# Tensor Networks for Lattice Gauge Theories with Continuous Groups 

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#### Abstract

We discuss how to formulate lattice gauge theories in the tensor-network language. In this way, we obtain both a consistent-truncation scheme of the Kogut-Susskind lattice gauge theories and a tensornetwork variational ansatz for gauge-invariant states that can be used in actual numerical computations. Our construction is also applied to the simplest realization of the quantum link models or gauge magnets and provides a clear way to understand their microscopic relation with the Kogut-Susskind lattice gauge theories. We also introduce a new set of gauge-invariant operators that modify continuously RokhsarKivelson wave functions and can be used to extend the phase diagrams of known models. As an example, we characterize the transition between the deconfined phase of the $Z_{2}$ lattice gauge theory and the RokhsarKivelson point of the $U(1)$ gauge magnet in 2D in terms of entanglement entropy. The topological entropy serves as an order parameter for the transition but not the Schmidt gap.


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## I. INTRODUCTION

Tensor-network (TN) techniques are starting to play an important role in our understanding of many-body quantum systems, both on the lattice and in the continuum. They can be used as a framework to classify the phases of quantum matter [1-3] or as a powerful numerical ansatz in actual computations of 1D [4,5] and 2D strongly correlated quantum magnets [6-8], fermionic systems [9,10], or anyonic systems [11,12]. They have also recently made their way into quantum chemistry as a computational tool to study the structure of molecules from the first principles [13,14].

While numerical simulations based on Monte Carlo (MC) techniques are still the most successful techniques in some of these fields, TNs start to provide viable alternatives to them, particularly in those contexts where MC simulations has troubles, such as the physics of frustrated antiferromagnets [15-17] and the real-time evolution of out-of-equilibrium systems [18-20].

At present, the main limitation of numerical TN techniques is that the cost of the simulations increases rapidly with the amount of correlations in the system (which is encoded in the bond dimension $D$ of the elementary tensors), and thus, TNs tend to be biased toward weakly correlated phases.

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However, the steady improvement of the TN algorithms [21,22] makes us confident that these limitations will soon be overcome, and as a consequence, TNs will become more and more useful in the physics of quantum many-body systems. Among interesting quantum many-body systems, we focus here on gauge theories, a context in which TNs have recently made a spectacular debut [23-27].

Gauge theories (GTs) [28] describe three of the four fundamental interactions (electromagnetic, weak, and strong interactions). In particular, strong interactions are described by an $\operatorname{SU}(3)$ gauge theory, called quantum chromodynamics (QCD) [29]. GTs also allow us to understand emergent phenomena at low energies in condensed-matter systems, e.g., antiferromagnets [30] and high-temperature superconductors [31-33].

The phase diagrams of GTs, similarly to those of the most strongly correlated many-body quantum systems, are still debated. Still, there are exactly solvable GTs that display topological phases. Recently, topological states have been proposed as possible hardware for quantum computers, and thus, there is an urgent need for clarification of generic GT phase diagrams where these states could appear [23,34-37].

Wilson's formulation of lattice gauge theories (LGTs) [38] was obtained by substituting the continuous spacetime with a discrete set of points (the lattice). It provided the breakthrough that has allowed us to develop numerical tools based on MC simulations that are able to address the strong-coupling regime of GTs. These tools are, as today, the main resource to compare various aspects of QCD at strong coupling with experiments [39]. Those aspects of QCD that are hard or impossible to address with MC are indeed in most of the cases still unclear. For example, the
mechanism of charge confinement [40], invented to explain the absence of isolated quarks [41], still stands as a conjecture in full QCD, four decades since it was first understood in Abelian models. Furthermore, MC simulations struggle to address hot and dense nuclear matter [42,43], probed by heavy nuclei collisions at CERN and RHIC [44,45]. In their current formulation, MC simulations of LGTs cannot be used to characterize the real-time out-ofequilibrium dynamics of GTs.

The aim of this paper is twofold. On one side, we develop the theory of TNs for LGTs with arbitrary groups. On the other, we provide a constructive approach to LGTs using the TN formalism (reviewed in Sec. III). In this framework, TNs are used as a model-building tool that, given a group $G$, allows us to design the most general gauge-invariant theory out of the simple knowledge of the group-representation matrices. The approach is based on reformulating very simple results about the theory of group representations (reviewed in Sec. IV) in the TN formalism.

In particular, we use the approach in which LGTs differ from standard many-body quantum systems due to the presence of a large amount of local symmetry constraints (Sec. V), which arise as a consequence of generalizations of the Gauss law. One of the guiding principles in designing LGTs will thus be the possibility of defining such local symmetry constraints (Secs. VA and V B). In particular, we identify the "physical Hilbert space" $\mathcal{H}_{P}$ as the space of states that fulfill those constraints (Sec. V C). We show how these constraints can be naturally embedded in a TN. We thus construct an exact projector onto $\mathcal{H}_{P}$ as a TN (Sec. VI). We rederive, with our formalism, the elementary gaugeinvariant operators (Sec. V D) necessary to describe the dynamics inside $\mathcal{H}_{P}$.

In the course of our discussion, we explain that continuous groups are associated with infinite-dimensional local Hilbert spaces, and thus, the TN network construction for them has infinite bond dimension $D$ and thus is computationally intractable.

In order to cure this problem, we introduce a scheme that allows us to truncate, in a gauge-invariant way, the infinitedimensional local Hilbert spaces. In this way, we obtain a version of the KS LGT for (compact) continuous groups defined on finite-dimensional Hilbert spaces (Sec. VII). The projector onto $\mathcal{H}_{P}$ for these models can be expressed as a TN with finite bond dimension and can thus be used in practical computations.

We review the alternative constructions of LGTs with continuous gauge symmetry and discrete local Hilbert spaces, called gauge magnets (or link models) [46-50] (Sec. VIII). We generalize it to arbitrary groups and we show that in the non-Abelian case, the gauge magnets are not equivalent to a local gauge-invariant truncation of the KS LGT.

At this stage, we are able to introduce a general TN variational ansatz for LGTs, with both discrete and
continuous groups, that again automatically embeds all local constraints dictated by the gauge symmetry. The states described by this ansatz are indeed gauge symmetric by construction. Gauge-symmetry constraints, indeed, allow us to restrict the attention to $\mathcal{H}_{P}$, which is still, however, exponentially large, as shown in Fig. 1. Low-energy states of local gauge-invariant Hamiltonians are expected to live only on a small region of $\mathcal{H}_{P}$, in the same way as lowenergy states of generic local Hamiltonians live in a small region of the unconstrained Hilbert space, since they fulfill the "area law" for the entanglement [51-54].

We explicitly construct a TN ansatz that allows us to explore this small corner of $\mathcal{H}_{P}$ (Sec. IX). In its simplest form, the TN ansatz requires the same bond dimension as the projector onto $\mathcal{H}_{P}$. In this case, $D \simeq \sqrt{d}$, with $d$ the dimension of the local Hilbert space, and it allows us to characterize the physics of generalized Rokhsar-Kivelson (RK) states [55]. By increasing the bond dimension, one can gradually explore all the space of gauge-invariant states $\mathcal{H}_{P}$, as represented in Fig. 1, by increasingly large orange circles.

The TN ansatz depends on several elementary tensors, each made by two distinguished parts. One part is completely determined by the gauge-symmetry constraints, while the other contains the free parameters to be used


FIG. 1. The Hilbert space $\mathcal{H}$ of a quantum many-body system (represented here by a 3D box) is exponentially large, since it is the tensor product of the Hilbert spaces of the constituents. Gauge symmetry allows us to identify a smaller space that we call the physical Hilbert space $\mathcal{H}_{P}$. This space is the subspace spanned by those states that fulfill all the local constraints imposed by the gauge symmetry and is represented by a membrane inside $\mathcal{H}$. $\mathcal{H}_{P}$ is smaller than the full $\mathcal{H}$ but it is still exponentially large. Lowenergy states of local gauge-invariant Hamiltonians, however, are expected to live in a small corner of $\mathcal{H}_{P}$, in the same way as lowenergy states of generic local Hamiltonians live in a small corner of $\mathcal{H}$ [51-54]. For this reason, we design a variational ansatz based on TNs that allows us to explore this small corner of $\mathcal{H}_{P}$ (orange oval). By increasing the bond dimension of the elementary tensors in the $\mathrm{TN}(D \uparrow)$, we can explore increasingly large regions of $\mathcal{H}_{P}$, and eventually, for $D \rightarrow \infty$, we can cover the whole $\mathcal{H}_{P}$. The projector on $\mathcal{H}_{P}$ and a family of interesting RK states are obtained exactly with a TN with minimal bond dimension that scales as $D \simeq \sqrt{d}$, where $d$ is the dimension of the local Hilbert space.
in variational calculations. As in the case of globally invariant TNs [56-58], our formalism allows us to unveil interesting connections between gauge-symmetric quantum states and spin networks [59].

As a further application, we define gauge-invariant vertex operators [60] for arbitrary gauge theories (Sec. X). In this way, we open new possibilities to use them as extensions of the standard Hamiltonians in order to explore extended phase diagrams of the known models.

We benchmark these new tools in the context of RK states. In particular, we focus on the recent proposals about the characterization of quantum phases based on the analysis of the entanglement scaling of the ground-state wave function [61-63]. We analyze the well-known transition between the eight-vertex and the six-vertex models. In the gauge-theory language, this transition is induced by applying the vertex operators [60] onto the $\mathrm{RK} \mathbb{Z}_{2}$ wave functions. In this way, we provide an example of a phase transition between a $\mathbb{Z}_{2}$ gapped spin liquid and a $U(1)$ algebraic spin liquid that is detected by the topological entropy but elusive for the lowest part of the entanglement spectrum (Sec. XI).

All the discussion about connections of our results with other works in the literature is postponed to Sec. XII, and we conclude with a summary of our results and an outlook on future developments in Sec. XIII.

## II. SUMMARY OF THE RESULTS

Here, we briefly summarize the most important results of our paper so that the reader interested in applying our formalism to specific models will easily find the relevant material. (i) We derive the TN representation of the standard KS LGT Hamiltonian for arbitrary compact groups in Eqs. (18), (19), and (23). (ii) We present an extra gauge-invariant operator that can be added to the KS Hamiltonian to explore generalized KS LGTs for arbitrary compact groups in Eq. (45). (iii) We provide the exact TN representation for the projector onto $\mathcal{H}_{P}$ of the KS LGT (represented as a hyperplane in Fig. 1) in Fig. 10. It is the contraction of several copies of elementary tensors $\mathcal{C}$ and $\mathcal{G}$ defined in Fig. 11 and Eq. (26). (iv) We describe a truncation of the Hilbert space of the KS LGT that is consistent with gauge symmetry in Sec. VII. This truncation allows us to study LGTs with continuous groups [i.e., $U(1)$ and $S U(N)$ ] with constituents leaving in finitedimensional Hilbert spaces. (v) In this way, we introduce an exact TN representation with finite bond dimension of the projector onto the $\mathcal{H}_{P}$ of an arbitrary LGT. It is encoded in the TN of Fig. 10 with the $\mathcal{C}$ and $\mathcal{G}$ tensors defined in Fig. 15. (vi) We discuss the relation of this truncation scheme with gauge magnets or quantum link models. We also build the exact TN representation of the projector onto $\mathcal{H}_{P}$ for gauge magnets. It is the TN of Fig. 10 with the $\mathcal{C}$ and $\mathcal{G}$ tensors defined in Fig. 19. (vii) We provide a variational ansatz for generic states of $\mathcal{H}_{P}$ of all the LGTs discussed as
the TNs of Fig. 20. The ansatz is the contraction of several copies of sparse tensors that unveils a connection between the LGT gauge-invariant Hilbert space and spin networks. (viii) We confirm that the topological entropy detects phase transitions elusive to standard local order parameters. In particular, we characterize the transition between two different topological phases: the 2D eight-vertex and sixvertex topological phases. These results are presented in Fig. 24. In Fig. 25, we show that the same phase transition does not affect the behavior of the lowest part of the entanglement spectrum.

## III. TENSOR NETWORKS

Tensors are multilinear maps $\bar{X}_{b c}^{a}$ acting among different Hilbert spaces. In particular, the coefficients of a state of a quantum many-body system are encoded in the element of a very large tensor $T^{i_{1}, \ldots, i_{N}}$

$$
\begin{equation*}
|\psi\rangle=\sum_{i_{1}, \ldots, i_{N}} T^{i_{1}, \ldots, i_{N}}\left|i_{1}, \ldots, i_{N}\right\rangle \tag{1}
\end{equation*}
$$

In general, the tensor $T$ is too large to be stored on a computer, and thus, it is useful to express it as the contraction of smaller elementary tensors. These contracted tensors are called TNs. When dealing with large TNs, the formulas become easily large and complex, and it is simpler to resort to a graphical notation. The graphical notation is explained already in the literature [6,64-66], but we also shortly review it here in order to fix the notation we use in the paper.

In the graphical notation, geometric shapes are associated with tensors and lines or "legs" attached to them represent their indexes. As an example, the upper panel of Fig. 2 represents a tensor with three indexes $X_{b c}^{a}$. A small dot on the shape allows us to keep track of the index ordering that is assumed to be clockwise starting from the dot. Incoming legs, or in-legs (upper indexes), are drawn with entering arrows, while outgoing legs, or out-legs, are drawn as outgoing arrows. The Hermitian conjugate of a tensor $X^{\dagger}$ involves taking the complex conjugate of the elements and exchanging the in-legs with the out-legs $X^{\dagger}=\bar{X}_{a}^{c b}$. This operation is represented graphically by mirroring the tensor and inverting the arrows on the lines (left-hand side of the upper panel of Fig. 2).

A leg connecting two tensors represents their multiplication through the contraction of the corresponding indexes (summation over all the values of those indexes). An example is represented on the left-hand side of the lower panel of Fig. 2, where the tensor $W^{\dagger}$ is contracted with the tensor $W$.

In our formulas, we sometime omit explicit summation and use the Einstein notation, where the summation is intended over repeated indexes.

A tensor can always be interpreted as a matrix by dividing its legs into two groups: One group identifies a


FIG. 2. Graphic representation of tensors and their contractions. Tensors are generalized vertexes (geometric shapes) whose legs are represented by dangling arrows. (a) A tensor with three legs $X_{b c}^{a}$. The upper indexes are incoming legs while lower indexes are outgoing legs. Complex conjugation is denoted by mirror reflection of the vertex. Different colors denote different tensors. Indexes are ordered clockwise, starting from the solid dot on the vertex. The contraction of two tensors is denoted by an arrow joining them. The dagger operation $X^{\dagger}$ involves both the complex conjugation of the elements of the tensor and the inversion of the arrows attached to its legs. As a consequence, the order of the legs also changes $X^{\dagger}=\bar{X}_{a}^{c b}$ [67]. (b) An isometric tensor (or simply isometry) is always represented by a triangular vertex. A tensor $W$ is isometric if there is a specific choice of legs, which identifies two vector spaces $U$ and $V$, such that when $W$ is contracted with $W^{\dagger}$ through the legs in $U$, the result is the identity tensor in $V$. In the figure, $U$ is spanned by the leg $a$ while $V$ is spanned by the two legs $b$ and $c$.
vector's spaces $U$ and the other $V$. In this way, the tensor becomes a map from $U$ to $V$. A natural choice is to interpret $U$ as the collection of in-legs and $V$ as the collection of outlegs, but this choice is not the only possibility.

An important class of tensors is isometric tensors. For a specific choice of $U$ and $V$, they fulfill

$$
\begin{equation*}
W W^{\dagger}=1_{V} \tag{2}
\end{equation*}
$$

In our drawings, triangular shapes always represent isometric tensors [68].

The lower panel of Fig. 2 illustrates a specific case of a three-leg isometric tensor $W$, where $U$ is spanned by the leg $a$ and $V$ is spanned by the two legs $b$ and $c$. This observation implies that $W^{\dagger}$ is defined as $W^{\dagger} \equiv \bar{W}_{a}^{c b}$ and Eq. (2) reads $\bar{W}_{a}^{k l} W_{b c}^{a}=\delta_{b}^{k} \delta_{c}^{l}$. (Remember that there is a sum over $a$.) Graphically, the contraction is represented by the line connecting the two tensors. The result of the
contraction is a four-leg tensor, explicitly written as the tensor product of two identity tensors represented by straight lines.

