Circuit-model representation of external- Q calculation

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A new representation of the external- Q (Q_{ext}) calculations for cavities coupled to waveguides is derived based on a circuit model. The Q_{ext} value is obtained by solving analytical equations of the input impedance of a cavity provided as a function of the frequency; large numerical simulations are not necessary. The derivation starts with a fundamental expression of the ratio of the stored energies of two loss-free circuits forming a parallel resonant circuit. By applying this expression to the input impedance of a cavity seen from the detuned-short-circuit plane, as well as to those of open- and short-end waveguides, a precise estimation of Q_{ext} is given. It is then explained that the result is almost independent of the choice of the cavity boundary. This method is an alternative to the electromagnetic representation of the Q_{ext} calculation [P. Balleyguier, in *Proceedings of the XIX International Linac Conference, Chicago, Illinois, USA*, edited by C. E. Eyberger, R. C. Pardo, and M. M. White (ANL-98/28, 1998), pp. 133–135.]. It is also shown why this estimation provides a good approximation when Q_{ext} is high. Formulas for the TEM, TE, and TM modes of waveguides are presented along with a simple example of a loop-coupled cavity.

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I. INTRODUCTION

The external- Q value (Q_{ext}) of a resonant mode of a cavity represents the strength of the electromagnetic coupling between this resonant mode and the transmission mode of a waveguide coupled to the cavity. Q_{ext} is high when the coupling is weak, which means that the stored energy in the cavity is not readily radiated into the waveguide.

The estimation of Q_{ext} is important for designing coupling devices for the cavities. It is also useful for developing dumping structures of higher modes of the cavities in high-beam-current accelerators. However, in the development process, construction of a scaled model is sometimes difficult and takes much time. Therefore, the numerical calculation of Q_{ext} is effective.

A traditional method for Q_{ext} calculations is based on Slater's tuning curve [1], where a number of numerical simulations are required to fit the calculated resonant frequencies to the theoretical curve of the coupled-cavity frequency. This method is only applicable to low Q_{ext} values because the tuning curves vary too markedly near the resonant point to be fitted precisely by the numerical data.

In 1998, a new method was proposed based on three-dimensional simulations [2]. This method is illustrated briefly as follows. First, using a computer code such as MAFIA [3], a resonant electromagnetic field is simulated for a resonant cavity coupled to a coaxial waveguide, as shown in Fig. 1(a), under the open-end boundary condition of the guide. A factor Q_1 is calculated with the simulated electric field $\mathbf{E}_1$ in the coupled resonator and is defined as

$$Q_1 \equiv \frac{\omega_1 \iiint_{\text{cavity}} |\mathbf{E}_1|^2 dv}{c \iint_{\text{open}} |\mathbf{E}_1|^2 ds}, \quad (1)$$

where c and ω_1 are the speed of light and resonant frequency, respectively. The volume integral in the numerator is performed in the cavity region, and the surface integral in the denominator is performed on the open end of the waveguide. Second, another simulation is performed using the same mesh model but under the short-end condition of the waveguide, as shown in Fig. 1(b), which leads to another factor Q_2 defined using the calculated magnetic field $\mathbf{H}_2$ as

$$Q_2 \equiv \frac{\omega_2 \iiint_{\text{cavity}} |\mathbf{H}_2|^2 dv}{c \iint_{\text{short}} |\mathbf{H}_2|^2 ds}, \quad (2)$$

where ω_2 is the resonant frequency. The surface integral in the denominator is performed on the short end of the waveguide. Usually, both ω_1 and ω_2 are very close to the resonant frequency of a cavity mode ω_a . Finally, in the proposed method, Q_{ext} is approximately equal to the sum of the two factors Q_1 and Q_2 :

$$Q_1 + Q_2 \simeq Q_{\text{ext}}. \quad (3)$$

A distinguishing advantage of this method is that only two simulations are required with the same mesh model for the Q_{ext} calculation. Another advantage is that Eq. (3) gives almost the same value independently of the length of the waveguide. This method is applicable to systems with high Q_{ext} values. The basis of this method is explained based on the physical properties of the electromagnetic field in the coupled cavities [2].

