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Assessing student expertise in introductory physics with isomorphic problems. II. Effect of some potential factors on problem solving and transfer

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In this paper, we explore the use of isomorphic problem pairs (IPPs) to assess introductory physics students' ability to solve and successfully transfer problem-solving knowledge from one context to another in mechanics. We call the paired problems "isomorphic" because they require the same physics principle to solve them. We analyze written responses and individual discussions for a range of isomorphic problems. We examine potential factors that may help or hinder transfer of problem-solving skills from one problem in a pair to the other. For some paired isomorphic problems, one context often turned out to be easier for students in that it was more often correctly solved than the other. When quantitative and conceptual questions were paired and given back to back, students who answered both questions in the IPP often performed better on the conceptual questions than those who answered the corresponding conceptual questions only. Although students often took advantage of the quantitative counterpart to answer a conceptual question of an IPP correctly, when only given the conceptual question, students seldom tried to convert it into a quantitative question, solve it, and then reason about the solution conceptually. Even in individual interviews when students who were given only conceptual questions had difficulty and the interviewer explicitly encouraged them to convert the conceptual question into the corresponding quantitative problem by choosing appropriate variables, a majority of students were reluctant and preferred to guess the answer to the conceptual question based upon their gut feeling. Misconceptions associated with friction in some problems were so robust that pairing them with isomorphic problems not involving friction did not help students discern their underlying similarities. Alternatively, from the knowledge-in-pieces perspective, the activation of the knowledge resource related to friction was so strongly and automatically triggered by the context, which is outside the conscious control of the student, that students did not look for analogies with paired problems or other aids that may be present.

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I. INTRODUCTION

In this paper, we explore the use of isomorphic problem pairs (IPPs) to assess introductory physics students' expertise in mechanics in a range of contexts. We call the paired problems isomorphic if they require the same physics principle to solve them. We investigate a few parameters as potential factors that may help problem solving and analyze the performance of students on the IPPs from the perspective of "transfer."[1–11] For example, we examine the effect of misconceptions about friction as a potential barrier for problem solving and transfer.[1–4] Transfer in physics is particularly challenging because there are only a few principles and concepts that are condensed into a compact mathematical form. Learning requires unpacking them and understanding their applicability in a variety of contexts that share deep features, e.g., the same law of physics may apply in different contexts. Cognitive theory suggests that transfer can be difficult especially if the "source" (from which transfer is intended) and the "target" (to which transfer is intended) do not share surface features. This difficulty arises because knowledge is encoded in memory with the context in which it was acquired, and solution of the source problem does not automatically manifest its deep similarity with the target problem.[1] The ability to transfer relevant knowledge from one context to another improves with expertise because an expert's knowledge is hierarchically organized and represented at a more abstract level in memory, which facilitates categorization and recognition based upon deep features.[2–9,12,13]

Students may find one problem in an IPP easier to tackle than its pair because context and representation are very important.[2–9,12,13] For example, if two equivalent groups of students are given only the quantitative or conceptual question from an IPP pairing a quantitative and conceptual question with similar contexts, one group may perform well on one of them but not the other. Some studies have shown that, if students are reasonably comfortable with the mathematical manipulation required to solve a quantitative problem, the group given the quantitative problem may perform better on it using an algorithmic approach than the group given the corresponding conceptual question.[14–16] In a study on student understanding of diffraction and interference concepts, the group that, was given a quantitative problem performed significantly better than the group given a similar conceptual question.[14] In another study, Kim *et al.*[15] examined the relation between traditional physics textbook-style quantitative problem solving and conceptual reasoning. They found that, although students in a mechanics course on average had solved more than 1000 quantitative problems and were facile at mathematical manipulations, they still had many common difficulties when answering conceptual questions on related topics. When Mazur[16] gave a group of Harvard students quantitative problems related to power dissipation in a circuit, students performed significantly better than when an equivalent group was given conceptual questions about the relative brightness of light bulbs in similar circuits. In solving the quantitative problems given by Mazur, students applied Kirchhoff's rules to write down a set of equations and then solved the equations algebraically for the relevant variables from which they calculated the power dissipated. When the conceptual circuit question was given to students in similar classes, many students appeared to guess the answer rather than reasoning about it systematically.[16] For example, if students are given quantitative problems about the power dissipated in each headlight of a car with resistance R when both bulbs are connected in parallel to a battery with an internal resistance r and then asked to repeat the calculation for the case when one of the headlights is burned out, the procedural knowledge of Kirchhoff's rules can help students solve for the power dissipated in each headlight even if they cannot conceptually reason about the current and voltage in different parts of the circuit.[14,16] To reason without resorting *explicitly* to mathematical tools (Kirchhoff's rules) that the single headlight in the car will be brighter when the other headlight is burned out, students have to reason in the following manner. The equivalent resistance of the circuit is lower when both headlights are working so that the current coming out of the battery is larger. Hence, more of the battery voltage drops across the internal resistance r and less of the battery voltage drops across each headlight and therefore each headlight will be less bright. If a student deviates from this long chain of reasoning required in conceptual understanding, the student may not make a correct inference.

II. HYPOTHESES AND GOALS

The experiments we describe here can broadly be classified into three categories. Experiment 1 involves IPPs that pair a quantitative question with a conceptual question. Experiment 2 involves IPPs in which both questions are conceptual. Experiment 3 addresses the effect of misconceptions about friction on students' ability to transfer relevant knowledge from a problem not involving friction to isomorphic problems involving friction.

