Space-charge limit for a finite-size bunched beam in a circular conducting pipe

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We derive the space-charge limit for a train of periodic finite-size charged bunches which are slightly displaced from the axis of a circular, perfectly conducting pipe. The general calculation assumes that the bunches, while periodic, may have arbitrary radial and longitudinal density distributions with respect to the beam axis. We present results for a few specific distributions, as well as the limits of unbunched and single-bunch beams. Finally, we show how the bunched beam space-charge limit along with the radial rms envelope equation for a bunched beam is applied to predict operational limits of periodic permanent magnet focusing klystrons, such as the SLAC 11.4 GHz 75 MW XP-3 device.

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I. INTRODUCTION

The dipolelike interaction of an off-axis charged particle beam with a surrounding conducting structure has been studied both electromagnetically for high-energy, low space-charge accelerator beams (also known as the beam breakup instability or BBU) [1,2], and electrostatically for low-energy, high space-charge beams in high-power microwave (HPM) sources [3,4]. The BBU instability inside of an accelerator cavity strongly depends on both the characteristic parameters associated with the beam, i.e., beam current, beam energy, and beam pulse length, as well as the characteristic parameters pertaining to the cavity, i.e., cavity size Q and transverse shunt impedance. In general, the starting current for a BBU instability can be computed in terms of the above parameters for which the growth rate of one or a few electromagnetic deflecting modes exceeds the rate of Ohmic loss in the accelerator structure [5].

In the high space-charge regime of HPM sources, such as high-power klystrons, the off-axis interaction of a bunched beam with the surrounding circular conductor pipe was successfully modeled, as shown in our earlier works [3,4], assuming that the conductor pipe has no Ohmic loss and the interaction is purely electrostatic. In particular, an electrostatic Green's function technique was used to compute the center-of-mass force on a periodic bunch train in which the bunches were assumed to be pointlike. If the bunches are in the presence of a uniform external magnetic field or a periodic permanent magnet (PPM) focusing field, then a critical space-charge limit for beam centroid confinement was derived and expressed in terms of the pipe radius, bunch spacing, and the root-mean-squared magnetic field strength.

The bunched beam space-charge limit predicted by the electrostatic model was found to be an excellent indicator for beam loss in six periodic permanent magnet kly-

trons: the SLAC 11.4 GHz 50 MW XL-PPM [6,7], 75 MW XP-1 [6,7], and 75 MW XP-3 [8] klystrons, the BINP/KEK 11.4 GHz 75 MW PPM-1 klystron [9], the Toshiba/KEK 5.7 GHz 50 MW klystron [10], and the SLAC 95 GHz Klystrino [11]. In these klystron experiments, the long copper drift tubes through which the beam bunches must propagate was found to be adequately modeled by the infinite-length and infinite-conductivity assumptions for the pipe.

In this paper, we investigate how the bunched beam space-charge limit is modified by including the effect of finite-size periodic azimuthally symmetric bunches with arbitrary radial and longitudinal distribution functions with respect to the axis of the bunch train. As in Refs. [3,4], we assume that the conducting pipe is infinitely long and has infinite conductivity. In order to utilize a Lorentz transformation to the beam rest frame, we assume that the longitudinal energy spread of the bunches is zero, i.e., every electron in a bunch is traveling with the same velocity. In the beam rest frame, we derive an expression for the self-consistent center-of-mass force on each bunch to first order in the bunch displacement from the axis using an electrostatic Green's function technique [3].

Using the center-of-mass force, we show how the space-charge limit for both uniform and periodic external magnetic fields derived from Hess and Chen [3,4] is revised due to the inclusion of the finite-sized bunch distribution. In particular, we find that an increase in the radial beam size leads to a decrease in the bunched beam space-charge limit, while an increase in the longitudinal beam size leads to an increase of the bunched beam space-charge limit. Finally, we show how to implement the space-charge limit for the parameters of five PPM klystrons: SLAC 11.4 GHz 50 MW XL-PPM [6,7], 75 MW XP-1 [6,7], and 75 MW XP-3 [8] klystrons, the BINP/KEK 11.4 GHz 75 MW PPM-1 klystron [9], and the Toshiba/KEK 5.7 GHz 50 MW klystron [10] using an estimate of the radial rms beam size from the bunched beam radial envelope equation [12,13].

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The paper is organized as follows. In Sec. II we present a formulation of the azimuthally symmetric periodic bunched beam model. Using the center-of-mass force expression, Eq. (9), which is calculated in the Appendix, we derive a general expression for the space-charge limit in Eq. (20). In Sec. III we calculate the space-charge limit explicitly for a few specific distribution functions with respect to the beam axis, and analyze the space-charge limit in the regime of unbunched and single-bunch beams. In Sec. IV we utilize the space-charge limit of the experimental parameters for the SLAC 11.4 GHz 50 MW XL-PPM [6,7], 75 MW XP-1 [6,7], and 75 MW XP-3 [8] klystrons, the BINP/KEK 11.4 GHz 75 MW PPM-1 klystron [9], and the Toshiba/KEK 5.7 GHz 50 MW klystron [10] along with the bunched beam radial envelope equation. In Secs. V and VI a discussion and a summary are presented, respectively.

II. SPACE-CHARGE LIMIT FOR FINITE-SIZED BUNCHES

In this section, we derive a space-charge limit for a periodic azimuthally symmetric finite-sized bunch train propagating in either a uniform magnetic focusing field,

$$\mathbf{B}^{\text{ext}} = B_0 \hat{\mathbf{z}}, \quad (1a)$$

or a periodic magnetic focusing field,

$$\mathbf{B}^{\text{ext}} = B_0 \cos(kz) \hat{\mathbf{z}} + \frac{B_0 k}{2} \sin(kz) (x \hat{\mathbf{x}} + y \hat{\mathbf{y}}), \quad (1b)$$

within a perfectly conducting, grounded circular pipe of radius a . The periodic spacing of the bunches in the z

FIG. 1. Schematic of a finite-size periodic bunched beam inside of a perfectly conducting circular pipe.

direction is given by L and the number of electrons per bunch is given by $N_b \gg 1$.

Within the system, the bunches are assumed to be azimuthally symmetric with respect to the beam axis which has a small offset from the axis of the circular pipe. They propagate with an average velocity $V_z \hat{\mathbf{e}}$, whose magnitude is much greater than all transverse velocities, as well, as any velocity fluctuation in the z direction. The density distribution for the beam as measured in the laboratory frame along with its periodicity and normalization properties are given by

$$f_{\text{lab}}(\mathbf{r}_{\perp}, z - V_z t) = \gamma_b \sum_{n=-\infty}^{\infty} g(|\mathbf{r}_{\perp} - \mathbf{R}_{\text{cm}}|, \gamma_b(z - V_z t + nL)), \quad (2a)$$

$$f_{\text{lab}}(\mathbf{r}_{\perp}, z - V_z t + L) = f_{\text{lab}}(\mathbf{r}_{\perp}, z - V_z t), \quad (2b)$$

$$\int_{\text{cell lab}} d^3 \mathbf{r} f_{\text{lab}}(\mathbf{r}_{\perp}, \gamma_b(z - V_z t)) = 1, \quad (2c)$$

where $\gamma_b = (1 - \beta_b^2)^{-1/2}$ and $\beta_b = V_z/c$, g is the single-bunch density distribution function with respect to the beam axis as measured in the average beam rest frame (i.e., a frame moving with velocity $V_z \hat{\mathbf{e}}$ with respect to the laboratory frame), $\mathbf{R}_{\text{cm}} = X_{\text{cm}} \hat{\mathbf{e}}_x + Y_{\text{cm}} \hat{\mathbf{e}}_y$ is the small transverse displacement of the beam center of mass from the pipe axis, i.e., $R_{\text{cm}} \ll a$, and the unit cell in the laboratory frame is defined by $r \leq a$ and $|z| \leq L/2$. Figure 1 illustrates the system. The center-of-mass equation of motion includes a term which is the total force on each bunch due to the space-charge forces of all the bunches as well as the induced surface charges on the wall of the conductor. This term will be explicitly derived to first order in R_{cm}/a for azimuthally symmetric bunches.

The Lorentz force equation for the i th electron in the $n = 0$ bunch is given by

$$\frac{d\mathbf{p}_i}{dt} = -e \left\{ \mathbf{E}^s(\mathbf{r}_i) + \frac{1}{c} \frac{d\mathbf{r}_i}{dt} [\mathbf{B}^s(\mathbf{r}_i) + \mathbf{B}^{\text{ext}}(\mathbf{r}_i)] \right\}, \quad (3)$$

where $\mathbf{p}_i$ is the momentum of the i th electron, $\mathbf{r}_i$ is the position of the electron, and $\mathbf{E}^s(\mathbf{r}_i)$ and $\mathbf{B}^s(\mathbf{r}_i)$ are the self-electric and self-magnetic fields at the electron, including those induced by the image charge and current. We note that choosing the $n = 0$ bunch is arbitrary since all of the bunches in our system are identical. For all further calculations in this manuscript, we will focus only on the $n = 0$ bunch.

The self-electric and self-magnetic fields are computed by Lorentz transforming to the average rest frame of the beam. Since the transverse velocities and fluctuations about the bunch average z motion are assumed to be nonrelativistic, the self-magnetic field is negligible in this frame and the self-electric field is given by $\mathbf{E}_{\text{rest}}^s$.

