 9(b) bottom right].

An intuitive explanation of the fact that only local contractions contribute for $q \rightarrow \infty$ is that any contraction between two dots in distant blocks will necessarily lead to a longer loop; as the total loop length is fixed, this implies a smaller number of loops and a lower order according to Eq. (18). This statement is proved in Appendix E.

Given the restriction to local contractions, a further simplification of the diagrammatic representation is possible. It is no longer necessary to represent the unitaries within a stack individually. Instead, we can simply depict the stack using a label to indicate the type of local contraction. We label the two types of local contraction that appear in a calculation of the purity A and B because they involve the same types of pairing as are induced by the trace structure in regions A and B of the system. The diagrammatic notation applied to

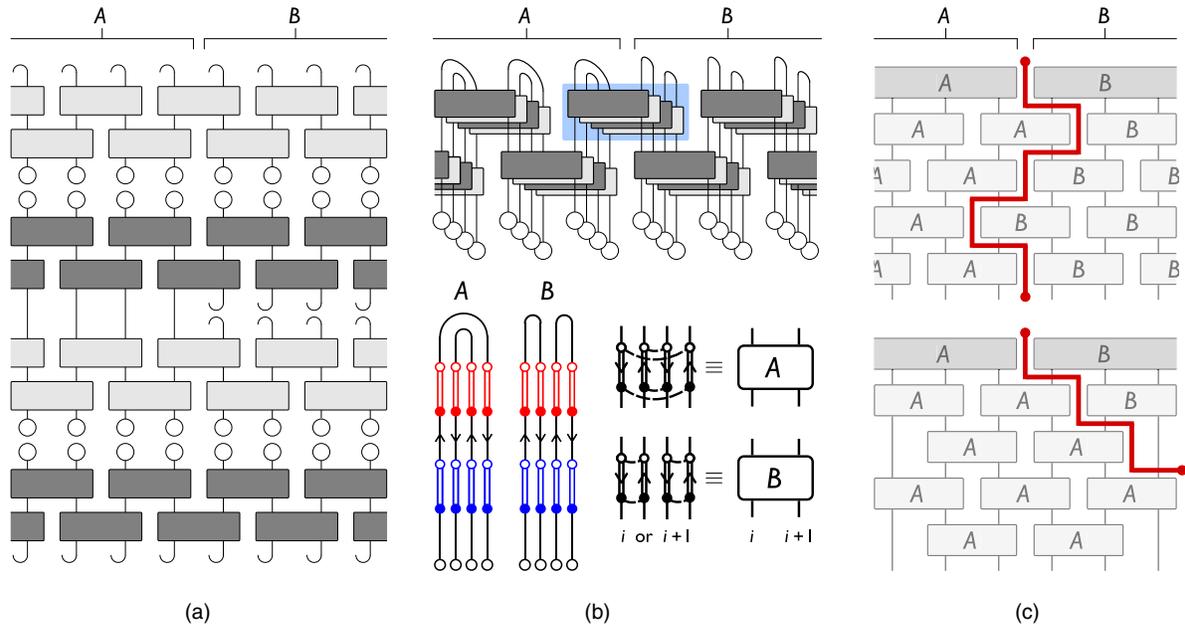


FIG. 9. Summary of the steps that convert the evaluation of the average purity into a combinatorial statistical mechanics problem. (a) Illustration of $e^{-S_2(t)}$ for $t = 1$. Open circles represent a product state in the site basis and other conventions are as in Fig. 2. (b) Top: Folded representation of $\exp[-S_2(t)]$ obtained by folding (a) so that timelines in $W(t)$ run upward and those in $W^\dagger(t)$ run downward. The large blue box highlights a stack of four unitary operators (see main text). (b) Bottom left: Site diagrams in regions A and B . Note in particular how the timelines of sites in the regions A and B connect differently at the top of these diagram because of the structure of traces in Eq. (9). (b) Bottom right: The two possible local contractions made within a block, which we label A and B . Only leading-order diagrams have such contractions. We represent these contractions in two alternative ways: either explicitly for a single site, as on the left side of the equivalences, or schematically for a pair of sites and a stack of unitaries, as on the right side of the equivalences. Note that the top boundary of the diagram can be treated as equivalent to blocks of such contractions. (c) Leading-order diagrams for $\langle e^{-S_2(t)} \rangle$ at large q are minimal-length domain-wall (DW) diagrams. Their weight is $q^{-h_{AB}}$ with h_{AB} being the length of horizontal DW between A and B contractions. The labels A and B on blocks in this figure indicate the type of local contraction. Top: A minimal-length DW diagram for $\langle \exp[-S_2(t)] \rangle$ at $t = 4$ and $L > 6$. Bottom: One of the two minimal-length DW diagrams at $t > 2$ and $L = 6$. These two diagrams have the DW directed solely to the left or right.

one stack is shown in Fig. 9(b) bottom right. The same notation is used for a larger system in Fig. 9(c).

We prove in Appendix E that the order of a diagram with only local contractions is given by $q^{-h_{AB}}$, where h_{AB} is the number of segments of horizontal wall of length one lattice unit between an A and a B block. Such walls are shown with horizontal red lines in Fig. 9(c). In this framework, the problem of finding the leading contributions admits a simple geometrical interpretation: They are represented by the set of minimal-length DW diagrams that separate the A and B contractions. Given the fixed boundary conditions for the blocks on the top of the diagram, a DW must connect the top center to either the side or the bottom of the diagram [Fig. 9(c)].

For $t \leq L/4$, a minimal-length DW can connect only the top center to the bottom edge of the diagram. Since there is a total of $2t$ rows, the minimal h_{AB} is $2t$, and the order is q^{-2t} . Moreover, there are two choices (for the DW to go left or right) below every row, so the number of leading-order diagrams is 2^{2t} . For $t > L/4$, the lengths of the DWs are minimized when they connect the top center and the side of the diagram with the shortest possible paths.

This observation implies that there are only two leading-order diagrams: those where the DW is directed exclusively towards either the left or the right. For these cases, $h_{AB} = L/2$, and the order is $q^{-L/2}$.