When TNs represent quantum states of many-body systems, the legs related to the constituents $\left[i_{1} \rightarrow i_{N}\right.$ in Eq. (1)] are called physical legs and are typically represented by latin letters, while all the others legs (those that are contracted) are legs called auxiliary legs and represented by greek letters.

## IV. GROUP THEORY IN THE TENSOR-NETWORK LANGUAGE

Here, we assume that the reader is familiar with basic concepts of the representation theory of both finite and continuous groups, and we list the relevant results for our paper in order to express them in the TN language. In particular, while this paper deals with gauge theories with continuous groups, we develop the formalism by using discrete groups $G$. The underlining ideas are indeed completely independent from whether the group is discrete or continuous, and we feel that discrete groups allow us to present these ideas in a simpler way.

Concretely, all the results we present, which involve the summation over group elements, can be rewritten for continuous compact groups by substituting the sums with integrals over the group, defined through the appropriate invariant measure (see, i.e., Chap. 4 of Ref. [69]).

Here, we remind readers that a collection of elements $\{g\}$ closed under multiplication forms a group $G$. We are mostly interested in matrix representations of the group $G$ that are obtained by associating with each group element $g$ a unitary matrix $\Gamma(g)$ acting on a given vector space. In this way, the group multiplication table is rephrased into specific relations between the representing matrices. In general, given a matrix representation of a group, there is a well-defined procedure to reduce it to a block-diagonal form, where each of the blocks constitutes a "smaller," independent representation of the same group $G$. If those blocks are not further reducible into smaller blocks, they define an irreducible representation of the group $G$ that in this paper will be labeled by $r$. The dimension and number of the irreducible representations depend on the group $G$, and their study is the subject of the theory of group representations. In the following, we denote by $\Gamma_{r}(g)$ the matrix representation of $g$ in the irreducible representation $r$.

One of the most important results of the theory of group representations is what is typically called the "great orthogonality theorem" [70]. It states that given a group $G$ with elements $g$, and given any pair of irreducible representations $r$ and $r^{\prime}$, the following relation holds:

$$
\begin{equation*}
\frac{\sqrt{n_{r} n_{r^{\prime}}}}{|G|} \sum_{g} \Gamma_{r}\left(g^{-1}\right)_{j}^{i} \Gamma_{r^{\prime}}(g)_{k}^{l}=\delta_{k}^{i} \delta_{l}^{j} \delta\left(r, r^{\prime}\right), \tag{3}
\end{equation*}
$$

where $\Gamma_{r}(g)_{k}^{l}$ are the $l$ and $k$ matrix elements of the irreducible representation (irrep) $r$ of $g,|G|$ is the number of elements of $G$, and $n_{r}$ and $n_{r^{\prime}}$ are the dimensions of the irreps $r$ and $r^{\prime}$. We now want to reinterpret this relation in terms of TN diagrams. As a first step, we need to identify two vector spaces $U$ and $V . U$ is spanned by the group elements $g$. This vector space is called group algebra $\mathbb{C}(g)$ and has dimension equal to $|G| . V$ is the vector space spanned by the direct sum $V=\oplus_{r}\left(V_{r} \otimes V_{\bar{r}}\right)$. Each $V_{r}$ is the defining space of the irrep $r . V_{\bar{r}}$ is the defining space of its conjugate representation, obtained by taking the Hermitian conjugation of the matrices $\Gamma_{r}^{\dagger}(g) \equiv \Gamma_{r}\left(g^{-1}\right)$, where we have used the property that we are dealing with unitary representations. The fact that the vector space $V$ has a direct sum structure is encoded in the $\delta\left(r, r^{\prime}\right)$ factor in the right-hand side of Eq. (3).

We start by focusing on the above formula in the case $r=r^{\prime}$. In this case, Eq. (3) tells us that the tensor $W_{r}$

$$
\begin{equation*}
W_{r}=\sqrt{\frac{n_{r}}{|G|}} \Gamma_{r}(g)_{k}^{l} \tag{4}
\end{equation*}
$$

is an isometry. The tensor $W_{r}$ is represented in Fig. 3(a). The fact that it is an isometry means that

$$
\begin{equation*}
W_{r}^{\dagger} W_{r}=1_{V_{r} \otimes V_{\bar{r}}} \tag{5}
\end{equation*}
$$

as represented in Fig. 3(b).
Each $W_{r}$ thus allows us to project a vector $|g\rangle$ in the group algebra $\mathbb{C}(G)$ onto a vector $\left|\left(e_{r}\right)_{l}^{m}\right\rangle$ of the tensor product $V_{r} \otimes V_{\bar{r}}$. This property can be used in two ways. Reading Eq. (4) from left to right, it tells us that if we know all the $\Gamma_{r}(g)$ for all the elements $g \in G$, we can collect them in a three-leg tensor and obtain an isometry that projects $\mathcal{C}(G)$ onto $V_{r} \otimes V_{\bar{r}}$. From right to left, we can obtain the irrep's matrices. If we are given the isometry $W_{r}$, by acting onto the vector $|\tilde{g}\rangle=\left(|G| / \sqrt{n_{r}}\right)|g\rangle$ of $\mathbb{C}(g)$, we obtain $\Gamma_{r}(g)$

$$
\begin{equation*}
\Gamma_{r}(g)=W_{r}|\tilde{g}\rangle, \tag{6}
\end{equation*}
$$

as we represent graphically in Fig. 3(c).
A peculiarity of our notation that we inherit from spin networks is that each leg of the tensor also carries a representation index $r$ that is typically superimposed to the line. In order to avoid confusion, after Fig. 4, we will drop all the letters labeling the indexes of the tensors (unless really needed) and keep only the letters related to the irrep $r$.

A second result of the theory of group representations is that $\sum_{r} n_{r}^{2}=|G|$, that is, the direct sum of all the $W_{r}$, is a unitary transformation, as encoded in following relations:


FIG. 3. (a) The orthogonality relation in Eq. (3) allows us to identify a set of isometries $W_{r}$ projecting $\mathbb{C}(g)$ onto $V_{r} \otimes \bar{V}_{r}$. If one knows the matrices $\Gamma_{r}(g)$ for all $g \in G, W_{r}$ is defined by collecting them inside a three-index tensor. The index $r$ identifying the irrep is written on the top of one of the two legs. (b) The isometric property of the tensor $W_{r}$. (c) Alternatively, if we know the $W_{r}$ isometry, we can obtain the matrices in the $r$ representation by acting with $W_{r}$ on a vector proportional to $|g\rangle$ (yellow circle in the figure). The resulting tensor has two indexes $i$ and $j$ and is the matrix $\Gamma_{r}(g)_{j}^{i}$.

$$
\begin{equation*}
W_{G}=\oplus W_{r}, \quad W_{G} W_{G}^{\dagger}=1_{\mathbb{C}(G)}, \quad W_{G}^{\dagger} W_{G}=1_{\oplus_{r}\left(V_{r} \otimes V_{\bar{r}}\right)} . \tag{7}
\end{equation*}
$$

A graphical representation of the direct sum of $W_{r}$ leading to $W_{G}$ is presented in Fig. 4.

## A. Symmetric tensors

In the context of many-body quantum systems, symmetries play a fundamental role in the classification of phases. Recently, in the context of TN states, symmetries have been used to classify gapped phases of 1D systems [1-3]. For this reason, a strong effort has been devoted to


FIG. 4. (a) The direct sum of $W_{r}$ is a unitary tensor. It allows us to change basis from $\mathbb{C}(G)$ to the direct sum of all the irreps times their conjugate $\oplus_{r}\left(V_{r} \otimes V_{\bar{r}}\right)$. On the upper part of the figure, we explicitly draw the collection of all the $W_{r}$ 's. A simplified picture is shown on the lower part, where the direct sum is implicit in the absence of the label $r$ attached to the legs of the tensor. We will use this notation in the following. When we want to represent $W_{r}$, we will attach $r$ to the leg, while we represent $W_{G}$ by exactly the same drawing but without the $r$. From now on, for simplicity, we will also omit to label the legs with letters. (b) The unitarity of $W_{G}$ is encoded in the fact that the contraction over the two legs acting on $\oplus\left(V_{r} \otimes V_{\bar{r}}\right)$ gives a delta function in $\mathbb{C}(G)$.
incorporate the appropriate exact symmetries even when studying many-body systems with an approximate variational ansatz. In the context of TNs, a sufficient condition in order to have symmetric states is that the constituent tensors are symmetric [57,58,71-73]. As an example, a symmetric tensor with respect to the group $G$ with one in-leg and two out-legs obeys the following equation:

$$
\begin{equation*}
T_{b^{\prime}, c^{\prime}}^{a^{\prime}}=T_{b, c}^{a} \Gamma_{r}^{\dagger}(g)_{a}^{a^{\prime}} \Gamma_{r^{\prime \prime}}(g)_{b^{\prime}}^{b} \Gamma_{r}(g)_{c^{\prime}}^{c} \tag{8}
\end{equation*}
$$

with $\Gamma_{r}(g)$ the matrix of the appropriate unitary representation of the group $G$. This relation is sketched graphically in Fig. 5.

Equation (8) can be satisfied by nonvanishing tensors only when the tensor product $R=\bar{r} \otimes r^{\prime} \otimes r^{\prime \prime}$ contains the trivial representation, that is, the representation where all group elements are mapped to the identity. It is thus important to be able to construct explicitly the trivial representation contained in a given tensor product of different representations. For continuous groups, this operation can be done by diagonalizing the corresponding


FIG. 5. A symmetric tensor is left invariant by the simultaneous rotation of all incoming legs by $\Gamma^{\dagger}(g)$ and its outgoing legs by $\Gamma(g)$ in the appropriate representation. Here, we exemplify the case of a three-leg tensor with an incoming leg that transforms in the irrep $r$ and two out-legs that transform under irreps $r^{\prime}$ and $r^{\prime \prime}$.

Casimir operators. Their zero eigenvalues, if present, identify the trivial factors. The way this operation is done in practice is explained in detail in Refs. [58,74].

An alternative way, which works both for discrete and continuous groups, is to explicitly build the projector onto the trivial representation as a group sum (or integral in the case of continuous groups). The projector is given by

$$
\begin{equation*}
P_{0}=\frac{1}{|G|} \sum_{g} \Gamma_{R}(g) \tag{9}
\end{equation*}
$$

as can be found, i.e., in Ref. [69]. We will mostly use this methods.

A third possibility entails disentangling the symmetry constraints following the ideas of Refs. [23,75,76]. We postpone the discussion about this technique until Sec. VI, where we will provide an explicit example of this procedure.

## V. HAMILTONIAN LATTICE GAUGE THEORIES FROM THE TENSOR-NETWORK PERSPECTIVE

Gauge theories originated, at a classical level, in the description of the electromagnetic interactions between charged particles and light. The physical processes, described by the Maxwell equations, depend only on the electric and the magnetic fields while Maxwell equations can be written in terms of a vector potential, and in this form, they show some redundancy. The vector potential can be modified by adding to it the gradient of an arbitrary scalar potential, without affecting the corresponding electric and magnetic field, so giving the same physical results.

At a quantum level, the vector potential becomes a full quantum field and the redundancy appears as a local symmetry in the action that drives its dynamics. The generalizations of these ideas to vector potentials describing non-Abelian "electric and magnetic" fields and their success in describing the hadron spectrum gave rise to the modern gauge theories and to our understanding of particle physics.

Actual calculations away from the perturbative regime are most of the time carried out numerically in the
framework of LGTs by discretizing the spacetime on a lattice. In this formalism, the vector potential is associated with the links of the lattice, while the charged matter fields live on the sites.

A Hamiltonian version of the system has been obtained by identifying one of the lattice directions as "time," fixing the temporal gauge, and constructing the Hamiltonian operator whose matrix elements coincide, in the timecontinuum limit, with those of the transfer matrix in the time direction [77,78].

In the Hamiltonian formulation, LGTs become manybody quantum systems, whose constituents are divided in two groups, gauge bosons attached to the links of an oriented lattice $\Lambda$ and matter constituents attached to the sites $s$ of $\Lambda$. Here, we will work on an oriented 2D square lattice, but what follows can easily be generalized to more complex orientable lattices.

The original local symmetry of the classical action is then mapped to a residual local symmetry of the quantum Hamiltonian. Symmetric Hamiltonians typically have symmetric eigenstates, and thus, one can decide to characterize the space of locally symmetric states. The residual local symmetry is defined in terms of a set of constraints that the quantum states and operators should fulfill. As we show in this section, both the operators used to define the symmetry constraints and the local constraints have a natural expression in terms of TN diagrams. In particular, in this section, we describe the Kogut-Susskind version of the Hamiltonian LGT (KS).

## A. Hilbert space of constituents

The Hilbert space for gauge bosons is the group algebra $\mathbb{C}(G)$. In this case, one can associate a state $|g\rangle$ with any group element $g \in G$. States representing different group elements are orthogonal $\langle g \mid h\rangle=\delta_{g}^{h}$. As a consequence, the dimension of the local Hilbert space is equal to $|G|$. In particular, continuous groups require dealing with infinite-dimensional Hilbert spaces.

The lattice $\Lambda$ is oriented, and it is made by $L$ links, so that the total Hilbert space is $\mathbb{C}(G)^{L}$. Changing the orientation of one link sends $|g\rangle \rightarrow\left|g^{-1}\right\rangle$. Acting with an operator $\mathcal{O}$ on the state $|g\rangle$, is equivalent to acting with an operator $\mathcal{O}^{\dagger}$ on the state $\left|g^{-1}\right\rangle$, obtained by reversing the link orientation.

A prerequisite for defining the action of the symmetry operators is to be able to define the action of left and right rotations of a state by arbitrary group elements $h$ and $k \in G$. This transformation is achieved by defining the operators $L\left(h^{-1}\right)$ and $R(k)$, which act on $|g\rangle$ and produce

$$
\begin{equation*}
L\left(h^{-1}\right) R(k)|g\rangle \equiv\left|h^{-1} g k\right\rangle . \tag{10}
\end{equation*}
$$

This operation is done first by using the $W_{G}$ of Eq. (7) to change basis from $\mathbb{C}(G)$ to $\oplus_{r}\left(V_{r} \otimes \bar{V}_{r}\right)$. At this stage, the rotation is performed through the direct sums of the rotation
matrices in each representation, and then, one rotates back to $\mathbb{C}(G)$ with $W_{G}^{\dagger}$. This operator is expressed graphically in Fig. 6(b) and reads

$$
\begin{align*}
L\left(h^{-1}\right) & =W_{G} \oplus_{r}\left[\Gamma_{r}\left(h^{-1}\right) \otimes 1_{r}\right], W_{G}^{\dagger},  \tag{11}\\
R(k) & =W_{G} \oplus_{r}\left[1_{r} \otimes \Gamma_{r}(k)\right] W_{G}^{\dagger} . \tag{12}
\end{align*}
$$

## B. Gauge transformations

Having the operators that perform the left and right rotations, we are now in the position to define the operators


FIG. 6. Gauge boson constituents are defined on the links of an oriented lattice. Links are represented by dashed lines, and constituents are small solid circles along these lines (which should not be confused with tensors, whose legs are solids lines). (a) In the Kogut-Susskind LGT, each constituent is described by a state $|g\rangle$ of the group algebra $\mathbb{C}(G)$. (b) Left and right rotations $L\left(h^{-1}\right)$ and $R(k)$ are introduced in Eqs. (11) and (12). They transform the state $|g\rangle$ into $\left|h^{-1} g k\right\rangle$. Both operators require an initial change of basis from $\mathbb{C}(G)$ to $\oplus_{r}\left(V_{r} \otimes \bar{V}_{r}\right)$, obtained through the $W_{G}$ of Eq. (7) and represented in Fig. 4 (the first horizontal triangle). One then applies to each of the legs the corresponding rotation matrix given by $\oplus_{r}\left[\Gamma_{r}\left(h^{-1}\right) \otimes \Gamma_{r}(k)\right]$, with the individual $\Gamma_{r}$ defined in Fig. 3(c) and represented here by the two vertical triangles. At last, $W_{G}^{\dagger}$, represented by the inverted triangle, allows us to go back to $\mathbb{C}(G)$.
$A_{s}(h)$, the building blocks of local gauge transformations. In particular, the local transformation rotates all the states of the links touching a site $s$ by an element $h \in G$. Since the lattice is oriented, the transformation induced by $A_{s}(h)$ is different for entering $s$ (in-links) and links leaving $s$ (outlinks). All in-links are rotated through $R(h)$, while all the out-links are rotated on the left by $L\left(h^{-1}\right)$. Concretely, $A_{s}(h)$ at site $s$, acting on links $s_{1}$ to $s_{4}$ ordered counterclockwise starting from the left, is defined as

$$
\begin{equation*}
A_{s}(h)=R(h)_{s_{1}} \otimes R(h)_{s_{2}} \otimes L\left(h^{-1}\right)_{s_{3}} \otimes L\left(h^{-1}\right)_{s_{4}} \tag{13}
\end{equation*}
$$

with $L$ and $R$ defined, respectively, in Eqs. (11) and (12). Notice that $\left[A_{s}(g), A_{s}{ }^{\prime}(h)\right]=0$ if $s \neq s^{\prime}$, as a consequence of the commutation between $L$ and $R$ operators defined on the same links. They are represented graphically in Fig. 7. A generic gauge transformation is then a product of local gauge transformations, where for each site $s$, one chooses a different element $g_{s}$ to perform the desired rotation. Given a lattice of $L_{s}$ sites and a choice of $L_{s}$ elements $h_{i}$, $i=1, \ldots, L_{s} \in G$, we obtain the transformation $\mathcal{T}$ :

$$
\begin{equation*}
\mathcal{T}\left(\left\{h_{1}, \ldots, h_{L_{s}}\right\}\right)=\prod_{s=1}^{L_{s}} A_{s}\left(h_{s}\right) \tag{14}
\end{equation*}
$$

## C. The gauge-invariant Hilbert space

The set of states in the Hilbert space that are invariant under all $A_{s}(g)$, the building blocks of the gauge transformations, defined by Eq. (13), constitutes the physical Hilbert space (or gauge-invariant Hilbert space) $\mathcal{H}_{p}$


FIG. 7. Graphical representation of the operator $A_{s}(h)$, the building block of gauge transformations in the Kogut-Susskind LGT. On every link entering a site $s$, it either rotates the state of the link through $R(h)$ or $L\left(h^{-1}\right)$, depending on whether the link enters or leaves the site. The forms of $L$ and $R$ are given in Fig. 6(b).

$$
\begin{align*}
\mathcal{H}_{p} & \equiv\left\{|\phi\rangle \in \mathbb{C}(G)^{L},\right. \\
A_{s}(g)|\phi\rangle & =|\phi\rangle \forall s \in \Lambda, g \in G\}, \tag{15}
\end{align*}
$$

where $s$ are the sites of the lattice $\Lambda, L$ is the number of links, and $g$ is an arbitrary group element.