A Lorentz transformation back to the laboratory frame yields $\mathbf{E}^s(\mathbf{r}_i)$ and $\mathbf{B}^s(\mathbf{r}_i)$ which are given by

$$\mathbf{E}^s = E_{\text{rest},z}^s \hat{\mathbf{e}}_z + \gamma_b (E_{\text{rest},x}^s \hat{\mathbf{e}}_x + E_{\text{rest},y}^s \hat{\mathbf{e}}_y) \quad (4a)$$

and

$$\mathbf{B}^s = \gamma_b (-\beta_b E_{\text{rest},x}^s \hat{\mathbf{e}}_x + \beta_b E_{\text{rest},y}^s \hat{\mathbf{e}}_y). \quad (4b)$$

Consistent with the assumption that the electron's average relativistic velocity in the z direction $V_z \hat{\mathbf{e}}_z$ is much greater than all other velocities, the self-magnetic force term in Eq. (3) can be approximated by

$$\frac{1}{c} \frac{d\mathbf{r}_i}{dt} \times \mathbf{B}^s \equiv \beta_b \hat{\mathbf{e}}_z \times \mathbf{B}^s = -\beta_b^2 \gamma_b (E_{\text{rest},x}^s \hat{\mathbf{e}}_x + E_{\text{rest},y}^s \hat{\mathbf{e}}_y). \quad (5)$$

Therefore, Eq. (3) can be written as

$$\mathbf{F}_{\text{cm}}^{\text{rest}} \equiv \mathbf{F}_{\text{cm}\perp}^{\text{rest}} + F_{\text{cm}\parallel}^{\text{rest}} \hat{\mathbf{e}}_z, \quad (9a)$$

$$F_{\text{cm}\parallel}^{\text{rest}} \equiv -e \sum_{i=1}^{N_b} E_{\text{rest},z}^s = 0, \quad (9b)$$

$$\mathbf{F}_{\text{cm}\perp}^{\text{rest}} \equiv -e \sum_{i=1}^{N_b} (E_{\text{rest},x}^s \hat{\mathbf{e}}_x + E_{\text{rest},y}^s \hat{\mathbf{e}}_y) = \frac{N_b^2 e^2 \alpha \mathbf{R}_{\text{cm}}}{\pi a^3} \left\{ 1 + \sum_{m=1}^{\infty} \frac{A_m m \alpha}{I_0(m\alpha) I_1(m\alpha)} \right\}, \quad (9c)$$

$$A_m = (2\pi)^2 \int_{-\infty}^{\infty} dz \int_0^a dr r \int_{-\infty}^{\infty} dz' \int_0^a dr' r' g(r', z') g(r, z) \cos[m(\hat{z} - \hat{z}')] I_0(m\hat{r}) I_0(m\hat{r}'), \quad (9d)$$

where $\alpha = 2\pi a / \gamma_b L$, the hat denotes normalization by $2\pi / \gamma_b L$, and $I_k(x)$ is the k th order modified Bessel function of the first kind. Therefore, we may simplify Eq. (8) to

$$\frac{d^2 \mathbf{X}_{\text{cm}}}{dt} = \frac{\mathbf{F}_{\text{cm}\perp}^{\text{rest}}}{\gamma_b^2 M} - \frac{e}{\gamma_b N_b m_e c} \sum_{i=1}^{N_b} \frac{d\mathbf{r}_i}{dt} \times \mathbf{B}^{\text{ext}}, \quad (10)$$

where $M = N_b m_e$ is the total mass of each bunch.

In Eq. (9), $\mathbf{F}_{\text{cm}}^{\text{rest}} = \mathbf{F}_{\text{cm}\perp}^{\text{rest}}$ depends on the bunch distribution. If we were to consider the complete dynamics of the bunches, i.e., centroid motion and distribution evolution, $f_{\text{lab}}(\mathbf{r}_{\perp}, z - V_z t, t)$, then the center-of-mass force would be explicitly time dependent, i.e., $\mathbf{F}_{\text{cm}}^{\text{rest}} = \mathbf{F}_{\text{cm}}^{\text{rest}}(t)$. While a fully self-consistent treatment of bunch density evolution is not tractable at present, we approximate the evolution of $f_{\text{lab}}(\mathbf{r}_{\perp}, z - V_z t, t)$ by solving the bunched beam envelope equations developed by Sacherer [12] and later refined by Allen and Reiser [13]. We will show an explicit example of how to implement the bunched beam envelope equations for computing the correct $\mathbf{F}_{\text{cm}}^{\text{rest}}$ for the

$$\begin{aligned} \frac{d\mathbf{p}_i}{dt} &= \gamma_b m_e \frac{d\mathbf{r}_i}{dt} \\ &= -e \left[E_{\text{rest},z}^s \hat{\mathbf{e}}_z + \frac{1}{\gamma_b} (E_{\text{rest},x}^s \hat{\mathbf{e}}_x + E_{\text{rest},y}^s \hat{\mathbf{e}}_y) \right. \\ &\quad \left. + \frac{1}{c} \frac{d\mathbf{r}_i}{dt} \times \mathbf{B}^{\text{ext}} \right], \end{aligned} \quad (6)$$

where m_e is the mass of the electron.

By summing over all of the electrons in the bunch and denoting the center-of-mass position as

$$\mathbf{X}_{\text{cm}} = \frac{1}{N_b} \sum_{i=1}^{N_b} \mathbf{r}_i, \quad (7)$$

Eq. (6) can be expressed as

$$\begin{aligned} \frac{d^2 \mathbf{X}_{\text{cm}}}{dt} &= -\frac{e}{\gamma_b m_e N_b} \sum_{i=1}^{N_b} \left[E_{\text{rest},z}^s \hat{\mathbf{e}}_z + \frac{1}{\gamma_b} (E_{\text{rest},x}^s \hat{\mathbf{e}}_x \right. \\ &\quad \left. + E_{\text{rest},y}^s \hat{\mathbf{e}}_y) + \frac{1}{c} \frac{d\mathbf{r}_i}{dt} \times \mathbf{B}^{\text{ext}} \right]. \end{aligned} \quad (8)$$

The following results will be explicitly derived in the Appendix:

case of the SLAC 50 MW XL-PPM klystron [6,7] in Sec. IV.

For *uniform magnetic focusing*, we can remove $\mathbf{B}^{\text{ext}} = B_0 \hat{\mathbf{e}}_z$ from under the summation in Eq. (10) to obtain

$$\frac{d^2 \mathbf{X}_{\text{cm}}}{dt} = \frac{\mathbf{F}_{\text{cm}\perp}^{\text{rest}}}{\gamma_b^2 M} - \frac{\omega}{\gamma_b} \frac{d\mathbf{X}_{\text{cm}}}{dt} \times \hat{\mathbf{e}}_z, \quad (11)$$

where $\omega_c = eB_0 / m_e c$. Equation (11) is a self-consistent equation of motion for the center of mass of a bunch, which depends explicitly on the displacement of the bunch from the pipe axis and implicitly on the bunch distribution function. Notice that since $\mathbf{F}_{\text{cm}}^{\text{rest}} = \mathbf{F}_{\text{cm}\perp}^{\text{rest}}$, the center-of-mass motion in the z direction is constant, i.e.,

$$\frac{d^2 Z_{\text{cm}}}{dt^2} = 0, \quad Z_{\text{cm}}(t) = V_z t + Z_{\text{cm}}(0) \quad (12a)$$

and, hence,

$$\frac{d^2 \mathbf{R}_{\text{cm}}}{dt^2} = \frac{\mathbf{F}_{\text{cm}\perp}^{\text{rest}}}{\gamma_b^2 M} - \frac{\omega_c}{\gamma_b} \frac{d\mathbf{R}_{\text{cm}}}{dt} \times \hat{\mathbf{e}}_z, \quad (12b)$$

where $Z_{\text{cm}}(0) = \int_{-L/2}^{L/2} dz \int d^2 \mathbf{r}_{\perp} z f_{\text{lab}}(\mathbf{r}_{\perp}, z)$.

For *periodic magnetic focusing*, the center-of-mass equation of motion, in general, is far more complicated than Eq. (11) for uniform magnetic focusing. For $kr_b \ll 1$ and $kz_b \ll 1$, where r_b is the characteristic radial size of the bunch and z_b is its characteristic longitudinal size, it can be shown that the center-of-mass equations of motion for periodic magnetic focusing are

$$\frac{d^2 Z_{\text{cm}}}{dt^2} = 0, \quad Z_{\text{cm}}(t) = V_z t + Z_{\text{cm}}(0) \quad (13a)$$

and

$$\frac{d^2 \mathbf{R}_{\text{cm}}}{dt^2} = \frac{\mathbf{F}_{\text{cm}\perp}^{\text{rest}}}{\gamma_b^2 M} - \frac{\omega \cos(kZ_{\text{cm}})}{\gamma_b} \frac{d\mathbf{R}_{\text{cm}}}{dt} \times \hat{\mathbf{e}}_z. \quad (13b)$$

In deriving Eq. (13b), we have used the fact that $\mathbf{B}^{\text{ext}} \cong B_0 \cos(kZ_{\text{cm}}) \hat{\mathbf{e}}_z$ when $kr_b \ll 1$ and $kz_b \ll 1$. Hence, $\mathbf{B}^{\text{ext}}$ may be removed from under the summation in Eq. (10) yielding Eq. (13b).