In summary, Gaussian diagrams give the leading behavior at large q :

$$\langle e^{-S_2(t)} \rangle \sim \begin{cases} 2^{2t} q^{-2t} & t \leq L/4, \\ 2q^{-L/2} & t > L/4. \end{cases} \quad (20)$$

Subleading terms eliminate the discontinuity in Eq. (20) at $t = L/4$. For larger α , one can repeat the construction of Fig. 9. Each block has now $\alpha!$ possible states corresponding to the possible local contraction inside a block of αU 's and αU^\dagger 's. So, generically the problem reduces to counting, but for large α , the procedure quickly becomes problematic. The case $\alpha = 3$ is discussed in Appendix F.

D. Out-of-time-order correlator

Consider the diagrammatic representation of the OTOC defined in Eq. (12). After the cancellation of U 's with U^\dagger 's wherever possible, we can distinguish two alternative

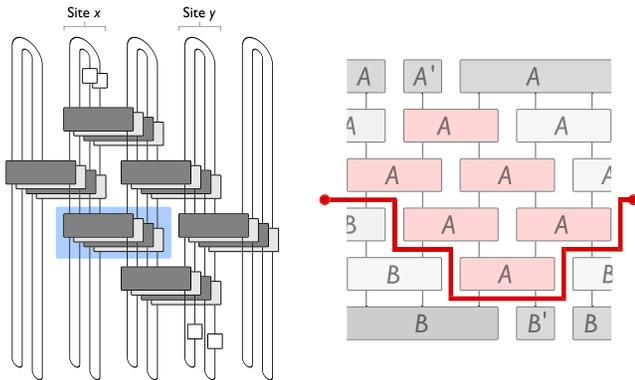


FIG. 10. Diagrammatic representation of OTOC. Left: Folded diagram for $\text{tr}[O(x,t)O(y)O(x,t)O(y)]$ with $|x-y|=2$ and $t=2$. The blue box highlights four unitary operators, which can be represented in the block representation in a similar way as the ones in Fig. 9(c). Right: An apparent minimal-length horizontal DW diagram that vanishes due to $\text{tr}O(y) \equiv 0$. The red blocks are unitary pairs that cannot be annihilated. The dark gray blocks represent the boundary conditions. A' represents an effective contraction A [Fig. 9(c)] with operators along the loops, and similar for B' .

scenarios. For $t < |x-y|/2$, all the unitary matrices are canceled and we simply have $\text{tr}[O(x,t)O(y)O(x,t)O(y)] = \text{tr}[O(x)^2]\text{tr}[O(y)^2] = 1$, independent of the realization. Substituting the OTOC in Eq. (11), we see that the commutator vanishes. This result is consistent with the fact that the light cone spreading from x with velocity $v=2$ has not yet reached the operator in y . Conversely, for $t \geq |x-y|/2$, there is a region of unitary operators that cannot be canceled.

The calculation of the OTOC for $t \geq |x-y|/2$ is shown in Fig. 10 left using a folded diagram. The diagram contains four layers of W 's, as for the purity, and we employ the block representation introduced in Sec. IV C. Similar to Sec. IV C, it can be shown that the leading-order diagrams necessarily involve only local contractions. Moreover, the trace structure of OTOC sets boundary conditions along the top of the diagram that act as A blocks and along the bottom that act as B blocks. From this feature, one would expect that the leading-order diagrams are minimal-length DWs that cross the diagram horizontally, separating an upper domain of A 's from a lower domain of B 's (Fig. 10 right). These diagrams, however, vanish because they involve factors of either $\text{tr}O(x) \equiv 0$ or $\text{tr}O(y) \equiv 0$. In Appendix G, we prove that the leading contributions to the OTOC for $t \geq |x-y|/2$ go to zero for $q \rightarrow \infty$, and hence, Eq. (13) is recovered in the large- q limit.

V. SUMMARY AND OUTLOOK

In summary, we introduce a minimal random-matrix model with extended spatial structure to study the chaotic Floquet dynamics of a many-body quantum system. We present a diagrammatic technique to compute several

quantities using a systematic and controlled expansion in the inverse of the local Hilbert space dimension q .

Our study of a minimal model and the techniques we develop are complementary to a variety of other recent works. In particular, the semiclassical approach to quantum chaos in few-body systems [25] has been extended to bosonic systems at high density by considering interfering paths in Fock space that arise from solutions to the Gross-Pitaevskii equation [26,27] and to periodically driven spin systems in the large-spin limit [28]. These semiclassical techniques have the attraction of applying directly to specific physical systems that are of wide interest rather than simply to a minimal model. They have been used so far to identify particular many-body interference phenomena but have not been developed to allow general computation of the dynamics of quantum information. In a separate advance, the Keldysh technique has been generalized to permit calculation of out-of-time-order correlators, with applications to a variety of microscopic models [29]. These calculations are well controlled in a quasiclassical regime, analogous to our large- q limit, while the augmented Keldysh contour of Ref. [29] is (unsurprisingly) mirrored quite closely by the structure of the diagrammatic calculations we describe here. Further topical research [30] addressing a spatially extended version of the Sachdev-Ye-Kitaev (SYK) model [31] is set apart from the results we present by the fact that the zero-dimensional SYK model exhibits much greater structure than the individual random matrices of our model.

There are some obvious and interesting directions for additional investigations using the techniques we set out. First, in contrast with random unitary circuits, our model has a well-defined Floquet operator, opening the possibility for the study of its spectral properties. It seems likely that further work in this direction will be useful, going beyond our evaluation of the spectral form factor in the random-matrix regime. Second, the results we present are averages over an ensemble of systems. It would be useful to understand the magnitude of sample-to-sample fluctuations by evaluating quantities such as $\langle (e^{-S_a(t)})^2 \rangle - \langle e^{-S_a(t)} \rangle^2$. More ambitiously, it would be appealing to use higher-order terms in the $1/q$ expansion to search for the expected differences between the naive light-cone velocity, the entanglement-spreading velocity, and the butterfly velocity, and to investigate broadening of the step in the OTOC given in Eq. (13) for the large- q limit.

There are also several generalizations. Our model and the techniques we develop can be naturally extended to higher dimensions. Analogous models could also be developed for the other symmetry classes in random-matrix theory. In the context of quantum transport, it would be interesting to incorporate the presence of conserved quantities by modifying the local structure of the Floquet operator, as has been done recently for random quantum circuits [32,33].

