Perturbation treatment of linear beam optics of combined function magnets

M. Seidl*

Institut für Kernphysik, Universität Mainz, D-55099 Mainz, Germany

(Received 23 October 2001; published 12 June 2002)

In this paper, a perturbation theory is developed for calculating the 6D transfer matrix of a combined function magnet. Starting from the linear part of the Hamiltonian, the transfer matrix M is decomposed into an infinite sum over matrices M_n , which represent an n -fold interaction of the beam with the quadrupole content of the magnet. Given the magnetic field in the vicinity of the reference path, each matrix M_n can be calculated either analytically or by numerical integration. As an example, the transfer matrix is calculated of the 90° bending magnet of the double sided microtron HDSM at the MAMI-C electron scattering facility, which so far has been possible only by means of particle tracking.

DOI: 10.1103/PhysRevSTAB.5.064001

PACS numbers: 29.20.-c, 41.85.Lc

I. INTRODUCTION

Particle tracking [1,2] and power series tracking [3] are two widely used tools for investigating the beam dynamics of combined function magnets. For general purpose fields, both approaches use sophisticated techniques to integrate the equation of motion, e.g., high-order Runge Kutta [4] integrators or differential algebra [5] integrators. Only in very rare cases can these integrators calculate the map between the initial and final coordinates in one step. Usually the transformations of several steps are concatenated. For example, a very rough estimate of the transfer matrix M of a combined function magnet is obtained by writing M as a product of matrices $M_c \cdot M_b \cdot M_a$, where M_b is the transfer matrix of a uniform-field sector dipole, and M_a and M_c , respectively, represent the entrance and exit fringe fields [6]. The disadvantage of this approach is that one must completely recalculate the map if, for example, the accuracy of the model is refined, or if the magnetic field is modified. For such advanced combined function magnets as the 90° bending magnets of the double sided microtron HDSM at MAMI-C (cf. Sec. IV), the map typically has to be calculated for several thousand different field profiles when designing the magnet. Only modest accuracy in accordance with manufacturing tolerance is required for M . Here perturbation theory offers several advantages. Most importantly, the maps for different field profiles can be derived in a systematic manner from a reference map, thereby reducing calculation time. Moreover, perturbation theory offers a clear physical picture of what kind of interactions dominate the particle motion and what kind of interactions can be neglected in practice.

In the following, the linear beam dynamics of a combined function magnet will be considered. In Sec. II, it will be shown that M can be expressed as an infinite sum over matrices M_n , which amounts to a perturbative solu-

tion of the Hamiltonian equation of motion. The matrices M_n successively increase the accuracy of M by an order of magnitude and can be calculated analytically or by numerical integration, given the magnetic field in the vicinity of the reference path. Implementation of perturbation theory on a computer is discussed in Sec. III. As an example, the matrices M_n are calculated for a 90° HDSM magnet and compared with the result from particle tracking in Sec. V. Furthermore, the effect of field errors on the map M is investigated within the framework of perturbation theory in Sec. VI. Finally, in Sec. VII, perturbation theory is combined with differential algebra techniques to calculate higher order aberrations.

The horizontal and vertical deviations of a particle from the reference path [7] are called x and y , respectively. The corresponding canonical momentum is p_x and p_y . In the following, it is assumed that the particle interacts only with the vector potential $\vec{A}$ of the magnet, and that its synchrotron radiation losses are negligible. Introducing the path length z instead of time t as an independent variable leads to the following Hamiltonian [8]:

$$H = -h\sqrt{\frac{p_t^2}{c^2} - m_0^2 c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2} - ehA_z. \quad (1)$$