From Eqs. (12a), (12b), (13a), and (13b), we find that the center of mass may be viewed as a quasiparticle with mass M and charge $Q = -N_b e$ under the influence of a transverse external force $\mathbf{F}_{\text{cm}\perp}^{\text{rest}}/\gamma_b^2$ as well as a magnetic field which is either uniform in the z direction or periodic in the z direction. An important space-charge limit may be derived from the transverse center-of-mass dynamics, as described by Eqs. (12b) and (13b), by recasting the equations in an effective center-of-mass Hamiltonian written in cylindrical coordinates. This is the approach utilized by Hess and Chen in Refs. [3,4]. The space-charge limit is then readily obtained from the center-of-mass Hamilton's equations of motion.

The effective center-of-mass Hamiltonian for a bunch interacting with a uniform or periodic magnetic field and the radial electrostatic force is given by

$$H = \frac{1}{2\gamma_b M} \left[P_r^2 + \left(\frac{P_\theta}{R_{\text{cm}}} + \frac{N_b e A_\theta}{c} \right)^2 \right] - \lambda R_{\text{cm}}^2, \quad (14)$$

where P_r is the center-of-mass canonical radial momentum, P_θ is the center-of-mass canonical angular momentum, $\mathbf{A} = A_\theta \hat{\mathbf{e}}_\theta = \frac{1}{2} R_{\text{cm}} B_0 \eta(Z_{\text{cm}}) \hat{\mathbf{e}}_\theta$ is the vector potential, $\eta(Z_{\text{cm}}) = 1$ for uniform magnetic focusing, $\eta(Z_{\text{cm}}) = \cos(kZ_{\text{cm}})$ for periodic magnetic focusing, and

$$\lambda = \frac{N_b^2 e^2 \alpha}{2\pi a^3 \gamma_b^2} \left\{ 1 + \sum_{m=1}^{\infty} \frac{A_m m \alpha}{I_0(m\alpha) I_1(m\alpha)} \right\}. \quad (15)$$

The equations of motion derived from H are

$$\frac{dR_{\text{cm}}}{dt} = \frac{P_r}{\gamma_b M}, \quad (16a)$$

$$\frac{dP_r}{dt} = \frac{P_\theta^2}{\gamma_b M R_{\text{cm}}^3} - \left(\frac{M \omega_c^2 \eta^2(Z_{\text{cm}})}{4\gamma_b^2} - 2\lambda \right) R_{\text{cm}}, \quad (16b)$$

$$\frac{d\theta}{dt} = \frac{P_\theta}{\gamma_b M R_{\text{cm}}^2} + \frac{\omega_c \eta(Z_{\text{cm}})}{2\gamma_b}, \quad (16c)$$

$$\frac{dP_\theta}{dt} = 0. \quad (16d)$$

For uniform magnetic focusing, Eqs. (16) need not be simplified any further. However, for periodic magnetic focusing, it is necessary to simplify the longitudinal dependency within the radial momentum equation due to the term with $\eta^2(Z_{\text{cm}})$. Notice that for our choice of periodic magnetic focusing, $\eta^2(Z_{\text{cm},z}) = \frac{1}{2}[1 + \eta(2Z_{\text{cm}})]$. If the characteristic time scale of radial motion through the magnetic field, $\tau_m = \pi/kV_z$, is much shorter than the time scale for transverse motion $\tau_\perp \sim (8\gamma_b^3)^{1/2}/\omega_c$, then the radial momentum equation may be averaged over a periodic focusing length, i.e.,

$$\begin{aligned} \left\langle \frac{dP_r}{dt} \right\rangle &= \frac{k}{2\pi} \int_0^{2\pi/k} dz \frac{dP_r}{dt} \\ &= \frac{P_\theta^2}{\gamma_b M R_{\text{cm}}^3} - \left(\frac{M \omega_{c,\text{rms}}^2}{8\gamma_b^2} - 2\lambda \right) R_{\text{cm}}. \end{aligned} \quad (17)$$

The averaged radial momentum equations for both uniform and periodic magnetic focusing fields can be expressed as a single equation using the root-mean-squared cyclotron frequency $\omega_{c,\text{rms}}$. This gives

$$\frac{dP_r}{dt} = \frac{P_\theta^2}{\gamma_b M R_{\text{cm}}^3} - \left(\frac{M \omega_{c,\text{rms}}^2}{4\gamma_b^2} - 2\lambda \right) R_{\text{cm}}, \quad (18)$$

where $\omega_{c,\text{rms}} = \omega_c$ for uniform magnetic focusing and $\omega_{c,\text{rms}} = \omega_c/\sqrt{2}$ for periodic magnetic focusing. Combining Eq. (18) with the radial motion equation as well as the conservation of angular momentum equations found in Eqs. (16) gives a closed set of equations for the radial orbits of the center-of-mass motion.

We note that the averaging technique used in Eq. (17) to obtain Eq. (18) has been numerically validated in a previous work [14] for typical PPM klystron parameters, i.e., $\tau_m/\tau_\perp \leq 0.37$, by demonstrating excellent agreement between the radial phase plots generated with and without averaging. When estimating $\tau_m/\tau_\perp$, we have used the initial beam energy which, in general, will be reduced near the output gap causing an increase in $\tau_m/\tau_\perp$. Computing the full range of values for $\tau_m/\tau_\perp$ in which the averaging technique is valid, however, is a difficult nonlinear dynamics problem and beyond the scope of this paper.

As discussed in Ref. [3], the upper bound on λ necessary for beam confinement, i.e., for closed orbits in radial phase space to exist, is a function of the two parameters α and P_θ . For all values of α , the maximum of the upper bound on λ occurs precisely at $P_\theta = 0$. Since P_θ is a constant [see Eq. (16d)], its value for any experiment would be set by the initial conditions at the cathode, i.e., the radial position $R_{cm}|_{t=0}$, the angular motion $d\theta/dt|_{t=0}$, as well as the magnitude of the magnetic field. The standard designs of PPM focusing klystrons use Pierce-type electron gun geometries, which are magnetically shielded; that is, $B = 0$ at the cathode. If we further impose the reasonable constraint that the center of mass of the beam is not rotating at the cathode, i.e., $d\theta/dt|_{t=0} = 0$, then $P_\theta = 0$. Hence, for the remainder of this paper we will focus only on the maximum space-charge limit for each α which occurs at $P_\theta = 0$.

The upper bound on λ for $P_\theta = 0$ is immediately seen from the radial momentum expression in Eq. (18). The quantity in parentheses must be positive in order to have beam centroid confinement. Hence,

$$\lambda \leq \frac{N_b m_e \omega_{c,rms}^2}{8}, \quad (19)$$

which can be rewritten in terms of the space-charge parameter

$$s_e = \frac{2\omega_p^2}{\omega_{c,rms}^2} \leq \frac{1}{1 + \sum_{m=1}^{\infty} \frac{A_m m \alpha}{I_0(m\alpha) I_1(m\alpha)}}, \quad (20)$$

where $\omega_p = (4\pi N_b n_b e^2/m)^{1/2}$ is the effective plasma frequency corresponding to the density of bunches in the pipe $n_b = (\pi a^2 \gamma_b L)^{-1}$. For pointlike bunches $A_m = 1$, and Eq. (20) recovers the result derived in Ref. [3].

III. EVALUATIONS OF SPACE-CHARGE LIMITS

In this section, we evaluate the space-charge limit, Eq. (20), analytically for a few important distribution functions and numerically for the uniform ellipsoidal distribution. We also discuss the properties of Eq. (20) in the limits of unbunched and single-bunch beams.

A. Separable distributions

The form factors A_m needed for calculating Eq. (20) may be computed numerically, in general. However, for a

few specific distribution functions, the factors A_m may be calculated analytically. In particular, we calculate A_m for separable distribution functions, i.e.,

$$g(r, z) = g_r(r) g_z(z), \quad (21)$$

where $2\pi \int_0^a dr r g_r(r) = 1$ and $\int_{-\infty}^{\infty} dz g_z(z) = 1$. For these specific types of distribution functions, we can express A_m as

$$A_m = B_m C_m, \quad (22a)$$

where

$$B_m = \left[2\pi \int_0^a dr r g_r(r) I_0(m\hat{r}) \right]^2 \quad (22b)$$

and

$$C_m = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz dz' g_z(z) g_z(z') \cos[m(\hat{z} - \hat{z}')]. \quad (22c)$$

We display the results of these calculations in Tables I and II. In particular, we show the B_m form factors for the delta, uniform, unimodal, and uniform annular radial distribution functions. We also provide the C_m form factors for the delta, uniform, Gaussian, and Lorentzian longitudinal distribution functions.

Using the form factors in Tables I and II and Eq. (20), we plot in Fig. 2 for $\alpha = 1.0$ the maximum space-charge parameter $s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$ as a function of $(\sigma_z/a, r_b/\sqrt{2}a)$, which corresponds to the rms longitudinal and radial dimensions normalized to a for uniform radial density distributions with characteristic radius r_b and Gaussian half-width σ_z . We note that $s_e(0, 0) \equiv 0.317440$ is the space-charge parameter corresponding to the point-charge distribution for $\alpha = 1.0$ [3,4]. From Fig. 2 we can qualitatively see that for a fixed longitudinal distribution, an increase in the radial size gives a decrease in the maximum $s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$. In other words, the center-of-mass force on the bunch due to the pipe is stronger as the radial bunch size increases. The center-of-mass force increase for larger radial bunch size is caused by the localization of image charges near the longitudinal center of the bunch, which effectively decreases the interaction distance between the bunch electrons and the induced surface charges. Likewise, we can also see that for a fixed radial distribution an increase in the longitudinal bunch size increases the maximum value for

TABLE I. B_m form factors for specific $g_r(r)$ functions.