Finally, and speculatively, it is possible that an understanding of the ergodic phase in our model for chaotic

many-body systems may play a role analogous to the treatment of a diffusive metal in the theory of disordered conductors and provide a starting point for a theory [34] of the many-body localization transition [35].

ACKNOWLEDGMENTS

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Note added.—Recently, we learned of recent related calculations of the spectral form factor for a many-body system in which the equivalent of Eq. (5) is derived for the orthogonal symmetry class [36]. There are, however, important differences between the two approaches. In particular, the result of Ref. [36] develops from a specific long-range Hamiltonian using ideas of periodic orbit theory. Instead, the present work utilizes a limit of large on-site Hilbert space and disorder averaging still retaining a local short-range structure. Therefore, one important potential use of our method is to address ergodicity in disordered systems [37].

APPENDIX A: RECURSIVE FORMULA FOR

$$V_{c_1, c_2, \dots, c_u}$$

The coefficient V_{c_1, \dots, c_k} appearing in Eq. (15) obeys the recursion relation [16,38,39]

$$\begin{aligned} \delta_{c_1,1} V_{c_2, \dots, c_k} &= N V_{c_1, \dots, c_k} + \sum_{p+q=c_1} V_{p, q, c_2, \dots, c_k} \\ &+ \sum_{j=2}^k c_j V_{c_1+c_j, c_2, \dots, c_{j-1}, c_{j+1}, \dots, c_k}, \end{aligned} \quad (\text{A1})$$

where $V_0 \equiv 1$, and $N = q^2$ is the dimension of the unitary group from which the Haar-distributed unitary operators are drawn.

APPENDIX B: ENUMERATION OF LEADING-ORDER DIAGRAMS

In the following, we describe two methods for efficiently enumerating the leading-order diagrams for the physical quantities we compute. The contraction addition method described in Appendix B 1 can be used to eliminate subleading single-site diagrams efficiently, which is particularly useful if the many-body diagram of interest has the same site diagrams across all sites (e.g., the spectral form factor). The DW approach described in Appendix B 2 allows us to obtain an upper bound to the order, a global property of a diagram, by making only local calculations between neighboring domains. We apply the ideas from Appendix B 1 in Appendices C, D, and G, and those from Appendix B 2 in Appendices E and G.

The foundation for both of these methods is Eq. (18), which gives the order $\mathcal{O}(G)$ of a diagram G as a product of factors $q^{\tau_i+u_i-n_i}$ from each site. As the numbers τ_i of T loops and u_i of U loops contribute to the order in the same way, we do not need to distinguish between T and U loops in the calculation of order discussed below.

1. Method of contraction addition

An ensemble-averaged many-body diagram consists of L site diagrams labeled by i , each with a number n_i of contractions. The idea of this approach is that, given a site diagram, we can first remove all the contractions and then reconstruct the diagram by adding contractions one at a time. The order can be evaluated by considering the effect of each contraction addition. The crucial point is that the procedure is independent of the sequence of contraction additions. Each addition either leaves the order in q unchanged (if it increases both $\tau_i + u_i$ and n_i by 1) or reduces the order (if the addition increases only n_i , or reduces $\tau_i + u_i$ and increases n_i). A convenient sequence (one that contains order-reducing contraction additions in the first few steps) can therefore be used to eliminate subleading diagrams efficiently.

To be more precise, given a diagram G , we consider the single-site configuration G_i on site i . We choose a sequence of diagrams $\{G_i^{(0)}, \dots, G_i^{(n_i)}\}$, such that $G_i^{(0)}$ has no contractions, $G_i^{(m+1)}$ is obtained from $G_i^{(m)}$ by adding one contraction, and $G_i^{(n_i)} = G_i$. The overall order is then

$$\mathcal{O}(G) = \prod_{i \in \text{site}} \mathcal{O}(G_i^{(0)}) \prod_{m=1}^{n_i} \Delta_i^{(m)}. \quad (\text{B1})$$

Here, $\Delta_i^{(m)} = \mathcal{O}(G_i^{(m)}) / \mathcal{O}(G_i^{(m-1)})$ is the change in the order when one contraction is added.

As we establish that we do not need to distinguish between T and U loops here, each contracted site diagram can be represented as a collection of loops using the same notation for both types of loops. The same representation can also be used for each intermediate $G_i^{(m)}$ despite the fact that the notion of distinct T and U loops may not apply to $G_i^{(m)}$. The addition of a contraction has the effect either of merging two loops into a single longer one or of breaking one loop into two. To see this, recall the prescription for construction of loops in a single-site configuration G_i that is implied by Fig. 6. It can be expressed as follows. Start at any point on a single (for T loops) or double (for U loops) line; follow the direction along this line; when a dashed line is encountered, go to the other end of the dashed line and continue along the single or double line in the same sense; repeat until the starting point is reached: the path traced is a loop. The addition of an extra contraction to this prescription simply reroutes the paths of the two loop segments that meet the contraction from either side. Moreover, since the

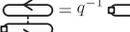
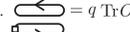
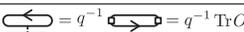
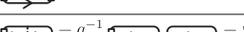
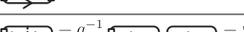
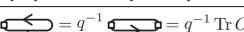
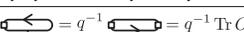
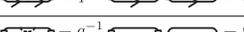
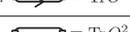
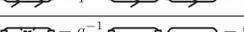
1.  = q	 = q^{-1}  = q	$\Delta = 1$
2.  = q^2	 = q^{-1}  = 1	$\Delta = q^{-2}$
3.  = $q \text{Tr} O$	 = q^{-1}  = $q^{-1} \text{Tr} O$	$\Delta = q^{-2}$
4.  = $q \text{Tr} O^2$	 = q^{-1}  = $q^{-1} \text{Tr} O^2$	$\Delta = q^{-2}$
5.  = $\text{Tr} O$	 = q^{-1}  = $\text{Tr} O$	$\Delta = 1$
6.  = $(\text{Tr} O)^2$	 = q^{-1}  = $q^{-1} \text{Tr} O^2$	$\Delta = q^{-1} \frac{\text{Tr} O^2}{(\text{Tr} O)^2}$
7.  = $\text{Tr} O^2$	 = q^{-1}  = $q^{-1} (\text{Tr} O)^2$	$\Delta = q^{-1} \frac{(\text{Tr} O)^2}{\text{Tr} O^2}$
8.  = $\text{Tr} O^2$	 = q^{-1}  = $\text{Tr} O^2$	$\Delta = 1$