An example of a state in $\mathcal{H}_{p}$ is given by

$$
\begin{equation*}
|\phi\rangle=|+\rangle^{\otimes L}=\left(\frac{1}{\sqrt{|G|}} \sum_{g}|g\rangle\right)^{\otimes L} \tag{16}
\end{equation*}
$$

since $\left[L\left(h^{-1}\right)|+\rangle R(k)\right]=|+\rangle$.
$\mathcal{H}_{p}$ is a subspace of the original tensor-product Hilbert space $\mathbb{C}(G)^{L}$, as sketched in Fig. 1. By construction, all states in $\mathcal{H}_{p}$ are also invariant under any global transformation $\mathcal{T}$, defined in Eq. (14).

## D. Gauge-invariant operators

We are now interested in introducing the dynamics of a LGT, and thus we need to characterize gauge invariant operators.

Gauge-invariant operators are operators $\mathcal{O}$ that commute with all the $A_{s}(g)$; that is,

$$
\begin{equation*}
\left[\mathcal{O}, A_{s}(g)\right]=0, \quad \forall g \in G, \quad s \in \Lambda \tag{17}
\end{equation*}
$$

We would like to find the "minimal" gauge-invariant operators, that is, those that constitute the building blocks for gauge-invariant Hamiltonians, and with as small support as possible. In order to find them, we take inspiration from the QED Hamiltonian, whose gauge part can be written as $H_{\mathrm{QED}}=E^{2}+B^{2}$, with $E$ and $B$ being the electric and magnetic fields. On the lattice, the $E^{2}$ term is mapped to a link operator. (Remember that $E=-\nabla V$, with $V$ being a scalar field, so that in a mathematical sense, $E$ is naturally a one form and thus geometrically attached to links.) The term $B^{2}$ is mapped to a plaquette operator, where plaquettes, the smallest possible closed loops made by links, are the elementary pieces of the lattice surface. (Again, remember that $B=\nabla \wedge A$, with $A$ being a vector field, and thus, $B$ is naturally identified with a two form, discretized on plaquettes.)

For this reason, we look for a generalization to an arbitrary group $G$ of the $E$ operator of electromagnetism, as a single-link operator, and of the $B$ operator, as an operator acting on plaquettes of the lattice. We start by the singlelink operator. In an Abelian LGT, any matrix representing the rotation by a group element commutes with all the others and thus fulfills Eq. (17). For this reason, we can choose any $\Gamma_{\text {reg }}(g)+$ H.c.. [79] as a link operator [80]. When dealing with non-Abelian LGTs, the only link operator that commutes with an arbitrary rotation, as requested by Eq. (17), is an operator proportional to the identity in each of the irreps, as a consequence of the Schur lemma [81]. A gauge-invariant link operator acting on a link $s_{n}$ in the KS LGT thus can be expressed as

$$
\begin{equation*}
\mathcal{E}_{s_{n}}^{2}=\left(W_{G}^{\dagger} \oplus_{r}\left[c^{r} \rrbracket_{r} \otimes \mathbb{\square}_{\bar{r}}\right] W_{G}^{\dagger}\right)_{s_{n}} \tag{18}
\end{equation*}
$$

with $c^{r}$ arbitrary numbers and $W_{G}$ defined in Eq. (7).
Plaquette operators can be defined as matrix-product operators whose elementary tensors $U$ act on the Hilbert space of a single link and an auxiliary Hilbert space. Specifically, $U$ acts on the tensor product of the $\mathbb{C}(G) \otimes V_{r_{m}} . V_{r_{m}}$ is the defining space of the irrep $r_{m}$ of $G$ that defines how the matter transforms under gauge transformation [77,78,82]. $U$ is defined as

$$
\begin{equation*}
U_{s_{n}}=\sum_{g}|g\rangle\left\langle\left. g\right|_{s_{n}} \otimes \Gamma_{r_{m}}(g)_{j}^{i} .\right. \tag{19}
\end{equation*}
$$

It acts diagonally on the Hilbert space of the link $s_{n}$ and performs a controlled rotation in the auxiliary space; in other words, it "reads" the state of the link and rotates accordingly the state on the auxiliary space $r_{m}$. The operator $U_{s_{n}}$ is represented graphically in Fig. 8.

From the definition, we can derive its covariance properties under the left and right rotations. Indeed, after the


FIG. 8. (a) The $U_{s_{n}}$ operator of Eq. (19) in the KS LGT. It acts on the tensor product of the physical space $\mathbb{C}(G)$ of the link $s_{n}$ and of an auxiliary space that defines the irrep $r_{m}: \mathbb{C}(G) \otimes V_{r_{m}}$. It is built from the contraction of a copy tensor $\mathcal{C} \equiv|g\rangle\langle g| \otimes\langle g|$ (red circle) and the corresponding rotation matrices $\Gamma_{r_{m}}(g)_{j}^{i}$ [defined in Eq. (6)]. (b) The above definition implies that the $U_{s_{n}}$ transmits the rotation onto the physical index, as induced by conjugation by $L$ and $R$, to the auxiliary indexes. This property allows us to use $U_{s_{n}}$ as a building block of the gauge-invariant plaquette operator in Eq. (22), where, thanks to the trace, the rotations attached to the auxiliary indexes cancel.
gauge transformation that sends $|g\rangle$ to $\left|h^{-1} g k\right\rangle$, a $U_{s_{n}}$ becomes

$$
\begin{align*}
U_{s_{n}}^{\prime} & \equiv\left[L\left(h^{-1}\right) R(k)\right] U\left[L\left(h^{-1}\right) R(k)\right]^{\dagger} \\
& =\sum_{g}\left|h^{-1} g k\right\rangle\left\langle\left. g\right|_{s_{n}} \otimes \Gamma_{r_{m}}(g)_{j}^{i},\right. \tag{20}
\end{align*}
$$

as illustrated in the lower panel of Fig. 8. By redefining $g^{\prime}=h^{-1} g k$, we obtain

$$
\begin{equation*}
U_{s_{n}}^{\prime}=\sum_{g^{\prime}}\left|g^{\prime}\right\rangle\left\langle\left. g\right|_{s_{n}} \otimes \Gamma_{r_{m}}\left(h g^{\prime} k^{-1}\right)_{j}^{i}\right. \tag{21}
\end{equation*}
$$

Equation (20) together with Eq. (21) show that the $U_{s_{n}}$ allows us to transfer the rotation on the physical Hilbert space of the link to corresponding rotations on the auxiliary space.

This relation is often considered as the defining relation of a LGT [82], since $U$ can be thought of the equivalent of the position operator in the group manifold, while $L$ and $R$ are equivalent to translation operators on the group manifold.

By appropriately building a close path out of $U$, we can get rid of the rotations on the auxiliary legs and thus obtain a gauge-invariant operator. In particular, we can now construct the simplest closed path that leads to the plaquette operator we are after:


FIG. 9. The plaquette operator $B_{p}$ of Eq. (22) constructed as a matrix-product operator from four $U_{p_{n}}$ 's, acting on the links around a plaquette in the KS LGT. As a consequence of the covariance of the $U_{p_{n}}$ under conjugation by $L$ and $R$ [see Fig. 8(b)], it transfers the rotation onto the physical legs to rotations onto the auxiliary legs. The operator commutes with the gauge transformations, and, thus, it can be used as a building block of a gauge-invariant Hamiltonian [see Eq. (23)].

$$
\begin{equation*}
B_{p}=\operatorname{tr}_{r_{m}}\left(U_{p_{1}} U_{p_{2}} U_{p_{3}}^{\dagger} U_{p_{4}}^{\dagger}\right) \tag{22}
\end{equation*}
$$

where we denote by $p$ a plaquette of $\Lambda$ and by $p_{1}, \ldots, p_{4}$ the links around it, ordered counterclockwise. The dagger is related to the fact that the orientation of the plaquette is in some cases opposite to the natural orientation of the lattice, and the trace is intended only over the auxiliary indexes. The equivalence between Eqs. (20) and (21) guarantees that the plaquette operator, represented in Fig. 9, commutes with the gauge transformations that have nontrivial overlap with it.

Having both gauge-invariant link operators and plaquette operators, we can now write the generalization of the $E^{2}+B^{2}$ Hamiltonian of QED for a LGT with gauge group $G$. The Hamiltonian reads

$$
\begin{equation*}
H_{\mathrm{LGT}}=\sum_{l} \mathcal{E}_{l}^{2}+\frac{1}{\alpha^{2}} \sum_{p}\left(B_{p}+B_{p}^{\dagger}\right) \tag{23}
\end{equation*}
$$

where $\alpha$ is the coupling constant, and the first sum runs over links $l \in \Lambda$ while the second runs over plaquettes $p \in \Lambda$.

## VI. PROJECTOR ONTO $\mathcal{H}_{P}$ AS A TENSOR NETWORK

The gauge-invariant Hilbert space $\mathcal{H}_{p}$ defined in Eq. (15) is made of those states that fulfill all constraints $A_{s}(g)|\phi\rangle=$ $|\phi\rangle$. This constraint can be obtained through a projector $\mathcal{P}$

$$
\begin{equation*}
\mathcal{P}: \mathbb{C}(G)^{L} \rightarrow \mathcal{H}_{p}, \quad \mathcal{P}^{2}=\mathcal{P} \tag{24}
\end{equation*}
$$

Here, we show that $\mathcal{P}$ has an exact TN representation. The idea is very general and requires the contraction of several copies of two types of elementary tensors: $\mathcal{C}$ tensors and $\mathcal{G}$ tensors. The two have different roles, as shown in Fig. 10. The $\mathcal{C}$ tensors copy the states from the physical legs to the auxiliary legs so that gauge constraints are decoupled and are imposed individually by $\mathcal{G}$ tensors. There is a $\mathcal{C}_{i, \beta}^{\alpha, j}$ for every link. They have all elements at 0 except those corresponding to $\alpha=i=\beta=j$ [83-85]. For every site, there is a $\mathcal{G}_{\alpha_{3} \alpha_{4}}^{\alpha_{1} \alpha_{2}}$ that only possesses auxiliary indexes. (All of its indexes are contracted [86].) It selects, among all states of the tensor product of the four auxiliary constituents around a site, only those that fulfill the gauge-symmetry requirements in Eq. (15).

Concretely, consider a horizontal link ( $s_{1}$ using the notation of Fig. 7) connecting sites $s-1$ and $s$. Its state is copied through the corresponding $\mathcal{C}$ to two auxiliary states, one located close to $s-1, \alpha$ and the other close to $s$, $\beta$ [see Fig. 10(b)]. In this way, we can treat the gauge constraint defined at the site $s-1$ as acting on $\alpha$ rather than on the link $s_{1}$; analogously, the gauge constraint at the site $s$ can be imposed on $\beta$ rather than on $s_{1}$. In this way, we have been able to completely decouple the gauge-symmetry constraints acting on sites $s-1$ and $s$. Before the copy tensors, they were acting on the same link $s_{1}$, while after it,


FIG. 10. The projector on the gauge-invariant states. (a) The corresponding TN is defined through the contraction of several copies of two elementary tensors $\mathcal{C}$, which copies the physical Hilbert space onto the auxiliary Hilbert space, and $\mathcal{G}$, which selects only configurations fulfilling the gauge invariance condition. (b) An example of gauge-constraint decoupling at sites $s-1$ and $s$ obtained through the insertion of the copy tensor $\mathcal{C}$. (c) The projector on the gauge-invariant states for a $4 \times 4$ square lattice with periodic boundary conditions.
they act on two different auxiliary sites $\alpha$ and $\beta$. This transformation allows us to address each gauge constraint individually.

By contracting as many copies of $\mathcal{C}$ as there are links on the lattice with as many copies of $\mathcal{G}$ tensors as there are sites on the lattice, following the pattern in Fig. 10, we obtain the desired projector $\mathcal{P}$.

Let us write explicitly the tensors $\mathcal{C}$ and $\mathcal{G}$. The only nonzero elements of the copy tensor are $C_{g, g}^{g, g}=1, \forall g$. Regarding $\mathcal{G}$, there are several ways to obtain them (all providing equivalent tensors). Here, we follow the one inspired by the known TN expressions for the ground states of quantum doubles [23,75,76,87]. The idea is to disentangle the gauge-symmetry constraint. It originally acts on four auxiliary sites, and we want to design an appropriate unitary transformation that transforms it to a single auxiliary site operator. In practice, we obtain this unitary operator by constructing $\mathcal{G}$ itself as the result of an elementary TN contraction, whose building blocks are unitary tensors that act on two constituents, and perform
controlled rotations. In particular, we define the tensors $C_{R}(\alpha, \beta)$ and $C_{L}(\alpha, \beta)$ as

$$
\begin{align*}
C_{R}(\alpha, \beta) & =\sum_{g}|g\rangle\left\langle\left. g\right|_{\alpha} \otimes R(g)_{\beta}\right. \\
C_{L}(\alpha, \beta) & =\sum_{g}|g\rangle\left\langle\left. g\right|_{\alpha} \otimes L(g)_{\beta}\right. \tag{25}
\end{align*}
$$

where $\alpha$ and $\beta$ specify the location of the constituents in the virtual lattice. The generic properties of these tensors are independent on the position of the constituents. They act on the tensor product $\mathbb{C}(G) \otimes \mathbb{C}(G)$ and transform the state $|g, h\rangle$ as $C_{R}:|g, h\rangle \rightarrow|g, h g\rangle$ and $C_{L}:|g, h\rangle \rightarrow|g, g h\rangle$. The two operators have the following properties:

$$
\begin{align*}
& C_{L}\left[R(h) \otimes L\left(h^{-1}\right)\right] C_{L}^{\dagger}=[R(h) \otimes I d] \\
& C_{R}^{\dagger}[R(h) \otimes R(h)] C_{R}=[R(h) \otimes I d] \tag{26}
\end{align*}
$$

These properties can be used in order to simplify the gaugesymmetry building blocks. Concretely, we now specify one possible arrangement of $C$ operators that allows us to simplify the gauge condition. If we define the following unitary operator $I_{s}=C_{R}^{\dagger}\left(\alpha_{1}, \alpha_{4}\right) C_{L}\left(\alpha_{4}, \alpha_{3}\right) C_{L}\left(\alpha_{1}, \alpha_{2}\right)$ by using the properties (26), we obtain

$$
\begin{equation*}
I_{s}^{\dagger} A_{s}(h) I_{s}=R(h)_{\alpha_{1}} \equiv A_{s}^{\prime}(h), \tag{27}
\end{equation*}
$$

where $A_{s}(h)$ are the building blocks of local gauge transformation defined in Eq. (13). The physical interpretation of this transformation is that we have concentrated a fourbody constraint onto a single-body constraint acting on $\alpha_{1}$ that is now easy to fulfill. On $\alpha_{1}$, gauge invariance (15) indeed requires us to pick the only state that is invariant under the rotation for an arbitrary group element. The corresponding state is the state $|+\rangle=1 / \sqrt{|G|} \sum_{g}|g\rangle$.