We developed several IPPs in the multiple-choice format (final version shown in the Appendix) with different contexts in mechanics. The problems spanned a range of difficulty. The correct solution to each question is italicized in the Appendix. We administered either one or both questions in an IPP to introductory physics students. We made hypotheses H1–H3 related to experiment 1, hypothesis H4 related to experiment 2, and hypothesis H5 related to experiment 3 as described below:

Experiment 1 with IPPs in which one question is more quantitative than the other: Although it is difficult to categorize physics questions as exclusively quantitative or conceptual, some of the IPPs had one question that required symbolic or numerical calculation while the other question could be answered by conceptual reasoning alone. The first five IPPs in the Appendix fall in this category (although questions 3 and 4 in the second IPP can both be classified as quantitative). We made the following hypotheses regarding these IPPs.

H1: Performance on quantitative questions of an IPP will be better when both the quantitative and conceptual questions are given than when only the quantitative question is given.

H2: Performance on conceptual questions of an IPP will be better when both the quantitative and conceptual questions are given than when only the conceptual question is given.

H3: The closer the match between the contexts of the quantitative and conceptual questions of an IPP, the better will students be able to discern their similarity and transfer relevant knowledge from one problem to another.

We note that our study is different from those mentioned earlier[14–16] because our goal is not to evaluate whether students perform better on the quantitative or conceptual question but rather to evaluate whether giving both questions together improves performance on each type of question compared to the case when only the conceptual or the quantitative question was given.

Hypothesis H1 is based on the assumption that solving the conceptual question of an IPP may encourage students to perform a qualitative analysis, streamline students' thinking, make it easier for them to narrow down relevant concepts, and thus help them solve the quantitative problem correctly. Prior studies show that introductory physics students are not systematic in using effective problem-solving strategies, and often do not perform a conceptual analysis while solving a quantitative problem.[17] They often use a "plug and chug" approach to solving quantitative problems which may prevent them from solving the problem correctly. The conceptual questions may provide an opportunity for reflecting upon the quantitative problem and performing a qualitative analysis and planning. This can increase the probability of solving the quantitative problem correctly. We note that, since the IPPs always had a quantitative question preceding the corresponding conceptual question, hypothesis H1 assumes that students will go back to the quantitative question if they got some insight from the corresponding conceptual question.

Hypothesis H2 is inspired by results of prior studies that show that introductory physics students often perform better on quantitative problems compared to conceptual questions on the same topic.[14–16] Students often treat conceptual questions as guessing tasks.[14–16] We hypothesized that students who are able to solve the quantitative problem in an IPP may use its solution as a hint for answering the conceptual question correctly if they are able to discern the similarity between the two questions. Since quantitative and conceptual questions of an IPP were given one after another, we hypothesized that students would likely discern their underlying similarity at least in cases where the contexts were similar. When reasoning without quantitative tools, it may be more difficult to create the correct chain of reasoning if a student is "rusty" about a concept.[18] Equations can provide a pivot point for constructing the reasoning chain. For example, if a student has forgotten whether the maximum safe driving speed while making a turn on a curved road depends on the mass of the vehicle, he/she will have great difficulty reasoning without equations that the maximum speed is not dependent on the mass. Similarly, a student with evolving expertise who is comfortable reasoning with equations may need to write down Newton's second law *explicitly* to conclude that the tension in the cable of an elevator accelerating upward is greater than its weight. An expert can use the same law *im-*

plicitly and conceptually argue that the upward acceleration implies that the tension exceeds the weight without writing down Newton's second law explicitly. Being able to reason conceptually without resorting to quantitative tools in a wide variety of contexts may be a sign of adaptive expertise, whereas conceptual reasoning by resorting to quantitative tools may be a sign of evolving expertise.[18,19]

Hypothesis H3 is based upon results of prior studies related to transfer.[1–11] For example, in teaching debugging in LOGO programming to children and investigating near and far transfer of debugging skills to other contexts, Carver *et al.* found that transfer of relevant knowledge is easier if the contexts of the problems are similar.[11] In the IPPs with paired questions with different contexts, transfer of relevant knowledge may be more difficult because students may have more difficulty discerning their underlying similarity. If the contexts are very different, discerning the underlying similarity of the problems in each pair can be considered a sign of adaptive expertise.[19] If students had difficulty discerning the underlying similarity of the IPPs with different contexts, we explore the aspects of the IPPs that made the transfer of relevant knowledge difficult. Among the first five IPPs, we identified the contexts of the questions in the first three IPPs to be closest, followed by the IPP pairing questions 9 and 10 and then the IPP pairing questions 7 and 8. The main difference between the contexts of questions in IPP (9) and (10) is that in one case a person is falling vertically into a boat moving horizontally and in the other case rain is falling vertically into a cart moving horizontally. The IPP with questions 7 and 8 was considered to be the one requiring the farthest transfer of relevant knowledge because the quantitative problem 7 asks about the time for a projectile to reach the maximum height and question 8 asks students to compare the time of flight for three projectiles launched with the same speed that achieved different heights and had different horizontal ranges. In order to transfer from problem 7 to problem 8, students need to know that the total time of flight for a projectile is twice the time to reach the maximum height. Moreover, in question 8, students should not get distracted by different horizontal ranges for the three projectiles, since the horizontal range is not a relevant variable for answering this question.

Experiment 2 involves IPPs with different contexts in which neither question is quantitative: Examples of three such IPPs are pairs in questions 11–16 in the Appendix. We made the following hypothesis.

H4: When both questions of an IPP are conceptual, performance will be better when both questions are given versus when only one is given.

Hypothesis H4 is based upon the assumption that one question in an IPP may provide a hint for the other question and may help students in converging their reasoning based upon relevant principles and concepts.

Experiment 3 involves IPPs or a problem triplet in which some questions involve distracting features, e.g., common misconceptions related to friction. IPPs involving questions 18 and 20, questions 24 and 25, and the triplet involving questions 21, 22, and 23 in the Appendix address this issue. We made the following hypothesis.

H5: In IPPs or problem triplets where some questions are related to friction for which misconceptions are prevalent, performance will be worse on the friction question than on the question that does not contain friction. Giving such problems involving friction with isomorphic problems not involving friction will not improve performance on the problems with friction.