FIG. 11. Enumeration of all distinct ways in which a contraction may be added to a site diagram. Columns from left to right are as follows. (i) Eight distinct combinations of one or two loops, before addition of a contraction, with associated contributions to $\mathcal{O}(G_i^{(m-1)})$. Three types of loop appear, containing zero, one, or two local operators O indicated using open squares. (ii) The same after adding a contraction shown as a dashed line and redrawn purely as loops, with associated contributions to $\mathcal{O}(G_i^{(m)})$. (iii) Values of $\Delta^{(m)}$ arising from each contraction obtained as the ratio of contributions in columns (ii) and (i).

two ends of the dashed line depicting a contraction are attached either to two separate loops or to two different portions of the same loop, we can evaluate $\Delta_i^{(m)}$ omitting all information except what concerns the one or two loops involved in the contraction. Once one focuses on the relevant details in this way, there are only eight distinct contraction-addition scenarios. We list them all in Fig. 11, using a single unbroken line to denote generic loops. A minor complication is that it is necessary to distinguish loops that pass through one or two operators representing local observables from ones that do not, since these operators may also affect the weight of a diagram.

We provide a simple example of how this approach to determining the order of a site diagram works in Fig. 12.

2. Domain-wall approach

In Sec. IV C, we set out a way of considering contributions to some quantities of interest in terms of space-time domains within which contractions are all of the same type. The power of this approach lies in the fact that one can associate a bound on the diagram's order with each DW

$$\begin{aligned} \mathcal{O}\left(\text{Diagram with two loops and a contraction}\right) &= \mathcal{O}\left(\text{Diagram with two loops}\right) \Delta\left(\text{Diagram with one loop and a contraction}\right) \Delta\left(\text{Diagram with two loops and a contraction}\right) \\ &= q \times 1 \times q^{-2} = q^{-1} \end{aligned}$$

where $\Delta\left(\text{Diagram with one loop and a contraction}\right) = \Delta\left(\text{Diagram with two loops and a contraction}\right) = q^{-2}$

FIG. 12. An example of order determination of a diagram using the method of contraction addition. Arrows giving an orientation to loops are omitted for clarity.

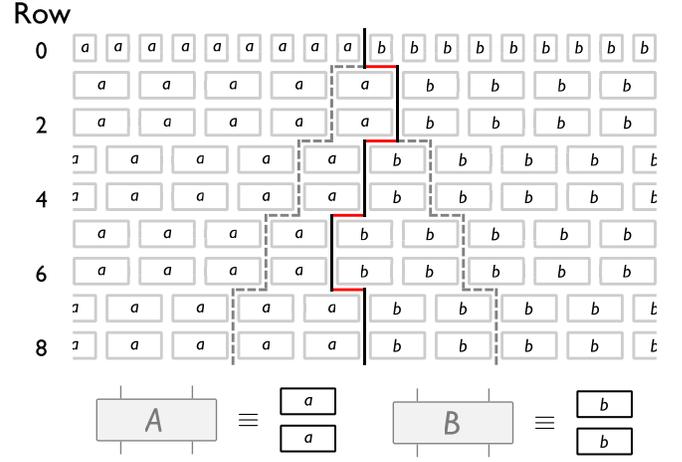


FIG. 13. Top: The block representation of a leading DW diagram for $\langle \exp[-S_2(t)] \rangle$. The top row (the zeroth row) denotes the effective blocks that represent the structure of traces in subregions A and B [Fig. 9(b) bottom left]. The dashed line represents a light cone outside of which all unitary pairs can be annihilated as described in Fig. 2. The red line represents costly boundaries (between blocks of different contraction types; see Appendix B 2 and Table I). Bottom: The correspondence between blocks and the symbols used in the bottom right of Fig. 9(b).

between two neighboring domains. Using DW, we can calculate a bound on the diagram's order, a global property, by making only local calculations between neighboring domains. We explain the details here, using for concreteness the second Rényi entropy (see Sec. IV C) and referring to the block representation introduced in Appendix E 1 and Fig. 13.

First, we rewrite the order of a diagram G in terms of loops. We define the length c_k of a loop k as half of the number of nondashed lines it contains. We make the convention that single lines at the top edge (where W and W^\dagger are multiplied) are counted twice so that lengths are always integer numbers (e.g., at the top edge of Fig. 5 there are four single lines in each single-site diagram). This construction generalizes the definition given in Sec. III to both U loops and T loops. We also define the total length $l_t = \sum_k c_k$ where the sum runs over all loops. Then the order of a diagram in Eq. (18) can be rewritten

$$\mathcal{O}(G) = q^{l_t - n} \prod_{k \in \text{loops}} \Gamma_{c_k}, \quad (\text{B2})$$

where $\Gamma_c = q^{1-c}$, and $n = \sum_i n_i$ is the total number of contractions. The values of l_t and n are fixed by the observable of interest. In particular, for the second Rényi entropy, $l_t = n$.

The intuition behind Eq. (B2) is as follows. Since a given diagram has a fixed number of length segments, its order is high (low) when there are many short (a few long) loops. The prefactor $q^{l_t - n}$ represents the theoretical maximum order when all length segments are used to form 1-loops. Γ_c

is the cost associated with a loop of length c , and its value decreases as c increases. For example, Γ_1 has value 1 since 1-loop is the shortest possible loop. Γ_2 has value q^{-1} because one could have formed two 1-loops [giving a factor of q^2 in Eq. (18)] instead of one 2-loop (giving a factor of q).