The rest mass of the particle with charge e is m_0 and its total energy is p_t . The bending radius R usually is a function of z and

$$h = 1 + \frac{x}{R}, \quad (2)$$

if the reference path lies entirely in the horizontal plane. The following two generating functions $F_3(Q, p, z)$ and $F_2(q, P, z)$

$$F_3(x, y, \delta\varphi, p_x, p_y, p_t, z) = -p_x x - p_y y + p_t \delta\varphi - \frac{z}{c} p_t, \quad (3)$$

$$F_2(x, y, \delta\varphi, p_x, p_y, \delta E, z) = p_x x + p_y y + \delta E \delta\varphi - E \delta\varphi, \quad (4)$$

*Present address: TÜV Süddeutschland Bau und Betrieb GmbH, D-80686 München, Germany.

replace z with $\delta\varphi$ and p_t with δE :

$$\delta\varphi = \frac{z}{c} - t, \quad (5)$$

$$\delta E = E - p_t, \quad (6)$$

where E is the reference energy. Neglecting m_0c , the new Hamiltonian is

$$H = \frac{E - \delta E}{c} - h\sqrt{\frac{1}{c^2}(E - \delta E)^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2} - ehA_z. \quad (7)$$

Considering only linear beam dynamics and particles at relativistic energies, it suffices to develop H to second order

$$H = \frac{1}{c}(E - \delta E)(1 - h) + \frac{hc}{2E} \times [(p_x - eA_x)^2 + (p_y - eA_y)^2] - ehA_z + \dots \quad (8)$$

Defining

$$\vec{v} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta\varphi \\ \delta E \end{pmatrix}; \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad (9)$$

yields the following equation of motion [9]:

$$\frac{d\vec{v}}{dz} = S \frac{\partial H}{\partial \vec{v}}. \quad (10)$$

Only taking the Jacobian matrix J of H into account, the solution of Eq. (10) can be written as

$$\vec{v}(z) = \vec{v}(0) + \int_0^z ds SJ(s)\vec{v}(s). \quad (11)$$

If J is not a function of z , the solution for M is trivially [10] found to be

$$\vec{v}(z) = e^{SJz}\vec{v}(0) := M\vec{v}(0). \quad (12)$$

II. PERTURBATION THEORY

To calculate the solution of Eq. (11) in a perturbative manner, it is assumed that J can be separated as follows:

$$J(z) = J_0 + \delta J(z), \quad (13)$$

where J_0 does not depend on z . If $\vec{v}(z)$ is a solution of Eq. (11), a new vector $\vec{v}(z)$ is defined according to

$$\vec{v}(z) = e^{-SJ_0z}\vec{v}(z). \quad (14)$$

The equation of motion for $\vec{v}$ is

$$\frac{d\vec{v}(z)}{dz} = \delta\tilde{J}(z)\vec{v}(z), \quad (15)$$

where

$$\delta\tilde{J}(z) = e^{-SJ_0z}[S\delta J(z)]e^{SJ_0z}. \quad (16)$$

Analogous to Eq. (11), the solution of (15) is

$$\vec{v}(z) = \vec{v}(0) + \int_0^z ds_1 \delta\tilde{J}(s_1)\vec{v}(s_1). \quad (17)$$

If the perturbation $\delta\tilde{J}$ is small, the integral equation Eq. (17) can be solved iteratively [11]:

$$\begin{aligned} \vec{v}(z) &= \left(1 + \int_0^z ds_1 \delta\tilde{J}(s_1) \right. \\ &\quad \left. + \int_0^z ds_1 \delta\tilde{J}(s_1) \int_0^{s_1} ds_2 \delta\tilde{J}(s_2) + \dots \right) \vec{v}(0) \\ &:= \mathcal{R}(z)\vec{v}(0). \end{aligned} \quad (18)$$