Every state of the other three auxiliary sites forming $\mathbb{C}(G)^{3}$ is by construction gauge invariant, since after the unitary transformation, the gauge constraint does not act on those sites. The projector operator is obtained by the equal superposition of all the gauge-invariant states. This equal superposition can be obtained by projecting each of the three $\mathbb{C}(G)$ onto $|+\rangle=\sum_{g}|g\rangle$. This construction is sketched in Fig. 11.

Before proceeding, let us summarize what we have obtained so far. By reexpressing the KS LGT in the language of TNs, we have been able to provide a TN prescription for the projector onto the gauge-invariant Hilbert space $\mathcal{H}_{p}$. This TN has bond dimension $D$ equal to the number of elements in the group $D=|G|$. This property means that for discrete groups, this TN can be used in actual computations since it has a finite bond dimension. Furthermore, the construction can be improved, as discussed in Sec. VII A, where we explain an alternative choice of $\mathcal{C}$ and $\mathcal{G}$ that allows us to express $\mathcal{P}$ with a TN


FIG. 11. The definition of the tensors $\mathcal{C}$ and $\mathcal{G}$ for the KogutSusskind LGT. The tensor $\mathcal{G}$ is obtained by composition of several copies of unitaries $C_{L}$ and $C_{R}$ described in the main text. They are followed by the projection onto the states $|+\rangle$ and $|+\rangle$ defined in the main text.
with bond dimension $D=\sum_{r} n_{r}$, where $n_{r}$ is the dimension of the irrep $r$. This number is of the order of the square root of $|G|$ since $|G|=\sum_{r} n_{r}^{2}$.

In any case, when we study LGTs with continuous groups, the bond dimension of the TN is infinite and thus is not useful for numerical simulations. It provides, nevertheless, an interesting analytical results since it encodes, in a TN, the exact projector onto the gauge-invariant Hilbert space.

We now generalize the KS LGT to models that are described by finite-dimensional local Hilbert spaces while invariant under continuous groups. We will generalize our TN construction and obtain a TN expression for $\mathcal{P}$ with finite bond dimension that can be used in actual numerical calculations.

## VII. TRUNCATED LGT

The constructive approach that we have followed so far allows us to generalize the original KS LGT. Here, we are departing from the Hamiltonian LGT whose Lagrangian formulation provides the standard Yang-Mills action in the continuum limit. However, we would still like to construct models that are related to the original KS LGT through a truncation of the local Hilbert space that commutes with local gauge transformations. In this way, what we have discussed so far applies as a whole to the truncated models [46].

Within our formalism, the truncation of the Hilbert space we are looking for is very natural. We use the $W_{G}$ of Eq. (7) to pass from $\mathbb{C}(G)$ to $\oplus_{r} W_{r}$ and to truncate the direct sum to an arbitrary finite set of irreps. The minimal choice
requires keeping at least two irreps $r \oplus r^{\prime}$ (the reason why we need at least two irreps will become clearer in the following), so that the projector can be written as

$$
\begin{equation*}
W_{T} \equiv W_{r} \oplus W_{r^{\prime}} \tag{28}
\end{equation*}
$$

where the $W_{r}$ are defined in Eq. (4).
After the projection, if the group $G$ is compact, the Hilbert space on a link becomes finite dimensional. It still preserves the property of the group algebra that we have used extensively; namely, in each block, it is the tensor product of $V_{r} \otimes V_{\bar{r}}$. We thus can write left and right rotations $L$ and $R$ as

$$
\begin{align*}
L\left(h^{-1}\right) & =\left[\Gamma_{r}\left(h^{-1}\right) \otimes 1_{r}\right] \oplus\left[\Gamma_{r^{\prime}}\left(h^{-1}\right) \otimes 1_{r^{\prime}}\right],  \tag{29}\\
R(k) & =\left[1_{r} \otimes \Gamma_{r}(k)_{k}^{l}\right] \oplus\left[1_{r^{\prime}} \otimes \Gamma_{r^{\prime}}(k)_{k}^{l}\right], \tag{30}
\end{align*}
$$

as represented graphically in Fig. 12(b).
As already mentioned, there is a lot of freedom on the choice of $r$ and $r^{\prime}$. A legitimate choice is the one that minimizes the dimension of the local Hilbert space, since this dimension takes part in the computational cost of TN algorithms. In this case, one should choose $r^{\prime}$ as the trivial representation and $r$ as the smallest faithful irrep of the group. For example, for $S U(N)$ groups, the dimension of the local


FIG. 12. (a) The Hilbert space of a gauge boson in the truncated KS LGT is obtained by truncating $\mathbb{C}(G)$ with the isometry $W_{T}$ defined in Eq. (28). It is isomorphic to $\left(V_{r} \otimes \bar{V}_{r}\right) \oplus\left(V_{r^{\prime}} \otimes \bar{V}_{r^{\prime}}\right)$. Each term of the direct sum is a tensor product of two constituents, so that we represent it by two solid circles on each link. (b) The operators implementing the left and right rotations defined in Eqs. (29) and (30) are obtained from those of Fig. 6 after conjugation with $W_{T}$.

Hilbert space with such a choice is $N^{2}+1$, where $N$ is indeed the dimension of the fundamental representation.

The building blocks of gauge transformations $A_{s}(h)$ are still defined by Eq. (13), with $L$ and $R$ given by Eqs. (29) and (30), as represented graphically in Fig. 13.

They allow us to define the gauge-invariant Hilbert space $\mathcal{H}_{p}$ by using Eq. (15).

Gauge-invariant operators have to commute with all the $A_{s}(g)$ defined above. Both single-link operators (electriclike) and plaquette operators (magneticlike) are just the truncation with the $W_{T}$ of the corresponding operators in the KS LGT. The single-link operator is the truncation of Eq. (18)

$$
\begin{align*}
\mathcal{E}_{T s_{n}}^{2} & =\left\{W_{T}^{\dagger} \oplus_{r}\left[c^{r}\left(\mathbb{\square}_{r} \otimes \mathbb{\square}_{\bar{r}}\right)\right] W_{T}\right\}_{s_{n}} \\
& =\left[c^{r}\left(\mathbb{\square}_{r} \otimes \mathbb{\square}_{\bar{r}}\right) \oplus c^{r \prime}\left(\square_{r^{\prime}} \otimes \mathbb{\square}_{\bar{r}^{\prime}}\right)\right]_{s_{n}} \tag{31}
\end{align*}
$$

and depends only on the two free parameters $c_{r}$ and $c_{r^{\prime}}$. Here is where it becomes clear that we need to keep in $W_{T}$ at least two irreps. The truncation of Eq. (18) to a single irrep is indeed proportional to the identity.

The truncated plaquette operator is built from $U_{s_{n}}$ in Eq. (19)

$$
\begin{equation*}
U_{T s_{n}}=W_{T}^{\dagger} U_{s_{n}} W_{T} \tag{32}
\end{equation*}
$$

and is represented graphically in Fig. 14.
Depending on the choice of $r$ and $r^{\prime}$ inside $W_{T}, U_{T s_{n}}$ could vanish. In order to see this effect, we have to remind readers that in the KS LGT, $V_{r_{m}}$ is chosen as the smallest faithful irrep. [For $S U(N)$, one chooses the fundamental irrep of dimension $N$.] From the above definition of $U_{T}$ in Eq. (32) and the definition of $U_{s_{n}}$ in Eq. (19), we see that


FIG. 13. The operator $A_{s}(h)$ that generates the gauge transformations in the truncated KS LGT. It is the result of the truncation of the operator $A_{s}(h)$ of Fig. 7 with $W_{T}$ defined in Eq. (28). On each link, the operator acts on the constituent closer to $s$ through a rotation $\Gamma_{r}(h)$ or $\Gamma_{r^{\prime}}(h)$, which is controlled by the irrep of the neighboring constituent on the same link. It thus performs controlled operations in between the two constituents of a link, as suggested by a dot on the controller leg and by an arrow that joins it with the controlled leg.


FIG. 14. (a) The $U_{T s_{n}}$ operator in the truncated KS LGT as defined in Eq. (32). (b) It has by construction the desired covariance properties, meaning that rotations on the physical legs (upper part of the panel) are transmitted to rotations on the auxiliary legs (lower part of the panel), as described by Eqs. (20) and (21).
$U_{T}$ entails terms of the kind $\sum_{g} \Gamma_{r}(g) \Gamma_{r^{\prime}}\left(g^{-1}\right) \Gamma_{r_{m}}(g)$. These terms are proportional to the projector onto the trivial representation defined in Eq. (9), where now the $R$ in Eq. (9) is given by $R=V_{r} \otimes V_{r^{\prime}} \otimes V_{r_{m}}$. This choice implies that the above terms will be nonvanishing only if the decomposition of $R$ in a direct sum of irreps contains the trivial representation.

This consideration implies that in order to have a nontrivial model, a certain care should be taken when choosing $r$ and $r^{\prime}$ and that their choice depends on the choice of $r_{m}$. In particular, the minimal prescription provides a valid truncation scheme with nontrivial dynamics, since in that case, $r_{m}=r$ and $r^{\prime}$ is by itself the trivial representation.

Once the $U_{T}$ is nontrivial, it automatically fulfills the desired commutation relations with the $L$ and $R$ operators defined in Eqs. (20) and (21), as illustrated once more in Fig. 14(b). In this way, it can be used to construct the desired plaquette operators, using the same formula of Eq. (22) where the $U$ are substituted with the $U_{T}$ just defined.

The Hamiltonian of the truncated LGT thus has the same form as the one of the KS LGT defined in Eq. (23), with the appropriate substitution of $\mathcal{E}^{2}$ by $\mathcal{E}_{T}^{2}$ and $U$ inside the plaquettes by the corresponding $U_{T}$ operators.

## A. $\mathcal{P}$ in the truncated LGT

In the truncated LGT, we can also consider the projector onto the Hilbert space of gauge-invariant states

$$
\begin{equation*}
\mathcal{P}_{T}:\left[\left(V_{r} \otimes V_{\bar{r}}\right) \oplus\left(V_{r^{\prime}} \otimes V_{r^{\prime}}\right)\right]^{\otimes L} \rightarrow \mathcal{H}_{p} \tag{33}
\end{equation*}
$$

with $\mathcal{H}_{p}$ defined as in Eq. (15) with the $A_{s}(g)$ of Eq. (13) containing the operators $L$ and $R$ of Eqs. (29) and (30). Even in this case, this projector can be written as an exact TN.

The construction is very similar to the one used for the KS LGT in Sec. VI. There, however, the specific form of $\mathcal{G}$ has been derived by exploiting that, in $\mathbb{C}(G)$, we are able to disentangle the symmetry requirements. In general, we are unable to perform this disentangling procedure explicitly, so that here, we introduce a generic approach (which can also be used for the full KS LGT and indeed provides a TN with a lower bond dimension). The TN structure of $\mathcal{P}_{T}$ is the same as the one of $\mathcal{P}$ in the KS. It consists of the contraction of various copies of $\mathcal{C}$ tensors (one per link) and $\mathcal{G}$ tensors (one per site) following the pattern in Fig. 10.

In particular, the four-leg tensor $\mathcal{C}$ copies the physical states to the auxiliary states, while the gauge-fixing tensor $\mathcal{G}$ selects among the auxiliary states only those that fulfill the gauge invariance condition $A_{s}(g)|\varphi\rangle=|\varphi\rangle$.

With a basis $\left|i_{r}, j_{r}\right\rangle,\left|k_{r^{\prime}} l_{r^{\prime}}\right\rangle$ chosen, with $\left\{i_{r}, j_{r}\right\}=$ $1, \ldots, d_{r} \quad$ and $\quad\left\{k_{r^{\prime}}, l_{r^{\prime}}\right\}=d_{r}+1, \ldots, d_{r}+d_{r^{\prime}} \quad$ of $\quad$ the Hilbert space of a link $\left[\left(V_{r} \otimes V_{\bar{r}}\right) \oplus\left(V_{r^{\prime}} \otimes V_{\bar{r}^{\prime}}\right)\right]$, the $\mathcal{C}$ tensor copies the left constituent to the left and the right constituent to the right:

$$
\begin{align*}
\mathcal{C}= & \left|i_{r} j_{r}\right\rangle_{s_{n}}\left\langle\left. i_{r} j_{r}\right|_{s_{n}} \otimes \mid i_{r}\right\rangle_{\alpha}\left\langle\left. j_{r}\right|_{\beta}\right. \\
& +\left|k_{r^{\prime}} l_{r^{\prime}}\right\rangle_{s_{n}}\left\langle\left. k_{r^{\prime}} l_{r^{\prime}}\right|_{s_{n}} \otimes \mid k_{r^{\prime}}\right\rangle_{\alpha}\left|l_{r^{\prime}}\right\rangle_{\beta} \tag{34}
\end{align*}
$$

$\mathcal{G}$ then acts nontrivially only on $V_{G} \simeq\left(V_{r} \oplus V_{r^{\prime}}\right)^{\otimes 2} \otimes$ $\left(V_{\bar{r}} \oplus V_{\bar{r}^{\prime}}\right)^{\otimes 2}$. On this space, the requirement of gauge invariance is equivalent to asking that $\mathcal{G}$ be a symmetric tensor with respect to rotations of elements in the group $G$; that is, it fulfills the requirements in Eq. (8), as explicitly shown in Fig. 15(a).

Furthermore, since we are interested in building the projector $\mathcal{P}_{T}$, we need to build $\mathcal{G}$ out of the equal superposition of all symmetric tensors acting on the above space.

As explained in Sec. IV A, there are two possible ways of building a symmetric tensor. The first one consists of applying the projector onto the trivial irrep defined in Eq. (9) to the space $V_{G}$. The projector acts separately in each block of irreps and involves terms of the type $(1 /|G|) \sum_{g} \Gamma_{r_{1}}(g) \otimes \Gamma_{r_{2}}(g) \otimes \Gamma_{r_{3}}\left(g^{-1}\right) \Gamma_{r_{4}}\left(g^{-1}\right)$ with $\left\{r_{i}\right\}=$ $r, r^{\prime}$. It is important to notice that not all of the blocks contain a trivial irrep, and the projector will then give 0 when acting on those blocks without it. After the action of the projector, we take the equal superposition of all symmetric tensors with equal weight. It is important at this point to find the rank of the previous projector and take an equal superposition of all possible normalized states on its support. This equal superposition is obtained by using the tensor $|+\rangle=|1 \underbrace{1, \ldots, 1}\rangle$, where $d_{0}$ is the number of $d_{0}$
copies of the trivial irrep (the rank of the projector), as is represented in Fig. 15(b).


FIG. 15. (a) The tensor $\mathcal{G}$ that is used to build the projector onto the physical Hilbert space is an invariant tensor under the action of $G$, as defined in Eq. (8). The legs of $\mathcal{G}$ carry an index of the irrep, since the Hilbert space on which they act has a direct sum structure labeled by the irrep $r$. (b) $\mathcal{G}$ can be obtained by using the explicit form of the projector onto the trivial irrep (9). The red circle is the copy tensor in the group algebra. After acting with the projector, we have to take an equal superposition of all vectors in the trivial irrep space, so as to obtain $\mathcal{P}_{T}$. This equal superposition requires finding the image of the projector, taking an equal superposition of vectors spanning such an image state that we represent by the cyan $|+\rangle$, and contracting on it all open legs. (c) Alternatively, one can use the construction introduced in Ref. [58] for building symmetric tensors. The tensor is then divided into a piece $Q$ that takes care of correctly matching the various irreps (made by the Clebsch-Gordan coefficients) and a degeneracy tensor $P$, made of free parameters, that assigns a four-leg tensor to each of the irreps which enter in composition of representations. In the example we draw, $P$ depends on the irreps $r_{5}$ that are contained into the tensor product of $r_{1}$ and $r_{2}$. Once more, the projector $\mathcal{P}_{T}$ is obtained by taking a uniform superposition of the possible values of $r_{5}$, that is, setting $P=|+\rangle$.