Next, we rewrite the order of a diagram in terms of boundaries that intersect the loops. We define a *boundary* as a *horizontal* segment of unit lattice spacing that separates two blocks (e.g., the red lines in Fig. 15 top) so that each block has four boundaries (except for the blocks representing the trace structure at the upper edge, e.g., zeroth row in Fig. 15 top). Every loop intersects two such boundaries per unit length. To each boundary w , we can therefore associate a cost

$$\mathcal{C}(w) = \prod_{\substack{k \in \text{loops} \\ k \dagger w}} \Gamma_{c_k}^{1/2c_k}, \quad (\text{B3})$$

where k labels the loops that the boundary w intersects, and c_k is the corresponding loop length. In this way, the order of a diagram G is expressed in terms of the cost of each boundary so that

$$\mathcal{O}(G) = q^{t-n} \prod_{w \in \text{walls}} \mathcal{C}(w). \quad (\text{B4})$$

Since $\Gamma_1 = 1$, loops of unit length do not contribute to $\mathcal{O}(G)$. We obtain an upper limit on the cost of a boundary by assuming longer loops to have length not more than $c_k = 2$. This procedure leads to

$$\mathcal{C}(w) = \prod_{k \dagger w, c_k \geq 2} \Gamma_{c_k}^{1/2c_k} \leq q^{-n_w/4}, \quad (\text{B5})$$

where n_w is the number of loops crossing the boundary w with length longer than 1. Referring to examples in Fig. 15, $\mathcal{C}(w)$ for the top subfigure is bounded by (and equal to) q^{-1} with $n_w = 4$, and $\mathcal{C}(w)$ for the bottom subfigure is bounded by $q^{-3/4}$ with $n_w = 3$. As we see, a costly boundary (one with $n_w > 0$) is always sandwiched between blocks with different contractions. It is therefore natural to call such a boundary a DW. With minor modifications, this method can also be applied to diagrams that contain local operators.

APPENDIX C: SPECTRAL FORM FACTOR

Here we use the method of contraction addition (Appendix B 1) to enumerate the leading-order diagrams for $\langle K(t) \rangle$. In Eq. (B1), we have $n_i = 4t$ and $\mathcal{O}(G_i^{(0)}) = q^2$. There are only two relevant contraction-addition scenarios illustrated in Fig. 11: (i) An intraloop addition (item 1 in the figure), where the two legs of the new contraction line land on the same loop. Then, $\Delta = q^0$ according to Eq. (18) because $\tau + u$ and n both increase by 1. (ii) An interloop addition (item 2 in figure), where the two contraction legs

land on two different bare loops. Since the two loops merge into a single bigger loop due to the new contraction, $\tau + u$ decreases by 1, and n increases by 1. So $\Delta = q^{-2}$.

The many-body diagram for $K(t)$ comprises site diagrams before contraction as in Fig. 7(a). For $t = 0$, we have $K(t) = q^{2L}$ since $W(0) = \mathbb{1}_{q^L}$. To compute the large- q limit of $K(t)$ for $t > 0$, we first note that there are diagrams with multiple contractions that are $O(1)$, such as Fig. 8(c). This is, in fact, a highest-order diagram for $t > 0$ according to Eq. (B1) because there must be at least one interloop contraction addition which is associated with $\Delta = q^{-2}$ per site, and because, from Fig. 11, the later contraction additions can be made without increasing the order of the diagram. So, any diagrams with order smaller than $O(1)$ are subleading.

There are t leading-order site diagrams on site i by the following argument. Using the fact that the order determination is independent of the sequence of contraction addition, we choose to contract on site i the (filled and unfilled) blue dots on the top layer from left to right. The first filled blue dot can be contracted with any one of the t -filled blue dots on the bottom layer. This interloop contraction costs q^{-2} . In order to obtain a leading diagram, the later contractions must all be of $O(1)$, i.e., intraloop contractions. A general feature of an $O(1)$ contraction is that it must partition a bigger loop into two smaller loops, such that the q factor associated with the extra loop (since τ or u increases by 1) cancels the q^{-1} factor of the contraction (since n increases by 1) (see Fig. 11). Furthermore, on each of the two smaller loops, there must be equal numbers of uncontracted blue dots on the top and bottom layers. Otherwise, there will be an interloop contraction which will render the diagram subleading. It is straightforward to see that there is a unique choice of contraction of the second blue dot that satisfies this requirement. Similarly, there are unique choices for the rest of the blue dots on this site. In order to not incur further cost on site i , each red dot on the top layer must be contracted with the only other red dot on the same loop on the bottom layer. So there are unique choices for the red dots as well.

Because of the bond constraint, the $(i + 1)$ th site diagrams inherit the choice of either the blue or red dot contractions on site i . We can repeat the above analysis site by site and conclude that there are t leading-order diagrams of $O(1)$ for $t > 0$ [Figs. 7(b)–7(d)].

Lastly, each of these t diagrams is translated algebraically to the factor 1, simply observing that it is Gaussian and for each site $\tau_i + u_i = n_i$ in Eq. (18). In other words, for $t > 0$, $K(t) = t$, and we arrive at Eq. (5).

APPENDIX D: RELAXATION OF LOCAL OBSERVABLES

Here we prove Eq. (7). For $t = 0$, the calculation is straightforward since $W(0) = \mathbb{1}_{q^L}$. For $t = 1$, there is only

one configuration at each site because there is only one pair of dots of each kind. Since we choose $\text{Tr}O_x = 0$, the algebraic term associated with this diagram vanishes. For $t > 1$, we use the same argument as in Appendix C to show that every possible diagram has an order that vanishes as $q \rightarrow \infty$. In the formulation of contraction addition (Appendix B 1), we have $n_i = 4t$ and $\mathcal{O}(G_i^{(0)}) = 1$. On site x , we choose to first contract the blue dot on the bottom left of the diagram [Fig. 8(b)]. This bottom left blue dot cannot be contracted with the bottom right blue dot nontrivially since $\text{Tr}O_x$ is zero. However, if this bottom left blue dot is contracted with any other blue dots, it will partition a bigger loop into two smaller loops, each of which has an unequal number of dots from U 's and U^\dagger 's. Consequently, there will be at least one addition of an interloop contraction with $\Delta \propto q^{-1}$ (item 6 in Fig. 11). Since all other contraction additions can further reduce only the order according to Fig. 11, we have $\mathcal{O}(G) < q^{-1}$ for any diagram G associated with $\langle [O(x, t)O(x)]_{\text{av}} \rangle$ when $t > 1$.