In this paper, however, an alternative approach is presented for the Q_{ext} calculation, based on a circuit model. Starting with a fundamental expression for the ratio of the

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FIG. 1. Straightforward method proposed by Balleyguier [2]. (a) and (b) illustrate the calculations of the factors Q_1 [Eq. (1)] and Q_2 [Eq. (2)], respectively. The Q_{ext} is approximated by the sum of Q_1 and Q_2 . The hatched area corresponds to the cavity region where the volume integrals of the denominators in Eqs. (1) and (2) are performed. The role of the detuned-short-circuit (DSC) plane is discussed in Sec. II D.

stored energies of two loss-free circuits at a resonance, a new representation is given for Q_{ext} using the susceptance of the cavity provided as a function of frequency. It will be explicitly shown why this estimation works well for cavity systems with high Q_{ext} values.

II. ANALYTICAL APPROACH TO EXTERNAL- Q CALCULATION

A. Basic concept

Consider a resonant circuit consisting of two loss-free circuits α and β , as shown in Fig. 2. A resonant frequency is a solution of the following equation of the admittances $Y_\alpha(\omega)$ and $Y_\beta(\omega)$:

$$Y_\alpha(\omega) + Y_\beta(\omega) = 0. \quad (4)$$

The same condition is expressed with the susceptances $B_\alpha(\omega)$ and $B_\beta(\omega)$:

$$B_\alpha(\omega) + B_\beta(\omega) = 0. \quad (5)$$

Let one of the solutions of Eq. (5) be ω_r .

At the resonant frequency $\omega = \omega_r$, the two circuits α and β interchange electromagnetic energies. The ratio of the time-averaged energies stored in the two circuits can be expressed as [4]

FIG. 2. Parallel resonant circuit consisting of two loss-free circuits α and β . The admittances and the susceptances are also indicated.

$$\frac{U_{\alpha,r}}{U_{\beta,r}} = \frac{dB_\alpha/d\omega(\omega_r)}{dB_\beta/d\omega(\omega_r)}, \quad (6)$$

where $U_{\alpha,r}$ and $U_{\beta,r}$ are the time-averaged energies stored in circuits α and β , respectively.

B. Circuit-model representation

The basic expression (6) is applied in the Q_{ext} calculation below. First, let circuit α be a resonant cavity and circuit β a coaxial waveguide of characteristic impedance Z_0 , as shown in Fig. 3. Let us assume that the boundary between the cavity and the waveguide coincides with a detuned-short-circuit (DSC) plane of a cavity mode of $\omega = \omega_a$. Under these conditions, the input impedance of the cavity seen from the boundary, $Z_{\text{DSC}}(\omega)$, can be expressed near ω_a as [1,5]

$$Z_{\text{DSC}}(\omega) = \frac{Z_0/Q_{\text{ext}}}{j(\omega/\omega_a - \omega_a/\omega)}. \quad (7)$$

Here, Q_{ext} is the external- Q value of the resonant mode of ω_a . Therefore, the susceptance $B_{\text{DSC}}(\omega)$ is written as

$$B_{\text{DSC}}(\omega) = \frac{Q_{\text{ext}}}{Z_0} \left(\frac{\omega}{\omega_a} - \frac{\omega_a}{\omega} \right). \quad (8)$$

Second, consider a coaxial waveguide of length l under an open-end condition connected to the cavity boundary. The impedance of the guide seen from the boundary, $Z_{\text{op}}(\omega)$, is given by

$$Z_{\text{op}}(\omega) = -jZ_0 \cot\left(\frac{l\omega}{c}\right), \quad (9)$$

and the susceptance $B_{\text{op}}(\omega)$ by

$$B_{\text{op}}(\omega) = \frac{1}{Z_0} \tan\left(\frac{l\omega}{c}\right). \quad (10)$$

By substituting the above expressions into Eq. (5), the resonant frequency ω_1 of this coupled system is given by

FIG. 3. Circuit-model representation of Q_{ext} calculation. The cavity boundary is chosen to be a DSC plane. The input impedance of the cavity seen from the DSC plane is given in Eq. (7). It is proved analytically that the sum of the factors q_1 and q_2 , defined by Eqs. (13) and (19), respectively, agrees with Q_{ext} accurately.

the solution of the analytical equation

$$Q_{\text{ext}} \left(\frac{\omega_1}{\omega_a} - \frac{\omega_a}{\omega_1} \right) = -\tan \left(\frac{l\omega_1}{c} \right). \quad (11)$$