$g_r(r)$	B_m
$\delta(r)/2\pi r$ (delta)	1
$\theta(r_b - r)/\pi r_b^2$ (uniform $r_b < a$)	$(2a/mar_b)^2 I_1^2(mar_b/a)$
$[(\mu + 1)(1 - r^2/r_b^2)^\mu]/\pi r_b^2$ ($\mu > 0, r < r_b < a$)	$\Gamma^2(\mu + 2)(2a/mar_b)^{2\mu+2} I_{\mu+1}^2(mar_b/a)$
$[\theta(r - r_i) - \theta(r - r_o)]/\pi(r_o^2 - r_i^2)$ (annular $r_i < r_o < a$)	$\{4a^2[r_o I_1(mar_o/a) - r_i I_1(mar_i/a)]^2\}/\alpha^2 m^2 (r_o^2 - r_i^2)^2$

TABLE II. C_m form factors for specific $g_z(z)$ functions.

$g_z(z)$	C_m
$\delta(z)$ (delta function)	1
$[\theta(z) - \theta(z - 2l_b)]/2l_b$	$(a/m\alpha l_b)^2 \sin^2(m\alpha l_b/a)$
$e^{-z^2/2\sigma_z^2}/\sqrt{2\pi\sigma_z^2}$ (Gaussian)	$e^{-m^2\alpha^2\sigma_z^2/a^2}$
$(l_b/\pi)[1/(z^2 + l_b^2)]$ (Lorentzian)	$e^{-m\alpha l_b/a}$

$s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$, which implies that an increase in the longitudinal bunch size decreases the center-of-mass force on the bunch. The center-of-mass force decrease for larger longitudinal distributions is caused by the delocalization, or spreading out, of the induced surface charges on the conductor resulting in a weaker beam-wall interaction.

An interesting property of $s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$ is that when $0.0 \leq \alpha \leq 1.5$ and $\sigma_z/a \leq 0.75$, $s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$ is weakly dependent on α . Specifically, in this parameter range $s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$ will vary by only 2%–3%. Hence, Fig. 2 would look essentially the same as a plot of $s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$ for any α in the range $0.0 \leq \alpha \leq 1.5$. Since $\alpha \rightarrow 0$ and $\gamma_b L \rightarrow \infty$ represent a single bunch interacting with its own induced surface charge, the weak dependency of $s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$ on α when $0.0 < \alpha \leq 1.5$ and $\sigma_z/a \leq 0.75$ implies that each bunch is sufficiently separated such that the interaction of a bunch with the induced surface charges due to other bunches is relatively small. We will discuss the single-bunch limit, i.e., $\alpha \rightarrow 0$, in greater detail in Sec. III C. For higher values of σ_z/a , say $\sigma_z/a = 1.5$, $s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$ can vary by 70%–80% in the range of $0.0 \leq \alpha \leq 1.5$. The explanation for this variation is that the bunches are now long enough to have a non-negligible interaction with the induced surface charges from other bunches.

FIG. 2. Contour plot of the maximum normalized space-charge parameter, $s_e(\sigma_z/a, r_b/a)/s_e(0, 0)$ in $(\sigma_z/a, r_b/\sqrt{2}a)$ space for uniform radial density distributions with characteristic radius r_b and Gaussian half-width σ_z with $\alpha = 1.0$.

B. Uniform ellipsoidal distribution

In the special case of a uniform ellipsoidal distribution in the beam rest frame, i.e.,

$$g(r, z) = \frac{3}{4\pi r_b^2 l_b} \theta\left(1 - \frac{r^2}{r_b^2} - \frac{z^2}{l_b^2}\right), \quad (23)$$

where $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x \geq 0$, it is readily shown that the form factors A_m are given by

$$A_m\left(\frac{l_b}{a}, \frac{r_b}{a}, \alpha\right) = \frac{9}{m^2 \alpha^2} \left(\frac{a^2}{r_b^2}\right) \left[\int_0^1 dx \cos\left(m\alpha \frac{l_b}{a} x\right) \times \sqrt{1 - x^2} I_1\left(m\alpha \frac{r_b}{a} \sqrt{1 - x^2}\right) \right]^2. \quad (24)$$

It can be readily proven that a train of uniform density spherical bunches, i.e., $r_b = l_b$, has exactly the same space-charge limit as a train of pointlike bunches. By using the following integral relation [15],

$$\int_0^b dy \sqrt{b^2 - y^2} \cos(y) I_1(\sqrt{b^2 - y^2}) = \frac{b^3}{3}, \quad (25)$$

it immediately follows from Eq. (24) that

$$A_m\left(\frac{r_b}{a}, \frac{r_b}{a}, \alpha\right) = 1.$$

Figure 3 shows a contour plot of the maximum space-charge parameter $s_e(l_b/a, r_b/a)/s_e(0, 0)$ as a function of the rms dimensions normalized to $a(l_b/\sqrt{5}a, \sqrt{2}r_b/\sqrt{5}a)$ for $\alpha = 1.0$. Since we have maintained the same α as in Fig. 2, the point-bunch normalization factor remains $s_e(0, 0) \cong 0.317440$. Notice that the space-charge limit

FIG. 3. Contour plot of the maximum normalized space-charge parameter $s_e(l_b/a, r_b/a)/s_e(0, 0)$ as a function of $(l_b/\sqrt{5}a, \sqrt{2}r_b/\sqrt{5}a)$ with uniform ellipsoidal distributions with $\alpha = 1.0$ and the operating points of five PPM klystrons: SLAC 11.4 GHz 50 MW XL-PPM (solid circle), SLAC 11.4 GHz 75 MW XP-1 (open square), SLAC 11.4 GHz 75 MW XP-3 (solid right-side-up triangle), the BINP/KEK 11.4 GHz 75 MW PPM-1 (open diamond), and the Toshiba/KEK 5.7 GHz 50 MW (solid upside-down triangle).

for uniform ellipsoidal bunches in Fig. 3 has very similar contours as those in Fig. 2 for uniform radial and Gaussian longitudinal bunches. As shown in Figs. 2 and 3, the space-charge limit for bunches with the same rms bunch dimensions varies slightly for different distribution functions when $0.0 < \alpha \leq 1.5$ and the bunch length is sufficiently small. In particular, for the rms bunch dimensions plotted in Figs. 2 and 3, the space-charge limit varies by 4%–5%. This illustrates an advantageous feature of this model, namely that rms equivalent bunches produce essentially the same space-charge limits. Since Figs. 2 and 3 are nearly the same due to rms bunch equivalency and since Fig. 2 varies only negligibly in the regime $0.0 \leq \alpha \leq 1.5$ as explained in the previous section, it follows that Fig. 3 is only slightly varying in the regime $0.0 \leq \alpha \leq 1.5$.

C. Unbunched and single-bunch beams

One important limit of Eq. (20) is the 2D unbunched beam limit. In this limit, $\gamma_b L \rightarrow 0$, $N_b \rightarrow 0$, and $-N_b e / \gamma_b L = \lambda_b$ is the finite charge per unit length. As $\gamma_b L \rightarrow 0$, $\alpha \rightarrow \infty$, the infinite series in Eq. (20) goes to zero exponentially fast, i.e., $A_m / I_0(m\alpha) I_1(m\alpha) \propto e^{-4\pi m(a-r_b)/\gamma_b L}$, where $r_b < a$ is a characteristic beam radius. Hence, the space-charge limit for an unbunched beam becomes

$$s_e = \frac{2\omega_p^2}{\omega_{c,\text{rms}}^2} \leq 1. \quad (27)$$

From Eq. (27), we see a significant result; that is, for a 2D radial distribution, the space-charge limit is independent of the radial distribution function.

A second important limit is the single-bunch limit, i.e., $\gamma_b L \rightarrow \infty$ and $\alpha \rightarrow 0$. Since $s_e \propto (\gamma_b L)^{-1}$, it is necessary to take the limit of s_e / α , i.e.,

$$\lim_{\alpha \rightarrow 0} \frac{s_e}{\alpha} = \frac{4N_b e^2}{m_e \pi a^3 \omega_{c,\text{rms}}^2} \leq \lim_{\alpha \rightarrow 0} \frac{1}{\alpha \left\{ \sum_{m=1}^{\infty} \frac{A_m m \alpha}{I_0(m\alpha) I_1(m\alpha)} \right\}}. \quad (28)$$

By performing the asymptotic substitutions,

$$\lim_{\alpha \rightarrow 0} \alpha \sum_{m=1}^{\infty} \rightarrow \int_0^{\infty} dx \quad (29a)$$

and

$$m\alpha \rightarrow x, \quad (29b)$$

Eq. (20) in the single-bunch limit with $\alpha \rightarrow 0$ is given by

$$\lim_{\alpha \rightarrow 0} \frac{s_e}{\alpha} = \frac{4N_b e^2}{m_e \pi a^3 \omega_{c,\text{rms}}^2} \leq \frac{1}{\int_0^{\infty} dx \frac{A(x)x}{I_0(x)I_1(x)}}, \quad (30a)$$

where

$$\begin{aligned} A(x) &= (2\pi)^2 \\ &\times \int_{-\infty}^{\infty} dz \int_0^a dr r \int_{-\infty}^{\infty} dz' \int_0^a dr' r' g(r', z') g(r, z) \\ &\times \cos \left[x \left(\frac{z}{a} - \frac{z'}{a} \right) \right] I_0 \left(x \frac{r}{a} \right) I_0 \left(x \frac{r'}{a} \right). \end{aligned} \quad (30b)$$

For example, the single-bunch space-charge limit for a delta function distribution may be computed using the relation $A(x) = 1$. Hence, for a single point bunch,

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{s_e}{\alpha} &\leq \frac{1}{\int_0^{\infty} dx \frac{x}{I_0(x)I_1(x)}} = \frac{1}{\pi(1.002\,735\,18\dots)} \\ &\equiv 0.317\,441\,55\dots \end{aligned} \quad (31)$$

In Ref. [3], the authors numerically computed the space-charge limit on the delta function distribution and found an approximate relationship $\lim_{\alpha \rightarrow 0} (s_e / \alpha) \leq 1/\pi \approx 0.318\,31$. Equation (31), however, is the exact expression for the delta function bunches in the single-bunch limit.