Introducing a time-ordered product

$$\begin{aligned} T[\delta\tilde{J}(s_1)] &= \delta\tilde{J}(s_1) \\ T[\delta\tilde{J}(s_1)\delta\tilde{J}(s_2)] &= \Theta(s_1 - s_2)\delta\tilde{J}(s_1)\delta\tilde{J}(s_2) \\ &\quad + \Theta(s_2 - s_1)\delta\tilde{J}(s_2)\delta\tilde{J}(s_1) \\ &\quad \vdots \end{aligned} \quad (19)$$

the matrix $\mathcal{R}(z)$ can be rewritten in more compact form as [12]

$$\mathcal{R}(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^z ds_1 \dots ds_n T[\delta\tilde{J}(s_1) \dots \delta\tilde{J}(s_n)]. \quad (20)$$

To gain some insight into the meaning of $\mathcal{R}(z)$, consider the vertical phase space (y, p_y) . In the absence of skew quadrupoles, it is decoupled from the horizontal (x, p_x) and longitudinal $(\delta\varphi, \delta E)$ phase spaces. The unperturbed Jacobian J_0 in this case generates a simple drift space (see Sec. V):

$$M_0(z) = e^{SJ_0z} = \begin{pmatrix} 1 & \frac{c}{E}z \\ 0 & 1 \end{pmatrix}. \quad (21)$$

The quadrupole strength of the combined function magnet leads to a perturbation of the form

$$S\delta J(z) = \begin{pmatrix} 0 & 0 \\ g(z) & 0 \end{pmatrix}. \quad (22)$$

This is just the transfer matrix of a thin lens quadrupole with the diagonal elements stripped away. The vertical transfer matrix M of the magnet according to Eqs. (14) and (20) is then

$$M = M_0(z)\mathcal{R}(z) = M_0 + M_1 + M_2 + \dots \quad (23)$$

The contributions M_n are

$$\begin{aligned} M_0 &= M_0(z) \\ M_1 &= \int_0^z ds_1 T[M_0(z - s_1)S\delta J(s_1)M_0(s_1)] \\ M_2 &= \frac{1}{2} \int_0^z ds_2 ds_1 T[M_0(z - s_2)S\delta J(s_2) \\ &\quad \times M_0(s_2 - s_1)S\delta J(s_1)M_0(s_1)] \\ &\quad \vdots \end{aligned} \quad (24)$$

Obviously the matrix M_n represents an n -fold interaction of the beam with the quadrupole perturbation δJ , contributing a correction on the order of g^n to M . The integrals in Eq. (24) average over all possible locations $s_1, s_2, \dots$, whereby the time-ordered product T preserves causality. Since the beam moves straight from 0 to z , an interaction at $s_2 > s_1$ is possible only if the beam has already passed s_1 . Just as in elementary particle physics, the contributions M_n can be ordered according to the number of interactions n of the beam with the quadrupole content of the magnet and can be visualized by some sort of Feynman diagrams, as shown in Fig. 1.

FIG. 1. (Color) Visualization of the contributions M_n from perturbation theory according to the number of interactions n of the beam with the quadrupole strength of the magnet.

III. IMPLEMENTATION OF PERTURBATION THEORY ON A COMPUTER

Perturbation theory can be implemented on a computer in the following way. Assume that $0 \leq s \leq z$, and define functions r_n according to

$$r_1(s) = \int_0^s d\sigma \delta\tilde{J}(\sigma), \quad (25)$$

and

$$r_n(s) = \int_0^s d\sigma \delta\tilde{J}(\sigma)r_{n-1}(\sigma); \quad n \geq 2. \quad (26)$$

Then according to Eq. (18)

$$\mathcal{R}(z) = 1 + r_1(z) + r_2(z) + \dots \quad (27)$$

If the value of $\delta\tilde{J}(\sigma)$ is evaluated and stored at k points in the interval $[0, z]$, the value of the function r_1 at these points can be calculated in k steps. Generally, if the value of the function r_{n-1} has been stored at these points, r_n can be calculated in k steps. Therefore the computational effort of calculating M_n is independent of n and the total effort of calculating M to n th order is proportional to n .