Also, by using Eq. (6), the ratio of the stored energies at $\omega = \omega_1$ is expressed as

$$\begin{aligned} \frac{U_{\text{DSC},1}}{U_{\text{op},1}} &= \frac{dB_{\text{DSC}}/d\omega(\omega_1)}{dB_{\text{op}}/d\omega(\omega_1)} \\ &= \frac{c}{l} Q_{\text{ext}} \left(\frac{1}{\omega_a} + \frac{\omega_a}{\omega_1^2} \right) \cos^2 \left(\frac{l\omega_1}{c} \right), \end{aligned} \quad (12)$$

where $U_{\text{DSC},1}$ and $U_{\text{op},1}$ are the stored energy in the cavity and that in the waveguide, respectively. Let us define a factor q_1 by

$$q_1 \equiv \frac{\omega_1}{2c} \cdot \frac{U_{\text{DSC},1}}{U_{\text{op},1}/l}. \quad (13)$$

The right-hand side of Eq. (13) can be written using Eq. (12) as

$$q_1 = \frac{Q_{\text{ext}}}{2} \left(\frac{\omega_1}{\omega_a} + \frac{\omega_a}{\omega_1} \right) \cos^2 \left(\frac{l\omega_1}{c} \right). \quad (14)$$

Third, the procedure above is also applied to the cavity coupled with a waveguide of length l under a short-end condition. The impedance $Z_{\text{sh}}(\omega)$ and susceptance $B_{\text{sh}}(\omega)$ of the guide seen from the cavity boundary are expressed, respectively, as

$$Z_{\text{sh}}(\omega) = jZ_0 \tan \left(\frac{l\omega}{c} \right), \quad (15)$$

$$B_{\text{sh}}(\omega) = -\frac{1}{Z_0} \cot \left(\frac{l\omega}{c} \right). \quad (16)$$

The resonant frequency ω_2 of the coupled system is given by solving

$$Q_{\text{ext}} \left(\frac{\omega_2}{\omega_a} - \frac{\omega_a}{\omega_2} \right) = \cot \left(\frac{l\omega_2}{c} \right). \quad (17)$$

The ratio of the stored energies at $\omega = \omega_2$ is expressed using Eq. (6) as

$$\frac{U_{\text{DSC},2}}{U_{\text{sh},2}} = \frac{dB_{\text{DSC}}/d\omega(\omega_2)}{dB_{\text{sh}}/d\omega(\omega_2)} = \frac{c}{l} Q_{\text{ext}} \left(\frac{1}{\omega_a} + \frac{\omega_a}{\omega_2^2} \right) \sin^2 \left(\frac{l\omega_2}{c} \right), \quad (18)$$

where $U_{\text{DSC},2}$ and $U_{\text{sh},2}$ are the stored energy in the cavity and that in the waveguide, respectively. Note that Eqs. (8) and (16) have been used to calculate the right-hand side of Eq. (18). Let us define a factor q_2 by

$$q_2 \equiv \frac{\omega_2}{2c} \cdot \frac{U_{\text{DSC},2}}{U_{\text{sh},2}/l}. \quad (19)$$

Substitution of Eq. (18) into the right-hand side of this definition yields an analytical expression for q_2 :

$$q_2 = \frac{Q_{\text{ext}}}{2} \left(\frac{\omega_2}{\omega_a} + \frac{\omega_a}{\omega_2} \right) \sin^2 \left(\frac{l\omega_2}{c} \right). \quad (20)$$

Finally, the following result is obtained from Eqs. (14) and (20):

$$\begin{aligned} q_1 + q_2 &= \frac{Q_{\text{ext}}}{2} \left[\left(\frac{\omega_1}{\omega_a} + \frac{\omega_a}{\omega_1} \right) \cos^2 \left(\frac{l\omega_1}{c} \right) \right. \\ &\quad \left. + \left(\frac{\omega_2}{\omega_a} + \frac{\omega_a}{\omega_2} \right) \sin^2 \left(\frac{l\omega_2}{c} \right) \right]. \end{aligned} \quad (21)$$

