Natural BNS damping of the fast ion instability

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In this paper we discuss the self-stabilization of the so-called fast ion instability of an electron beam in the case when coherent oscillations of bunches can be described in the linear approximation. Such a possibility occurs due to an increase in the number of ions along the electron beam and a proportional increase in the betatron tune shifts of electrons in subsequent bunches. The tune variation along the beam results in the BNS damping of the coherent oscillations of electron bunches. [S1098-4402(99)00038-5]

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I. INTRODUCTION

Starting from the paper [1] many authors attacked the problem of the so-called fast ion instability in high current electron rings. This is a beam breakup-type multibunch coherent instability, which may occur due to temporary collection in the electron beam of the ions produced due to ionization of the residual gas atoms by the beam electrons. It is assumed that the beam forms a train of bunches with a gap, which is long enough to let ions leave the beam to the next turn. As a result, at each turn the beam interacts with fresh ions, so that the head-on bunches of the beam deflecting the produced ions can excite the oscillations of subsequent bunches. Previous calculations of this phenomenon predict that in the linear approximation on the transverse deflections of the bunches the amplitudes of coherent oscillations of bunches increase in time asymptotically according to

$$A_n \propto \exp(n\sqrt{t/t_0}), \quad A_n \ll \sigma_{\perp}. \quad (1)$$

Here, A_n is the oscillation amplitude of the n th bunch in the beam, $\sigma_{\perp}$ is the bunch radius, n is the number of the bunch in the train, and t_0 is the instability time scale. The predicted instability growth rates increase with an increase in the density of the electron beam and become especially high for the high current electron storage rings. For these reasons, the fast ion instability is considered as a phenomenon, which can seriously limit the operational performance of such machines.

Such an asymptotic behavior of the oscillation amplitudes due to the beam breakup instability could be expected, if the tunes of the betatron oscillations of particles coincide in all bunches of the beam. However, the number of ions produced at a given longitudinal position at the closed orbit increases with an increase in the number of the passed bunches. That results in a proportional increase in the particle betatron tunes in the electron beam

towards the end of the train. In these conditions, the interactions of the subsequent and foregoing bunches via produced ions become nonresonant ones, so that the growth of the oscillation amplitudes can be affected by the so-called BNS damping [2], in particular, that can eliminate the instability. The reader should remember, however, that a suppression of the beam breakup instability differs from the suppression of the conventional type of instabilities in the particle accelerators and storage rings. In the last case, the damping reduces the values of the bunch eigenmodes increments. If the reduction becomes strong enough, the increments change their signs, so that the excitation of oscillations is eliminated. The beam breakup instability occurs due to excitations of the oscillations of the tail bunches by the foregoing ones. The initial blowup of the excited oscillations cannot be eliminated unless the excitation forces vanish. For this reason, the stabilization of the beam breakup instability can occur only asymptotically. The importance of the BNS damping for the instabilities due to ion production in the beam was already mentioned in our previous paper [3].

Surprisingly, all papers calculating the linear stage of the instability use the equations which contain the term describing an increase in the betatron tune shift along the train. However, the obtained solutions do not contain any trace of the BNS damping. We think that the reason is in using nonadequate approximations when solving equations describing linear coherent oscillations of bunches.

In this paper we solve the equations describing the linear period of the fast ion instability in the smoothed focusing approximation for both electrons and ions, assuming that the oscillations of electron bunches are developed slowly enough to enable neglecting the coupling of the beam betatron modes. That condition usually holds well for the most coherent instabilities in particle accelerators and storage rings. Within the framework of this assumption, all other equations are solved exactly. Resulting solutions describe the most features of the BNS damped coherent oscillations. In particular, these solutions describe the amplification of coherent oscillations along the beam, but they never grow with time asymptotically.