Hypothesis H5 is based upon the assumption that distracting features such as misconceptions can divert students' attention away from the central issue and may mask the similarity between questions in an IPP. From the perspective of knowledge in pieces, problem context with distracting features can trigger the activation of knowledge that a student thinks is relevant but which is not actually applicable in that context. The student may feel satisfied applying the activated knowledge resource and may not look further for analogies to paired problems or other aids. Thus, transfer of relevant knowledge in these cases may be challenging. One common misconception about the static frictional force is that it is always at its maximum value, because students have difficulty with the mathematical inequality that relates the magnitude of the static frictional force with the normal force.[21] Students overgeneralize the inequality regarding the static frictional force and think that, since we so often set static friction to its maximum value, it must be maximum all the time. Students also have difficulty in determining the direction of the frictional force. Another difficulty students have is in determining when static vs kinetic friction is relevant for a problem.

III. METHODOLOGY

Students in nine college calculus-based introductory physics courses participated in the study. The questions were asked after instruction in relevant concepts and after students had an opportunity to work on their homework on related topics. Some students were given both questions of an IPP (or all three questions in triplet questions 21–23), which were asked back to back, while others were given only one of the two questions. When students were given both questions of an IPP back to back, the questions were always given in the order given in the Appendix. For example, in the first five IPPs, the quantitative questions preceded the corresponding conceptual question. However, students were free to go back and forth between them if they wished and could change the answer to the previous question if they acquired additional insight for solving the previous question by answering a later question. Students who were given both questions of an IPP were not told explicitly that the questions given were isomorphic. Students were given 2.5 min on an average to answer each question.

Not all of the IPPs were used for all of the eight courses due to logistical difficulties. In particular, instructors of the courses often were concerned about the time it would take to administer all of the questions and they ultimately determined which questions from the IPPs they gave to their classes. In some cases, depending upon the consent of the course instructor (due to the time constraint for a class), students were asked to explain their reasoning in each case to obtain full credit. The questions contributed to students' grades in all courses. In some of the courses, we discussed the responses individually with several student volunteers. In one of these courses, students were given a survey after they had worked on the IPPs to evaluate the extent to which they realized that the questions were isomorphic and how often they took advantage of their response to one of the questions to solve its pair. Because the patterns of student responses are similar for different classes, we discuss the responses collectively here.

IV. RESULTS AND DISCUSSION

For the isomorphic problems given in the multiple-choice format in the Appendix, Table I summarizes the numbers of students who were given both questions or one of the questions of an IPP (or all three questions 21–23), and students' average performance. Table I also shows the results of a χ^2 test with both the χ^2 and p values for comparison between cases when both questions in an IPP (or all three questions 21–23) were given vs only one of the questions was given. Students can make appropriate connections between the questions in an IPP only if they have a certain level of expertise that helps them discern the connection between the isomorphic questions. Improved student performance when both questions of an IPP were given vs when only one of the questions was given was taken as one measure of transfer of relevant knowledge from one problem to another. Below we discuss the findings and analyze student performance in light of our hypotheses H1–H5.

A. Experiment 1: IPPs with quantitative-conceptual pairs

Table I shows that, contrary to our hypothesis H1, student performance on quantitative questions was not significantly different when both quantitative and conceptual questions were given back to back (with the quantitative question preceding the conceptual question) than when only the corresponding quantitative question was given. In some cases, the performance on the conceptual question was better than the performance on the quantitative question (problem pair 9 and 10), but students could not leverage their conceptual knowledge for gain on the corresponding quantitative problem. As noted earlier, the two questions in an IPP were always given in the same order, although students could go back and forth if they wanted. It is possible that students overall did not go back to the questions they had already answered, especially due to the time constraint, even if the question that followed provided a hint for it. Future research will evaluate the effect of switching the order of the quantitative and conceptual questions in an IPP when both questions are given.

On the other hand, in support of hypothesis H2, students who worked on both questions of the IPPs involving a conceptual and a quantitative problem performed better on the conceptual questions at least for three of the five IPPs than when they were given only the conceptual questions. Table I shows that, for three of the IPPs, if one question in an IPP was quantitative and the other conceptual (the first five problem pairs), students often performed better on the conceptual question when both questions were given rather than the corresponding conceptual question alone. The fact that many students took advantage of the quantitative problem to solve the conceptual question points to their evolving expertise. For example, many students who were given both questions 1 and 2 recognized that the final momenta of the ships are independent of their masses under the given conditions by solving the quantitative problem. Written responses and individual discussions suggest that some students who answered the conceptual question 2 correctly were not completely sure about whether the change in momentum in

TABLE I. Summary of results. For the isomorphic problems given in the multiple-choice format (see the Appendix), the first column lists the problem numbers, the second column gives the percentage of students who chose the correct answer when only one of the questions was given to them, and the third column gives the percentage of students who chose the correct answer when both questions (triplet for questions 21–23) were given. The numbers in parentheses in the second and third columns refer to the number of students who answered the question. The last two columns for all questions list the p value and χ^2 for comparison of student performance between cases when only one of the isormorphic questions was given vs when the question was given with its isomorphic pair. In experiment 3, we test only for significant differences for questions involving friction when they were given alone vs with an isomorphic question not involving friction. Questions 18 and 20 are isomorphic but they are not consecutive because, for the results presented in the table, they were given with the corresponding free-body diagrams (i.e., students who answered both questions 18 and 20, actually answered questions 17–20 in that order).