Let us assume here that Q_{ext} is high. Since the coupling between the cavity and the waveguide is weak, both ω_1 and ω_2 are very close to ω_a for almost all values of l :

$$\omega_1 \simeq \omega_a \simeq \omega_2. \quad (22)$$

Therefore, from Eq. (21), the sum of q_1 and q_2 is approximately equal to Q_{ext} , which is independent of l :

$$q_1 + q_2 \simeq Q_{\text{ext}}. \quad (23)$$

C. Relation to electromagnetic representation

Let us consider the right-hand side of Eq. (13) from the viewpoint of the electromagnetic field. The volume integral in the numerator of Eq. (13) represents the stored energy in the cavity. Taking the cavity boundary into account, the volume integral is written as

$$\begin{aligned} U_{\text{DSC},1} &= \frac{\epsilon_0}{4} \iiint_{\text{DSC}} |\mathbf{E}_1|^2 dv + \frac{\mu_0}{4} \iiint_{\text{DSC}} |\mathbf{H}_1|^2 dv \\ &= \frac{\epsilon_0}{2} \iiint_{\text{DSC}} |\mathbf{E}_1|^2 dv, \end{aligned} \quad (24)$$

where the suffix DSC represents the cavity region shown in Fig. 3. The right-hand side of this equation has been obtained using a property of the resonant electromagnetic field: the time-averaged electric energy and the magnetic energy are equal to each other in the cavity [5]. The denominator of Eq. (13) is the stored energy per unit length in the waveguide, where the total (electric + magnetic) energy distributes uniformly along the length at resonance. The energy per unit length expressed only by the electric energy at the open end is

$$\frac{U_{\text{op},1}}{l} = \frac{\epsilon_0}{4} \iint_{\text{open}} |\mathbf{E}_1|^2 ds. \quad (25)$$

Therefore, the right-hand side of Eq. (13) can be written by substituting Eqs. (24) and (25) as

$$q_1 = \frac{\omega_1 \iiint_{\text{DSC}} |\mathbf{E}_1|^2 dv}{c \iint_{\text{open}} |\mathbf{E}_1|^2 ds}. \quad (26)$$

This is nothing but Eq. (1) under the condition that the cavity boundary is chosen to be a DSC plane.

In the same way, the right-hand side of Eq. (19) is related to the electromagnetic representation as follows. The stored energy per unit length in the waveguide is expressed by the magnetic field at the short end,

$$\frac{U_{\text{sh},2}}{l} = \frac{\mu_0}{4} \iint_{\text{short}} |\mathbf{H}_2|^2 ds, \quad (27)$$

and the stored energy in the cavity is also expressed by the magnetic field,

$$U_{\text{DSC},2} = \frac{\mu_0}{2} \iiint_{\text{DSC}} |\mathbf{H}_2|^2 dv. \quad (28)$$

Therefore,

$$q_2 = \frac{\omega_2 \iiint_{\text{DSC}} |\mathbf{H}_2|^2 dv}{c \iint_{\text{short}} |\mathbf{H}_2|^2 ds}. \quad (29)$$

This is the same as Eq. (2) under the condition that the cavity boundary is chosen to be a DSC plane.

D. Effect of cavity boundary

The circuit-model explanation above assumes that the cavity boundary coincides with a DSC plane of the resonant mode of ω_a . It is, however, impossible to choose the

DSC plane before the calculation. Nevertheless, it will be shown below that the estimation of Q_{ext} is almost independent of the position of the cavity boundary when Q_{ext} is high.

Let us choose the cavity boundary to be a distance $|d|$ from the DSC plane. From the definition of the DSC plane, $|d|$ can be chosen as $|d| < \lambda_a/2$, where $\lambda_a = c/(2\pi\omega_a)$. For simplicity, let the cavity boundary be closer to the cavity than the DSC plane (Fig. 1). The volume integral in Eq. (1) is estimated under this condition as

$$\begin{aligned} \iiint_{\text{cavity}} |\mathbf{E}_1|^2 dv &= \iiint_{\text{DSC}} |\mathbf{E}_1|^2 dv - \iiint_{\text{line:d}} |\mathbf{E}_1|^2 dv \\ &> \iiint_{\text{DSC}} |\mathbf{E}_1|^2 dv - |d| \iint_{\text{open}} |\mathbf{E}_1|^2 ds. \end{aligned} \quad (30)$$