APPENDIX E: SECOND RÉNYI ENTROPY

The diagrammatic representation used in Fig. 9(c) requires a straightforward extension when it is employed in the presentation of detailed proofs, and we start by describing this extension. It arises for the following reason. The rectangles of Fig. 9(c) represent local contractions for a stack of two U 's and two U^\dagger 's. Each contraction involves linking both open dots with open dots, and closed dots with closed dots. To allow for the possibility that different pairings are made for open dots and for closed dots, each rectangle appearing in Fig. 9(c) is divided horizontally into two in this Appendix.

1. Block representation of $\langle \exp[-S_2(t)] \rangle$

In detail, we introduce the *block representation* as follows. Beginning from the folded representation that we introduce in Sec. IV C, we consider a stack of four unitary operators (blue box in Fig. 9). We use Fig. 3 to represent each unitary operator in terms of dots and double lines. A *block* is defined as a region within this box that encloses only filled (or only empty) dots. In this way, to each stack of four unitaries, we associate two blocks containing, respectively, filled and empty dots [so that the four-legged symbol in the bottom right of Fig. 9(b) corresponds to two blocks in Fig. 13 bottom]. For $\langle \exp[-S_\alpha(t)] \rangle$, there are $4t$ rows of such blocks, and drawing each of them as an empty rectangle leads to the representation in Fig. 13.

We categorize the possible contractions of the dots within a block into seven types $\mathcal{T} = \{a, b, a_1, a_2, b_1, b_2, x\}$. As we explain in the main text, a and b involve only local contractions (within the block), while x involves only non-local contractions. The other types are of mixed local and nonlocal character and are defined in Fig. 14. An upper bound $\omega(c, c')$ for the cost of the boundary between two

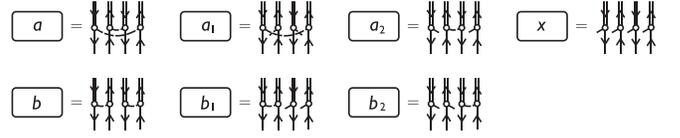


FIG. 14. Contractions in a portion of a site diagram for $\langle \exp[-S_2(t)] \rangle$ showing all seven possible block types.

neighboring block configurations $c, c' \in \mathcal{T}$ is shown in Table I. Two examples of the evaluation of $\omega(c, c')$ are illustrated in Fig. 15 using the method introduced in Appendix B 2.

2. Evaluation of $\langle \exp[-S_2(t)] \rangle$

We discuss diagrams for $\langle \exp[-S_2(t)] \rangle$ in terms of different block pairings. We first fix our convention as follows. We label the rows of blocks from the top as in Fig. 13. We refer to the boundaries immediately above the p th row of blocks as the p th row of boundaries.

Because of the simplification illustrated in Fig. 2, pairs of U and U^\dagger can be canceled outside of the light cone originating from the subsystem boundary (shown with dashed lines in Fig. 13). Alternatively, instead of canceling them, it is equivalent to assume that the blocks on the far left and far right have, respectively, a and b contractions, as long as we remain in the large- q limit. Consistently, this choice implies costless boundaries according to Table I. Therefore, in particular, a configuration for any given odd row of boundaries can be parametrized as in Fig. 16, and we denote this as $(ad_1d_2d_3, \dots, d_kb)_{\text{wall}}$ with variables $d_1 \neq a$, $d_k \neq b$, and $d_i \neq d_{i+1}$ for $i = 1, \dots, k-1$. a , b , and d_i represent (connected) domains of blocks with the same type of contractions. Note that for every change of domain, there is a DW associated with a factor of q^{-m} with $m > 0$.

We claim the following for the leading diagrams. (i) Each odd row of walls has the form $(ab)_{\text{wall}}$. (ii) Each even row of walls is sandwiched between two rows of blocks with identical configurations. (iii) The leading diagrams have order q^{-2t} .

TABLE I. Upper bounds for the cost associated with the boundaries between the seven possible types of block contraction. The matrix is symmetric, and so only the upper triangle is written explicitly.

ω	a	b	a_1	a_2	b_1	b_2	x
a	q^0	q^{-1}	$q^{-1/2}$	$q^{-1/2}$	q^{-1}	q^{-1}	q^{-1}
b		q^0	q^{-1}	q^{-1}	$q^{-1/2}$	$q^{-1/2}$	q^{-1}
a_1			q^0	q^{-1}	$q^{-3/4}$	$q^{-3/4}$	$q^{-1/2}$
a_2				q^0	$q^{-3/4}$	$q^{-3/4}$	$q^{-1/2}$
b_1					q^0	q^{-1}	$q^{-1/2}$
b_2						q^0	$q^{-1/2}$
x							q^0

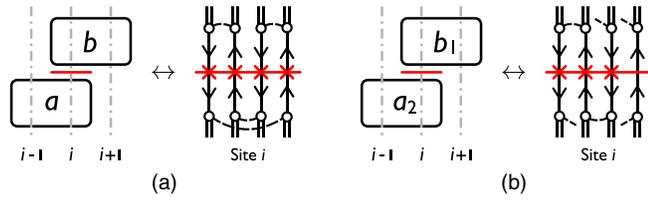


FIG. 15. Examples of contractions in a portion of a site diagram for $\langle \exp[-S_2(t)] \rangle$. Left sides of (a) and (b): Two neighboring blocks along site i shown in the space-time plane with the time axis vertical. Right sides of (a) and (b): Equivalent site diagrams with the time axis vertical; open-ended dashed lines are contracted nonlocally with an external block. Red lines represent sections of boundary (of unit length in the left-hand pictures). (a) Right: The boundary crosses four times a loop of length at least 2 at the red crosses, so $\omega(a, b) = q^{-1}$. (b) Right: The boundary has three costly crossings, so $\omega(b_1, a_2) = q^{-3/4}$.

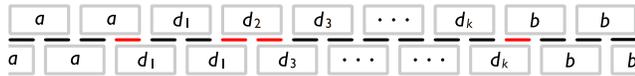


FIG. 16. A generic configuration of an odd row of boundaries denoted as $(ad_1d_2d_3, \dots, d_kb)_{\text{wall}}$. Costly boundaries (DWs) are drawn in red. Costless boundaries are drawn in black.