Problem no.	Only one	Both	p value	χ^2
1	59 (138)	54 (289)	0.40	0.8
2	31 (215)	58 (289)	0.00	36.0
3	34 (138)	38 (289)	0.45	0.6
4	23 (215)	30 (289)	0.07	3.3
5	81 (138)	76 (289)	0.26	1.4
6	55 (215)	80 (289)	0.00	36.3
7	52 (138)	56 (289)	0.47	0.6
8	44 (150)	51 (289)	0.19	1.9
9	49 (138)	49 (289)	1.00	0.0
10	53 (150)	71 (289)	0.00	13.4
11	53 (81)	65 (289)	0.05	3.9
12	23 (65)	52 (289)	0.00	17.7
13	50 (81)	52 (289)	0.9	0.0
14	33 (65)	58 (289)	0.00	14.2
15	65 (81)	74 (289)	0.16	2.3
16	64 (65)	86 (289)	0.0	16.1
17		90 (479)		
19		67 (479)		
18		72 (479)		
20	20 (81)	28 (479)	0.14	2.4
21		77 (150)		
22	24 (190)	30 (150)	0.27	1.4
23	18 (81)	16 (150)	0.71	0.2
24		71 (150)		
25	30 (81)	32 (150)	0.77	0.1

question 1 was given by option (a) or (e). However, since the answer in either case is independent of the mass of the object, these students chose the correct option (c) for question 2. The students who chose the incorrect option (a) for question 1 but the correct option (c) for question 2 often assumed that both ships in question 2 must have traveled the same distance, although that is not correct. In individual discussions, several students explicitly noted that the mass cancels out in question 5 so the answer to question 6 cannot depend on mass.[20] Similarly, discussions with individual students and students' written work suggest that solving the quantitative question 9 helped many students formulate their solution to question 10. Although some students were not able to solve the quantitative question, e.g., due to algebraic error or not realizing that, when the conservation of the horizontal component of momentum is considered, Batman's vertical velocity should not be included, it was easier for them to answer the conceptual question after thinking about the quantitative one. Most of them realized that the boat would slow down after Batman lands in it.

Previous research shows that conceptual questions can sometimes be more challenging for students than quantitative ones, if the quantitative problems can be solved algorithmically and students' preparation is sufficient to perform the mathematical manipulations.[14–16,22] If a student knows which equations are involved in solving a quantitative problem or how to find the equations, he or she can combine them in *any order* to solve for the desired variables even without a deep conceptual understanding of relevant concepts. On the contrary, while reasoning without equations, the student must usually proceed in a *particular order* in the reasoning chain to arrive at the correct conclusion.[14–16,22] Therefore, the probability of deviating from the correct reasoning chain increases rapidly as the chain becomes long. We note, however, that our hypothesis H2 is not about whether students will perform better on the quantitative or conceptual question of an IPP when the two questions are given separately (especially because the wording is not parallel for the quantitative and conceptual questions in an IPP). Rather, our hypothesis relates to whether students will recognize the similarity of the quantitative and conceptual questions in an IPP, and take advantage of their solution to one question to answer the corresponding paired question. Our finding suggests that students can leverage their quantitative solutions to correctly answer the corresponding conceptual questions.

The fact that students often performed better on conceptual questions when they were paired with quantitative questions brings up the following issue. If students could turn the conceptual questions into analogous quantitative problems themselves when only the conceptual questions were given, they may have solved the quantitative problem algorithmically if they were comfortable with the level of mathematics needed, and then reasoned qualitatively about their results to answer the original conceptual question. Almost without exception, students did not do this. One can hypothesize that students have not thought seriously about the fact that a conceptual question can be turned into a quantitative problem, or that a mathematical solution can provide a tool for reasoning conceptually. Without explicit guidance, students may not realize that this conversion route may be more productive than carrying out long conceptual reasoning without mathematical relations. However, we find that students avoided turning conceptual questions into quantitative ones, even when explicitly encouraged to do so. In one-on-one interview situations, when students were given only the conceptual questions, they also tried to guess the answer based upon their gut feeling. More research is required to understand why students are reluctant to transform a conceptual question into a quantitative problem even if the mathematical manipulations required after such a conversion and making correct conceptual inferences are not too difficult for them. One possible explanation for such reluctance is that such a transformation from a conceptual to a quantitative problem is cognitively demanding for a typical introductory physics student and may cause a mental overload.[23] According to Simon's theory of bounded rationality, an individual's rationality in a particular context is constrained by his/her expertise and experience and an individual will choose only one of the few options consistent with his/her expertise that does not cause a cognitive overload.[24]

Consistent with hypothesis H3, student performance on question 8 did not improve significantly when it was given together with question 7. Discussions with individual students who answered both questions 7 and 8 suggest that, after solving the quantitative problem, some students were unsure whether the horizontal component of motion was important or not. Students also needed to know that the time to reach the maximum height is half of the time of flight. However, students' performance on question 10 improved significantly when it was given with question 9 rather than alone, despite the fact that the contexts were somewhat different; in particular, in one case a person is falling vertically and in the other case rain is falling vertically. In this case, many students were able to transfer relevant knowledge from question 9 to 10.

We note that, for the IPP in questions 3 and 4, the quantitative problem itself was very challenging. Most interviewed students and those who wrote something on their answer sheet did not use conservation of energy correctly and forgot to take into account both the rotational and translational kinetic energies in their analysis. Thus, it is not surprising that there is no significant difference between cases when only one of the questions was given vs both questions were given.

Some of the quantitative questions asked for numerical answers while others asked for symbolic answers. Individual discussions suggest that students were often able to take advantage of their process for quantitative solution in either case to tackle the conceptual question more successfully (e.g., question 1 asked for a symbolic answer whereas question 5 asked for a numerical answer) than if they were given only the conceptual question. Future research will further investigate the differences in numerical vs symbolic answers by giving identical questions requiring numerical answers from some students and symbolic answers from others.