Therefore, Q_1 is estimated to be

$$\begin{aligned} Q_1 &> \frac{\omega_1}{c} \frac{\iiint_{\text{DSC}} |\mathbf{E}_1|^2 dv - |d| \iint_{\text{open}} |\mathbf{E}_1|^2 ds}{\iint_{\text{open}} |\mathbf{E}_1|^2 ds} \\ &= q_1 - \frac{\omega_1 |d|}{c} > q_1 - \frac{\omega_1}{\omega_a} \pi. \end{aligned} \quad (31)$$

In the same way, the following estimation holds for Q_2 :

$$Q_2 > q_2 - \frac{\omega_2}{\omega_a} \pi. \quad (32)$$

A combination of the two estimations above yields

$$Q_1 + Q_2 > q_1 + q_2 - \frac{\omega_1 + \omega_2}{\omega_a} \pi \simeq Q_{\text{ext}} - 2\pi. \quad (33)$$

Similar estimations are obtained for the case where the cavity boundary is outside of the DSC plane. As a result, the upper limit of the accuracy of Eq. (3) is given by

$$\left| \frac{Q_1 + Q_2}{Q_{\text{ext}}} - 1 \right| \leq \frac{2\pi}{Q_{\text{ext}}}. \quad (34)$$

Therefore, Q_{ext} can be estimated at an arbitrary position of the waveguide. This is the reason that the straightforward method [2] works reasonably well for high Q_{ext} .

III. CIRCUIT-MODEL FORMULAS FOR EXTERNAL- Q CALCULATION

A. TEM mode

As shown in Sec. IID, the cavity boundary can be chosen arbitrarily. This means that, when an input impedance (or admittance) of a cavity seen from a certain plane of the waveguide is given as a function of the frequency ω , Q_{ext} can be calculated analytically based on Eqs. (5) and (6), as follows.

Let us represent the input admittance of the cavity as $Y_{\text{in}}(\omega) = jB_{\text{in}}(\omega)$ and take a coaxial waveguide of length l with a characteristic impedance of Z_0 . First, one solves the following equations numerically to get the resonant fre-

quency ω_1 for the coupled system of the cavity and the open-end waveguide [Fig. 4(a)]:

$$B_{\text{in}}(\omega_1) = -\frac{1}{Z_0} \tan\left(\frac{l\omega_1}{c}\right) = -B_{\text{op}}(\omega_1). \quad (35)$$

Then, the factor Q_1 is defined by

$$Q_1 \equiv \frac{\omega_1}{2c} \cdot \frac{U_{\text{cavity},1}}{U_{\text{op},1}/l}. \quad (36)$$

The right-hand side of this equation can be calculated using Eq. (6) as

$$Q_1 = \frac{l\omega_1}{2c} \cdot \frac{dB_{\text{in}}/d\omega(\omega_1)}{dB_{\text{op}}/d\omega(\omega_1)} = \frac{\omega_1 Z_0}{2} \cdot \frac{dB_{\text{in}}}{d\omega}(\omega_1) \cos^2\left(\frac{l\omega_1}{c}\right). \quad (37)$$

Second, one solves the following equation to get the resonant frequency ω_2 for the coupled system of the cavity and the short-end waveguide [Fig. 4(b)]:

$$B_{\text{in}}(\omega_2) = \frac{1}{Z_0} \cot\left(\frac{l\omega_2}{c}\right) = -B_{\text{sh}}(\omega_2). \quad (38)$$

Then, the factor Q_2 is defined as

$$Q_2 \equiv \frac{\omega_2}{2c} \cdot \frac{U_{\text{cavity},2}}{U_{\text{sh},2}/l}. \quad (39)$$

The right-hand side of this equation can be calculated using

FIG. 4. Representation of Q_{ext} calculation based on circuit model. (a) and (b) illustrate the calculations of the factors Q_1 [Eq. (37)] and Q_2 [Eq. (40)], respectively. Q_{ext} is approximated by the sum of Q_1 and Q_2 , as shown in Eq. (41). The cavity boundary can be chosen to be any plane in the waveguide.