IV. MAMI-C

A double sided microtron HDSM is currently being built at the MAMI-C electron scattering facility [13]. Up to now, three racetrack microtrons have been cascaded to provide a cw electron beam at 855 MeV [14]. In a fourth stage, the HDSM will raise the beam energy to 1500 MeV in 43 cycles.

The magnetic field B_y in the midplane of the 90° HDSM dipoles (cf. Fig. 2) depends only on z_0 and is perpendicular

FIG. 2. (Color) Fringe field and main field regions of a 90° HDSM bending magnet.

FIG. 3. (Color) Magnetic field $B_y(z_0)$ in the midplane of a 90° HDSM bending magnet.

to the reference path. Thus the matrix element M_{21} of the transfer matrix of the whole magnet is independent of $B_y(z_0)$ and is consistently zero. To nullify the strong vertical defocusing forces of the fringe field, the magnets have an additional field gradient [15] in the main field region (cf. Fig. 3). Given the fringe field, the field $B_y(z_0)$ in the main field region is determined by the condition that the matrix element M_{43} of the transfer matrix of the whole magnet be zero [16] for all beam energies between 855 and 1500 MeV. Therefore the HDSM dipoles are designed to have no focusing properties in both the horizontal and vertical phase spaces.

V. PERTURBATION TREATMENT OF THE HDSM MAGNETS

To calculate M , the vector potential $\vec{A}$ is needed. In the vicinity of the reference path, the magnetic field $\vec{B}$ is equal to the gradient of a scalar function Ψ [17]. Assuming that the magnetic field changes only slowly with z , we have

$$\begin{aligned} \Psi = & a_{1,0}x + a_{0,1}y \\ & + a_{1,1}xy + \frac{1}{2} \left(a_{0,2} + \frac{a_{1,0}}{2R} \right) (y^2 - x^2) \\ & - \frac{a_{1,0}}{4R} (y^2 + x^2) + \dots, \end{aligned} \quad (28)$$

where

$$a_{0,1} = B_y(z); \quad a_{1,1} = \frac{\partial B_y}{\partial x}(z), \quad (29)$$

and

$$a_{1,0} = B_x(z); \quad a_{0,2} = \frac{\partial B_y}{\partial y}(z). \quad (30)$$

Neglecting the solenoid fields, the vector potential can be found by partial integration of Ψ

$$\begin{aligned} \vec{A} = & 0 \cdot \hat{x} + 0 \cdot \hat{y} \\ & + \frac{1}{h} \left(\int_0^y ds \left[h \frac{\partial \Psi}{\partial x} \right] (0, s, z) \right. \\ & \left. - \int_0^x ds \left[h \frac{\partial \Psi}{\partial y} \right] (s, y, z) \right) \hat{z}, \end{aligned} \quad (31)$$

and thus

$$\begin{aligned} h\vec{A} = & \left[-a_{0,1}x \left(1 + \frac{x}{2R} \right) + \frac{a_{1,1}}{2} (y^2 - x^2) \right] \hat{z} \\ & + \left[a_{1,0}y - a_{0,2}xy \left(1 + \frac{x}{2R} \right) \right] \hat{z} + \dots \end{aligned} \quad (32)$$

In the following, it is assumed that both $a_{1,0}$ and $a_{0,2}$ are zero. Field errors of this kind are discussed in Sec. VI. Under the condition that

$$eB_y(z) = \frac{E}{cR(z)}, \quad (33)$$

the Jacobian matrix $J_0 + \delta J$ of H becomes

$$SJ_0 = \begin{pmatrix} 0 & \frac{c}{E} & 0 & 0 & 0 & 0 \\ -\frac{E}{cR_z} & 0 & 0 & 0 & 0 & -\frac{1}{cR_\infty} \\ 0 & 0 & 0 & \frac{c}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{cR_\infty} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (34)$$

and

$$S\delta J = \begin{pmatrix} -\frac{E}{c} \left(\frac{1}{R^2} - \frac{1}{R_z^2} \right) - e\partial_x B_y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{Rc} + \frac{1}{R_\infty c} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e\partial_x B_y & 0 & 0 & 0 & 0 \\ \frac{1}{Rc} - \frac{1}{R_\infty c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (35)$$

where R_∞ is defined by

$$\frac{\pi}{2} R_\infty = \int_{\theta=45^\circ}^{\theta=135^\circ} dz. \quad (36)$$