B. Experiment 2: IPPs that do not mix quantitative and conceptual questions

Table I shows that, in support of hypothesis H4, students' performance often improved when both questions of an IPP were given compared to when only one of the two questions was given. For example, Table I shows that the performance on both questions 11 and 12 improved when both questions were given. Individual interviews and written responses suggest that students sometimes got confused about the distinction between angular momentum and angular speed. However, students who answered question 11 correctly were often able to extend their argument to question 12 and they were able to identify that the angular momentum does not change and angular speed increases when the star collapses. For example, during an interview, a student who answered both questions 11 and 12 first narrowed down the possible correct choices for question 12 to (a) or (e), noting that the angular momentum does not change here, similarly to the skater problem. Then the student noted that, since the angular speed must increase as the star shrinks, the correct choice must be (e). This student took clues from the question about the ice skater and answered the white dwarf question correctly, explicitly making the comparison between the paired questions and quickly eliminating options (b), (c), and (d) in question 12, which sheds light on this student's expertise and his ability to transfer relevant knowledge from one context to another.

Similarly, in question 14, in which ball B is unconventional in that the density is not uniform and the inner core is denser than the outer shell, student performance improved when it was given with question 13. Individual discussions and the difference between the correct responses for the cases where students answered both questions on the IPP vs only question 14 suggest that students took advantage of the scaffolding provided by the paired problem. In particular, question 13 specifically helped students to consider whether the mass and the radius are the relevant variables, or the moment of inertia (the distribution of mass). Many students appeared to have the expertise to transfer this knowledge to question 14. On the other hand, question 14 does not provide any hints for question 13 and appears not to be helpful for answering question 13 when both questions were given as a pair.

In the IPP involving questions 15 and 16, students performed significantly better on question 16 when both questions of the IPP were given. It is somewhat surprising that students did better on the turntable problem than the bandit problem when both were given. Some students who were given both questions 15 and 16 claimed that the speed of the cart would be unchanged after the bandit lands in it. This may be due to the fact that students did not recognize the bandit question as a completely inelastic collision problem, for which the object slows down after the collision. This is not surprising considering the *subtle* fact that, in the bandit problem, there must be a frictional force between the bandit and the cart which will slow down the cart and bring the bandit to the same horizontal speed as the cart. In particular, after the bandit falls in the cart, the frictional force that the cart exerts on the bandit's shoes is equal in magnitude but opposite in direction to the force that the bandit's shoes exert on the cart from Newton's third law. These forces will slow the cart down and speed up the bandit so that they both have the same final horizontal velocity. In a typical inelastic collision problem given to students, objects do not move perpendicular (but parallel) to each other before the collision, as in the case of two cars colliding head on and sticking to each other. In the bandit problem, some interviewed students and those providing written explanation reasoned incorrectly that the vertical motion of the bandit cannot affect the horizontal motion of the cart. This misconception originates from the decoupling of the vertical and horizontal motions, e.g., for a projectile. Interviews suggest that for the turntable problem students used their intuition and experience about this problem to predict that the turntable will slow down when the putty falls on it and did not explicitly invoke conservation of angular momentum. Future research will involve giving these problems in the opposite order to evaluate the ordering effect.

C. Experiment 3: Influence of distracting features and misconceptions about friction

Consistent with hypothesis H5, students had difficulty in seeing the deep connection between the isomorphic problems not involving friction and those involving friction,[21] even though they were given back to back, and in transferring relevant knowledge to the problem involving friction. The fact that students did not take advantage of the easier problems not involving friction to answer the questions involving friction suggests that the misconceptions about friction were quite robust.[25] Many students believed that (i) the static friction is always at the maximum value, (ii) the kinetic friction is responsible for keeping the car at rest on an incline, or (iii) the presence or absence of friction must affect the work done by you even if you apply the same force over the same distance.

In the IPP involving questions 18 and 20, the weight of the car and the normal force exerted on the car by the inclined surface are the same in both problems. The only other force acting on the car (which is the tension force in one problem and the static frictional force in the other problem) must be the same. Consistent with the common misconception about the static frictional force that it must be at its maximum value $f_s^{max}=\mu_s N$ (where μ_s is the coefficient of static friction and N is the magnitude of the normal force), the most common incorrect response to question 20 was $\mu_s N = 11\ 700$ N ($\sim 40\%$). Giving both questions 18 and 20 did not improve student performance on question 20 compared to when it was given alone.

In order to help students discern the similarity between questions 18 and 20, we later introduced two additional questions 17 and 19 that asked students to identify the correct free-body diagrams for questions 18 and 20. We wanted to assess whether forcing students to think about the free-body diagram in each case would help them focus on the similarity of the problems. Although the performance improved somewhat when students were also asked about the free-body diagrams (Table I presents data for the case when students were given questions 17–20), it is not significantly different from when they were only asked question 20. The strong misconception prevented transfer of relevant knowledge from the problem not involving friction to the one involving friction, even when students were explicitly asked for the free-body diagrams in the two cases. The most common incorrect response to question 19 was choice (a), because these students believed that the frictional force should be pointing down the incline. Approximately, 40% believed that friction had a magnitude $\mu_s N$ and approximately 30% believed it was $\mu_k N$. In individual interviews, students often noted that the problem with friction must be solved differently from the problem involving tension because there is a special formula for the frictional force. Even when the interviewer drew students' attention to the fact that the other forces (normal force and weight) were the same in both questions and they are both equilibrium problems, only some of the students appeared concerned. Others used convoluted reasoning and asserted that friction has a special formula which should be used whereas tension does not have a formula, and therefore, a free-body diagram must be used.

In the earlier administration, questions 21 and 22 were given as an IPP but in the later administration whose results are given in Table I, either three questions 21–23 were given together as a triplet, or question 22 or question 23 involving friction was given alone. Table I shows that the performance on questions 22 or 23 did not improve significantly when they were given with question 21. The most common incorrect response in question 22 was $\mu_s N = 600$ N ($\sim$40%), with or without question 21.