Eq. (6) as

$$Q_2 = \frac{l\omega_2}{2c} \cdot \frac{dB_{\text{in}}/d\omega(\omega_2)}{dB_{\text{sh}}/d\omega(\omega_2)} = \frac{\omega_2 Z_0}{2} \cdot \frac{dB_{\text{in}}}{d\omega}(\omega_2) \sin^2\left(\frac{l\omega_2}{c}\right). \quad (40)$$

Finally, Q_{ext} is approximated reasonably precisely by

$$\begin{aligned} Q_{\text{ext}} &\approx Q_1 + Q_2 \\ &= \frac{Z_0}{2} \left[\omega_1 \frac{dB_{\text{in}}}{d\omega}(\omega_1) \cos^2\left(\frac{l\omega_1}{c}\right) \right. \\ &\quad \left. + \omega_2 \frac{dB_{\text{in}}}{d\omega}(\omega_2) \sin^2\left(\frac{l\omega_2}{c}\right) \right]. \end{aligned} \quad (41)$$

The equations in this section can be solved and calculated easily with commercial software such as MATHEMATICA [6].

Note that Eqs. (37) and (40) correspond to Eqs. (1) and (2), respectively.

B. TE and TM modes

Q_{ext} in TE and TM modes can be estimated in almost the same way as that in the TEM mode. In the TE and TM modes, the input susceptances of the short-end guide and the open-end guide are represented, respectively, as

$$\tilde{B}_{\text{op}}(\omega) = \frac{1}{Z_0} \tan\left[\xi(\omega) \frac{l\omega}{c}\right], \quad (42)$$

$$\tilde{B}_{\text{sh}}(\omega) = -\frac{1}{Z_0} \cot\left[\xi(\omega) \frac{l\omega}{c}\right]. \quad (43)$$

The factor $\xi(\omega)$ is expressed using the cutoff angular frequency ω_c as

$$\xi(\omega) = \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}. \quad (44)$$

First, as in the TEM-mode calculation, one solves the following equation to get the resonant frequency ω_1 :

$$B_{\text{in}}(\omega_1) = -\tilde{B}_{\text{op}}(\omega_1). \quad (45)$$

The factor Q_1 should be slightly modified from that in the TEM-mode calculation to

$$Q_1 \equiv \frac{\omega_1}{2c} \cdot \frac{U_{\text{cavity},1}}{U_{\text{op},1}/l} \cdot \frac{1}{\xi(\omega_1)}. \quad (46)$$

The right-hand side of this equation is transformed using Eqs. (6) and (42) into

$$\begin{aligned} Q_1 &= \frac{l\omega_1}{2c} \cdot \frac{dB_{\text{in}}/d\omega(\omega_1)}{d\tilde{B}_{\text{op}}/d\omega(\omega_1)} \cdot \frac{1}{\xi(\omega_1)} \\ &= \frac{\omega_1 Z_0}{2} \cdot \frac{dB_{\text{in}}}{d\omega}(\omega_1) \cos^2\left[\xi(\omega_1) \frac{l\omega_1}{c}\right]. \end{aligned} \quad (47)$$

Second, one solves the following equation to get the resonant frequency ω_2 :

$$B_{\text{in}}(\omega_2) = -\tilde{B}_{\text{sh}}(\omega_2). \quad (48)$$

Then, Q_2 is defined in a similar manner as Q_1 :

$$Q_2 \equiv \frac{\omega_2}{2c} \cdot \frac{U_{\text{cavity},2}}{U_{\text{sh},2}/l} \cdot \frac{1}{\xi(\omega_2)}. \quad (49)$$

The right-hand side of this equation is transformed using Eqs. (6) and (43) into

$$\begin{aligned} Q_2 &= \frac{l\omega_2}{2c} \cdot \frac{dB_{\text{in}}/d\omega(\omega_2)}{d\tilde{B}_{\text{sh}}/d\omega(\omega_2)} \cdot \frac{1}{\xi(\omega_2)} \\ &= \frac{\omega_2 Z_0}{2} \cdot \frac{dB_{\text{in}}}{d\omega}(\omega_2) \sin^2 \left[\xi(\omega_2) \frac{l\omega_2}{c} \right]. \end{aligned} \quad (50)$$

Finally, Q_{ext} is estimated as

$$\begin{aligned} Q_{\text{ext}} &\simeq Q_1 + Q_2 \\ &= \frac{Z_0}{2} \left\{ \omega_1 \frac{dB_{\text{in}}}{d\omega}(\omega_1) \cos^2 \left[\xi(\omega_1) \frac{l\omega_1}{c} \right] \right. \\ &\quad \left. + \omega_2 \frac{dB_{\text{in}}}{d\omega}(\omega_2) \sin^2 \left[\xi(\omega_2) \frac{l\omega_2}{c} \right] \right\}. \end{aligned} \quad (51)$$