Alternatively, one can use the technology developed in Ref. [58] for constructing invariant tensors. The idea is to decompose the tensor $\mathcal{G}$ into a $Q$ part (the part that is fully dictated by the symmetry constraints) and a $P$ part, the part that contains the variational parameters. The $Q$ part, in our case, is built out from the Clebsch-Gordan coefficients that decompose the tensor product of the two irreps $r_{1}$ and $r_{2}$ into the direct sum of irreps $r_{5}$. Then, every $r_{5}$ is again decomposed into the tensor product of two irreps $r_{3}$ and $r_{4}$.

In general, this observation implies that the tensor $P$ has an extra index corresponding to $r_{5}$. The projector is obtained by taking the equal superposition of all the $Q$ tensors with different $r_{5}$; that is, $P=|+\rangle$. Figure 15(c) contain the graphical representation of the construction.

For concreteness, we discuss how to construct $\mathcal{G}$ for the KS LGT with $G=S U(2)$ truncated to the sum of the trivial irrep, plus the product of two $J=1 / 2$ irreps. $\mathcal{C}$ copies the trivial irrep on both sides, while it copies one of the two $J=1 / 2$ to the left and the other to the right. $\mathcal{G}$ is then used to project onto the invariant states. Each auxiliary site then lives on $V_{J=0} \oplus V_{J=1 / 2}$ that has dimension $d=3$. The possible blocks of the fourfold tensor product $\left(V_{J=0} \oplus V_{J=1 / 2}\right)^{\otimes 4}$ that contain the trivial irrep are those with an even number of $J=1 / 2$ factors. It follows that there is one block $V_{J=0}^{\otimes 4}$, six blocks with $V_{J=0}^{\otimes 2} \otimes V_{J=1 / 2}^{\otimes 2}$, and one block $V_{J=1 / 2}^{\otimes 4}$. This last block furthermore leads to two values of $r_{5}\left(r_{5}=0\right.$ and $\left.r_{5}=1\right)$ that need to be equally taken into account. The equal superposition of all these blocks gives the $\mathcal{G}$ necessary to build the projector $\mathcal{P}_{T}$. The explicit form of $\mathcal{G}$ is given in the Appendix.

We conclude this section by discussing the bond dimension of the $\mathrm{TN} \mathcal{P}_{T}$. It clearly depends on the choice of irreps that one decides to consider in the $W_{T}$ of Eq. (28). In the minimal case for an $S U(N)$ LGT, it is $D=N+1$.

## VIII. GAUGE MAGNETS AND QUANTUM LINK MODELS

The truncated LGT we have just presented is not the only LGT with continuous groups, defined on a finitedimensional Hilbert space. In particular, a set of models has been proposed in the literature that has the same features and is known as gauge magnets (GMs) [47,48]. Here, we briefly recall their construction using the tools that we have described in the previous sections.

In gauge magnets, the local Hilbert space is the direct sum $V_{r} \oplus V_{r}$. By rewriting the direct sum into a tensor product of $\mathbb{C}^{2} \otimes V_{r}$, we obtain a natural basis spanned by $\{|0\rangle,|1\rangle\} \otimes\{|v\rangle\}$ in terms of a position qubit times a spin vector in $V_{r}$ (more details are given in Sec. VIII A). In this case, the constituents live in a space of dimension $d_{\mathrm{GM}}=2 n_{r}$, where $n_{r}$ is the dimension of the irrep $r$. In general, $d_{\mathrm{GM}}$ is, smaller than $d_{t \mathrm{KS}}=n_{r}^{2}+1$, the Hilbert space of the KS LGT truncated to the same irrep $r$. For this reason, GMs, in their simplest version, can be considered the "minimal LGTs," that is, the LGT with the smallest local Hilbert space.

However, the two terms in the direct sum still allow us to define left and right rotations of the state of a link for an arbitrary element of the group that, as we have seen, is the prerequisite for being able to define gauge transformations. The left and right rotations for the elements $h$ and $k$ in $G$ are defined through

$$
\begin{equation*}
L(h) R(k) \equiv\left[\Gamma_{r}(h) \oplus 1\right]\left[1 \oplus \Gamma_{r}(k)\right]=\Gamma_{r}(h) \oplus \Gamma_{r}(k) . \tag{35}
\end{equation*}
$$

(a)

$$
\begin{gathered}
\\
\text { (a) } \\
V_{r} \oplus V_{r}
\end{gathered}
$$


(b)


FIG. 16. (a) Gauge boson constituents in gauge magnets are states of the direct sum of two irreps $V_{r} \oplus V_{r}$, which can be identified as the left and the right constituents at the ends of each link. Graphically, we represent this local Hilbert space as a single circle embracing the two copies of $V_{r}$, each of them being a smaller circle identified by a different color and position inside the bigger circle. (b) The legs of the tensors acting on this Hilbert space are represented by bands rather than lines so that we can specify operators that only act on one sector as acting on half of the band. Each subsector is colored differently. (c),(d) The left and right rotations in the gauge magnets only act on half of the direct sum, as depicted here and discussed in Eq. (35).

The above equation is represented graphically in Figs. 16(c) and $16(\mathrm{~d})$ and can be rewritten in the tensor-product basis as $L(h)=|0\rangle\langle 0| \otimes \Gamma_{r}(h)+|1\rangle\langle 1| \otimes 1$ and $R(k)=|0\rangle\langle 0| \otimes$ $1+|1\rangle\langle 1| \otimes \Gamma_{r}(k)$. In this notation, it is clear that $|0\rangle$ represents the left end of the link while $|1\rangle$ represents the right end of the link. The above equation allows us to identify the gauge boson with a boson that can occupy one of the two extremes of the link. (For a physical implementation of these ideas with cold atoms, please see Refs. [88,89].)

The definition of local gauge transformations at a site slightly differs from the one in the KS LGT of Eq. (13)

$$
\begin{equation*}
A_{s}(h)=R(h)_{s_{1}} \otimes R(h)_{s_{2}} \otimes L(h)_{s_{3}} \otimes L(h)_{s_{4}} \tag{36}
\end{equation*}
$$

since it rotates all the links by the element $h$, independently if they are entering or leaving the site $s$. In terms of left and right constituents, $A_{s}(h)$ only acts nontrivially on those constituents of the links that are located close to $s$, as illustrated in Fig. 17.

Once the building blocks of gauge transformations are defined, the discussion parallels the one for the other LGTs, and, in particular, we can use the $A_{s}$ 's in order to define the physical Hilbert space of gauge-invariant states $\mathcal{H}_{p}$. Furthermore, we use the new $A_{s}(h)$ in order to define gauge-invariant operators, as from Eq. (17). As a result, all gauge-invariant link operators are defined as $\mathcal{E}_{G s_{n}}^{2}=$ $\left(c_{0} \rrbracket_{l} \oplus c_{1} \rrbracket_{r}\right)_{s_{n}}$ with $c_{0}$ and $c_{1}$ arbitrary numbers. The equivalent of the $U_{s_{n}}$ operator is used to build gaugeinvariant plaquette operators. In the literature, it is possible to find the specific form for the $U_{s_{n}}$ operators for $\operatorname{SU}(N)$, $S P(N)$, and $G_{2}$ groups [50,90,91]. Here, we provide a recipe to generalize it to an arbitrary group $G$, either discrete or continuous.

We use a similar construction to the one used for the KS LGT and define an operator that acts on $\left(V_{r} \oplus V_{r}\right) \otimes V_{r_{m}}$

$$
\begin{align*}
U_{s_{n}}^{G} & =\sum_{g}\left\{\left[|0\rangle\langle 1| \otimes \Gamma_{r}(g)+|1\rangle\langle 0| \otimes \Gamma_{r}\left(g^{-1}\right)\right]\right. \\
& \left.\otimes \Gamma_{r_{m}}(g)\right\} . \tag{37}
\end{align*}
$$

If we now study how $U_{s_{n}}^{G}$ changes under a left rotation, we immediately see that

$$
\begin{align*}
& L(h) U_{s_{n}}^{G} L\left(h^{-1}\right) \\
& \left.\quad=\sum_{g}\left\{|0\rangle\langle 1| \otimes \Gamma_{r}(g)+|1\rangle\langle 0| \otimes \Gamma_{r}\left(g^{-1}\right)\right] \otimes \Gamma_{r_{m}}\left(h^{-1} g\right)\right\} \tag{38}
\end{align*}
$$



FIG. 17. The operator that generates gauge transformations in the gauge magnet LGT defined in Eq. (35). It rotates both incoming and outgoing links by the same group elements. On every link, it only acts on the constituent that is closer to the site.
and that under a right rotation,

$$
\begin{align*}
& R(k) U_{s_{n}}^{G} R\left(k^{-1}\right) \\
& \quad=\sum_{g}\left\{\left[|0\rangle\langle 1| \otimes \Gamma_{r}(g)+|1\rangle\langle 0| \otimes \Gamma_{r}\left(g^{-1}\right)\right] \otimes \Gamma_{r_{m}}(g k)\right\} \tag{39}
\end{align*}
$$

The definition and the transformation properties of $U_{s_{n}}^{G}$ are illustrated in Fig. 18. Once more, $U_{s_{n}}^{G}$ allows us to transmit the rotations on the link to rotations onto the auxiliary space. Thus, we can use it as the building block for a plaquette operator analogous to the one of the KS LGT in Eq. (22).

The Hamiltonian of a gauge magnet has thus the same structure as the Hamiltonian for the KS LGT defined in Eq. (23), where this time, $\mathcal{E}^{2}$ is substituted by $\mathcal{E}_{G}^{2}$ and plaquette operators are built from the $U_{s_{n}}^{G}$ operators just described.

As a final comment, it is important to notice that, in general, for non-Abelian groups, the gauge magnets and the


FIG. 18. (a) The $U_{s_{n}}^{G}$ operator in the gauge magnet LGT defined in Eq. (37). It can be defined, once more, as the contraction of three $\Gamma_{r}(g)$ 's and a copy tensor in $\mathbb{C}(G)$ (red circle). (b),(c) From the definition, one can check that it allows us to pass the $L$ and $R$ rotations on the physical legs, induced by the gauge transformation of Fig. 17, to analogous rotations onto the auxiliary legs. The covariance of $U_{S_{n}}^{G}$ allows us to build a gauge-invariant plaquette operator, defined as in Eq. (22), where the rotation on the auxiliary legs cancels by taking the trace onto the auxiliary legs.
truncated LGTs are different models. Indeed, there is no consistent truncation at the level of a single link that allows us to map the KS LGT onto the GM. Indeed, such a truncation would require us to decompose $\mathbb{C}(g)$ into a direct sum of irreps and keep only two of them. Unfortunately, such decomposition cannot be performed in an invariant way. The group algebra $\mathbb{C}(G)$ can be decomposed in an invariant way only as a direct sum of tensor products of irreps [as we have done through the $W_{G}$ of Eq. (7) in the previous sections]. Any further decomposition of each of the terms $V_{r} \otimes V_{\bar{r}}$ into a direct sum of irreps requires a choice of basis in one of the two factors and thus cannot be invariant under rotations by elements of $G$ [69].

There is an exception to this rule in the case of Abelian LGTs where $n_{r}=1$, and consequently, $d_{\mathrm{GM}}=d_{t \mathrm{KS}}$. In Ref. [80], we have shown how to construct Abelian GMs as a specific truncation of the Abelian KS LGT.

Gauge magnets have been independently reformulated in Ref. [49] as quantum link models. The same authors generalized them to arbitrary groups and representations [50,90,91], introducing the concept of rishons. Both the truncated LGT and the gauge magnets can be understood as specific quantum link model constructions, where the GM corresponds to a quantum link model with a single rishon per link while the truncated LGT corresponds to a quantum link model with two rishons per link [92]. It is, however, interesting to point out that some of the link models can be obtained as a consistent truncation of the KS LGT and others cannot.

## A. $\mathcal{P}_{\text {GM }}$ for gauge magnets

We now discuss how to obtain the projector $\mathcal{P}_{\text {GM }}$ onto the physical Hilbert space $\mathcal{H}_{P}$ for gauge magnets as a TN. The construction is similar to the one used for the other LGT. In particular, $\mathcal{P}_{\mathrm{GM}}$ is obtained by contracting as many copies of tensors $\mathcal{C}$ as there are links on the lattice and as many copies of tensors $\mathcal{G}$ as there are sites, following the patterns of Fig. 10.

The tensor $\mathcal{C}$ is a four-leg tensor that copies the state of physical Hilbert space half into the left auxiliary space and half into the right auxiliary space. Notice that both auxiliary spaces need to be extended so that $V_{r}$ is embedded in a larger vector space that has at least one extra orthogonal direction that we call $|\phi\rangle$. Concretely, having chosen a basis $|l\rangle|v\rangle$ with $|l\rangle=\{|0\rangle,|1\rangle\}$ and $|v\rangle \in V_{r}, \mathcal{C}$ has elements

$$
\begin{equation*}
\mathcal{C}=|0 v\rangle\left\langle\left. 0\right|_{i j} \otimes \mid v\right\rangle_{\alpha}\left\langle\left.\phi\right|_{\beta}+\mid 1 v\right\rangle\left\langle\left. 1\right|_{i j} \otimes \mid \phi\right\rangle_{\alpha}\left\langle\left. v\right|_{\beta},\right. \tag{40}
\end{equation*}
$$

where $|v\rangle \in V_{r}$. (Remember that $\langle\phi \mid v\rangle=0$ for all $|v\rangle \in V_{r}$.) It follows that the bond dimension of the $\mathcal{C}$ tensor is $n_{r}+1$, with $n_{r}$ the dimension of $V_{r}$. We can now define the operator $\mathcal{G}$ that acts on such Hilbert space. Once more, $\mathcal{G}$ needs to be an invariant tensor; that is, it needs to
fulfill Eq. (8). Interestingly, the auxiliary Hilbert space is isomorphic to the auxiliary Hilbert space of the truncated LGT TN. Nevertheless, since $A_{s}(h)$ is defined differently in the two models, $\mathcal{G}$ is different. The symmetry requirements induced by the gauge transformation on $A_{s} h$ are shown in Fig. 19(a).

The explicit form of $\mathcal{G}$ can be obtained once more using different techniques, either by using the projector onto the trivial irrep of Eq. (9), as done in Fig. 19(b), or by using the standard recipes for constructing symmetric tensors of Ref. [58], as shown in Fig. 19(c). In the latter case, one first fuses the irreps $r_{1}$ and $r_{2}$ to $r_{5}$, then $r_{3}$ and $r_{4}$ to $r_{6}$, and then fuses $r_{5}$ and $r_{6}$ to the trivial irrep. In this way, the


FIG. 19. (a) In gauge magnet LGT, the tensor $\mathcal{G}$ is an invariant $G$ tensor; i.e., it is left invariant under multiplying all its legs by $\Gamma_{r}(g)$. (b) $\mathcal{G}$ is obtained, for a finite group, by summing all the representation matrices through the projector on the trivial representation. After projecting, the uniform superposition of all the states of the trivial representation is obtained by closing the free legs with the product of $|+\rangle$. (c) For continuous gauge groups, the projector can either be written as an integral over the group elements with the appropriate invariant measure or it can be obtained by using the Clebsch-Gordan coefficients, which transform the product of the four representations into a direct sum of irreps, and then by projecting onto to the trivial irrep.
$Q$ part of the tensor is well defined, while the $P$ part depends explicitly on all $r_{6}$ and $r_{5}$ compatible with the incoming irrep. Furthermore, it should be chosen as such to provide a uniform superposition of all symmetric tensors.

In the Appendix, we provide the explicit tensors for both $U(1)$ and $S U(2)$ gauge magnets.