IV. APPLICATION OF SPACE-CHARGE LIMIT TO PPM FOCUSING KLYSTRONS

In this section we show how the bunched beam space-charge limit along with Sacherer's rms radial envelope equation [12,13] can be applied to predict an operational limit on PPM focusing klystrons. Specifically, by assuming that the bunches are given by uniform density distributions with a reasonable value for the bunch length, e.g., 60° with respect to the rf, we compute the matched rms bunch radius from the radial envelope equation. Equation (20) can be utilized thereafter to calculate an upper bound on the critical space-charge parameter $2\omega_p^2 / \omega_c^2$. As examples, we will use five PPM klystrons designed in support of the Next Linear Collider and Global Linear Collider experiments: SLAC 11.4 GHz 50 MW XL-PPM [6,7], 75 MW XP-1 [6,7], and 75 MW XP-3 [8] klystrons, the BINP/KEK 11.4 GHz 75 MW PPM-1 klystron [9], and the Toshiba/KEK 5.7 GHz 50 MW klystron [10].

In Sacherer's theory of the rms beam envelope which is applicable for typical accelerator beams, there are two coupled envelope equations: one for the radial envelope and one for the longitudinal envelope. These equations can be solved by specifying the initial bunch shape and total charge, as well as the focusing forces and emittances in the radial and longitudinal directions. For klystron beams, however, the longitudinal bunching process is due to the interaction of fast- and slow-space-charge waves, and is further enhanced by the presence of intermediate bunching cavities. This type of process is difficult to describe in terms of rms envelope equations. We therefore use a reasonable size for the klystron bunch length,

60° with respect to the rf, which has been simulated by particle-in-cell codes [7].

For a uniform density bunch, Eq. (23), with an rms half-length in the laboratory frame of $z_{\text{rms}}^{\text{lab}} = l_b/\gamma_b\sqrt{5}$ and radius $r_{\text{rms}} = r_b\sqrt{2/5}$, the radial rms envelope equation is given by [12,13]

$$\frac{d^2 r_{\text{rms}}}{dz^2} + \kappa_r(z)r_{\text{rms}} - \frac{K}{r_{\text{rms}}^2} W_r\left(\frac{z_{\text{rms}}^{\text{lab}}}{r_{\text{rms}}}\right) - \frac{\varepsilon^2}{r_{\text{rms}}^3} = 0, \quad (32)$$

where $\kappa_r(z) = e^2 B_z^2(z)/4m_e^2 \gamma_b^2 \beta_b^2 c^4$, $K = 3N_b e^2/5\sqrt{5} m_e c^2 \gamma_b^3 \beta_b^2$ is the beam perveance, $\varepsilon = 16\varepsilon_{\text{rms}}/5$ with ε_{rms} is the rms unnormalized emittance, and

$$W_r(x) = \begin{cases} \frac{\sqrt{2}}{(1-2x^2)^{3/2}} \tan^{-1}\left[\frac{(1-2x^2)^{1/2}}{\sqrt{2}x}\right] - \frac{2x}{(1-2x^2)}, & 0 < x < \frac{1}{\sqrt{2}}, \\ \frac{2x}{(2x^2-1)} - \frac{\sqrt{2}}{(2x^2-1)^{3/2}} \tanh^{-1}\left[\frac{(2x^2-1)^{1/2}}{\sqrt{2}x}\right], & \frac{1}{\sqrt{2}} < x < \infty. \end{cases} \quad (33)$$

For a fixed choice of bunch length, Eq. (32) may be solved for the matched radial envelope solution, which, in general, spans a range of radii corresponding to the envelope oscillation. Therefore, the space-charge limit will also encompass a range of values. However, for typical PPM klystron parameters the radial envelope oscillation is small for each focusing period implying that only the average radial envelope is important. In using Eq. (32), we implicitly assume that each bunch maintains a uniform distribution function as it propagates through the PPM focusing structure.

Table III shows the key parameters for the five PPM klystrons: operating frequency f , average beam current I_b , beam energy γ_b , the rms magnetic field strength B_{rms} , and the pipe radius a . Table III also shows the values for $\alpha = 2\pi a/\gamma_b L$ and the experimental space-charge parameter $s_{e,\text{exp}}$, which can be computed using the formulas $L = \beta_b c/f$ and $N_b = I_b/ef$. The last two rows of Table III show the critical space-charge limit of point

bunches $s_e(0, 0)$ for each klystron's α value and the ratio $s_{e,\text{exp}}/s_e(0, 0)$.

We can now estimate the beam bunch size by solving Eq. (32). By assuming that the full bunch length is approximately 60° with respect to the rf, then $l_b/\gamma_b L = 1/12$. To solve Eq. (32), we also use the magnetic focusing wave number $k = 2.38 \text{ cm}^{-1}$ [6,7], and an estimated unnormalized rms emittance from thermal spread at the cathode of $\varepsilon_{\text{rms}} = 5.4 \times 10^{-6} \text{ mrad}$ [16]. We note that the values used for the magnetic focusing wave number and emittance correspond to the SLAC 50 MW XL-PPM klystron. However, since the radial envelope oscillation is small, only the rms averaged magnetic field strength is important, not the focusing wave number. Moreover, each klystron is highly space-charge dominated in this regime, and hence, the emittance term is small.

The application of the space-charge limit to the experimental klystron parameters can be demonstrated using Fig. 3. In Sec. III we showed the useful property that $s_e(l_b/a, r_b/a)/s_e(0, 0)$ has only a negligible variation in the regime $0.0 \leq \alpha \leq 1.5$. Hence, Fig. 3, which is plotted for $\alpha = 1.0$, can be utilized for comparison to PPM klystron data which, as shown in Table III, are in the range $0.74 \leq \alpha \leq 0.79$. In Fig. 3, we plot the average bunch size operating points for each of the five klystrons in Table III by solving Eq. (32).

A klystron is below (above) the space-charge limit if its operating point is to the right (left) of the contour with the value corresponding to $s_{e,\text{exp}}/s_e(0, 0)$. For example, the SLAC 11.4 GHz 50 MW XL-PPM klystron's operating point (solid circle) is to the right of the contour with the value $s_{e,\text{exp}}/s_e(0, 0) \cong 0.798$. Therefore, this klystron is operating below the space-charge limit which is in agreement with its successful operation and observation of negligible beam power loss, i.e., $\sim 0.8\%$. The measured beam power loss of 0.8% was attributed to halo formation [16]. From Fig. 3 and Table III we find that the SLAC 11.4 GHz 75 MW XP-3 (solid right-side-up triangle) and Toshiba/KEK 5.7 GHz 50 MW (solid upside-down tri-

TABLE III. Parameters of five PPM klystron devices.

Parameter	50 MW XL-PPM (SLAC)	75 MW XP-1 (SLAC)	75 MW XP-3 (SLAC)	50 MW (Toshiba/KEK)	75 MW PPM-1 (BINP/KEK)
f (GHz)	11.4	11.4	11.4	5.7	11.4
I_b (A)	190	257	260	317	266
γ_b	1.83	1.96	2.00	1.69	1.94
B_{rms} (T)	0.20	0.16	0.21	0.14	0.17
a (cm)	0.48	0.54	0.54	0.90	0.55
α	0.75	0.77	0.74	0.79	0.79
$s_{e,\text{exp}}$	0.19	0.28	0.16	0.20	0.25
$s_e(0, 0)$	0.238	0.244	0.234	0.251	0.251
$s_{e,\text{exp}}/s_e(0, 0)$	0.798	1.148	0.684	0.796	0.996
$s_e/s_e(0, 0)$	0.90	0.79	0.95	0.85	0.82
Beam power loss	0.8%	Significant but not measured	$\sim 1.0\%$	Small but not measured	30%

angle) klystrons are also below the space-charge limit, which is in agreement with their successful operations and observation of negligible beam loss. Likewise, we see that the SLAC 11.4 GHz 75 MW XP-1 (open square) and BINP/KEK 11.4 GHz 75 MW PPM-1 (open diamond) klystrons are operating above the space-charge limit which is in agreement with their observation of significant beam power loss.

By using a reasonable estimate for the beam size, either through direct simulation or through the beam envelope equations, we can immediately verify when the space-charge limit in Eq. (20) becomes important. The authors are currently in the process of investigating beam distributions for other klystron designs using particle-in-cell simulations.