The right-hand side of Eq. (46) can be expressed by the electromagnetic fields as follows. The stored energy per unit length in the waveguide is given in the following form for the TE and TM modes:

$$\frac{U_{\text{op},1}}{l} = \begin{cases} \frac{\epsilon_0}{4} \iint_{\text{open}} |\mathbf{E}_1|^2 ds & \text{for TE} \\ \xi(\omega_1)^2 \frac{\epsilon_0}{4} \iint_{\text{open}} |\mathbf{E}_1|^2 ds & \text{for TM.} \end{cases} \quad (52)$$

Therefore Q_1 is expressed as

$$Q_1 = \begin{cases} \xi(\omega_1) \cdot \frac{\omega_1 \iiint_{\text{cavity}} |\mathbf{E}_1|^2 dv}{c \iint_{\text{open}} |\mathbf{E}_1|^2 ds} & \text{for TE} \\ \frac{1}{\xi(\omega_1)} \cdot \frac{\omega_1 \iiint_{\text{cavity}} |\mathbf{E}_1|^2 dv}{c \iint_{\text{open}} |\mathbf{E}_1|^2 ds} & \text{for TM.} \end{cases} \quad (53)$$

In the same way, the right-hand side of Eq. (49) is written as

$$\frac{U_{\text{sh},2}}{l} = \begin{cases} \xi(\omega_2)^2 \frac{\mu_0}{4} \iint_{\text{short}} |\mathbf{H}_2|^2 ds & \text{for TE} \\ \frac{\mu_0}{4} \iint_{\text{short}} |\mathbf{H}_2|^2 ds & \text{for TM.} \end{cases} \quad (54)$$

Therefore,

$$Q_2 = \begin{cases} \frac{1}{\xi(\omega_2)} \cdot \frac{\omega_2 \iiint_{\text{cavity}} |\mathbf{H}_2|^2 dv}{c \iint_{\text{short}} |\mathbf{H}_2|^2 ds} & \text{for TE} \\ \xi(\omega_2) \cdot \frac{\omega_2 \iiint_{\text{cavity}} |\mathbf{H}_2|^2 dv}{c \iint_{\text{short}} |\mathbf{H}_2|^2 ds} & \text{for TM.} \end{cases} \quad (55)$$

C. Example

The Q_{ext} of a cavity coupled through an inductive loop to a coaxial waveguide is derived using the circuit-model formula Eq. (41) presented above. The equivalent circuit under consideration is illustrated in Fig. 5(a), where the cavity is represented with a parallel LCR circuit. The inductance of the loop and the mutual inductance of the

FIG. 5. (a) Equivalent circuit of loop-coupled cavity. (b) Loss-free circuit of loop-coupled cavity.

coupling are L_p and M , respectively. The input impedance is represented by

$$\begin{aligned} Z_{\text{in}}(\omega) &= j\omega L_p + \frac{\omega^2 M^2}{\omega_a^2 L^2} \frac{1}{\omega_a C} \\ &\quad \times \frac{1}{j(\omega/\omega_a - \omega_a/\omega) + 1/Q_0}, \end{aligned} \quad (56)$$

where

$$\omega_a = 1/\sqrt{LC}, \quad (57)$$

and Q_0 is the internal- Q of the cavity:

$$Q_0 = \omega_a RC. \quad (58)$$

The matching condition is given by

$$R = \left(\frac{L}{M} \right)^2 \cdot Z_0 \cdot \left[1 + \left(\frac{\omega_a L_p}{Z_0} \right)^2 \right], \quad (59)$$

where Z_0 is the characteristic impedance of the coaxial waveguide.