## IX. TENSOR-NETWORK VARIATIONAL ANSATZ FOR GAUGE-INVARIANT STATES

So far, we have discussed how to construct the projector onto the gauge-invariant Hilbert space $\mathcal{H}_{P}$ defined in Eq. (15) as a TN. The existence of the projector implies that any state of the gauge-invariant Hilbert space can be constructed by acting with $\mathcal{P}$ on a generic state $|\phi\rangle \in \mathcal{H}$. The challenging problem is still how to express $|\phi\rangle$, since, in general, it is a state of an exponentially large Hilbert space. One possibility would be to express $|\phi\rangle$ itself as a TN and then project it with $\mathcal{P}$. However, even in the best scenario, in which the two states share the same structure as a TN , this procedure rapidly becomes computationally intractable since the bond dimension of the combined TN would be the sum of the bond dimensions for $|\phi\rangle$ and $|P\rangle$.

The alternative is to construct directly all states in $\mathcal{H}_{P}$ from a symmetric variational TN ansatz. This approach has the big advantage of being a sparse TN so that computations are much cheaper than in full TNs having the same bond dimension.

The idea is represented schematically in Fig. 1, where we show that $\mathcal{H}_{P}$ is embedded in $\mathcal{H}$ by drawing it as a membrane inside a 3D box. States described by the variational ansatz belong to $\mathcal{H}_{P}$ and are represented as orange ovals on it. Their size increases by increasing the bond dimension of the elementary tensors, as represented by the label $D \uparrow$.

The simplest gauge-invariant states are the ground states of the Hamiltonian (23) in the strong-coupling limit, which typically are product states. As we increase slightly the complexity of the ground-state wave function, we find another simple class of gauge-invariant states. They consist of uniform superpositions of all gauge-invariant states and are obtained by acting with $\mathcal{P}$ onto a product state $|\varphi\rangle=|+\rangle^{\otimes L}$

$$
\begin{equation*}
|\phi\rangle=1 / \sqrt{Z} \mathcal{P}\left(\prod_{i}^{\otimes L}|+\rangle_{i}\right), \tag{41}
\end{equation*}
$$

where $Z$ is a normalization constant such that $\langle\phi \mid \phi\rangle=1$ and
 of generalized RK states for arbitrary gauge groups [55,93] and are described by TNs with the same bond dimension as $\mathcal{P}$.

Slight generalizations still allow us to describe a larger family than RK states by a TN with the same bond dimension. This result can be done, for example, by projecting different states than $|+\rangle$ with $\mathcal{P}$ with the net effect of moving away from the equal superposition of all gauge-invariant states by changing the matrix elements
inside $\mathcal{C}$. Alternatively, one could change the weights in the linear superposition of gauge-invariant states by changing the tensors $\mathcal{G}$. (We will discuss the operators that allow us to deform the RK state by acting onto the physical Hilbert space in Sec. X.) At this point, we have exhausted all possible gauge-invariant states that can be obtained without extending the TN.

For this reason, in order to describe all other states in $\mathcal{H}_{P}$, we have to introduce a more complex variational ansatz, which consists of a superposition of spin networks [59]. It is again formed by two types of tensors $\mathcal{C}$ and $\mathcal{G}$, and as before, every bond of the TN is decorated by an irrep index. There is one $\mathcal{C}$ tensor for each link $s_{n}$ of the lattice and one $\mathcal{G}$ tensor for each site of the lattice. The elementary tensors are contracted following the usual pattern of Fig. 10. The structure imposed by gauge invariance is exactly the same as the one discussed for the projector $\mathcal{P}$; we will refer to it as the "symmetry part" of the TN. Now, however, we add to each tensor a "degeneracy part." We promote every element of the elementary tensors in $\mathcal{P}$ to a full tensor acting on the appropriate degeneracy space. In practise, we attach to every irrep $r$ a degeneracy space (unconstrained by the symmetry) of dimension $D_{r}$ (chosen independently for each irrep $r$ ). Such degeneracy space contains the variational parameters of the tensor.

We illustrate the construction for the KS LGT, but the same construction is applicable to all the other cases we have discussed so far. On each link, the Hilbert space is isomorphic to $\sum_{r} V_{r} \otimes V_{\bar{r}}$. The tensor $\mathcal{C}$ is a three-leg tensor composed by a symmetry part and a degeneracy part. The tensor $\mathcal{C}$ is represented in Fig. 20(a), where the symmetry part is just made of lines representing identity matrices, while the degeneracy part is made by a tensor of free parameters. Its symmetry part is indeed the same as the $\mathcal{C}$ tensor of an RK state discussed above. Inside each block defined by the irrep $r$, it copies the states in $V_{r}$ to the left and those in $V_{\bar{r}}$ to the right. Its degeneracy part is novel with respect to the previous examples of RK states. It is made of matrices of size $D_{r} \times D_{r}$. These matrices (one per block) are shown in cyan in Fig. 20(a) and contain the variational parameters. Turning to tensor $\mathcal{G}$ in Fig. 20(b), it is also made of two pieces, a symmetry part and a degeneracy part. The symmetry part [once more, the lower piece in the graphical representation of the tensor in Fig. 20(b)] does not contain any free variational parameter but simply takes cares of correctly matching the irreps in such a way that the obtained tensor is symmetric; that is, it fulfills Eq. (8). In the graphical representation, the symmetric part is expressed in terms of the Clebsch-Gordan tensors represented by the small black circles. With every irrep label, one now associates a degeneracy tensor (shown in orange) in Fig. 20(b), which is populated by the variational parameters. Importantly, the degeneracy tensor is obtained as a sum over the irreps corresponding to the internal lines of the symmetric part ( $r_{5}$ in the figure). The variational state is thus


FIG. 20. (a) The $\mathcal{C}$ tensor used to build a variational ansatz for a gauge-invariant state. The tensor acts on a link $s_{n}$ and embeds its state onto the auxiliary space. It is composed by two parts, a "symmetry part" that does not contain any variational parameter (the two lower lines) and a degeneracy part that contains the variational parameters (the cyan tensors). The tensor has several blocks labeled by the irrep $r$. (b) The $\mathcal{G}$ tensor only acts on the auxiliary space, and it has again a block structure. It is divided in two pieces. The first is responsible for the correct symmetry properties of $\mathcal{G}$ [see Fig. 15(a)]. This part does not contain any free parameters and is given in terms of the Clebsch-Gordan coefficients of the group (small black dots). The second part is a degeneracy part that is formed by a sum of several tensors (one for each allowed value of $r_{5}$ ) acting on $D_{r_{1}} \otimes D_{r_{2}} \otimes D_{r_{3}} \otimes D_{r_{4}}$. These tensors store the variational parameters of the $\mathcal{G}$ tensor in the appropriate symmetry blocks. (c) Variational ansatz for gauge-invariant states on a lattice of $4 \times 4$ sites and periodic boundary conditions. The network contains one $\mathcal{C}$ per link of the lattice and one $\mathcal{G}$ per site. The double lines connecting the tensors are used to remind readers that each of the elementary tensors has a double structure, one part dictated by the symmetry and the other one containing the actual variational parameters. In the figure, we are assuming the sum over all the irreps $r$ on every bond of the TN, so that the specific irrep label is omitted.
expressed by the contraction of the various $\mathcal{C}$ and $\mathcal{G}$ given by Fig. 20(c), where it is represented for the specific case of a lattice made by $4 \times 4$ sites and periodic boundary conditions are assumed in both directions.

As a second example, we can consider the $U(1)$ GM, whose $\mathcal{P}$ is defined in the Appendix. We start with the $\mathcal{C}$ tensor for a generic gauge-invariant state. In this case, the physical Hilbert space of a link is two dimensional and involves only two blocks labeled by the irreducible representations 0 and 1 . On the other hand, the auxiliary Hilbert space has two blocks with arbitrary dimension $D_{0}$ and $D_{1}$, so that the auxiliary space is $\mathcal{H}_{\text {aux }}=\mathcal{H}_{0} \oplus \mathcal{H}_{1}$. The nonzero elements of $\mathcal{C}$ are $\mathcal{C}^{\alpha_{0}, \beta_{0}, 0}$, a full matrix that acts on the sub-block $\mathcal{H}_{0} \otimes \mathcal{H}_{0}$ of the Hilbert space $\mathcal{H}_{\text {aux }} \otimes \mathcal{H}_{\text {aux }}$

$$
\begin{equation*}
\mathcal{C}^{\alpha_{0}, \beta_{0}, 0}: \mathcal{H}_{0} \rightarrow \mathcal{H}_{0} \tag{42}
\end{equation*}
$$

Similarly, $\mathcal{C}^{\alpha_{1} \beta_{1}, 1}$ is an independent matrix that acts on a different block of $\mathcal{H}_{\text {aux }} \otimes \mathcal{H}_{\text {aux }}$

$$
\begin{equation*}
\mathcal{C}^{\alpha_{1}, \beta_{1}, 1}: \mathcal{H}_{1} \rightarrow \mathcal{H}_{1} \tag{43}
\end{equation*}
$$

The same idea applies to $\mathcal{G}$ that now can be thought of as a collection of six tensors, each of them acting on one of the only six blocks of the fourfold tensor product $\prod_{i=1}^{\otimes 4}\left(\mathcal{H}_{0} \oplus \mathcal{H}_{1}\right)_{i}$ (which has 16 blocks) allowed by the symmetry constraints. Just to give an example, one of the allowed blocks is $\mathcal{H}_{0} \otimes \mathcal{H}_{0} \otimes \mathcal{H}_{1} \otimes \mathcal{H}_{1}$, where $\mathcal{G}$ has entries $\mathcal{G}^{\alpha_{0}, \beta_{0}, \gamma_{1}, \delta_{1}}$.

Before continuing, we summarize the results of this section. We have proposed a variational ansatz for states of the physical Hilbert space $\mathcal{H}_{P}$. The ansatz involves the contraction of several copies of two families of tensors, $\mathcal{C}$ tensors (one per link) and $\mathcal{G}$ tensors (one per site). Each of those tensors has two components, one completely determined by the requirements of gauge symmetry and a second one that contains the free variational parameters. The resulting TN has bond dimension

$$
\begin{equation*}
D=\sum_{r} n_{r} D_{r}, \tag{44}
\end{equation*}
$$

where $n_{r}$ is the dimension of the irrep $r, D_{r}$ is the dimension of the degeneracy space associated with the irrep $r$ (which is a free parameter), and the sum over $r$ extends to all the irreps one needs to consider.

There are several advantages in dealing with such a symmetric ansatz. On one side, the ansatz can be manipulated with a cost smaller than the cost involved in manipulating a nonsymmetric ansatz with the same $D$, since one can work separately in each block [58]. Also, the ansatz can be used to obtain approximations of interesting gauge-invariant states (such as eigenstates of gauge-invariant Hamiltonians) while exactly preserving the gauge symmetry. Furthermore, the ansatz allows us to target not only the invariant states but also covariant states belonging to separate symmetry sectors. A typical application of this scenario is the characterization of the effects of background charges on the physics of the gauge bosons.

## X. GAUGE-INVARIANT VERTEX OPERATORS

We have just discussed how one can change, given the projector $\mathcal{P}$ on $\mathcal{H}_{P}$ as a TN , the parameters defining $\mathcal{C}$ and $\mathcal{G}$, and thus, one can define a family of RK states. Here, we want to address the question about what the operators are that allow us to modify the RK wave function by acting on the physical Hilbert space. Those operators, in principle, can be added to the standard LGT Hamiltonian of Eq. (23) so as to extend it and allow us to explore extended phase diagrams. The operators that allow us to change the entries of $\mathcal{C}$ are just the $\mathcal{E}^{2}$ operators already discussed. On the other hand, the operators that allow us to modify the entries of $\mathcal{G}$ act on crosses and thus are codiagonal with $\mathcal{G}$.

In particular, $\mathcal{G}$ consists in an isometric tensor followed by a uniform projection onto the state $\prod_{\otimes}|+\rangle$, that is, $\mathcal{G} \equiv \tilde{\mathcal{G}} \prod_{\otimes}|+\rangle$, as is explicit in Fig. 11. We can now use $\tilde{G}^{\dagger}$ to get to the correct basis of the gauge-invariant configurations and weight each of them differently through a diagonal tensor $\Sigma$

$$
\begin{equation*}
\mathcal{V}=\tilde{G} \Sigma \tilde{G}^{\dagger} \tag{45}
\end{equation*}
$$

In the KS construction of $\mathcal{G}$ given in Sec. VI, $\Sigma$ has $|G|_{\sim}^{3}$ elements that can be chosen arbitrarily. Both the isometry $\tilde{\mathcal{G}}$ and the vertex operator $\mathcal{V}$ are represented graphically in Fig. 21.

In particular, we will use these operators in order to characterize the transition from the eight-vertex to the sixvertex model in the next section.


FIG. 21. Left: The isometry $\tilde{\mathcal{G}}$ can be used to go to the gaugeinvariant states. Now, a diagonal operator in this space commutes with the gauge transformations and is thus gauge invariant. Right: The vertex operator described in Eq. (45) is built by concatenating the tensor $\tilde{\mathcal{G}}$, a diagonal tensor $\Sigma$ (a red circle) [acting on $\mathcal{C}(G)^{3}$ ], and then $\tilde{\mathcal{G}}^{\dagger}$. It acts as a potential for different gaugeinvariant configurations and allows us to favor one with respect to another.

## XI. BENCHMARK NUMERICAL RESULTS

In this section, we benchmark our proposal against known analytic results. We start by describing the RK wave function of the $\mathbb{Z}_{2}$ LGT, which is the exact ground state of the Hamiltonian (23) for $\alpha=0$. In this case, the $U$ operator is just the standard $\sigma^{z}$ Pauli matrix, while $\mathcal{E}^{2}$ is the $\sigma_{x}$ Pauli matrix. The Hamiltonian reads

$$
\begin{equation*}
H_{z 2}=\sum_{l} \sigma_{l}^{x}+\frac{1}{\alpha} \sum_{p} \sigma_{p_{1}}^{z} \sigma_{p_{2}}^{z} \sigma_{p_{3}}^{z} \sigma_{p_{4}}^{z}, \tag{46}
\end{equation*}
$$

where $p$ are the plaquettes and $l$ the links of the lattice $\Lambda$. In the limit $\alpha \rightarrow \infty$, the ground state becomes the RK state compatible with the symmetry constraints

$$
\begin{equation*}
\sigma_{s_{1}}^{x} \sigma_{s_{2}}^{x} \sigma_{s_{3}}^{x} \sigma_{s_{4}}^{x}|\psi\rangle=|\psi\rangle, \quad \forall s \in \Lambda, \tag{47}
\end{equation*}
$$

where, as usual, $s_{1}, \ldots, s_{4}$ are the links around a site $s$. The above RK state is obtained by contracting a TN with $D=2$ of the form of the one in Fig. 10 , where $\mathcal{C}^{0,0,0}=1$, $\mathcal{C}^{1,1,1}=1, \quad$ and $\quad \mathcal{G}^{0,0,0,0}=\mathcal{G}^{1,1,1,1}=1, \quad \mathcal{G}^{0,1,1,0}=$ $\mathcal{G}^{1,0,0,1}=1, \quad \mathcal{G}^{1,0,1,0}=\mathcal{G}^{0,1,0,1}=1, \quad \mathcal{G}^{0,0,1,1}=\mathcal{G}^{1,1,0,0}=1$, where, as always, we denote the indexes by $s_{1}, \ldots, s_{4}$, following the pattern of Fig. 7.

As written explicitly in Sec. I of the Appendix, the $\mathcal{P}$ on $\mathcal{H}_{P}$ for the $U(1)$ gauge magnet has a very similar tensor structure. In that case, however, the tensor $\mathcal{G}$ misses the last two entries since $\mathcal{G}^{0,0,1,1}=\mathcal{G}^{1,1,0,0}=0$. The $U(1)$ RK state is the ground state of the following Hamiltonian:

$$
\begin{align*}
H_{\mathrm{GM}}= & \sum_{p}\left[\left(a_{p_{1}} a_{p_{2}} a_{p_{3}}^{\dagger} a_{p_{4}}^{\dagger}+\text { H.c. }\right)\right. \\
& \left.-\left(a_{p_{1}} a_{p_{2}} a_{p_{3}}^{\dagger} a_{p_{4}}^{\dagger}+\text { H.c. }\right)^{2}\right] \tag{48}
\end{align*}
$$

with $a=|0\rangle\langle 1|$; that is, it is the ground state of the gauge magnet Hamiltonian at its RK point [94-96].