The unperturbed matrix M_0 is [7]

$$M_0(z) = e^{SJ_0 z} = \begin{pmatrix} \cos\phi & \beta \sin\phi & 0 & 0 & 0 & \frac{\beta}{c}(\cos\phi - 1) \\ -\frac{1}{\beta} \sin\phi & \cos\phi & 0 & 0 & 0 & -\frac{1}{c} \sin\phi \\ 0 & 0 & 1 & \frac{c}{E}z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{c} \sin\phi & \frac{\beta}{c}(1 - \cos\phi) & 0 & 0 & 1 & \frac{\beta}{c^2}(\sin\phi - \phi) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (37)$$

where

$$\beta = \frac{R_\infty c}{E}; \quad \phi = \frac{z}{R_\infty}. \quad (38)$$

To calculate the matrices $M_1, M_2, \dots$, the partial derivative $\partial_x B_y$ is expressed as

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_y}{\partial x_0} \frac{dx_0}{dx} + \frac{\partial B_y}{\partial z_0} \frac{dz_0}{dx} = -\frac{\partial B_y}{\partial z_0} \sin\theta(z_0). \quad (39)$$

Since

$$Rd\theta = dz, \quad (40)$$

it follows that

$$\sin\theta(z_0) = \sin(45^\circ) + \frac{ec}{E} \int_0^{z_0} dz_0 B_y(z_0). \quad (41)$$

Finally, the path length $z(z_0)$ can be calculated from

$$z(z_0) = \int_0^{z_0} \frac{dz_0}{\cos\theta(z_0)}. \quad (42)$$

Therefore, if $B_y(z_0)$ is given as in Fig. 2, the matrices M_n can be calculated by numerical integration of Eq. (24) after a change of variables from z to z_0 . Figure 4 shows the vertical matrix elements of M up to third order in perturbation theory at 1018 MeV beam energy. There the abscissa is defined as follows:

$$\begin{aligned} z'_0(\theta) &= z_0(\theta); & 45^\circ \leq \theta \leq 90^\circ, \\ z'_0(\theta) &= 2z_0(90^\circ) - z_0(\theta); & 90^\circ < \theta \leq 135^\circ. \end{aligned}$$

To test the accuracy of perturbation theory, the field $B_y(z_0)$ in the main field region was recalculated given the field in the fringe field region. The optimization procedure

FIG. 4. (Color) Vertical matrix elements of a 90° HD SM magnet.

FIG. 5. (Color) $B_y(z_0)$ in the main field region of a 90° HDSM magnet calculated by particle tracking and by perturbation theory.

was identical to that used by Ratschow [16], but particle tracking was replaced by perturbation theory. In short, assume that at $E_1 = 855$ MeV the particle trajectory reaches its apex ($\theta = \pi/2$) at $z_{0,1}$. The energy is then slightly increased by the amount $\Delta E = 5$ MeV to E_2 . The field in the interval between $z_{0,1}$ and $z_{0,2}$ is taken to be

$$B_y(z_0) = B_y(z_{0,1}) + \epsilon \frac{\partial B_y}{\partial z_{0,1}} (z_0 - z_{0,1}), \quad (43)$$

and ϵ is varied until the matrix element M_{43} is zero for E_2 . These steps are repeated until the maximum beam energy of 1500 MeV is reached. Figure 5 shows $B_y(z_0)$ in the main field region for calculations up to third order in perturbation theory. The third order calculation deviates less than 0.07% from the results of particle tracking, which is sufficient for most practical purposes.