An example of order q^{-2t} can be found by constructing a single DW dividing two domains of blocks with contraction a and b (Fig. 13), such that on the odd rows of boundaries, the order is $\omega(a, b) = q^{-1}$, and on the even rows of boundaries, the order is 1. So any diagram with order lower than q^{-2t} is subleading.

To prove these statements, we note that an odd row $(ab)_{\text{wall}}$ has only one DW segment with cost q^{-1} . Conversely, by using the costs in Table I, it is easy to verify that any row configuration $(ad_1d_2d_3, \dots, d_kb)_{\text{wall}}$ with $k \geq 1$ always has an order strictly smaller than q^{-1} . Note that the order of a row with $k \geq 2$ is straightforwardly smaller than q^{-1} since each wall costs at least $q^{-1/2}$.

An even row of walls that is sandwiched by two rows of blocks with different configurations is associated with an order of q lower than 1 because there is a costly wall sandwiched by two blocks of different types (see Appendix E 1).

Leading diagrams are obtained maximizing the order on odd and even rows, thus, proving (i) and (ii). Statement (iii) follows trivially from statements (i) and (ii). Finally, (ii) implies that these diagrams are Gaussian (see Sec. III) and translate into positive algebraic factor. Hence, we recover Eq. (20).

APPENDIX F: THIRD AND HIGHER RÉNYI ENTROPIES

We can also represent the leading contributions to $\langle \exp[-2S_3(t)] \rangle$ by using a block representation. An extra

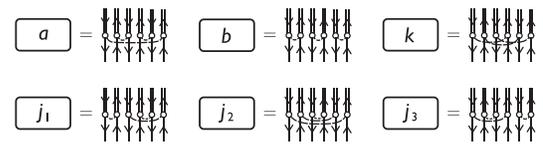


FIG. 17. The six local contractions for a block appearing in $\langle \exp[-S_3(t)] \rangle$ with the same notation as the one in Fig. 14.

feature in this case compared with $\alpha = 2$ is that there are six possible types of local contractions within a block. We label these contractions a , b , k , and j_i for $i = 1, 2, 3$ (Fig. 17). The blocks at the top edge of the diagram are fixed to be of type a and b by the structure of traces in $\exp[-S_3(t)]$.

Repeating the approach for $\alpha = 2$, we find that leading diagrams involve only local contractions. Equivalent statements to (i)–(iii) in Appendix E 2 apply. For (i), we find that the leading-order odd rows of boundaries have order q^{-2} , and they are of the forms $(ab)_{\text{wall}}$ or $(aj_i b)_{\text{wall}}$. All other odd row configurations $(ad_1d_2d_3, \dots, d_kb)_{\text{wall}}$ are associated with factors smaller than q^{-2} . For $k > 4$, the upper bound is trivially smaller than q^{-2} because each wall costs at least $q^{-1/2}$. For $1 \leq k \leq 4$, we enumerate all cases using symbolic computation to complete the proof. Statement (ii) for $\alpha = 3$ is the same as the one for $\alpha = 2$, and (iii) follows trivially for time $t \leq L/4$. An example of a leading-order $[O(q^{-4t})]$ diagram is given in Fig. 19.

The main difference with respect to $\alpha = 2$ is that the leading contributions also include diagrams with regions of j_i contractions, as well as regions of a and b contractions. We classify the leading diagrams at t by the width r of the j_i region on the bottom row of blocks. To count the number of leading diagrams, we use an inductive approach from time t to $t + 1$. Every leading diagram at $t + 1$ can be generated from a leading diagram at t by adding four rows of blocks at the bottom of the diagram at t . A newly added even row must have a configuration identical to the one above. A newly added odd row has configuration depending on the one above according to the following rules (Fig. 18). For each even row in which the width of the j_i region is zero, there are five possible configurations for the odd row below: two with width 0 and three with width 1. For each

Row p	Configuration						Width r	Configuration			Width
	a	j_i	\dots	\dots	j_i	b		a	a	b	
$p+1$	a	a	j_i	\dots	j_i	b	$r-1$	a	a	b	0
$p+1$	a	a	j_i	\dots	j_i	j_i	r	a	b	b	0
$p+1$	a	j_i	j_i	\dots	j_i	b	r	a	j_i	b	1
$p+1$	a	j_i	j_i	\dots	j_i	j_i	$r+1$	a	j_i	b	1

FIG. 18. The recursive rules for enumerating the $p + 1$ th row of the leading-order diagrams given the p th row of the leading-order diagram. These rules are derived from the fact that, in a leading diagram, the difference between the DW positions at even row p and odd row $p + 1$ is exactly one lattice spacing.