Misconceptions about friction were so strong that students who were given both problems did not fully discern their similarity and take advantage of their responses to question 21 to analyze the horizontal forces in question 22. An alternative knowledge-in-pieces view can also be used to explain these findings in terms of students activating different resources to deal with somewhat different contexts which experts view as equivalent. Smith *et al.*[26] argue that student responses should be considered as "resources rather than flawed" and note that "Persistent misconceptions, if studied in an even-handed way, can be seen as novice's efforts to extend their existing useful conceptions to instructional contexts in which they turn out to be inadequate. Productive or unproductive is a more appropriate criterion than right or wrong, and final assessments of particular conceptions will depend on the contexts in which we evaluate their usefulness." From this point of view, the problem context triggers activation of knowledge that students think is relevant and they reach a conclusion that is incorrect but that nevertheless makes sense to them. Therefore, students do not feel the need to look further for analogies to the paired problem. In question 23, the coefficient of friction was not provided, and similar to the common misconception in question 20, almost 50% of the students believed that it is impossible to determine the resultant force on the crate without this information.

Similarly, although the frictional force in question 25 is irrelevant for the question asked, it was a distracting feature for a majority of students. Common incorrect reasoning for question 25 was based on the assumption that friction must play a role in determining the work done by the person, and the angle of the ramp was required to calculate this work, even though the distance by which the box was moved along the ramp was given. Interviews suggest that many students had difficulty distinguishing between the work done on the box by the person and the total work done. They asserted that the work done by the person cannot be the same in the two problems because friction must make it more difficult for the person to perform the work.

D. Survey about the effectiveness of the IPPs

In one of the courses in which students were given many of the IPPs, they were also given a multiple-choice survey with the following questions.

(1) Did you notice a pairing between the problems on the quiz? [Choices: (a) Yes, it was obvious, (b) yes, after a while, (c) a few questions seemed paired, (d) maybe one, (e) not at all.]

(2) Did the paired problems cause you to reconsider any answers? [Choices: (a) Yes, all of them, (b) several of them, (c) a few, (d) maybe one, (e) not at all.]

(3) Were the paired problems helpful? [Choices: (a) The first problem in a pair helped me with the second, (b) the second problem helped me with the first, (c) the problems helped me with each other, (d) they did not help me at all, (e) they were actually confusing.]

(4) Which type of problems were most helpful, if any? [Choices: (a) Algebraic answer, (b) numerical answer, (c) comparison (more/less), (d) simple question, with a reason, (e) scaling question, e.g., if you double the radius, … .]

(5) You had a chance to explain multiple-choice answers for partial credit. Did you find that this helped you formulate the answer better? [Choices: (a) Yes, very much so, (b) helped somewhat, (c) so so, no effect, (d) didn't help at all, and used up time, (e) it was actually confusing.]

The survey data are self-reported and should be interpreted with this fact in mind. In response to survey question 1, more than 50% of students chose (a), claiming to notice the pairing immediately. In response to survey question 2, more than 40% of students noted that the paired problems caused them to reconsider at least a few answers [choice (c)], and in response to question 3, about 40% of students noted that the paired problems helped them with each other [choice (c)]. These responses again suggest that students were actually trying to make sense of the problems to the best of their ability and taking advantage of the IPPs. In response to question 4, choices (a), (b), and (d) were all equally popular. These responses are consistent with the fact that algebraic or numerical problems gave students some confidence and provided them with tools to make sense of the paired conceptual problems. In response to question 5, more than 30% of students noted that explaining the multiple-choice answers helped somewhat in better formulating their responses while 20% noted that it helped very much.

V. SUMMARY

Student performance on the quantitative problems did not improve significantly when they were paired with the corresponding conceptual questions compared to when quantitative problems were given alone. However, students often performed significantly better on the conceptual questions when both quantitative and conceptual questions were given than when the conceptual question alone was given. Individual discussions and written responses suggest that many students were able to recognize the isomorphisms between problems, reason about their quantitative solution, and transfer that knowledge to the conceptual solution.

While students often took advantage of the quantitative problem to answer the corresponding conceptual question of an IPP, those who were only given the corresponding conceptual question did not automatically convert it into a quantitative problem as an aid for reasoning correctly. Examination of students' scratch work suggests that they seldom attempted such conversion by choosing appropriate variables. One-on-one discussions suggest that students often used gut feeling to reason about the conceptual questions. This tendency persisted even when the interviewer explicitly encouraged students to convert a conceptual question into a quantitative one. It is possible that converting the conceptual questions to quantitative ones was too cognitively demanding for introductory students and may have caused mental overload.

Even in IPPs that did not pair quantitative and conceptual questions but one question provided a hint for the other, students could sometimes exploit the reasoning for one of the questions to answer the other question when both questions in a pair were given. The fact that many students could discern the similarity between the problems and take advantage of their solution to one problem to answer the other one suggests that their expertise is evolving. In a survey given to students in one course, they noted that they often realized the similarity of the paired problems and sometimes tried to make a connection between the problem pairs to answer the question that was more difficult in each pair.

In this research, isomorphic problems were given back to back, and the more quantitative question always preceded the conceptual question in an IPP. The three IPPs in questions 11–16 were also always given in the same order. It is possible that the order in which questions were asked and the proximity of the paired questions in an IPP are major factors in whether students recognize their similarity and transfer relevant knowledge from one problem to another. In future research, one can explore the effect of spacing the isomorphic problems and changing the order, e.g., of the quantitative and conceptual questions, on students' ability to benefit from having both questions of an IPP. Changing the order in future research would also be insightful for the IPPs in which both questions were relatively conceptual (e.g., the three IPPs involving questions 11–16). This research may be helpful in understanding whether one question in such IPPs provides a better hint by explicitly mentioning relevant variables for answering the corresponding paired question.