We will see below that this matching condition is reproduced by the analytical method. To this end, let us take the following values for the circuit parameters:

$$L = 47.20 \times 10^{-9} [\text{H}], \quad C = 4.860 \times 10^{-12} [\text{F}],$$

$$R = 1.056 \times 10^6 [\Omega], \quad Z_0 = 50 [\Omega],$$

$$L_p = 20.70 \times 10^{-9} [\text{H}], \quad M = 429.1 \times 10^{-12} [\text{H}].$$

These values are taken from a 1/3-scaled model of a rebuncher cavity designed for the RIKEN RI-Beam

Factory project [7]. Note that these values satisfy the matching condition Eq. (59). The parameters above give the resonant frequency and internal- Q , respectively, as

$$\omega_a = 2.08791 \times 10^9 [\text{rad/s}], \quad Q_0 = 10715. \quad (60)$$

In order to use the analytical method, the input impedance is modified by taking $Q_0 \rightarrow \infty$ in Eq. (56), which corresponds to a circuit without the shunt resistance R , as shown in Fig. 5(b). Therefore, the susceptance of the loss-free circuit is given by

$$B_{\text{in}}(\omega) = \left[-\omega L_p + \frac{\omega^2 M^2}{\omega_a^2 L^2} \cdot \frac{1}{\omega_a C} \cdot \frac{1}{(\omega/\omega_a - \omega_a/\omega)} \right]^{-1}. \quad (61)$$

The procedure represented by Eqs. (35), (37), (38), and (40), has been performed with a MATHEMATICA program where l and c were assumed to be $0.5[\text{m}]$ and $2.998 \times 10^8 [\text{m/s}]$, respectively. The results are

$$\omega_1 = 2.08782 \times 10^9 [\text{rad/s}], \quad Q_1 = 2626, \quad (62)$$

$$\omega_2 = 2.08805 \times 10^9 [\text{rad/s}], \quad Q_2 = 8102. \quad (63)$$

Therefore,

$$Q_{\text{ext}} = Q_1 + Q_2 = 10728, \quad (64)$$

which is similar to the value of Q_0 in Eq. (63). The matching condition has been thus reproduced. It has also been confirmed that the result is almost independent of l .

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APPENDIX: RELATION TO THE TUNING-CURVE METHOD

A closer look at the results in Sec. IIB provides a clear relation between the circuit-model approach and the traditional tuning-curve method. The tuning curve is obtained by solving Eq. (11). It is represented by

$$l_1(\omega_1) = -\frac{c}{\omega_1} \arctan \left[Q_{\text{ext}} \left(\frac{\omega_1}{\omega_a} - \frac{\omega_a}{\omega_1} \right) \right]. \quad (A1)$$

One obtains the slope of the curve by differentiating l_1 with respect to ω_1 :

$$\begin{aligned} \frac{dl_1}{d\omega_1}(\omega_1) &= -\frac{c}{\omega_1} Q_{\text{ext}} \left(\frac{1}{\omega_a} + \frac{\omega_a}{\omega_1^2} \right) \cos^2 \left(\frac{l\omega_1}{c} \right) \\ &\quad - \frac{1}{\omega_1} l_1(\omega_1). \end{aligned} \quad (A2)$$

By comparing Eq. (A2) with Eq. (14),

$$q_1 = -\frac{\omega_1}{2c} \left[\omega_1 \frac{dl_1}{d\omega_1}(\omega_1) + l_1(\omega_1) \right]. \quad (A3)$$

The length l which gives the open-end resonance is also represented by a function of ω_2 by solving Eq. (17):

$$l_2(\omega_2) = \frac{c}{\omega_2} \arctan \left[\frac{1}{Q_{\text{ext}}(\omega_2/\omega_a - \omega_a/\omega_2)} \right]. \quad (A4)$$

By differentiating this with respect to ω_2 , one gets

$$\begin{aligned} \frac{dl_2}{d\omega_2}(\omega_2) &= -\frac{c}{\omega_2} Q_{\text{ext}} \left(\frac{1}{\omega_a} + \frac{\omega_a}{\omega_2^2} \right) \sin^2 \left(\frac{l\omega_2}{c} \right) \\ &\quad - \frac{1}{\omega_2} l_2(\omega_2). \end{aligned} \quad (A5)$$

Therefore, Eq. (20) can be written as

$$q_2 = -\frac{\omega_2}{2c} \left[\omega_2 \frac{dl_2}{d\omega_2}(\omega_2) + l_2(\omega_2) \right]. \quad (A6)$$

In the tuning-curve method, the Q_{ext} is derived from the maximum value of the right-hand side of Eq. (A3) [1], which is actually Q_{ext} in Eq. (14). On the other hand, the circuit-model and the straightforward method [2] make good use of the fact that the sum of the right-hand sides of Eqs. (14) and (20) is almost constant, which is independent of the waveguide length, as shown in Eq. (41).

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