V. DISCUSSION

Within the framework of the present model we have calculated a space-charge limit for azimuthally symmetric bunches only. The reason for making this ansatz is two fold. First, it allowed for a simplified expression for the force in Eqs. (9c) and (9d). Second, this assumption yields a force which is in the same direction as the displacement, which allows for a simplified analysis of the bunched beam space-charge limit in Eq. (20).

In general, however, a nonazimuthally symmetric beam which is slightly displaced off axis will have a force which is not only difficult to analytically evaluate, but will be in a different direction than the displacement. The dynamics for beams in which the center-of-mass force is not parallel to the displacement is far more complicated than the dynamics described in Sec. II. In particular, the effective center-of-mass canonical angular momentum in Eq. (14) will no longer be conserved for nonaxisymmetric distributions, which implies that the total canonical angular momentum will be shared between the center-of-mass motion and the internal bunch evolution as a function of time. While the results reported in this paper may be useful for azimuthally symmetric beams, we plan to extend the analysis to the regime of nonaxisymmetric beams. Such an analysis would be quite useful in describing the center-of-mass dynamics of non-axisymmetric beams which are of recent interest, such as elliptical or sheet beams.

Within our analysis in Sec. II resulting in the center-of-mass equations of motion for periodic magnetic focusing, Eqs. (13a) and (13b), we have ignored the effect of the bunch size relative to the variation length of the magnetic field. This assumption greatly simplifies the analysis, because including the effect requires extensive magnetic field averaging methods using individual electron orbits in a bunch. Electron orbits, in turn, require a knowledge of the evolution for the bunch distribution function. We are pursuing this calculation at present.

Finally, our model assumes that the bunch shape is constant for the center-of-mass dynamics in Sec. II. Obviously, an actual beam would undergo a shape deformation due to internal bunch forces, magnetic focusing, particle-wave interactions, etc. In Ref. [3] it was argued that the time scale of the internal bunch dynamics could be of the same order as the center-of-mass dynamics. An interesting extension of this paper would be to compute the center-of-mass dynamics in Eq. (10) for a bunched beam which has an evolving shape, i.e., $f_{lab}(\mathbf{r}_\perp, z - V_z t, t)$. As an approximation, the bunch evolution could be solved using a 2D particle-in-cell code, thereby obtaining $f_{lab}(\mathbf{r}_\perp, z - V_z t, t)$. Equation (10) could then be solved separately. We note that by implementing the bunched beam envelope equation in Sec. IV to estimate the variation in the radial size due to the periodic magnetic field, the internal space charge, and the emittance, we did not account for how the variation couples to the center-of-mass motion as a function of time.

VI. SUMMARY

In the present paper, we derived an expression for the space-charge limit, Eq. (20), on a bunch train which is slightly off axis in a perfectly conducting circular pipe. The physical origin of the force which results in the limit is due to the electrostatic interaction of each bunch with the induced surface charges on the pipe. The space-charge limit was far more general than the one derived in Refs. [3,4] for pointlike bunches, since the bunches in the present model are assumed to have arbitrary radial and longitudinal distribution functions with respect to the axis of the bunch train. Qualitatively, for a given longitudinal distribution function, the beam current limit decreases as the radial bunch profile is increased in size. For a given radial bunch profile, however, the beam current limit increases as the longitudinal beam size is increased.

We derived explicit expressions for the space-charge limit in the cases of separable distribution functions corresponding to delta, uniform, unimodal, and uniform annular in radial space, as well as for delta, uniform, Gaussian, and Lorentzian distributions in longitudinal space. We provided an explicit expression for the space-charge limit, up to an integral which must be evaluated numerically, corresponding to the uniform ellipsoidal distribution. Expressions for the space-charge limit in the special cases of unbunched and single-bunch beams were also derived.

We showed how to apply Eq. (20) to the experimental parameters of five PPM klystrons: SLAC 11.4 GHz 50 MW XL-PPM [6,7], 75 MW XP-1 [6,7], and 75 MW XP-3 [8] klystrons, the BINP/KEK 11.4 GHz 75 MW PPM-1 klystron [9], and the Toshiba/KEK 5.7 GHz 50 MW klystron [10]. Using a uniform ellipsoidal distribution function with a reasonable estimate of the rms

bunch length of 60° with respect to the rf, we computed each klystron's radial beam envelope size from the bunched beam envelope equation [12,13]. The SLAC 11.4 GHz 50 MW XL-PPM and 75 MW XP-3 klystrons, as well as the Toshiba/KEK 5.7 GHz 50 MW klystron were found to be operating below the space-charge limit, which is in agreement with their observation of negligible beam power loss and successful operations. The SLAC 11.4 GHz 75 MW XP-1 and BINP/KEK 11.4 GHz 75 MW PPM-1 klystrons, however, were found to be operating above the space-charge limit which is in agreement with their observations of significant beam power loss.

The results reported in this paper are relevant for relativistic bunched beams propagating in a long, circular conductor pipe, such as the bunched beams found in high-power klystrons. This paper is useful in understanding the nature of beam confinement for finite-sized high-intensity bunched beams found in rf accelerators and high-power microwave sources.

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APPENDIX: CALCULATION OF THE CENTER-OF-MASS BUNCH FORCE

We derive Eqs. (9), which includes the expression for $\mathbf{F}_{\text{cm}}^{\text{rest}}$. In the derivation, we utilize a Green function method in order to correctly account for the presence of the circular conductor pipe. In the average beam rest frame, we assume that each bunch can be represented within a unit cell, $r \leq a$ and $|z| \leq \gamma_b L/2$, by an azimuthally symmetric distribution function which has been offset from the pipe axis by a transverse displacement $\mathbf{R}_{\text{cm}}$, i.e.,

$$f(\mathbf{r}_\perp, z) = \sum_{n=-\infty}^{\infty} g(|\mathbf{r}_\perp - \mathbf{R}_{\text{cm}}|, z + n\gamma_b L), \quad (\text{A1a})$$

where z is the coordinate measured in the rest frame. The distribution function satisfies the normalization con-

straint and the periodicity conditions

$$\int_{\text{cell}} d^3\mathbf{r} f(\mathbf{r}_\perp, z) = 1 = 2\pi \int_{-\infty}^{\infty} dz \int_0^a dr r g(r, z), \quad (\text{A1b})$$

$$f(\mathbf{r}_\perp, z + \gamma_b L) = f(\mathbf{r}_\perp, z). \quad (\text{A1c})$$

The center-of-mass force for a finite-sized periodic beam is calculated from the Green function generated by a periodic point-charge beam with the same periodicity. Inside the circular conductor pipe, the Green function for a train of periodic unit point charges with periodicity $\gamma_b L$ and charge density,

$$\rho(\mathbf{r}) = \frac{1}{r} \delta(r - r') \delta(\theta - \theta') \sum_{m=-\infty}^{\infty} \delta(z - z' - m\gamma_b L), \quad (\text{A2})$$

is given by [3]

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}') &= \frac{1}{L_r} G_{2D} + \frac{4}{L_r} \sum_{m=1}^{\infty} \cos[m(\hat{z} - \hat{z}')] \frac{I_0(n\hat{r}_{<})}{I_0(n\alpha)} \\ &\quad \times \{I_0(n\alpha)K_0(n\hat{r}_{>}) - K_0(n\alpha)I_0(n\hat{r}_{>})\} + \frac{8}{L_r} \\ &\quad \times \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \cos[l(\theta - \theta')] \cos[m(\hat{z} - \hat{z}')] \frac{I_l(n\hat{r}_{<})}{I_l(n\alpha)} \\ &\quad \times \{I_l(n\alpha)K_l(n\hat{r}_{>}) - K_l(n\alpha)I_l(n\hat{r}_{>})\}, \end{aligned} \quad (\text{A3})$$

where $\hat{r}_{>(<)}$ denotes the greater (lesser) of $\hat{r}$ and $\hat{r}'$, and G_{2D} is the Green function for a two-dimensional rod distribution given by [17]

$$G_{2D} = \ln \left[\frac{\alpha^2 + (\hat{r}\hat{r}'/\alpha)^2 - 2\hat{r}\hat{r}'\cos(\theta - \theta')}{\hat{r}^2 + \hat{r}'^2 - 2\hat{r}\hat{r}'\cos(\theta - \theta')} \right]. \quad (\text{A4})$$

For the train of periodic unit point charges, the electric field is given by

$$\mathbf{E}(\mathbf{r}, \mathbf{r}') = -\nabla_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}'). \quad (\text{A5})$$