From a misconceptions standpoint, strong alternative views about friction related to the context of some of the problems often prevented students from seeing the underlying similarities between the problems involving friction and an isomorphic problem that students found easier to solve. For example, many students believed that the static friction is always at the maximum value, or that the kinetic friction is responsible for keeping the car at rest on an incline, or that the presence of friction must affect the work done by you even if you apply the same force over the same distance. In such cases, students appeared to frame the isomorphic problems involving friction and not involving friction differently and traversed different problem spaces while solving them. From a knowledge-in-pieces perspective, the context when friction is present and prominent triggers activation of knowledge that students think is relevant (e.g., the formula for maximum static friction) and they run with it and reach a conclusion that makes sense to them but that is not correct. Thus, there is no need to look further to similarities to the paired problem or to anything else. This latter view is similar to Simon's theory of "satisficing" where individuals will select only a few of the large number of possible paths in the problem space that are consistent with their expertise in the area, which satisfies them and does not cause a cognitive overload. If an individual is not an expert in a domain, it is likely that these paths in the problem space are not the ones that will lead to success. When students satisfice, there is no need to discern the deep similarity of the paired problems and transfer their analysis for the problem not involving friction to the one involving friction, because within their world view the solution strategy that comes to their mind after understanding the problem makes sense.

VI. INSTRUCTIONAL IMPLICATIONS

Although student performance on quantitative problems did not improve significantly when such problems were paired with conceptual questions, students benefited from quantitative and conceptual problem pairs in answering conceptual questions. Presenting quantitative and conceptual isomorphic pairs helped students make conceptual inferences using quantitative tools. Such problem pairs as part of instruction may help students go beyond the plug-and-chug strategy for quantitative problem solving and may give them an opportunity to reflect upon their solution and develop reasoning and metacognitive skills. Solving these paired problems can force students to reflect upon the problem solving process and improve their metacognitive skills. Helping students develop metacognitive skills can also improve transfer of relevant knowledge from one problem to another.

In cases where the strong alternative views about friction prevented transfer of relevant knowledge, students may benefit from paired problems only after they are provided the opportunity to repair their knowledge structure so that there is less room for these alternative views. Instructional strategies embedded within a coherent curriculum that force students to realize that the static frictional force does not have to be at its maximum value or that the work done on a box by a person who is applying a fixed force over a fixed distance will not depend on friction may be helpful; students may be attempting to include all information in a problem statement to answer a question when only some information is relevant. Asking students to predict what should happen in concrete situations, helping them realize the discrepancy between their predictions and what actually happens, and then providing guidance and support to enhance their expertise is one such strategy.[27,28]

Isomorphic problems can be exploited as useful tools for teaching and learning. One strategy is to give isomorphic problems similar to those in this study and then discuss their isomorphism later with students to help them learn to discern the underlying similarities of the problems. Another strategy is to tell students that the problems are isomorphic and ask them to justify the isomorphism. Using these strategies with a variety of isomorphic problems with varying difficulty can help develop expertise and improve students' ability to transfer relevant knowledge from one context to another. Also, the simplest level of isomorphic problems where the same problem is asked with different parameters can be a useful tool for teaching students to do symbolic manipulation. Unlike the expert strategy, some students may trade the symbols for numbers in the equations at the beginning while solving problems because they may not recognize the advantage of symbolic manipulation[22] or may not have the mathematical skills to carry out algebraic manipulations with symbols. One hypothesis for future testing is that, if students are consistently given homework problems where they have to solve problems with different sets of numerical parameters and they are told that, if they obtain a correct symbolic answer, they will get full credit without inserting each set of parameters, they will be motivated to perform symbolic manipulation. Another important issue often is one of extracting meaning from symbolic manipulations because some students can manipulate symbolic equations and yet are not able to interpret the physical meaning once the answer is reached. Students can be rewarded for identifying isomorphic problems in their homework problems, e.g., if they explain why two problems are isomorphic they can solve only one of them in great detail and can simply lay out the plan for solving the other one. Such reward policies can motivate students to perform a conceptual analysis and planning before jumping into the implementation phase of problem solving and can help them extract meaning from mathematical manipulations.

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APPENDIX: MULTIPLE-CHOICE QUESTIONS

See separate auxiliary material for multiple-choice questions for the isomorphic problems.

[1] M. Gick and K. Holyoak, The cognitive basis of knowledge transfer, in *Transfer of Learning: Contemporary Research andAapplications*, edited by S. M. Cormier and J. D. Hagman (Academic, New York, 1987); M. Gick and K. Holyoak, Schema induction and analogical transfer, Cogn. Psychol. **15**, 1 (1983); K. Holyoak, The pragmatics of analogical transfer, in *The Psychology of Learning and Motivation*, edited by G. Bower (Academic, New York, 1985), Vol. 19; K. Holyoak and P. Thagard, *Mental Leaps: Analogy in Creative Thought*, (MIT Press, Cambridge, MA, 1995).

[2] R. Dufresne, J. Mestre, T. Thaden-Koch, W. Gerace, and W. Leonard, Knowledge representation and coordination in the transfer process, in *Transfer of Learning from a Modern Multidisciplinary Perspective*, edited by J. P. Mestre (Information Age Publishing, Greenwich, CT, 2005), pp. 155–215.

[3] J. Lobato, How design experiments can inform a rethinking of transfer and vice versa, Educ. Res. **32**, 17 (2003); J. Lobato, Transfer Strand: Alternative perspectives on the transfer of learning: History, issues and challenges for future research, J. Learn. Sci. **15**, 431 (2006).

[4] D. J. Ozimek, P. V. Engelhardt, A. G. Bennett, and N. S. Rebello, Retention and transfer from trigonometry to physics, in *Proceedings of the 2004 Physics Education Research Conference*, edited by J. Marx, P. Heron, and S. Franklin, AIP Conf. Proc. No. 790 (AIP, Melville, NY, 2005), pp. 173–176.