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II. INITIAL EQUATIONS

We start from equations which were derived in Yokoya's note [4], and we shall use the same assumptions and definitions. These equations coincide with those used in other papers describing the fast ion instability in an electron beam (see, for example, Ref. [1]). Like in all papers on this subject, we assume that the electron bunches are equipopulated and replace the sequence of bunches by a single quasicontinuous bunch, which has length equal to the train length and constant linear density equal to the average linear density in the beam. We also shall use the smoothed focusing approximation when solving equations of motion both for electrons and ions. We take that the ions are produced at the rest. Then, after the reduction of the ion variables from the equations of motion for electrons, the equation describing the evolution of the (vertical) centroid of the electron beam reads [1,4]

$$\frac{\partial^2 y}{\partial s^2} + k^2 y = -2Ck \left\{ \tau y(\tau, s) - \int_0^\tau d\tau' h(\tau, \tau') y(\tau', s) \right\}. \quad (2)$$

Here,

$$h(\tau, \tau') = \cos \Omega(\tau - \tau') + \Omega \tau' \sin \Omega(\tau - \tau'), \quad (3)$$

and¹

$$C = \frac{n_i r_e}{2\gamma L \sigma_x \sigma_y k}, \quad \Omega = \sqrt{\frac{N_b r_p}{AL \sigma_x \sigma_y}}, \quad (4)$$

$$\sigma_x \gg \sigma_y,$$

where $n_i = N_b n_A \sigma_i$ is the number of ions produced by an electron bunch in the beam per the unit length, L is the bunch-to-bunch length, k is the betatron wave number, A is the atomic number of the ions, n_A is the density of the residual gas, $\sigma_{x,y}$ are the rms horizontal and vertical electron bunch sizes, $r_{e,p}$ are the classical radii of the electron and of the proton, respectively, s is the path along the closed orbit (equivalent to the time), and $\tau = nL$ is the distance inside the electron beam measured from its head. Equation (2) can also be rewritten in the form

$$\frac{\partial^2 y}{\partial s^2} + k^2 y = -2Ck \int_0^\tau d\tau' \tau' \cos \Omega(\tau - \tau') \frac{\partial y(\tau', s)}{\partial \tau'}, \quad (5)$$

or in Yokoya's form

¹Particular expressions for C and Ω depend on the model, which is used to describe coherent oscillations. Within the framework of the rigid bunch model which is used in this paper, the presented expressions hold for Gaussian distributions with the same transverse rms sizes in the electron bunch and in the ion cloud. Details of these expressions are not important for the future analysis and they are given here rather for illustrative purposes.

$$\left(\frac{\partial^2}{\partial \tau^2} + \Omega^2 \right) \left(\frac{\partial^2}{\partial s^2} + k^2 \right) y(\tau, s) = -2Ck \frac{\partial}{\partial \tau} \left(\tau \frac{\partial y}{\partial \tau} \right). \quad (6)$$

All these equations contain the terms describing an increase in the (vertical) betatron tune in the electron beam due to an increase in the number of the produced ions. For that reason, we may expect that solutions of these equations will have the features which are specific for the beam breakup instability with the BNS damping. In particular, the oscillations may grow, or decay, depending on the distribution of initial centroid deflections in the beam (see, for example, Ref. [5]). One stable solution of Eq. (6) is seen from this equation directly. If initial deflections are constant along the beam $y(\tau, 0) = Y$, Eqs. (6) or (5) have the solution [4]

$$y(\tau, s) = Y \cos(k[s - s_0]). \quad (7)$$

III. SOLVING OF THE CENTROID EQUATION

We want to solve Eq. (6). It is useful to define the dimensionless variables

$$x = \Omega \tau, \quad u = ks. \quad (8)$$

Then, Eq. (6) reads

$$\left(\frac{\partial^2}{\partial x^2} + 1 \right) \left(\frac{\partial^2}{\partial u^2} + 1 \right) y(x, u) = -2Q \frac{\partial}{\partial x} \left(x \frac{\partial y}{\partial x} \right), \quad (9)$$

$$Q = \frac{C}{\Omega k}.$$

For typical conditions the value of Q is small. However, the value in the right-hand side of Eq. (9) contains the derivatives of y with respect to τ . For that reason, it can be large, if y may vary along the beam rapidly. Such a possibility is a specific feature of the beam breakup solutions (see, for example, Ref. [5]). To solve Eq. (9) we define two new functions

$$\left(\frac{\partial^2}{\partial u^2} + 1 \right) y(x, u) = P(x, u), \quad (10)$$

and

$$\xi_m(x, u) = \frac{\partial y}{\partial u} - imy, \quad m = \pm 1. \quad (11)$$