VI. INVESTIGATION OF FIELD ERRORS

If the magnetic field is not symmetrical with respect to the midplane of the magnet, the derivative $\partial_y B_y$ is not equal to zero as assumed in Sec. V. Since the two poles of a HDSM magnet cannot be manufactured completely identical, it is of interest to note how $\partial_y B_y$ changes the transfer matrix. Thus an additional perturbation of

$$S \delta J' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -e \partial_y B_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -e \partial_y B_y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (44)$$

has to be added to Eq. (35). The first order contribution of $\delta J'$ to M is

$$M'_1 = \int_0^z ds M_0(z-s) S \delta J'(s) M_0(s). \quad (45)$$

To investigate its impact on a 90° HDSM magnet, we define

$$g_q = -e \frac{R_\infty}{2} \int_0^{\pi/2} d\phi \frac{\partial B_y}{\partial y} e^{iq\phi}; \quad g'_q = \frac{dg_q}{dq}, \quad (46)$$

where $q = -1, 0, 1$. If one makes the simplifying assumption that

$$\frac{\partial B_y}{\partial y}(z) = \frac{\partial B_y}{\partial y} \left(\frac{\pi}{2} R_\infty - z \right), \quad (47)$$

then one obtains

$$\begin{aligned} M'_1 &= \begin{pmatrix} 0 & G_{12} & 0 \\ G_{21} & 0 & G_{23} \\ 0 & G_{32} & 0 \end{pmatrix}, \\ G_{12} &= \begin{pmatrix} \beta(g_1 + g_{-1}) & -i\beta^2(g'_1 + g'_{-1}) \\ -i(g_1 - g_{-1}) & -\beta(g'_1 - g'_{-1}) \end{pmatrix}, \\ G_{21} &= \begin{pmatrix} -\beta(g'_1 - g'_{-1}) & -i\beta^2(g'_1 + g'_{-1}) \\ g_1 + g_{-1} & -i\beta(g_1 - g_{-1}) \end{pmatrix}, \\ G_{23} &= \begin{pmatrix} 0 & \frac{\beta^2}{c}(2ig'_0 - g'_1 + g'_{-1}) \\ 0 & (-2g_0 + g_1 + g_{-1})\frac{\beta}{c} \end{pmatrix}, \\ G_{32} &= \begin{pmatrix} \frac{\beta}{c}(2g_0 + ig_1 - ig_{-1}) & \frac{\beta^2}{c}(-2ig_0 + g'_1 - g'_{-1}) \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (48)$$

Obviously, the skew quadrupole content of the magnet leads to a coupling between the horizontal, vertical, and longitudinal phase spaces. A characteristic feature of the coupling is that an energy deviation δE at the entrance of the magnet excites only vertical betatron oscillations, if

observed in a region with zero horizontal dispersion. This can be helpful to distinguish between azimuthal alignment errors of quadrupoles used for beam focusing and skew quadrupole contributions stemming from the bending magnets. Moreover, if one wants to model the skew quadrupole

content of the magnet without assuming a certain field distribution and without recalculating the design map M , all possible first order effects can be accounted for, if the matrix M'_1 is added to M .

VII. PERTURBATION THEORY AND DIFFERENTIAL ALGEBRA TECHNIQUES

In this section, we dispense with the assumption that the perturbation $\delta\tilde{J}$ is a linear function of $\tilde{\nu}$. Then Eq. (17) becomes

$$\tilde{\nu}(z, \tilde{\nu}_0) = \tilde{\nu}_0 + \int_0^z ds_1 \delta\tilde{J}[s_1, \tilde{\nu}(s_1, \tilde{\nu}_0)]. \quad (49)$$

