Reply to “Comment on ‘Centroid theory of transverse electron-proton two-stream instability in a long proton bunch’”

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The authors disagree with Dr. D.V. Pestrikov’s assertion that “the results obtained in the commented paper are not true,” based on how Volterra’s integral equations are treated in the paper. The authors would agree, for clarity, that the equal sign ($=$) in Eqs. (34), (36), and (38) in the paper be replaced by some special symbol or the approximate sign ($\approx$) to indicate the omission of initial conditions. These and similar changes together with the revision to an error found by the authors have been published in a recent erratum.

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The authors would like to thank Dr. Pestrikov for his comments [1]. The authors disagree with his assertion that “the results obtained in the commented paper are not true,” based on the lemma that the homogeneous Volterra’s integral equation of the second kind has only zero solution. However, Dr. Pestrikov’s comment did stimulate the authors to check their calculation more carefully and to find a different error in [2] which indeed leads to a correction to the earlier result [3].

The essence of the paper is to explore the general characteristics of the e - p instability based on the asymptotic solution of two coupled differential equations that should be formulated into an inhomogeneous Volterra’s integral equation of the second kind (which is known to have a nonzero solution) if one solves the initial value problem via the integral-equation approach. Examining the evolution of any specific initial perturbation is not the interest of this paper [2]. Thus, the authors are actually looking for an approximate solution of an inhomogeneous Volterra’s integral equation of the second kind at the stage where the instability has grown into oscillations with amplitude much larger than the small initial perturbation. For the “one-pass” interaction (similar to the beam breakup model) we are considering, this occurs when $t \gg \tau$ and $z' \gg v/f_e$, where τ is the instability e -folding time, v is the axial speed of the protons, and f_e is the typical electron bounce frequency inside the proton bunch. The homogeneous integral equation occurs in the process of excluding the direct contribution of initial conditions at large t/τ for easier access to the approximate functional form of the asymptotic solution. For this kind of procedure, the claim that zero is the only solution of the homogeneous Volterra’s integral equation of the second kind is not relevant to how the problem is treated, except for the preciseness of using correct mathematical language, or for the case where the system remains in

equilibrium because there is no initial perturbation. The authors would agree, for clarity, that the equal sign ($=$) in Eqs. (34), (36), and (38) in the paper be replaced by a special symbol or the approximate sign ($\approx$) to indicate the omission of initial conditions as shown in a recent erratum [3].

To describe our approach more precisely, we start by taking a Laplace transformation of Eq. (32) in [2] to derive the inhomogeneous Volterra’s integral equation

$$\hat{Y}(z', \omega) = \frac{1}{v^2 W(\omega_\beta^2 - \omega^2)} \int_0^{z'} \Omega^2(x) G(x) [\Phi(z') \Psi(x) - \Psi(z') \Phi(x)] \hat{Y}(x, \omega) dx + \frac{i\omega Y(z', 0) + Y'(z', 0)}{G(z')(\omega_\beta^2 - \omega^2)} e^{-(\Delta_p - \Delta_e - i\omega)z'/v}, \quad (1)$$

where $Y(z', 0)$ and $Y'(z', 0)$ are the initial conditions at $t = 0$, $\omega = -is$, and s is the Laplace transformation variable. Equation (1) is the same as Eq. (36) in the paper except for the second term on the right-hand side that contains the initial conditions. Following the same procedure in deriving Eq. (41) in the paper, we obtain

$$\zeta(z', \omega) \approx \int_0^{z'} \frac{\Omega^2(x) G(x) R^2(x)}{v^2 W(\omega_\beta^2 - \omega^2)} \zeta(x, \omega) dx + H(z'), \quad (2)$$

where

$$H(z') = \frac{\omega_\beta^3 [i\omega Y(z', 0) + Y'(z', 0)] e^{-(\Delta_p - \Delta_e - i\omega)z'/v}}{G(z') \Phi(z') (\omega_\beta^2 - \omega^2)}. \quad (3)$$

Equation (41) in the paper can be obtained by neglecting the second term on the right-hand side of Eq. (2). The equivalent differential equation for Eq. (2) is

$$\frac{d\zeta(z', \omega)}{dz'} \approx \frac{\Omega^2(z')G(z')R^2(z')}{v^2W(\omega_\beta^2 - \omega^2)}\zeta(z', \omega) + \frac{dH(z')}{dz'}. \quad (4)$$

In the region of $z' \gg v/f_e$, i.e., in the middle and the later part of the bunch where ζ can grow to become much larger than the small initial perturbation when $t \gg \tau$, the right-hand side of Eq. (4) is dominated by the first term. That is, after some growth time, the evolution of the instability in the later part of a long proton bunch depends on how the beam interacts with the electrons much more than on the details of the small initial perturbation. This property can be seen in the numerical solutions of Eqs. (18)–(21) in the paper. Hence to obtain the approximate asymptotic solution, we can neglect the second term on the right-hand side of Eq. (4) to yield

$$\frac{d\zeta(z', \omega)}{dz'} \approx \frac{\Omega^2(z')G(z')R^2(z')}{v^2W(\omega_\beta^2 - \omega^2)}\zeta(z', \omega), \quad (5)$$

which has a *possible* solution

$$\zeta(z', \omega) \approx \exp\left[\frac{i\omega_\beta^2 J(z')}{\omega^2 - \omega_\beta^2}\right], \quad (6)$$

given by Eq. (42) in the paper. However, the authors have

lately found that the correct asymptotic solution for $\zeta(z', \omega)$ should be

$$\zeta(z', \omega) \approx \left(\frac{1}{\omega^2 - \omega_\beta^2}\right) \exp\left[\frac{i\omega_\beta^2 J(z')}{\omega^2 - \omega_\beta^2}\right], \quad (7)$$

instead of the one shown in Eq. (6) [Eq. (42) in the paper]. The factor $1/(\omega^2 - \omega_\beta^2)$ is needed for the inverse Fourier transformation to yield the correct betatron-oscillation solution for Y_p . Indeed, the authors have noted that in the absence of electrons, $J(z') = 0$, Eq. (6) leads to an undefined solution instead of the betatron-oscillation solution in Eq. (45) in the paper. This revision and the related changes to the relevant results have also been included in the erratum [3].

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- [1] D.V. Pestrikov, preceding Comment, Phys. Rev. ST Accel. Beams **7**, 119201 (2004).
 - [2] T.F. Wang, P.J. Channell, R.J. Macek, and R.C. Davidson, Phys. Rev. ST Accel. Beams **6**, 014204 (2003).
 - [3] T.F. Wang, P.J. Channell, R.J. Macek, and R.C. Davidson, Phys. Rev. ST Accel. Beams, **7**, 069901 (2004).