Apparently, the function $P(x, u)$ is proportional to the right-hand side in Eq. (5),

$$P = \frac{-2C}{k} \int_0^\tau d\tau' \tau' \cos \Omega(\tau - \tau') \frac{\partial y(\tau', s)}{\partial \tau'}. \quad (12)$$

The functions P and ξ_m obey the following equations:

$$\frac{\partial^2 P}{\partial x^2} + P = iQ \frac{\partial}{\partial x} \left(x \frac{\partial}{\partial x} [\xi_{-1} - \xi_1] \right), \quad (13)$$

and

$$\frac{\partial \xi_m}{\partial u} + im\xi_m = P, \quad m = \pm 1. \quad (14)$$

The functions $\xi_{1(-1)}$ describe the dipole betatron modes of the beam. According to Eq. (5), for nonsingular centroid distributions along the beam ($\partial y / \partial \ln \tau = 0$, if $\tau = 0$) the function P obeys the border conditions

$$P(0, u) = 0, \quad \frac{\partial P(0, u)}{\partial \tau} = 0. \quad (15)$$

Since

$$y = \frac{1}{2i} [\xi_{-1} - \xi_1], \quad (16)$$

we can also rewrite the function P in Eq. (12) in the following form:

$$\begin{aligned} P &= P_{-1} + P_1 \\ &= \frac{C}{ik} \int_0^\tau d\tau' \tau' \cos \Omega(\tau - \tau') \sum_{m=\pm 1} m \frac{\partial \xi_m(\tau', s)}{\partial \tau'}. \end{aligned} \quad (17)$$

Assuming that coherent oscillations are developed slower than the periods of the betatron oscillations, we omit the coupling of the modes with $m = \pm 1$. Then, equations for the modes $\xi_{\pm 1}$ read

$$\begin{aligned} \frac{\partial^2 P_m}{\partial x^2} + P_m &= -imQ \frac{\partial}{\partial x} \left(x \frac{\partial \xi_m}{\partial x} \right), \\ \frac{\partial \xi_m}{\partial u} + im\xi_m &= P_m, \quad m = \pm 1, \end{aligned} \quad (18)$$

while the border conditions in Eq. (15) for the functions P_m hold separately.

Equations (18) can be solved exactly using the Fourier transform. We define

$$\xi_m = e^{-imu} \zeta_m(x, u), \quad P_m = e^{-imu} F_m(x, u). \quad (19)$$

Substituting these expressions in Eqs. (18) we find that the functions ζ_m and F_m will depend on m only in the combination mQ . Since m takes only two values $m = \pm 1$, we shall write below mQ like Q remembering that Q changes the sign, when m is replaced for $-m$. Correspondingly, we shall omit the index m in the amplitudes ζ_m and F_m , assuming the substitutions $\zeta_m \rightarrow \zeta$ and $F_m \rightarrow F$. After these transformations Eqs. (18) can be rewritten in the following form:

$$\frac{\partial \zeta}{\partial u} = F, \quad \frac{\partial^2 F}{\partial x^2} + F = -iQ \frac{\partial}{\partial x} \left(x \frac{\partial \zeta}{\partial x} \right). \quad (20)$$

Multiplying here both sides of the first equation by $e^{i\omega u}$ and integrating them over $0 \leq u < \infty$, we find

$$\int_0^\infty du e^{i\omega u} \frac{\partial \zeta}{\partial u} = -i\omega \zeta(x, \omega) - \zeta(x, 0) = F(x, \omega). \quad (21)$$

So that, if we know $F(x, \omega)$, the function $\zeta(x, u)$ is found using

$$\zeta(x, u) = \int_{-\infty}^\infty \frac{d\omega}{-2\pi i \omega} e^{-i\omega u} [\zeta(x, 0) + F(x, \omega)], \quad \text{Im} \omega > 0. \quad (22)$$

The Fourier transform in the second equation in Eq. (20) simply replaces the functions F and ζ by their Fourier amplitudes. Substituting there the value $\zeta(x, \omega) = i[\zeta(x, 0) + F(x, \omega)]/\omega$, we find the following equation for the amplitude $F(x, \omega)$:

$$\frac{d^2 F}{dx^2} + F = \frac{Q}{\omega} \frac{d}{dx} \left(x \frac{dF}{dx} + x \frac{d\zeta(x, 0)}{dx} \right). \quad (23)$$

Note that the right-hand side in this equation dramatically increases for the amplitudes, which are in the resonance with betatron oscillations ($\omega \rightarrow 0$).