- [1] M. L. Mehta, *Random Matrices* (Academic Press, New York, 2004).
- [2] F. J. Wegner, *Disordered System with n Orbitals per Site: $n = \infty$ Limit*, *Phys. Rev. B* **19**, 783 (1979).
- [3] J. M. Deutsch, *Quantum Statistical Mechanics in a Closed System*, *Phys. Rev. A* **43**, 2046 (1991).
- [4] M. Srednicki, *Chaos and Quantum Thermalization*, *Phys. Rev. E* **50**, 888 (1994).
- [5] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, *From Quantum Chaos and Eigenstate Thermalization to Statistical Mechanics and Thermodynamics*, *Adv. Phys.* **65**, 239 (2016).
- [6] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Colloquium*, *Rev. Mod. Phys.* **83**, 863 (2011).
- [7] E. H. Lieb and D. W. Robinson, *The Finite Group Velocity of Quantum Spin Systems*, *Commun. Math. Phys.* **28**, 251 (1972).
- [8] H. Kim and D. A. Huse, *Ballistic Spreading of Entanglement in a Diffusive Nonintegrable System*, *Phys. Rev. Lett.* **111**, 127205 (2013).
- [9] W. W. Ho and D. A. Abanin, *Entanglement Dynamics in Quantum Many-Body Systems*, *Phys. Rev. B* **95**, 094302 (2017).
- [10] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, *Quantum Entanglement Growth under Random Unitary Dynamics*, *Phys. Rev. X* **7**, 031016 (2017).
- [11] A. Nahum, S. Vijay, and J. Haah, *Operator Spreading in Random Unitary Circuits*, *Phys. Rev. X* **8**, 021014 (2018).
- [12] C. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. Sondhi, *Operator Hydrodynamics, OTOCs, and Entanglement Growth in Systems without Conservation Laws*, *Phys. Rev. X* **8**, 021013 (2018).
- [13] A. Hamma, S. Santra, and P. Zanardi, *Quantum Entanglement in Random Physical States*, *Phys. Rev. Lett.* **109**, 040502 (2012); *Ensembles of Physical States and Random Quantum Circuits on Graphs*, *Phys. Rev. A* **86**, 052324 (2012); P. Zanardi, *Local Random Quantum Circuits: Ensemble Completely Positive Maps and Swap Algebras*, *J. Math. Phys. (N.Y.)* **55**, 082204 (2014).
- [14] A. I. Larkin and Yu. N. Ovchinnikov, *Quasiclassical Method in the Theory of Superconductivity*, *Zh. Eksp. Teor. Fiz.* **55**, 2262 (1969) [*Sov. Phys. JETP* **28**, 1200 (1969)].
- [15] J. Maldacena, S. H. Shenker, and D. Stanford, *A Bound on Chaos*, *J. High Energy Phys.* **16** (2016) 106.
- [16] P. W. Brouwer and C. W. J. Beenakker, *Diagrammatic Method of Integration over the Unitary Group, with Applications to Quantum Transport in Mesoscopic Systems*, *J. Math. Phys. (N.Y.)* **37**, 4904 (1996).
- [17] A. Nahum, J. Ruhman, and D. A. Huse, *Dynamics of Entanglement and Transport in 1D Systems with Quenched Randomness*, *Phys. Rev. B* **98**, 035118 (2018).
- [18] A. Chan, A. De Luca, and J. T. Chalker, *Spectral Statistics in Spatially Extended Chaotic Quantum Many-Body Systems*, [arXiv:1803.03841](https://arxiv.org/abs/1803.03841).
- [19] K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, England, 1996).
- [20] M. R. Zirnbauer, *Supersymmetry for Systems with Unitary Disorder: Circular Ensembles*, *J. Phys. A* **29**, 7113 (1996).
- [21] D. N. Page, *Average Entropy of a Subsystem*, *Phys. Rev. Lett.* **71**, 1291 (1993).
- [22] C. Nadal, S. N. Majumdar, and M. Vergassola, *Phase Transitions in the Distribution of Bipartite Entanglement of a Random Pure State*, *Phys. Rev. Lett.* **104**, 110501 (2010).
- [23] T. Zhou and A. Nahum, *Emergent Statistical Mechanics of Entanglement in Random Unitary Circuits*, [arXiv:1804.09737](https://arxiv.org/abs/1804.09737).
- [24] P. Calabrese and J. Cardy, *Evolution of Entanglement Entropy in One-Dimensional Systems*, *J. Stat. Mech.* (2005) P04010; *Quantum Quenches in Extended Systems*, *J. Stat. Mech.* (2007) P06008; *Quantum Quenches in 1 + 1 Dimensional Conformal Field Theories*, *J. Stat. Mech.* (2016) 064003.
- [25] F. Haake, *Quantum Signatures of Chaos* (Springer, New York, 2010).
- [26] T. Engl, J. Dujardin, A. Argüelles, P. Schlagheck, K. Richter, and J. D. Urbina, *Coherent Backscattering in Fock Space: A Signature of Quantum Many-Body Interference in Interacting Bosonic Systems*, *Phys. Rev. Lett.* **112**, 140403 (2014).
- [27] T. Engl, J. D. Urbina, and K. Richter, *The Semiclassical Propagator in Fock Space: Dynamical Echo and Many-Body Interference*, *Phil. Trans. R. Soc. A* **374**, 1 (2016).
- [28] M. Akila, D. Waltner, B. Gutkin, P. Braun, and T. Guhr, *Semiclassical Identification of Periodic Orbits in a Quantum Many-Body System*, *Phys. Rev. Lett.* **118**, 164101 (2017).
- [29] I. L. Aleiner, L. Faoro, and L. B. Ioffe, *Microscopic Model of Quantum Butterfly Effect: Out-of-Time-Order Correlators and Traveling Combustion Waves*, *Ann. Phys. (Amsterdam)* **375**, 378 (2016).
- [30] Y. Gu, X.-L. Qi, and D. Stanford, *Local Criticality, Diffusion and Chaos in Generalized Sachdev-Ye-Kitaev Models*, *J. High Energy Phys.* **17** (2017) 125.
- [31] A. Kitaev, *A Simple Model of Quantum Holography*, in *Proceedings of KITP Strings Seminar and Entanglement*, 2015 [<http://online.kitp.ucsb.edu/online/entangled15/kitaev/>; <http://online.kitp.ucsb.edu/online/entangled15/kitaev2/>].
- [32] V. Khemani, A. Vishwanath, and D. A. Huse, *Operator Spreading and the Emergence of Dissipation in Unitary Dynamics with Conservation Laws*, *Phys. Rev. X* **8**, 031057 (2018).
- [33] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, *Diffusive Hydrodynamics of Out-of-Time-Ordered Correlators with Charge Conservation*, *Phys. Rev. X* **8**, 031058 (2018).
- [34] A. Altland and T. Micklitz, *Field Theory Approach to Many-Body Localization*, *Phys. Rev. Lett.* **118**, 127202 (2017).
- [35] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, *Metal-Insulator Transition in a Weakly Interacting Many-Electron System with Localized Single-Particle States*, *Ann. Phys. (Amsterdam)* **321**, 1126 (2006).

- [36] P. Kos, M. Ljubotina, and T. Prosen, *Many-Body Quantum Chaos: The First Analytic Connection to Random Matrix Theory*, *Phys. Rev. X* **8**, 021062 (2018).
- [37] A. Chan, A. De Luca, and J. T. Chalker, *Spectral Statistics in Spatially Extended Chaotic Quantum Many-Body Systems*, *Phys. Rev. Lett.* **121**, 060601 (2018).
- [38] S. Samuel, *$u(n)$ Integrals, $1/n$, and the Dewit-'t Hooft Anomalies*, *J. Math. Phys. (N.Y.)* **21**, 2695 (1980).
- [39] P. A. Mello, *Averages on the Unitary Group and Applications to the Problem of Disordered Conductors*, *J. Phys. A* **23**, 4061 (1990).