It is useful to decompose $\mathbf{E}(\mathbf{r}, \mathbf{r}')$ into two parts [18],

$$\mathbf{E}(\mathbf{r}, \mathbf{r}') = \mathbf{E}_{\text{free}}(\mathbf{r}, \mathbf{r}') + \mathbf{E}_{\text{surf}}(\mathbf{r}, \mathbf{r}'), \quad (\text{A6})$$

where

$$\begin{aligned} \mathbf{E}_{\text{free}}(\mathbf{r}, \mathbf{r}') &= -\nabla \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} + \sum_{m=1}^{\infty} \left(\frac{1}{|\mathbf{r} - \mathbf{r}' + m\mathbf{L}_r|} + \frac{1}{|\mathbf{r} - \mathbf{r}' - m\mathbf{L}_r|} - \frac{2}{mL_r} \right) \right] \\ &= -\nabla \left[-\frac{C}{2L} - \frac{1}{L_r} \ln[\hat{r}^2 + \hat{r}'^2 - 2\hat{r}\hat{r}'\cos(\theta - \theta')] + \frac{4}{L_r} \sum_{m=1}^{\infty} \cos[m(\hat{z} - \hat{z}')] I_0(m\hat{r}_{<}) K_0(m\hat{r}_{>}) + \frac{8}{L_r} \right. \\ &\quad \left. \times \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \cos[l(\theta - \theta')] \cos[m(\hat{z} - \hat{z}')] I_l(m\hat{r}_{<}) K_l(m\hat{r}_{>}) \right] \end{aligned} \quad (\text{A7})$$

is the free-space contribution of the real charges in Eq. (A2), and

$$\mathbf{E}_{\text{surf}}(\mathbf{r}, \mathbf{r}') = -\nabla \left\{ \frac{C}{2L} + \frac{1}{L_r} \ln[\alpha^2 + (\hat{r}\hat{r}'/\alpha)^2 - 2\hat{r}\hat{r}'\cos(\theta - \theta')] \right\} - \frac{4}{L_r} \sum_{m=1}^{\infty} \cos[m(\hat{z} - \hat{z}')] \frac{I_0(m\hat{r}_{<})I_0(m\hat{r}_{>})K_0(m\alpha)}{I_0(m\alpha)} \\ - \frac{8}{L_r} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \cos[l(\theta - \theta')] \cos[m(\hat{z} - \hat{z}')] \frac{I_l(m\hat{r}_{<})I_l(m\hat{r}_{>})K_l(m\alpha)}{I_l(m\alpha)} \quad (\text{A8})$$

is the contribution of the induced surface charges. Here, $C = 0.5772\dots$ is the Euler-Mascheroni constant.

For a train of periodic finite-size bunches with the distribution function in Eqs. (A1a)–(A1c), the total center-of-mass force on each bunch can be expressed as

$$\mathbf{F}_{\text{cm}}^{\text{rest}} = (N_b e)^2 \int_{-\infty}^{\infty} dz \int d^2\mathbf{r}_{\perp} g(|\mathbf{r}_{\perp} - \mathbf{R}_{\text{cm}}|, z) \\ \times \int_{-\infty}^{\infty} dz' \int d^2\mathbf{r}'_{\perp} g(|\mathbf{r}'_{\perp} - \mathbf{R}_{\text{cm}}|, z') \mathbf{E}(\mathbf{r}, \mathbf{r}'). \quad (\text{A9})$$

The integral over the primed coordinates in Eq. (A9) computes the electric field at a point $\mathbf{r}$ due to all of the bunches as well as the induced surface charge on the conductor. The integral over the unprimed coordinates in Eq. (A9) yields the total electrostatic force on each bunch due to the total electric field.

Because of the periodicity of the bunches, the center-of-mass force on each bunch due to all bunches through free space interaction is precisely zero, i.e.,

$$(N_b e)^2 \int_{-\infty}^{\infty} dz \int d^2\mathbf{r}_{\perp} g(|\mathbf{r}_{\perp} - \mathbf{R}_{\text{cm}}|, z) \int_{-\infty}^{\infty} dz' \\ \times \int d^2\mathbf{r}'_{\perp} g(|\mathbf{r}'_{\perp} - \mathbf{R}_{\text{cm}}|, z') \mathbf{E}_{\text{free}}(\mathbf{r}, \mathbf{r}') = 0. \quad (\text{A10})$$

We note that the integral in Eq. (A10) can be proven analytically by first showing that the free space center-of-mass force is zero when each bunch is formed by a set of discrete particles. Equation (A10) is then verified by taking the limit that the discrete particles form the continuous distribution $g(|\mathbf{r}_{\perp} - \mathbf{R}_{\text{cm}}|, z)$.

Hence, the center-of-mass force in Eq. (A9) is only due to the induced surface charges on the conductor, i.e.,

$$\mathbf{F}_{\text{cm}}^{\text{rest}} = (N_b e)^2 \int_{-\infty}^{\infty} dz \int d^2\mathbf{r}_{\perp} g(|\mathbf{r}_{\perp} - \mathbf{R}_{\text{cm}}|, z) \\ \times \int_{-\infty}^{\infty} dz' \int d^2\mathbf{r}'_{\perp} g(|\mathbf{r}'_{\perp} - \mathbf{R}_{\text{cm}}|, z') \mathbf{E}_{\text{surf}}(\mathbf{r}, \mathbf{r}'). \quad (\text{A11})$$

While Eq. (A11) is an exact expression for the center-of-mass force on any charge bunch in a periodic distribution within a conducting pipe, for the remainder of this paper we will now employ our assumption of an azimuthally symmetric bunch with $Z_{\text{cm}} = 0$ and a slight off-axis displacement $\mathbf{R}_{\text{cm}} = R_{\text{cm}} \hat{\mathbf{e}}_x$. We note that our choice of transverse displacement in the x direction is arbitrary, and is used only to simplify the following calculation.

The dipole force is calculated to order R_{cm} by expanding $g(|\mathbf{r}_{\perp} - \mathbf{R}_{\text{cm}}|, z)$ to the same order. In particular, the $g(|\mathbf{r}_{\perp} - \mathbf{R}_{\text{cm}}|, z)$ to first order in R_{cm} is

$$g(|\mathbf{r}_{\perp} - \mathbf{R}_{\text{cm}}|, z) \cong g(r, z) - R_{\text{cm}} \cos\theta \frac{\partial g(r, z)}{\partial r}, \quad (\text{A12})$$

where $r = (x^2 + y^2)^{1/2}$ and $x = r \cos\theta$. The center-of-mass force is given approximately by

$$\mathbf{F}_{\text{cm}}^{\text{rest}} = (N_b e)^2 \int_{-\infty}^{\infty} dz \int d^2\mathbf{r}_{\perp} \int_{-\infty}^{\infty} dz' \int d^2\mathbf{r}'_{\perp} \left[g(r, z) \right. \\ \left. - R_{\text{cm}} \frac{\partial g}{\partial r} \cos\theta \right] \left[g(r', z') - R_{\text{cm}} \frac{\partial g}{\partial r'} \cos\theta' \right] \\ \times \mathbf{E}_{\text{surf}}(\mathbf{r}, \mathbf{r}'). \quad (\text{A13})$$

However, the zeroth order term is precisely zero, i.e.,

$$(N_b e)^2 \int_{-\infty}^{\infty} dz \int d^2\mathbf{r}_{\perp} g(r, z) \int_{-\infty}^{\infty} dz' \\ \times \int d^2\mathbf{r}'_{\perp} g(r', z') \mathbf{E}_{\text{surf}}(\mathbf{r}, \mathbf{r}') = 0, \quad (\text{A14})$$

since an azimuthally symmetric distribution will induce an azimuthally symmetric surface charge, which yields zero force contribution. To first order in R_{cm} , the center-of-mass force is approximately

$$\mathbf{F}_{\text{cm}}^{\text{rest}} = -R_{\text{cm}\perp} (N_b e)^2 \\ \times \int_{-\infty}^{\infty} dz \int d^2\mathbf{r}_{\perp} \int_{-\infty}^{\infty} dz' \int d^2\mathbf{r}'_{\perp} h(\mathbf{r}, \mathbf{r}') \mathbf{E}_{\text{surf}}(\mathbf{r}, \mathbf{r}'), \quad (\text{A15})$$

where

$$h(\mathbf{r}, \mathbf{r}') = g(r, z) \frac{\partial g(r', z')}{\partial r'} \cos\theta' + g(r', z') \frac{\partial g(r, z)}{\partial r} \cos\theta. \quad (\text{A16})$$

Azimuthal integration of Eq. (A15) immediately shows that the center-of-mass force has a zero component in the z direction which yields Eq. (9b). Hence, the center-of-mass force can only be in the transverse direction, and by symmetry, or equivalently by azimuthal integration, can only be in the direction of displacement, $\mathbf{R}_{\text{cm}} = R_{\text{cm}} \hat{\mathbf{e}}_x$. Therefore, we only need to compute $\mathbf{E}_{\text{surf}}(\mathbf{r}, \mathbf{r}')$ in the $\hat{\mathbf{e}}_x$ direction, i.e.,

$$\mathbf{E}_{\text{surf}}(\mathbf{r}, \mathbf{r}') \cdot \hat{\mathbf{e}}_x = \frac{4\pi}{\gamma_b^2 L^2} \left[\frac{\hat{r}' \cos \theta' - \hat{r}(\hat{r}'/\alpha)^2 \cos \theta}{\alpha^2 + (\hat{r}\hat{r}'/\alpha)^2 - 2\hat{r}\hat{r}' \cos(\theta - \theta')} \right] + \mathbf{E}_2(\mathbf{r}, \mathbf{r}'), \quad (\text{A17})$$

where

$$\begin{aligned} \mathbf{E}_2(\mathbf{r}, \mathbf{r}') = & \frac{8\pi}{\gamma_b^2 L^2} \sum_{m=1}^{\infty} \frac{m I_0(m\hat{r}') I_1(m\hat{r}) K_0(m\alpha)}{I_0(m\alpha)} \cos[m(\hat{z} - \hat{z}')] \cos \theta + \frac{16\pi}{\gamma_b^2 L^2} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{m I_l(m\hat{r}') K_l(m\alpha)}{I_l(m\alpha)} \cos[m(\hat{z} - \hat{z}')] \\ & \times \left\{ m I_{l+1}(m\hat{r}) \cos[l(\theta - \theta')] \cos \theta + \frac{I_l(m\hat{r})}{\hat{r}} (\cos[l(\theta - \theta')] \cos \theta + \sin[l(\theta - \theta')] \sin \theta) \right\}. \end{aligned} \quad (\text{A18})$$