Solutions of Eq. (23) can be found directly. Substituting

$$\chi = \frac{2\omega\sqrt{(1 - Qx/\omega)}}{Q}, \quad \frac{d}{dx} = -\frac{2\omega}{Q\chi} \frac{d}{d\chi}, \quad (24)$$

we transform Eq. (23) into the following:

$$\begin{aligned} \frac{1}{\chi} \frac{d}{d\chi} \left(\chi \frac{dF}{d\chi} \right) + F &= \Phi, \\ \Phi &= \frac{Q}{\omega} \frac{d}{dx} \left(x \frac{d\zeta(x, 0)}{dx} \right). \end{aligned} \quad (25)$$

Correspondingly, solutions are expressed in terms of the combinations of the Bessel functions:

$$F = A(x, \omega) J_0(\chi) + B(x, \omega) N_0(\chi). \quad (26)$$

Here, $J_0(x)$ and $N_0(x)$ are the Bessel and Neumann functions of the zero order (see, for example, Ref. [6]). Simple calculations, together with a requirement that

$$A' J_0 + B' N_0 = 0, \quad F' = A J_0' + B N_0',$$

$$A' = dA/d\chi,$$

result in the following equations for the functions A and B :

$$A' J_0 + B' N_0 = 0, \quad A' J_0' + B' N_0' = \Phi.$$

Using [6] the identity $J_0(\chi) N_0'(\chi) - J_0'(\chi) N_0(\chi) = 2/(\pi\chi)$, we find

$$\frac{dA}{dx} = \pi N_0(\chi) \frac{d}{dx} \left(x \frac{d\zeta(x, 0)}{dx} \right),$$

and similarly

$$\frac{dB}{dx} = -\pi J_0(\chi) \frac{d}{dx} \left(x \frac{d\zeta(x, 0)}{dx} \right).$$

Solving these equations with the border condition from

Eq. (15) [$A(x = 0) = 0$ and $B(x = 0) = 0$], we obtain

$$F(x, \omega) = \pi \int_0^x dx' \frac{d}{dx'} \left(\frac{d\zeta(x', 0)}{d \ln x'} \right) \times \begin{vmatrix} J_0(\chi), & N_0(\chi) \\ J_0(\chi'), & N_0(\chi') \end{vmatrix}. \quad (27)$$

Here, $\chi' = 2\sqrt{\omega(\omega - Qx')}/Q$. Using the relationships between various Bessel functions we can rewrite Eq. (27) in many convenient forms. In particular, we may replace

$N_0 = i[J_0 - H_0^{(1)}]$, where $H_\nu^{(1,2)}$ are the Hankel functions (see, for example, Ref. [6]). This yields

$$F(x, \omega) = -i\pi \int_0^x dx' \frac{d}{dx'} \left(\frac{d\zeta(x', 0)}{d \ln x'} \right) \times \begin{vmatrix} J_0(\chi), & H_0^{(1)}(\chi) \\ J_0(\chi'), & H_0^{(1)}(\chi') \end{vmatrix},$$

or, assuming that $xd\zeta/dx = 0$, if $x = 0$,

$$F = i\pi \int_0^x dx' x' \frac{d\zeta(x', 0)}{dx'} \frac{d}{dx'} [J_0(\chi)H_0^{(1)}(\chi') - H_0^{(1)}(\chi)J_0(\chi')],$$

$$= \frac{i\pi w}{\sqrt{w(w - 2x')}} \int_0^x dx' x' \frac{d\zeta(x', 0)}{dx'} \begin{vmatrix} J_0(\chi), & H_0^{(1)}(\chi) \\ J_1(\chi'), & H_1^{(1)}(\chi') \end{vmatrix}. \quad (28)$$