By applying the vertex operators defined in Eq. (45) to the $\mathbb{Z}_{2}$ RK state, we can switch off the two extra elements in $\mathcal{G}$, thus effectively interpolating between the $\mathbb{Z}_{2} \mathrm{RK}$ state and the $U(1)$ RK state. In particular, in order to study the transition between the two models, we parametrize the elements $\mathcal{G}^{0,0,1,1}=\mathcal{G}^{1,1,0,0}=\cos (\theta)$, with $0 \leq \theta \leq \pi / 2$. At $\theta=0$, we have the $\mathbb{Z}_{2}$ RK state, while at $\theta=\pi / 2$, we have the $U(1)$ RK state.

We characterize the corresponding RK wave functions for 2D infinite cylinders with circumference $L$, as sketched in Fig. 22(a).

For each value of $\theta$, the norm of the state obtained from the above tensors gives the partition function of the eightvertex model, whose phase diagram was uncovered by Baxter [97]. In particular, in the language of the eightvertex model, we follow the line at $a=b=c=1$ and vary $d$ in the range $0 \leq d \leq 1$. Such a line has also been studied in Fig. 5 of Ref. [60].

The interest in this specific line stands in the fact that along it, the model approaches a transition between two


FIG. 22. (a) The setup used in our numerical calculations. Each of the elementary tensors $\mathcal{C}$ (cyan) and $\mathcal{G}$ (orange) has bond dimension $D=2$, and its matrix elements are those described in the main text. The elementary tensors are contracted, so as to provide the quantum states $|\psi\rangle$ of a 2D infinitely large cylinder of circumference $L$. In the drawing, one has to assume periodic boundary conditions along the vertical direction. Here, we represent the norm of the state that defines the partition function $Z$ of a 2D classical model. (b) We compute the spectrum of the TM across the cylinder (sketched on the left), which characterizes the decay of the correlations addressed in Sec. XI A, and the spectrum of the reduced density matrix of half of the cylinder $\rho_{1 / 2}$ (shown on the right), which gives access to the entanglement characterization of the states addressed in Sec. XI B. Both calculations are performed via sparse exact diagonalization, and their costs increase exponentially with $L$, so that we can address at most systems with $L=20$.
topological phases. At $d=1$, the eight-vertex model is in a $\mathbb{Z}_{2}$ deconfined phase. This phase is the paradigm of a $\mathbb{Z}_{2}$ gapped spin liquid. At $d=0$, there is the transition between the eight- and six-vertex models, so that the system enters in an algebraic spin-liquid phase. Here, we analyze how to characterize the two phases and the transition between them by using our numerical ansatz.

## A. Decay of correlations

The correlations across the cylinder are mediated by the transverse transfer matrix (TM), made by the contraction of all $\mathcal{C}$ 's and $\mathcal{G}$ 's along a transverse slice of the cylinder, as sketched in Fig. 22(b). In particular, at the $\mathbb{Z}_{2}$ LGT point $\theta=0$, the TM has only two degenerate nonvanishing eigenvalues $t_{1}$ and $t_{2}$. The first gap $\Delta_{1}=-\log \left(t_{2} / t_{1}\right)=$ 0 , while all others are infinite. The model thus has zero correlation length. As we start departing from $\theta=0$, the two degenerate eigenvalues start to split so that $\Delta_{1}$ starts to diverge as $\Delta_{1} \propto \exp (L)$. A family of new eigenvalues starts to appear, and the model acquires a nonzero correlation length. The decay of correlation functions is thus exponential in all this region. The new eigenvalues tend to approach $t_{1}$. The bigger of them $t_{3}$ is separated from $t_{1}$ by a gap $\Delta_{2}=-\log \left(t_{3} / t_{1}\right)$. Such gap vanishes when approaching $d=0$ (that is, at $\theta=\pi / 2$ ) as $\Delta_{2} \propto 1 / L$, (cf. 23). The model thus develops algebraic correlations at $\theta=\pi / 2$. Our benchmark numerical results agree with this exact picture. All the eigenvalues are computed by exact sparse diagonalization of
the TM with a cost that increases exponentially with $L$. These results give a first check that our numerical technique works, and we can now apply it to characterize the phase transition in terms of two of the proposed order parameters based on the scaling of the entanglement entropy: the topological entropy and the Schmidt gap. Both quantities could be relevant in understanding the phase diagram of gauge theories. Gauge theories indeed present phases that cannot be distinguished by using a local order parameter. A legitimate question is whether the scaling of entanglement allows us to discern those eluding phases.

## B. Order parameters based on entanglement

We compute the entanglement entropy of the reduced density matrix of half of the infinite cylinder $\rho_{1 / 2}$, sketched in Fig. 22(b). The spectrum $\left\{\lambda_{n}\right\}, n=1, \ldots, D^{L}$, of $\rho_{1 / 2}$ is computed using sparse exact diagonalization of the eigenvectors of the TM, as explained in detail in Ref. [98]. The cost of the computation increases exponentially with $L$. We are interested in characterizing the scaling of the entanglement entropy as a function of $L$ since we want to extract the topological entropy $\gamma_{T}$. In a gapped spin liquid, the entanglement entropy $S_{A}$ of a region $A$ with boundaries of length $L$ scales as

$$
\begin{equation*}
S=c_{1} * L+\gamma_{T}+c_{2} / L+\cdots, \tag{49}
\end{equation*}
$$

where the dots stand for the omission of higher-order corrections starting with $(1 / L)^{2}$. The constant $\gamma_{T}$, in the topological phases, is negative and universal and encodes the topological entropy [61,62]. In the specific case of a $\mathbb{Z}_{2}$ spin liquid, it is known to be

$$
\begin{equation*}
\gamma_{T}=-\log (2) . \tag{50}
\end{equation*}
$$

Equation (49) holds also for the gapless spin-liquid phase described by the six-vertex model. In a series of seminal works, Stéphan and co-workers [99,100] have shown that it is indeed possible to get an exact expression for $\gamma_{T}$ for the whole phase diagram of the six-vertex model. In particular, they have expressed it in the language of the $X X Z$ spin chain defined on a spacelike section of the cylinder. The eigenvectors of the Hamiltonian of the $X X Z$ model (51) are indeed equal to those of the transfer matrix of the six-vertex model [97]. The entanglement entropy of half of the infinite cylinder corresponds, in the $X X Z$ model, to the Shannon entropy of the ground-state wave function of a chain with periodic boundary conditions and length $L$. The $X X Z$ Hamiltonian is given by

$$
\begin{equation*}
H=\sum_{i} \sigma_{i}^{x} \sigma_{i+1}^{x}+\sigma_{i}^{y} \sigma_{i+1}^{y}+\Delta \sigma_{i}^{z} \sigma_{i+1}^{z} \tag{51}
\end{equation*}
$$

In the range $-1<\Delta \leq 1$, the Hamiltonian is critical and the low-energy physics is described by a conformal field theory with $c=1$ describing a free boson compactified
on a circle with radius $R=\sqrt{2-(2 / \pi) \arccos (\Delta)}$. In the whole phase, the topological entropy is given by [99]

$$
\begin{equation*}
\gamma_{T}=\log (R)-\frac{1}{2} \tag{52}
\end{equation*}
$$

The specific point we are studying, called the ice point of the six-vertex model, corresponds to $\Delta=-1 / 2$.

The numerical results we have obtained are presented in Fig. 24. We have only access to modest sizes in the transverse direction $L=4, \ldots, 20 . \gamma_{T}$ is easy to extract close to $\theta=0$ and $\theta=\pi / 2$, where we are able to recover, from our numerics, its exact analytical value. As a crosscheck, we have further reduced the size of the system and considered only the smaller cylinders from $L=4, \ldots, 10$, $L=10, \ldots, 16, L=12, \ldots, 18$, and $L=14, \ldots, 20$. The results, obtained with those sets of data (represented by different symbols in Fig. 24), still provide estimates of $\gamma_{T}$ in agreement with the theory close to $\theta=0$, where finitesize corrections are completely negligible, and close to $\theta=\pi / 2$, where they are reasonably small (see the rightmost inset of Fig. 24).

The situation is very different for intermediate values of $\theta$. Especially in the region between $0.7 \leq \theta \leq 1.5$, we observe strong crossover effects. The values of $\gamma_{T}$ extracted from different series of $L$ do not agree, as shown by the fact that curves made by different symbols are distinct. This


FIG. 23. Upper panel: The second gap of the TM represented in Fig. 22(b) in the $\mathbb{Z}_{2}$ spin-liquid phase closes as $\theta$ approaches $\pi / 2$, where there is the transition from the $\mathbb{Z}_{2}$ spin-liquid phase to the $U(1)$ algebraic spin-liquid phase. The collapse of the data for the $\Delta_{2} L$ close to $\pi / 2$, for the values of $L$ in the range $L=10, \ldots, 16$, confirms that the gap at the transition closes as $1 / L$ as expected. Thus, what we are studying is a transition from a gapped phase with an exponential decay of correlations to a gapless phase governed by an algebraic decay of correlations. Lower panel: On the other hand, the first gap of the $\mathrm{TM} \Delta_{1}$, representing the gap between the two different topological sectors appearing on the cylinder, opens exponentially with $L$ as $\theta$ tends to $\pi / 2$. This behavior is again confirmed by the collapse of our numerical data for $\log \left(\Delta_{1}\right) / L$, with $L$ in the range $L=10, \ldots, 16$.


FIG. 24. The topological entropy $\gamma_{T}$ defined in Eq. (49) and extracted from the scaling of the entropy of half infinite cylinders with respect to their circumference $L$. The red dots are obtained by considering the scaling of the entropy in the interval $L=4, \ldots, 10$, blue up-facing triangles $L=10, \ldots, 16$, green squares $L=12, \ldots, 18$, and the yellow triangles pointing to the right $L=14, \ldots, 20$. The solid orange line represents the exact value of $\gamma_{T}$ for the $\mathbb{Z}_{2}$ spin liquid in Eq. (50), while the cyan dashed line represents its value for the $U(1)$ spin liquid from Eq. (52) with $R$ associated with $\Delta=-1 / 2$. We see that for small $\theta, \gamma_{T}$ coincides, independently of the size of the cylinder considered, with the expected exact value $\mathbb{Z}_{2}$ (left inset). As we move toward the transition, $\gamma_{T}$ shows a transient oscillation that tends to become sharper and deeper and move toward $\pi / 2$ for larger $L$. At $\pi / 2$, it attains again the expected analytical value with very small corrections induced by considering the two different set of data (right inset). In the main text, we provide a discussion of these results.
effect could be related to the subleading corrections that become more important when we approach the transition. We also observe that for larger systems, the crossover region shrinks and moves toward the phase transition at $\theta=\pi / 2$. It looks like that, if we were able to reach the thermodynamic limit, $\gamma_{T}$ would present a very sharp jump between the two asymptotic values.

These results give further evidence that, provided one is able to address large enough systems, $\gamma_{T}$ can be used as an order parameter, away from a relatively small crossover region (which furthermore shrinks with increasing system size). In our specific case, indeed, it allows us to discern the gapped $\mathbb{Z}_{2}$ spin-liquid phase from the algebraic $U(1)$ spin-liquid phase at $\theta=\pi / 2$.

We now analyze another possible order parameter based on the scaling of the entanglement. Li and Haldane in Ref. [63] suggested that phases could be easier to identify by considering the scaling of the full entanglement spectrum rather than focusing on a single number as the topological entropy. The entanglement spectrum is the collection of the logarithm of the eigenvalues of the reduced density matrix $\log \left(\lambda_{n}\right)$.

In particular, numerical studies of 1D systems have provided a precise characterization of the scaling of the
lowest part of the entanglement spectrum, the one associated with the largest eigenvalues of the reduced density matrix [101,102]. In many cases, the lowest gap of the entanglement spectrum, called the Schmidt gap, vanishes when approaching a quantum phase transition following universal scaling laws. The authors have thus proposed to use the Schmidt gap as an order parameter. This idea is further supported by the recent results that show that for conformal invariant critical points, several gaps in the entanglement spectrum close in a way that allows us to identify the critical exponents of the underlying conformal field theory [102]. It is still unclear how general these results are.

For this reason, we have decided to analyze the behavior of the lowest part of the entanglement spectrum of $\rho_{1 / 2}$ of half cylinders, when approaching the transition at $\theta=\pi / 2$. In particular, we address the crossover region between $1 \leq \theta \leq \pi / 2$, where the analysis of the topological entropy is unreliable.

The results are presented in Fig. 25. The main panel shows the first 100 values of the entanglement spectrum $\log \left(\lambda_{n}\right)$ as a function of $\theta$ in the crossover region. Surprisingly, nothing strange seems to happen. The spectrum presents the plateau structure characteristic of the RK wave functions. The structure of the first plateau seems quite stable, and the only effect of increasing $\theta$ toward $\pi / 2$ is to shift the relative height of the plateau to accommodate the appearance of new ones in the tails. In particular, the first two eigenvalues are degenerate for all the intervals considered. The Schmidt gap is indeed constantly 0 and


FIG. 25. The behavior of the first 100 eigenvalues $\lambda_{n}$ ( $n$ is on the $y$ axis) of the reduced density matrix of half infinite cylinders $\rho_{1 / 2}$ defined in Fig. 22(b) with $L=16$. We plot them as a function of $\theta$ ( $x$ axis). $\theta$ varies in the crossover range $1 \leq \theta \leq \pi / 2$ identified during the analysis of the topological entropy in Fig. 24. The entanglement spectrum presents clear plateaux, footprints of RK wave functions. Nevertheless, its lower part does not seem to detect the phase's transition. The first Schmidt gap is identically 0 everywhere (there are two degenerate eigenvalues), and the second Schmidt gap increases while approaching the transition (inset in the figure) for all the $L=10, \ldots, 16$.
does not detect the transition. Even from the plot of the second Schmidt gap, which increases monotonically with $\theta \rightarrow \pi / 2$, we are unable to appreciate that we are approaching a phase transition, as shown in the inset of Fig. 25 for several values of $L$. We thus conclude that the low-energy part of the entanglement spectrum seems to fail to detect the phase transition between the gapped $\mathbb{Z}_{2}$ spin-liquid phase and the algebraic $U(1)$ spin liquid occurring at $\theta=\pi / 2$.

## XII. PREVIOUS WORK ON THE SUBJECT

Here, we try to list the works that have contributed to our understanding of various topics and to the final formulation of our proposal, as we have outlined.

There are several good references for group theory in a diagrammatic notation. In particular, the literature [67] deals with continuous groups. Continuous groups and the specific case of $S U(2)$ have been extensively studied in the literature of TNs [56-58,72,103]. The reader not familiar with the elementary concepts in the theory of group representations would benefit from studying the first few chapters of some of the standard textbooks [69,81]. A nice summary of relevant material can also be found in the Appendixes of Ref. [104].

A nice introduction to LGT in the Lagrangian formulation can be found in Ref. [105] and in any of the standard textbooks on the subject, i.e., Refs. [82,106]. The standard Hamiltonian formulation was obtained by Kogut and Susskind [77] and by Creutz [78].

Truncated LGTs were discussed independently by many authors $[46,47,49]$ that have called them gauge magnet or quantum link models. Horn originally introduced a model similar to the one described here in Sec. VII, which we generalize here to arbitrary continuous and discrete groups. Later, in Ref. [50], the quantum link models were generalized to several continuous gauge groups with a different strategy from the one presented here. In the same work, they were also given an interpretation in terms of rishons. In that language, the truncated LGT we have discussed in Sec. VII when dealing with continuous groups is a specific quantum link model with two rishons per link, while the one in Sec. VIII is a quantum link model with one rishon per link [107]. Still, the results we present here can also be applied to discrete groups and thus provide a further generalization of quantum link models. Also, they allow us to easily distinguish between those quantum link models that can be obtained as a consistent truncation of the KS LGT and those that cannot.

The study of lattice gauge theories with matrix-product states, the simplest 1D TN structure, has, by now, a quite long tradition [24-26,108-111]; 2D LGT with TNs, on the other hand, have been less studied. LGTs with discrete groups have been addressed with entanglement renormalization in Ref. [23]; some aspects of them have been studied in Ref. [85] with categorical TNs, a construction that has strong connections to the present proposal. There, however, the emphasis is on Abelian discrete groups and
the LGTs are addressed at the exactly solvable point obtained at $g=0$ while here we discuss a construction that allows us to tackle generic groups both Abelian and non-Abelian, discrete and continuous, at any point in the phase diagram. LGTs have also recently been addressed by using tensor-renormalization schemes [27,112-114].