Integrating over θ and θ' on the terms in $\mathbf{E}_2$, we can compute its contribution to $\mathbf{F}_{\text{cm}}^{\text{rest}}$, which we call $\mathbf{F}_2$, to the center of force in Eq. (A15). In particular,

$$\begin{aligned} \mathbf{F}_2 = & -\frac{16\pi^3 \mathbf{R}_{\text{cm}} (N_b e)^2}{\gamma_b^2 L^2} \int_{-\infty}^{\infty} dz \int_0^a dr r \int_{-\infty}^{\infty} dz' \int_0^a dr' r' g(r', z') \frac{\partial g(r, z)}{\partial r} \sum_{m=1}^{\infty} m \cos[m(\hat{z} - \hat{z}')] \\ & \times \left\{ \frac{I_0(m\hat{r}') I_1(m\hat{r}) K_0(m\alpha)}{I_0(m\alpha)} + \frac{I_1(m\hat{r}) K_0(m\alpha)}{I_0(m\alpha)} [I_2(m\hat{r}') + \frac{2}{m\hat{r}'} I_1(m\hat{r}')] \right\}. \end{aligned} \quad (\text{A19})$$

The expression on the right-hand side of Eq. (A19) can be simplified by using the Bessel function identity $I_0(x) = I_2(x) + 2I_1(x)/x$ and the Wronskian relation $I_0(x)K_1(x) + I_1(x)K_0(x) = 1/x$. The simplified expression is

$$\mathbf{F}_2 = -\frac{16\pi^3 \mathbf{R}_{\text{cm}} (N_b e)^2}{\alpha \gamma_b^2 L^2} \int_{-\infty}^{\infty} dz \int_0^a dr r \int_{-\infty}^{\infty} dz' \int_0^a dr' r' g(r', z') \left\{ \frac{\partial g(r, z)}{\partial r} \sum_{m=1}^{\infty} \frac{\cos[m(\hat{z} - \hat{z}')] I_0(m\hat{r}') I_1(m\hat{r})}{I_0(m\alpha) I_1(m\alpha)} \right\}. \quad (\text{A20})$$

The contribution to $\mathbf{F}_{\text{cm}}^{\text{rest}}$ due to the term in brackets in Eq. (A17), which we call $\mathbf{F}_1$, can be computed with well-known trigonometric integrals. The θ' integral over the bracketed term multiplied by $\cos \theta$ gives zero since

$$\begin{aligned} & \int_0^{2\pi} d\theta' \cos \theta \left[\frac{\hat{r}' \cos \theta' - \hat{r}(\hat{r}'/\alpha)^2 \cos \theta}{\alpha^2 + (\hat{r}\hat{r}'/\alpha)^2 - 2\hat{r}\hat{r}' \cos(\theta - \theta')} \right] \\ & = \int_0^{2\pi} d\phi \cos \theta \left[\frac{\hat{r}' \cos(\phi + \theta) - \hat{r}(\hat{r}'/\alpha)^2 \cos \theta}{\alpha^2 + (\hat{r}\hat{r}'/\alpha)^2 - 2\hat{r}\hat{r}' \cos(\phi)} \right] \\ & = \int_0^{2\pi} d\phi \frac{\hat{r}'}{\alpha^2} \left[\frac{\cos^2 \theta (\cos \phi - \hat{r}\hat{r}'/\alpha^2) - \sin \theta \cos \theta \sin \phi}{1 + (\hat{r}\hat{r}'/\alpha^2)^2 - 2(\hat{r}\hat{r}'/\alpha^2) \cos(\phi)} \right] = 0. \end{aligned} \quad (\text{A21})$$

In line 3 of Eq. (A21), we made use of the two integral relations [19,20],

$$\int_0^{2\pi} d\phi \left[\frac{\cos \phi - x}{1 + x^2 - 2x \cos \phi} \right] = 0, \quad 0 < x < 1, \quad (\text{A22a})$$

$$\int_0^{2\pi} d\phi \left[\frac{\sin \phi}{1 + x^2 - 2x \cos \phi} \right] = 0. \quad (\text{A22b})$$

The θ integral over the bracketed term in Eq. (A17) multiplied by $\cos \theta'$ yields

$$\begin{aligned} & \int_0^{2\pi} d\theta \cos \theta' \left[\frac{\hat{r}' \cos \theta' - \hat{r}(\hat{r}'/\alpha)^2 \cos \theta}{\alpha^2 + (\hat{r}\hat{r}'/\alpha)^2 - 2\hat{r}\hat{r}' \cos(\theta - \theta')} \right] \\ & = \int_0^{2\pi} d\phi \left[\frac{\hat{r}' \cos^2 \theta' - \hat{r}(\hat{r}'/\alpha)^2 \cos(\phi + \theta') \cos \theta'}{\alpha^2 + (\hat{r}\hat{r}'/\alpha)^2 - 2\hat{r}\hat{r}' \cos(\phi)} \right] \\ & = \int_0^{2\pi} d\phi \frac{\hat{r}'}{\alpha^2} \left[\frac{\cos^2 \theta' - \hat{r}\hat{r}'/\alpha^2 \cos \phi \cos^2 \theta' + \hat{r}\hat{r}'/\alpha^2 \sin \theta' \cos \theta' \sin \phi}{1 + (\hat{r}\hat{r}'/\alpha^2)^2 - 2(\hat{r}\hat{r}'/\alpha^2) \cos(\phi)} \right] = \frac{2\pi \hat{r}'}{\alpha^2} \cos^2 \theta'. \end{aligned} \quad (\text{A23})$$

In line 3 of Eq. (A23), we made use of Eqs. (A22) as well as the relation [21]

$$\int_0^{2\pi} d\phi \left[\frac{1 - x \cos \phi}{1 + x^2 - 2x \cos \phi} \right] = 2\pi. \quad (\text{A24})$$

Therefore, $\mathbf{F}_1$ is readily found to be

$$\mathbf{F}_1 = -\frac{8\pi^3 \mathbf{R}_{\text{cm}} (N_b e)^2}{\alpha \gamma_b^2 L^2 a} \int_{-\infty}^{\infty} dz \int_0^a dr r \int_{-\infty}^{\infty} dz' \int_0^a dr' r'^2 g(r, z) \frac{\partial g(r', z')}{\partial r'}, \quad (\text{A25})$$

and hence the center-of-mass force becomes

$$\begin{aligned} \mathbf{F}_{\text{cm}}^{\text{rest}} &= \mathbf{F}_1 + \mathbf{F}_2 \\ &= -\frac{8\pi^3 \mathbf{R}_{\text{cm}} (N_b e)^2}{\alpha \gamma_b^2 L^2} \int_{-\infty}^{\infty} dz \int_0^a dr r \int_{-\infty}^{\infty} dz' \int_0^a dr' r' g(r', z') \frac{\partial g(r, z)}{\partial r} \left\{ \frac{r}{a} + 2 \sum_{m=1}^{\infty} \frac{\cos[m(\hat{z} - \hat{z}')] I_0(m\hat{r}') I_1(m\hat{r})}{I_0(m\alpha) I_1(m\alpha)} \right\}, \end{aligned} \quad (\text{A26})$$

where in Eq. (A26) we have combined $\mathbf{F}_1$ and $\mathbf{F}_2$ under the common integrand $g(r', z') \partial g(r, z) / \partial r$ since the primed and unprimed coordinates are dummy variables.

By exploiting the fact that the charge density at the conductor boundary is zero, i.e., $g(r = a, z) = 0$, then integration by parts may be utilized to transform Eq. (A26)) into

$$\mathbf{F}_{\text{cm}}^{\text{rest}} = \frac{8\pi^3 (N_b e)^2 \mathbf{R}_{\text{cm}}}{\alpha \gamma_b^2 L^2} \int_{-\infty}^{\infty} dz \int_0^a dr r \int_{-\infty}^{\infty} dz' \int_0^a dr' r' g(r', z') g(r, z) \left\{ \frac{2r}{a} + 2 \sum_{m=1}^{\infty} \frac{\cos[m(\hat{z} - \hat{z}')] I_0(m\hat{r}')}{I_0(m\alpha) I_1(m\alpha)} \frac{\partial}{\partial r} [r I_1(m\hat{r})] \right\}. \quad (\text{A27})$$

Finally, using the normalization constraint on $g(r, z)$, as well as the relation $\partial [r I_1(m\hat{r})] / \partial r = m\hat{r} I_0(m\hat{r})$, we obtain the result as shown in Eqs. (9c) and (9d),

$$\mathbf{F}_{\text{cm}}^{\text{rest}} = \frac{N_b^2 e^2 \alpha \mathbf{R}_{\text{cm}}}{\pi a^3} \left\{ 1 + \sum_{m=1}^{\infty} \frac{A_m m \alpha}{I_0(m\alpha) I_1(m\alpha)} \right\}, \quad (\text{A28a})$$

where

$$A_m = (2\pi)^2 \int_{-\infty}^{\infty} dz \int_0^a dr r \int_{-\infty}^{\infty} dz' \int_0^a dr' r' g(r', z') g(r, z) \cos[m(\hat{z} - \hat{z}')] I_0(m\hat{r}) I_0(m\hat{r}'). \quad (\text{A28b})$$

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