Defining $t = Qu/2$ and substituting F from Eq. (28) in Eq. (22), we obtain

$$\zeta(x, u) = \zeta(x, 0) + \Delta\zeta, \quad (29)$$

where we defined

$$\Delta\zeta = \int_{-\infty}^{\infty} \frac{d\omega e^{-i\omega u}}{-2\pi i \omega} F = \frac{-1}{2} \int_0^x dx' x' \frac{d\zeta(x', 0)}{dx'} R(x, x', t) \quad (30)$$

and $R(x, x', t)$ as the resolvent of Eq. (29),

$$R(x, x', t) = \int_{-\infty}^{\infty} \frac{dw e^{-i\omega t}}{\sqrt{w(w - 2x')}} \begin{vmatrix} J_0(\sqrt{w(w - 2x)}), & H_0^{(1)}(\sqrt{w(w - 2x)}) \\ J_1(\sqrt{w(w - 2x')}), & H_1^{(1)}(\sqrt{w(w - 2x')}) \end{vmatrix}. \quad (31)$$

The integration contour in this equation goes above the singularities of the integrand [$\text{Im}(w) > 0$]. We deform the integration contour to transform Eq. (31) to a more convenient form. During these deformations the new contour should not cross the singularities of the integrand in Eq. (31). These are the branch points at $w = 2x'$ and $w = 2x$, as well as two simple poles at the points $w = 0$ and $w = 2x'$. To make the integrand a single-valued function we cut the plane w along the real half-axes $w \geq 0$. Since

$$\begin{vmatrix} J_0(\chi), & H_0^{(1)}(\chi') \\ J_1(\chi), & H_1^{(1)}(\chi') \end{vmatrix} = \begin{vmatrix} H_0^{(2)}(\chi), & H_0^{(1)}(\chi') \\ H_1^{(2)}(\chi), & H_1^{(1)}(\chi') \end{vmatrix},$$

and

$$H_0^{(2,1)}(\sqrt{w(w - 2x)})H_1^{(1,2)}(\sqrt{w(w - 2x')}) \simeq \frac{2}{\pi w} e^{\pm i(x-x')}, \quad |w| \rightarrow \infty,$$

the integration contour can be deformed in such a way that it starts from $w = \infty$, returns to $w = 0$ below the cut, and, after detouring this point in the negative direction, the contour returns to the point $w = \infty$ above the cut (see Fig. 1).

After these transformations we note that the contour integrals along the upper and lower sides of the cut on the real axes of w between the points $w = 0$ and $w = 2x'$ cancel each other. Similarly, the integrals along the upper and lower sides of the cut between the points $w = 2x$ and $w = \infty$ cancel each other. The calculations of the remaining contour integrals reduce those to the

FIG. 1. The deformation of the integration contour in Eq. (31). The dashed line shows the part of the cut in the beam spectrum. C_{in} shows the initial integration contour.

pole contributions² and to the integral along the betatron oscillations spectrum of the beam $x' \leq w/2 \leq x$,

$$R(x, x', t) = \frac{2}{x'} [1 - e^{-2ix't} I_0(2\sqrt{x'(x-x')})] + 2 \int_{x'}^x dw e^{-2iwt} \frac{I_0(2\sqrt{w(x-w)}) J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}}. \quad (32)$$

Here, $I_n(x)$ are the Bessel functions of the imaginary argument

$$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!(k+n)!}.$$

For the mode $m = -1$ the resolvent for Eq. (30) can be calculated similarly, but the cut in the plane of the complex variable w should be directed along the negative half-axes of w ($\text{Re } w \leq 0$). This expression can also be obtained directly from Eq. (32), if we recall that indeed Q is mQ ($m = \pm 1$), so that we can find R_{-1} using the following substitutions $R_m \rightarrow mR$, $x \rightarrow mx$, and $x' \rightarrow mx'$ ($m = \pm 1$):

$$R_m(x, x', t) = \frac{2m}{mx'} [1 - e^{2imx't} I_0(2\sqrt{x'(x-x')})] + 2m \int_{mx'}^{mx} dw e^{-2iwt} \frac{I_0(2\sqrt{w(mx-w)}) J_1(2\sqrt{w(w-mx')})}{\sqrt{w(w-mx')}} ,$$

or ($w \rightarrow -w$)

$$R_{-1}(x, x', t) = \frac{2}{x'} [1 - e^{2ix't} I_0(2\sqrt{x'(x-x')})] + 2 \int_{x'}^x dw e^{2iwt} \frac{I_0(2\sqrt{w(x-w)}) J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}}. \quad (33)$$

Expressions for $R_{\pm 1}$ differ by the complex conjugation of exponential functions. Such a difference for oscillations near the upper and the lower sidebands is a specific feature of the beam breakup solutions describing the excitation of oscillations of subsequent (here) bunches in the beam by the foregoing ones.