The linear part of the map M between $\tilde{\nu}_0$ and $\tilde{\nu}(z)$ is

$$\frac{\partial \tilde{\nu}(z, \tilde{\nu}_0)}{\partial \tilde{\nu}_0} = 1 + \int_0^z ds_1 D_2 \delta\tilde{J}[s_1, \tilde{\nu}(s_1, \tilde{\nu}_0)] \frac{\partial \tilde{\nu}(s_1, \tilde{\nu}_0)}{\partial \tilde{\nu}_0}, \quad (50)$$

where

$$D_2 \delta\tilde{J} = \frac{\partial}{\partial \tilde{\nu}} \delta\tilde{J}(s_1, \tilde{\nu}). \quad (51)$$

The integral equation (50) is again solved iteratively

$$\begin{aligned} \frac{\partial \tilde{\nu}(z, \tilde{\nu}_0)}{\partial \tilde{\nu}_0} = & \left(1 + \int_0^z ds_1 D_2 \delta\tilde{J}[s_1, \tilde{\nu}(s_1, \tilde{\nu}_0)] \right. \\ & + \int_0^z ds_1 D_2 \delta\tilde{J}[s_1, \tilde{\nu}(s_1, \tilde{\nu}_0)] \\ & \times \left. \int_0^{s_1} ds_2 D_2 \delta\tilde{J}[s_2, \tilde{\nu}(s_2, \tilde{\nu}_0)] + \dots \right). \end{aligned} \quad (52)$$

Because of the special choice of the reference path, we know that for all z

$$\tilde{\nu}(z, 0) = \vec{0}. \quad (53)$$

Therefore the map M can be calculated in the vicinity of the reference path by inserting Eq. (53) into (52). Nonlinear terms of M can now be investigated by considering $\tilde{\nu}$ as a DA vector [18], which in the following is written as $[\tilde{\nu}]$. The n th order derivative of $\tilde{\nu}$ with respect to $\tilde{\nu}_0$ is symbolically written as $\partial_{\nu_0}^n \tilde{\nu}$. It can be calculated iteratively with the help of Eq. (52). To this end we assume $[\tilde{\nu}_0]$ to be the following n -dimensional DA vector

$$[\tilde{\nu}_0] = (0, 1, 0, \dots, 0). \quad (54)$$

Insertion of Eq. (54) into $\tilde{\nu}(z, \tilde{\nu}_0)$ and into the left-hand side of Eq. (52) yields

$$\tilde{\nu}(z, [\tilde{\nu}_0]) = (0, \partial_{\nu_0}^1 \tilde{\nu}(z, 0), \dots, \partial_{\nu_0}^{n-1} \tilde{\nu}(z, 0)), \quad (55)$$

and

$$\partial_{\nu_0}^1 \tilde{\nu}(z, [\tilde{\nu}_0]) = (\partial_{\nu_0}^1 \tilde{\nu}(z, 0), \partial_{\nu_0}^2 \tilde{\nu}(z, 0), \dots, \partial_{\nu_0}^n \tilde{\nu}(z, 0)). \quad (56)$$

Therefore the n th order derivative $\partial_{\nu_0}^n \tilde{\nu}(z, 0)$ can be calculated from the knowledge of all lower order derivatives, if Eq. (55) is inserted into the right-hand side of Eq. (52) and

FIG. 6. (Color) Total octupole strength of a 90° HDSM dipole calculated by particle tracking and by perturbation theory using differential algebra.

the result is compared with Eq. (56). The actual computation of the integrals can be done, for example, with the DA package cosy [19].

Because of its symmetry, the overall sextuple strength of a 90° HDSM dipole is zero. As an application, its overall octupole strength m_3 was calculated according to

$$m_3 = \frac{1}{p_z} \frac{\partial^3 p_y}{\partial y_0^3}. \quad (57)$$

For this purpose the potential Ψ in Eq. (28) was expanded to fourth order and the unperturbed solution was again chosen to be the transfer matrix of a hard edge 90° dipole as given by Eq. (37). In Fig. 6, the result for m_3 from a third order perturbation theory calculation using differential algebra is shown and compared with the result from particle tracking. Both results agree within 6%.