[5] M. Bassok and K. Holyoak, Interdomain transfer between isomorphic topics in algebra and physics, J. Exp. Psychol. Learn. Mem. Cogn. **15**, 153 (1989).

[6] J. D. Bransford and D. Schwartz, Rethinking transfer: A simple proposal with multiple implications, Rev. Res. Educ. **24**, 61 (1999).

[7] *Transfer on trial: Intelligence, cognition and instruction*, edited by D. K. Detterman and R. J. Sternberg (Ablex, Norwood, NJ, 1993); R. J. Sternberg, Component processes in analogical reasoning, Psychol. Rev. **84**, 353 (1977).

[8] L. Novick, Analogical transfer, problem similarity and expertise, J. Exp. Psychol. Learn. Mem. Cogn. **14**, 510 (1988).

[9] A. Brown, Analogical learning and transfer: What develops? in *Similarity and Analogical Reasoning*, edited by S. Vosniadu and A. Ortony (Cambridge University Press, New York, 1989), pp. 369–412; E. M. Kurz and R. D, Tweney, The practice of math and science: From calculations to the clothesline problems, in *Rational Models of Cognition*, edited by M. Oalesfard and N. Chater (London: Oxford University Press, 1998), pp. 415–438; P. Adey and M. Shayer, An exploration of long-term far-transfer effects following an extended intervention program in the high school science curricula, Cogn. Instruct. **11**, 1 (1993); D. Genter and C. Toupin, Systematicity and surface similarity in the development of analogy, Cogn. Sci. **10**, 277 (1986).

[10] G. Hendrickson and W. Schroeder, Transfer training in learning to hit a submerged target, J. Educ. Psychol. **32**, 205 (1941).

[11] D. Klahr and S. M. Carver, Cognitive objectives in a LOGO debugging curriculum: Instruction, learning and transfer, Cogn. Psychol. **20**, 362 (1988).

[12] M. T. H. Chi, P. J. Feltovich, and R. Glaser, Categorization and representation of physics problems by experts and novices, Cogn. Sci. **5**, 121 (1981).

[13] P. T. Hardiman, R. Dufresne, and J. P. Mestre, The relation between problem categorization and problem solving among novices and experts, Mem. Cognit. **17**, 627 (1989).

[14] L. C. McDermott, Oersted Medal Lecture 2001: Physics education research–The key to student learning, Am. J. Phys. **69**, 1127 (2001); B. S. Ambrose, P. S. Shaffer, and L. C. McDermott, An investigation of student understanding of single-slit diffraction and double-slit interference, *ibid.* **67**, 146 (1999).

[15] E. Kim and S.-J. Pak, Students do not overcome conceptual difficulties after solving 1000 traditional problems, Am. J. Phys. **70**, 759 (2002).

[16] E. Mazur, *Peer Intruction: A User's Manual*, Prentice-Hall Series in Educational Innovation (Prentice-Hall, Upper Saddle River, NJ, 1997).

[17] C. Singh, Interactive video tutorials for enhancing problem solving, reasoning, and meta-cognitive skills of introductory physics students, in *Proceedings of Physics Education Research Conference, Madison, WI*, edited by S. Franklin, K. Cummings, and J. Marx AIP Conf. Proc. No. 720, (AIP, Melville, NY, 2004), pp. 177–180.

[18] F. Reif, Millikan Lecture 1994: Understanding and teaching important scientific thought processes, Am. J. Phys. **63**, 17 (1995).

[19] G. Hatano and K. Inagaki, Two courses in expertise, in *Child Development and Education in Japan*, edited by H. Stevenson, J. Azuma, and K. Hakuta (W. H. Freeman and Co., New York, 1986), pp. 262–272.

[20] C. Singh and D. Rosengrant, Multiple-choice test of energy and momentum concepts, Am. J. Phys. **71**, 607 (2003).

[21] A. A. diSessa and B. Sherin, What changes in conceptual change?, Int. J. Sci. Educ. **20**, 1155 (1998).

[22] F. Reif, Teaching problem solving—A scientific approach, Phys. Teach. **19**, 310 (1981); J. H. Larkin and F. Reif, Understanding and teaching problem solving in physics, Eur. J. Sci. Educ. **1**, 191 (1979).

[23] J. Sweller, Cognitive load during problem solving: Effects on learning, Cogn. Sci. **12**, 257 (1988); J. Sweller, R. Mawer, and M. Ward, Development of expertise in mathematical problem solving, J. Exp. Psychol. Gen. **112**, 639 (1983).

[24] H. A. Simon, *Reason in Human Affairs* (Basil Blackwell, Oxford, 1983); A. Tversky and D. Kahneman, Judgment under uncertainty: Heuristics and biases, Science **185**, 1124 (1974); J. Reason, *Human Error* (Cambridge University Press, Cambridge, U.K., 1990).

[25] C. Singh, Effect of misconceptions on transfer in problem solving, in *Proceedings of the 2007 Physics Education Research Conference*, edited by L. Hsu, C. Henderson, L. McCullough, AIP Conf. Proc. No. 951 (AIP, Melville, NY, 2007), pp. 196–199.

[26] J. Smith, A. diSessa, and J. Rochelle, Misconceptions reconceived: A constructivist analysis of knowledge in transition, J. Learn. Sci. **3**, 115 (1994).

[27] H. Ginsberg and S. Opper, *Piaget's Theory of Intellectual Development* (Prentice Hall, Englewood Cliffs, NJ, 1969); R. Gorman, *Discovering Piaget: A Guide for Teachers* (Merrill, Columbus, 1972).

[28] L. Vygotsky, *Mind in Society: The Development of Higher Psychological Processes* (Harvard University Press, Cambridge, MA, 1978); J. Wertsch, *Mind in Action* (Oxford University Press, New York, 1998).