Since $R(x, x', t = 0) = 0$, we should expect

$$\frac{2}{x'} [1 - I_0(2\sqrt{x'(x-x')})] + 2 \int_{x'}^x \frac{I_0(2\sqrt{w(x-w)}) J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}} dw = 0,$$

or

$$\int_{x'}^x \frac{I_0(2\sqrt{w(x-w)}) J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}} dw = \frac{1}{x'} [I_0(2\sqrt{x'(x-x')}) - 1]. \quad (34)$$

The integral in the left-hand side of this expression can be calculated directly (see Appendix A) confirming the identity in Eq. (34). Using this expression we can also rewrite Eq. (32) in the equivalent form

$$R(x, x', t) = \frac{2}{x'} [1 - e^{-2ix't}] I_0(2\sqrt{x'(x-x')}) - 2 \int_{x'}^x dw [1 - e^{-2iwt}] \frac{I_0(2\sqrt{w(x-w)}) J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}}. \quad (35)$$

Substituting this formula into Eq. (30), we find

$$\begin{aligned} \zeta(x, t) = & \zeta(x, 0) - \int_0^x dx' \frac{d\zeta(x', 0)}{dx'} [1 - e^{-2ix't}] I_0(2\sqrt{x'(x-x')}) \\ & + \int_0^x dx' x' \frac{d\zeta(x', 0)}{dx'} \int_{x'}^x dw [1 - e^{-2iwt}] \frac{I_0(2\sqrt{w(x-w)}) J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}}. \end{aligned} \quad (36)$$

One can easily prove by direct calculations that expressions in either Eq. (36), or its equivalent, which holds for nonsingular functions $d\zeta(x, 0)/dx$,

$$\zeta_m(x, t) = \zeta_m(0, 0) + \int_0^x dx' e^{-2imx't} I_0(2\sqrt{x'(x-x')}) \left[\frac{d\zeta_m(x', 0)}{dx'} - J_\zeta(x') \right], \quad m = \pm 1, \quad (37)$$

²In order to avoid confusion, we note that for the Bessel functions of the integer order the branch point and the pole singularities contribute in additive terms. For example, we can write $\pi N_1(z) = 2J_1(z)(\ln \frac{z}{2} + C) - \frac{2}{z} - \Delta_1(z)$, where C is the Euler constant and $\Delta(z)$ is a nonsingular function of z .

where

$$J_{\zeta}(w) = \int_0^w dy y \frac{d\zeta_m(y, 0)}{dy} \frac{J_1(2\sqrt{w(w-y)})}{\sqrt{w(w-y)}}$$

satisfies Eqs. (18) (see also Appendix B). According to Eq. (36), the distribution of the dipole momentum along the beam to a given time $t = Qks/2$ strongly depends on its initial distribution along the beam. In particular, if all bunches initially have the same offsets in the phase space [$\zeta(x, 0) = \text{const}$], no instability occurs. Correspondingly, we may expect that the instability will result in smaller excitation amplitudes for smoother initial distribution of the bunch dipole moments along the beam.

Since the result is expressed in terms of the Fourier integral over a finite frequency interval, it never grows asymptotically (i.e., when $t \rightarrow \infty$). For the same reason, we may also expect an echoing time dependence of the signals. That behavior is a consequence of the BNS damping of the solutions due to increases in the betatron tunes of particles along the beam, which is a specific feature of the discussed problem [see Eq. (2)]. As was already mentioned in the Introduction, the stabilization of the beam breakup instability may occur only asymptotically. It means that oscillations initially will blowup until the damping will overcome the excitation. For the fast ion instability a rough estimation of this moment can be done in the following way. According to Eqs. (2) and (3) and neglecting the BNS damping we, generally, might expect $\zeta \propto \exp[\sqrt{s/s_0}]$ (see, for example, Ref. [5]) with the excitation growth rate estimated by the parameter