VIII. CONCLUSION

Perturbation theory offers yet another way to calculate the transfer matrix M of combined function magnets. Its main advantage over particle tracking is its ability to derive the map between the initial and final coordinates in a systematic manner from a known reference map, thereby reducing calculation time. It is most useful when the magnetic field in the vicinity of the reference path can be calculated without the help of particle tracking, as has been demonstrated for the 90° dipoles of the double sided microtron HDSM at MAMI-C. Here the calculation time to optimize the field profile $B_y(z_0)$ is significantly reduced by the fact, for example, that the overall deflection angle of 90° is independent of the field profile and is already determined by the reference map M_0 . While the established integrators seek to solve the equation of motion step by step along the reference trajectory, perturbation theory uses an integral approach. Since each contribution M_n is expressed as an integral along the whole reference trajectory, all field properties of the magnet are taken into account at

every stage of the calculation. Therefore the sum over M_n can approximate M even at a low n with reasonable accuracy, e.g., $n = 3$ is sufficient for the HDSM magnets for almost all practical purposes. Finally, perturbation theory can even be used in conjunction with differential algebra techniques to effectively calculate higher order aberrations.

IX. ACKNOWLEDGMENT

I am indebted to K.-H. Kaiser and U. Ludwig-Martin at the Institut für Kernphysik, Universität Mainz for their cooperation. This work was supported by DFG (Graduiertenkolleg “Physik und Technik von Beschleunigern”).

-
- [1] K.-H. Kaiser, Institut für Kernphysik Universität Mainz MAMI Report No. 8, 1986.
 - [2] S. Kowalski and H. A. Enge, LNS-MIT Technical Report No. 156, 1986.
 - [3] M. Berz, Nucl. Instrum. Methods Phys. Res., Sect. A **258**, 431 (1987).
 - [4] G. W. E. Hairer and S. P. Norsett, *Solving Ordinary Differential Equations* (Springer-Verlag, Berlin, 1993), Vol. I.
 - [5] M. Berz and K. Makino, *Reliable Computing* **4**, 361 (1998).
 - [6] S. Humphries, *Principles of Charged Particle Acceleration* (Wiley, New York, 1986).
 - [7] H. Wiedemann, *Particle Accelerator Physics* (Springer-Verlag, Berlin, 1993), Vol. I.
 - [8] G. Ripken, DESY Report No. 85-084, 1985.
 - [9] V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, Berlin, 1989).
 - [10] V.I. Arnold, *Ordinary Differential Equations* (Springer-Verlag, Berlin, 1992).
 - [11] R. Courand and D. Hilbert, *Methods of Mathematical Physics* (Wiley, New York, 1989), Vol. I.
 - [12] F. Dyson, *Phys. Rev.* **75**, 486 (1949); **75**, 1736 (1949).
 - [13] DFG Collaborative Research Centre No. 443, 2001.
 - [14] H. Herminghaus, A. Feder, K.-H. Kaiser, W. Manz, and H. v. d. Schmitt, *Nucl. Instrum. Methods* **138**, 1 (1976).
 - [15] U. Ludwig-Martin, H. Euteneuer, K.-H. Kaiser, H.-J. Kreidel, and S. Ratschow, in *Proceedings of the 6th European Particle Accelerator Conference*, edited by J. Pool and C. Petit (Institute of Physics, Bristol, U.K., 1998).
 - [16] S. Ratschow, Ph.D. thesis, Institut für Kernphysik Universität Mainz, 2000.
 - [17] H. Wiedemann, *Particle Accelerator Physics* (Springer-Verlag, Berlin, 1998), Vol. II.
 - [18] M. Berz, *Nucl. Instrum. Methods Phys. Res., Sect. A* **363**, 100 (1995).
 - [19] K. Makino and M. Berz, *Nucl. Instrum. Methods Phys. Res., Sect. A* **427**, 338 (1999).