There have been several proposals on how to obtain the gauge-invariant Hilbert space. The original proposal by KS was to act on a reference state with all possible gaugeinvariant operators. However, there, the states generated are not necessarily orthogonal, and thus, special care needs to be taken. Osborne has pointed out Refs. $[115,116]$ to us, which are related (implicitly) to our work, and we also found the discussion in Ref. [117] very illuminating. Osborne himself is working along similar ideas [87], and, in particular, he has independently worked out operators similar to the ones we have presented in Eqs. (19) and (25).

In the context of characterizing a family of states (such as the RK states we have discussed), Osborne has also pointed out Refs. [118,119]. In condensed matter, in particular, the recent results presented in Refs. [120,121] have been obtained by applying similar ideas to the characterization of singlet states.

At last, we have also used the available literature about TNs and topological order in order, whenever possible, to make connections between our ideas and the one presented in that context. In particular, we have found particularly useful Refs. [83,104,122].

To our knowledge, the only previous mention of vertex operators in the context of LGT is the one of Ref. [60]; however, there, the analysis is limited to $Z_{2}$ and $U(1)$ LGT, while here, we give a prescription for arbitrary groups.

It is also worth mentioning that there is an alternative connection between LGT and TN through a map of the lowenergy physics of QCD in the chiral limit to the physics of specific spin chains, as pursued in Refs. [123,124].

Finally, several groups have recently addressed the experimental implementation of truncated LGTs $[80,88,89$, 96,125-130].

With respect to the absence of the closure of the Schmidt gap across a phase transition, a similar observation was made in Ref. [100], and while we were preparing the final version of our manuscript, in Ref. [131], the authors have provided a plausible argument to understand that this nonclosure is a quite general phenomenon for phase transitions between different RK states.

## XIII. CONCLUSIONS

In this paper, we have defined a TN framework for studying LGTs. It allows us to use TNs as both a LGT model-building tool (and, as such, we have used them to construct the minimal consistent-truncation scheme for the KS LGT) and as a practical tool to numerically explore LGTs, their phase diagrams, and their emerging properties.

The ansatz we have proposed follows the same spirit as the one proposed in Ref. [23]. The TN indeed has a symmetric part that allows us to exactly encode the constraints imposed by the gauge-symmetry conditions and a variational part that can be optimized numerically in order to characterize interesting physical states such as the low-energy states of gauge-invariant Hamiltonians.

The new framework is also very powerful from the theoretical point of view. In this paper, we have indeed been able to derive through it a consistent truncation of the local Hilbert space of the KS LGT with a continuous group to finite-dimensional Hilbert spaces. We have also obtained an explicit alternative construction of gauge magnets and of their $U$ operators for arbitrary gauge groups [Eq. (37)] that is also applicable to discrete groups, and the construction of gauge-invariant vertex operators for arbitrary gauge groups [Eq. (45)]. We have also been able to show that, differently from the Abelian case, the non-Abelian gauge magnets cannot be obtained as a consistent local truncation of the KS LGT. For this reason, they stand as an alternative microscopic formulation of LGT. This result does not exclude the possibility that both gauge magnets and the KS LGT can encode the same emergent physics. It excludes, however, that they are locally (where, by locally, we mean at the level of a single link) unitarily equivalent. Furthermore, the distinct form of their projectors onto $\mathcal{H}_{P}$ in terms of TN (given explicitly in Secs. 2 and 3 of the Appendix) also points to the fact that their RK states are probably different (as we will analyze in a subsequent paper). This finding is not particularly surprising since the relation between the low-energy physics of quantum link models and of standard LGT was already discussed in Ref. [90] and required the use of dimensional reduction arguments [so that $(D+1)$-dimensional quantum link models are expected in some limit to be equivalent to $D$-dimensional standard LGT].

The tools that we have developed here can be used to analyze the entanglement content of interesting LGT states. The entanglement, studied in a basis of states belonging to the original Hilbert space tensor product of the constituents, does not have a direct physical meaning (since the only measurable operators in a real GT are gauge-invariant operators; see the recent discussion in Ref. [132]). However, it still provides an estimate of the computational cost of simulating such states using a TN. (See the related discussion in Ref. [23].)

As a benchmark, we have considered the transition between the eight-vertex and the six-vertex models in terms of the RK wave function of the corresponding LGT. We have shown that, while the transition is correctly detected by the behavior of the entanglement entropy, it is hard to detect it by observing the behavior of the Schmidt gap and the lower part of the entanglement spectrum.

We envisage that the tools that we have developed will play an important role in the characterization of the realtime dynamics in LGT and in the quest for finding a model
displaying stable topological phases even at finite temperature. All the recent developments about the characterization of topological phases in terms of 2D TN such as the ones of Refs. [83,122] can be easily applied to our construction, as we plan to in the near future.

Recently, gauge magnets have received a lot of attention from the Atomic, Molecular, and Optical Physics community, due to the possibility of implementing them in experiments based on the emerging new quantum technologies such as cold atoms, trapped ions, etc. [80,88,89,96,125-130]. There is still a great deal of room for improvement on these first proposals, and the tools we have designed will help in this task. We have indeed just become aware that a new proposal for simulating $S U(2)$ LGT along the lines of our discussion has already been independently designed [133].

Furthermore, our analysis is just the starting point in the development of a TN approach to LGT. In particular, questions relevant for high-energy LGT like, e.g., taking the continuous limit, have not been addressed here and constitute a logical next step to be done. Some such questions are the subject of an ambitious collaborative project, coordinated by Osborne, that is open to contributions and available online [87].

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## APPENDIX: THE EXPLICIT FORM OF THE TENSORS FOR RK STATES FOR $\boldsymbol{U}(\mathbf{1})$ AND $S U(2)$ LGT

## 1. $U(1)$

In the case of $U(1)$, both the local Hilbert space and the auxiliary space are two dimensional and the tensor $\mathcal{C}$ is given in the computational basis as

$$
\begin{equation*}
\mathcal{C}=|0\rangle|0\rangle\langle 0|+|1\rangle|1\rangle\langle 1| \tag{A1}
\end{equation*}
$$

while the tensor $\mathcal{G}$ is given in the computational basis

$$
\begin{align*}
\mathcal{G}= & |0\rangle_{s_{1}}|0\rangle_{s_{2}}\left\langle0 | _ { s _ { 3 } } \left\langle\left. 0\right|_{s_{4}}\right.\right. \\
& +|1\rangle_{s_{1}}|1\rangle_{s_{2}}\left\langle\left. 1\right|_{s_{3}}\left\langle\left. 1\right|_{s_{4}}+\mid 1\right\rangle_{s_{1}} \mid 1\right\rangle_{s_{2}}\left\langle1 | _ { s _ { 3 } } \left\langle\left. 1\right|_{s_{4}}\right.\right. \\
& +|1\rangle_{s_{1}}|0\rangle_{s_{2}}\left\langle\left. 1\right|_{s_{3}}\left\langle\left. 0\right|_{s_{4}}+\mid 0\right\rangle_{s_{1}} \mid 1\right\rangle_{s_{2}}\left\langle0 | _ { s _ { 3 } } \left\langle\left. 1\right|_{s_{4}}\right.\right. \\
& +|1\rangle_{s_{1}}|0\rangle_{s_{2}}\left\langle\left. 0\right|_{s_{3}}\left\langle\left. 1\right|_{s_{4}}+\mid 0\right\rangle_{s_{1}} \mid 1\right\rangle_{s_{2}}\left\langle1 | _ { s _ { 3 } } \left\langle\left. 0\right|_{s_{4}} .\right.\right. \tag{A2}
\end{align*}
$$

## 2. Truncated $\boldsymbol{S U}(\mathbf{2})$ LGT

The truncated $S U(2)$ LGT has Hilbert space of dimension five. In it, we call the vectors as $|0\rangle,|11\rangle,|12\rangle,|21\rangle$, and $|22\rangle$, in order to remember that we have two blocks, the irrep $j=0$ and the irrep $j=1 / 2$, which are a direct sum, one of dimension one and the other of dimension four, that is, the tensor product of two two-dimensional spaces.

The TN that encodes the RK states can be highly simplified by noting that only a part of the Hilbert space needs to be copied on the left and another part on the right. In particular, we can write the $\mathcal{C}$ tensor as

$$
\begin{align*}
\mathcal{C}= & |0\rangle|2\rangle\langle 2| \\
& +|11\rangle|0\rangle\langle 0|+|12\rangle|0\rangle\langle 1|+|21\rangle|1\rangle\langle 0|+|22\rangle|1\rangle\langle 1| . \tag{A3}
\end{align*}
$$

From the second line, we immediately recognize that $\mathcal{C}$ in the four-dimensional block, that is, the tensor product, copies the left factor to the left and the right factor to the right. In this way, the auxiliary dimension is only $D=3$. We now need to select gauge-invariant configurations on the auxiliary links by using $\mathcal{G}$. We give the expression of the blocks individually:

$$
\begin{aligned}
\mathcal{G}_{1 / 2,1 / 2,1 / 2,1 / 2}= & 1 / 2\left(| 0 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \langle 0 | _ { s _ { 3 } } \langle 0 | _ { s _ { 4 } } + | 1 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \langle 1 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + | 0 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \langle 0 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + | 1 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \left\langle\left.1\right|_{s_{3}}\left\langle\left. 0\right|_{s_{4}}\right)\right.\right. \\
& +1 /(2 \sqrt{3})\left(| 0 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \langle 0 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + | 1 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \langle 1 | _ { s _ { 3 } } \langle 0 | _ { s _ { 4 } } + - | 0 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \left\langle0 | _ { s _ { 3 } } \left\langle\left.0\right|_{s_{4}}\right.\right.\right. \\
& +-|1\rangle_{s_{1}}|1\rangle_{s_{2}}\left\langle\left. 1\right|_{s_{3}}\left\langle\left. 1\right|_{s_{4}}\right)+-1 /(\sqrt{3})\left(| 1 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \langle 0 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + | 0 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \left\langle\left.1\right|_{s_{3}}\left\langle\left. 0\right|_{s_{4}}\right),\right.\right.\right.
\end{aligned}
$$

$$
\begin{equation*}
\mathcal{G}_{0,0,0,0}=1 / 2\left(| 2 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \left\langle\left.2\right|_{s_{3}}\left\langle\left. 2\right|_{s_{4}}\right),\right.\right. \tag{A5}
\end{equation*}
$$

$$
\mathcal{G}_{0,0,1 / 2,1 / 2}=1 / \sqrt{2}\left(| 2 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \left\langle0 | _ { s _ { 3 } } \left\langle\left.1\right|_{s_{4}}+\right.\right.\right.
$$

$$
-|2\rangle_{s_{1}}|2\rangle_{s_{2}}\left\langle\left. 1\right|_{s_{3}}\left\langle\left. 0\right|_{s_{4}}\right)\right.
$$

$$
\begin{align*}
\mathcal{G}_{1 / 2,0,0,1 / 2}= & 1 / \sqrt{2}\left(| 0 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \left\langle2 | _ { s _ { 3 } } \left\langle\left.0\right|_{s_{4}}\right.\right.\right. \\
& +|1\rangle_{s_{1}}|2\rangle_{s_{2}}\left\langle\left. 2\right|_{s_{3}}\left\langle\left. 1\right|_{s_{4}}\right),\right.  \tag{A10}\\
\mathcal{G}_{1 / 2,0,1 / 2,0}= & 1 / \sqrt{2}\left(| 0 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \left\langle0 | _ { s _ { 3 } } \left\langle\left.2\right|_{s_{4}}\right.\right.\right. \\
& +|1\rangle_{s_{1}}|2\rangle_{s_{2}}\left\langle\left. 1\right|_{s_{3}}\left\langle\left. 2\right|_{s_{4}}\right) .\right. \tag{A11}
\end{align*}
$$

$$
\mathcal{G}_{0,1 / 2,0,1 / 2}=1 / \sqrt{2}\left(| 2 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \left\langle2 | _ { s _ { 3 } } \left\langle\left.0\right|_{s_{4}}\right.\right.\right.
$$

$$
+|2\rangle_{s_{1}}|1\rangle_{s_{2}}\left\langle\left. 2\right|_{s_{3}}\left\langle\left. 1\right|_{s_{4}}\right),\right.
$$

## 3. The $\boldsymbol{S U}(2)$ gauge magnet

$$
\mathcal{G}_{0,1 / 2,1 / 2,0}=1 / \sqrt{2}\left(| 2 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \left\langle0 | _ { s _ { 3 } } \left\langle\left.2\right|_{s_{4}}\right.\right.\right.
$$

Similarly to the truncated $S U(2), \mathcal{P}$ for the $S U(2)$ gauge magnet is written as a TN with $D=3$. The local Hilbert

$$
+|2\rangle_{s_{1}}|1\rangle_{s_{2}}\left\langle\left. 1\right|_{s_{3}}\left\langle\left. 2\right|_{s_{4}}\right)\right.
$$ space with dimension-four tensor $\mathcal{C}$ reads

$$
\begin{equation*}
\mathcal{G}_{1 / 2,1 / 2,0,0}=1 / \sqrt{2}\left(| 0 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \left\langle2 | _ { s _ { 3 } } \left\langle\left.2\right|_{s_{4}}\right.\right.\right. \tag{A12}
\end{equation*}
$$

$$
\mathcal{C}=\sum_{j=0,1}|0, j\rangle|j\rangle\langle 2|+|1, j\rangle|2\rangle\langle j| .
$$

$$
+-|1\rangle_{s_{1}}|0\rangle_{s_{2}}\left\langle\left. 2\right|_{s_{3}}\left\langle\left. 2\right|_{s_{4}}\right)\right.
$$

The tensor $\mathcal{G}$, on the other hand, reads

$$
\begin{align*}
\mathcal{G}= & 1 / \sqrt{2}\left(| 2 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \langle 0 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + - | 2 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \left\langle\left.1\right|_{s_{3}}\left\langle\left. 0\right|_{s_{4}}\right)+1 / \sqrt{2}\left(| 2 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \langle 2 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + - | 2 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \left\langle\left.2\right|_{s_{3}}\left\langle\left. 0\right|_{s_{4}}\right)\right.\right.\right.\right. \\
& +1 / \sqrt{2}\left(| 2 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \langle 1 | _ { s _ { 3 } } \langle 2 | _ { s _ { 4 } } + - | 2 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \left\langle\left.0\right|_{s_{3}}\left\langle\left. 2\right|_{s_{4}}\right)+1 / \sqrt{2}\left(| 0 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \langle 2 | _ { s _ { 3 } } \langle 2 | _ { s _ { 4 } } + - | 1 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \left\langle\left.2\right|_{s_{3}}\left\langle\left. 2\right|_{s_{4}}\right)\right.\right.\right.\right. \\
& +1 / \sqrt{2}\left(| 0 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \langle 2 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + - | 1 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \left\langle\left.2\right|_{s_{3}}\left\langle\left. 0\right|_{s_{4}}\right)+1 / \sqrt{2}\left(| 0 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \langle 1 | _ { s _ { 3 } } \langle 2 | _ { s _ { 4 } } + - | 1 \rangle _ { s _ { 1 } } | 2 \rangle _ { s _ { 2 } } \left\langle\left.0\right|_{s_{3}}\left\langle\left. 2\right|_{s_{4}}\right)\right.\right.\right.\right. \\
& +1 / 2\left(| 1 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \langle 1 | _ { s _ { 3 } } \langle 0 | _ { s _ { 4 } } + | 0 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \langle 0 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + - | 1 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \langle 0 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + - | 0 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \left\langle\left.1\right|_{s_{3}}\left\langle\left. 0\right|_{s_{4}}\right)\right.\right. \\
& +1 /(2 \sqrt{3})\left(| 0 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \langle 0 | _ { s _ { 3 } } \langle 1 | _ { s _ { 4 } } + | 1 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \langle 1 | _ { s _ { 3 } } \langle 0 | _ { s _ { 4 } } + | 1 \rangle _ { s _ { 1 } } | 1 \rangle _ { s _ { 2 } } \langle 0 | _ { s _ { 3 } } \langle 0 | _ { s _ { 4 } } + | 0 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \left\langle\left.1\right|_{s_{3}}\left\langle\left. 1\right|_{s_{4}}\right)\right.\right. \\
& +-1 /(\sqrt{3})\left(| 0 \rangle _ { s _ { 3 } } \langle 0 | _ { s _ { 4 } } + | 1 \rangle _ { s _ { 1 } } | 0 \rangle _ { s _ { 2 } } \left\langle\left.0\right|_{s_{3}}\left\langle\left. 1\right|_{s_{4}}\right) .\right.\right. \tag{A13}
\end{align*}
$$

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