$$\frac{1}{s_0} \approx 2Cx\tau = \frac{2Cx^2}{\Omega}. \quad (38)$$

On the other hand, the variation of the betatron wave number along the beam is determined by [see Eq. (2)]

$$\Delta k = C\tau = \frac{Cx}{\Omega}. \quad (39)$$

For rough estimations we may assume that the BNS damping due to this tune variation results in an exponential decay of the coherent signal-like Landau damping of conventional coherent oscillations ($\zeta \propto \exp[-\Delta ks] = \exp[-Cxs/\Omega]$). A total dependence of ζ on time will be then determined by the expression

$$\zeta \propto \exp\left(-\frac{Cxs}{\Omega} + x\sqrt{\frac{2Cs}{\Omega}}\right). \quad (40)$$

This expression shows that the damping of oscillations will overcome their excitation after

$$\frac{Cs}{\Omega} \geq 2. \quad (41)$$

This very rough estimation of the suppression starting moment coincides surprisingly well with that obtained below using accurate numeric calculations for several particular initial distributions of the bunch centroids along the beam.

IV. PARTICULAR EXAMPLES

In this section we shall study several examples of when initial distributions of the beam dipole momenta along the beam enable some simplifications of the integrals in Eqs. (36) or (37). Usually, the resulting expressions in these cases can be reduced to single integrals, which can be easily tabulated. Apart from the illustrative worth, these examples enable an estimation of the influence of the shape of the initial distribution of the dipole momentum along the beam on its asymptotic behavior.

A. Stepwise initial centroid distribution

Although the electron bunch centroids do not grow in the time asymptotically, the interaction of bunches via produced ions may result in their initial blowup and significant amplifications of the oscillation amplitudes along the beam. That is another specific feature of the discussed phenomenon. As an example we consider the case when $m = 1$ and $\zeta_1(x, 0) = \zeta(x, 0)$ is a steplike function so that the first bunches in the beam with $x \leq a$ are deflected with, say, a unit amplitude, while all others are not deflected, and, therefore

$$\frac{d\zeta(x, 0)}{dx} = -\delta(x - a).$$

For this case, Eq. (36) yields [$x \geq a$; it is obvious that for the discussed case in the region $x < a$, $\zeta(x, s) = 1$]

$$\zeta(x, s) = \zeta_a - \zeta_w, \quad x \geq a, \quad (42)$$

where

$$\zeta_a(x, t) = \left[1 - \exp\left(-\frac{iCas}{\Omega}\right)\right] I_0(2\sqrt{a(x-a)}), \quad (43)$$

and

$$\begin{aligned} \zeta_w = a \int_{a/x}^1 dw \left[1 - \exp\left(-iw \frac{Cxs}{\Omega}\right)\right] \\ \times \frac{I_0(2x\sqrt{w(1-w)})J_1(2x\sqrt{w(w-a/x)})}{\sqrt{w(w-a/x)}}. \end{aligned} \quad (44)$$

Though the integrals in these equations are still complicated for their analytic calculations, the function $\zeta(x, t)$ can be easily tabulated and plotted. Inspections of the dependencies of ζ on x indicate that, although its character generally depends on time, the amplitudes of oscillations are amplified along the beam, and at large x the amplification coefficient indicates exponential dependence on x [see Fig. 2(a)].

As is visible in Fig. 2(b), the resonantlike dependencies of the phases of oscillations on x occur due to an artificial reduction of the data for this plot to the interval $-\pi \leq \arg(\zeta) \leq \pi$.

The time dependencies of the amplitude of the coherent signal in the discussed example are not simple and, generally, depend on both x and a (see Figs. 3 and 4).

FIG. 2. Dependences of the amplitude $|\zeta|$ [(a) dimensionless] and phase $\arg(\zeta)$ (b) of oscillations on the distance along the beam. For (a) from top to bottom $t = Cs/\Omega = 1, 0.5$, and 0.1 ; for (b) solid dots: $t = 0.1$; solid line: $t = 0.5$; dashed line: $t = 1$; $a = 1$.

It does not indicate any quasiexponential law, which was found in previous papers. In the region where $x \gg 1$ and $x \gg a$, the dependence of the oscillation amplitude on time has a shape of a bump with a maximum at $Cs/\Omega \approx 2$ and then decays to some saturation level (Fig. 4). More generally, this dependence is determined by the ratio of functions ζ_a and ζ_w from Eqs. (43) and (44). If ζ_a dominates (small x , close values of x and a), the amplitude time dependence indicates beatings with the period approximately $2\pi\Omega/Ca$ (Fig. 3). An increase in the amplitude of ζ_w can decrease the beating amplitude

FIG. 3. Dependence of the [dimensionless] amplitude of the coherent signal (solid line) on time ($t = Cs/\Omega$). In both cases, $a = x/2$, (a) $x = 2$, (b) $x = 6$. The dashed line and solid dots on these pictures show the contributions from ζ_a and ζ_w , correspondingly.

[Figs. 3(b) and 4]. An interesting feature of these time dependencies is a nonlinear dependence on time at the start of oscillations (Figs. 3 and 4). If the betatron tunes of the bunches along the beam would be constant, this dependence should be a linear one, since the oscillations of all bunches would be in exact resonance. Due to deviations of the betatron tunes along the beam, the excitations of the subsequent bunches becomes a nonresonant one, which results in slower increase in the oscillation amplitude with time.

FIG. 4. Same as in Fig. 3, but $x = 10$ and $a = 2$.

B. Linear initial displacement

Another example which enables the calculation of the beam betatron modes in a more or less simple way corresponds to the case when initial values of the beam bunch dipole momenta grow, or decrease, along the beam linearly. For example, we take ($m = 1$, $\zeta_m \rightarrow \zeta$)

$$\zeta(x, t = 0) = \frac{x}{L\Omega}, \quad (45)$$

so that the last bunch in the train has a unit displacement. Substituting this expression for $\zeta(x, 0)$ in Eq. (37), we find

$$\zeta_m(x, t) = \frac{1}{L\Omega} \int_0^x dw e^{-2imwt} I_0(2\sqrt{w(x-w)}) \times \left[1 - \int_0^w dy y \frac{J_1(2\sqrt{w(w-y)})}{\sqrt{w(w-y)}} \right]. \quad (46)$$

Using

$$\begin{aligned} \int_0^w dy y \frac{J_1(2\sqrt{w(w-y)})}{\sqrt{w(w-y)}} &= \frac{1}{w} \int_0^w dy y \frac{d}{dy} J_0(2\sqrt{w(w-y)}) \\ &= 1 - \int_0^w \frac{dy}{w} J_0(2\sqrt{w(w-y)}), \end{aligned}$$

we rewrite Eq. (46) in the following form:

$$\zeta_m(x, t) = \frac{1}{L\Omega} \int_0^x dw e^{-2imwt} I_0(2\sqrt{w(x-w)}) \int_0^w \frac{dy}{w} J_0(2\sqrt{w(w-y)}). \quad (47)$$

Now, we note that

$$\int_0^w \frac{dy}{w} J_0(2\sqrt{w(w-y)}) = \int_0^1 dy J_0(2w\sqrt{y}) = \frac{J_1(2w)}{w},$$

and hence

$$\zeta_m(x, t) = \frac{1}{L\Omega} \int_0^x \frac{dw}{w} e^{-2imwt} I_0(2\sqrt{w(x-w)}) J_1(2w). \quad (48)$$

In particular, for the initial distribution of the beam centroid we find

$$x = \int_0^x \frac{dw}{w} I_0(2\sqrt{w(x-w)}) J_1(2w). \quad (49)$$

The integral in the right-hand side of this expression coincides with the integral in Eq. (A1), if we take there $x' = 0$. Using Eq. (A1) and

$$\lim_{x' \rightarrow 0} \left\{ \frac{1}{x'} [I_0(2\sqrt{x'(x-x')}) - 1] \right\} = x, \quad (50)$$

we prove the identity in Eq. (49).

Although the analytic calculation of the right-hand side in Eq. (48) is not very simple, this formula can be easily tabulated. As it could be expected beforehand, the interactions of bunches along the beam result in the amplification of the oscillation amplitude towards the end

of the train. As seen from Fig. 5 for the linear initial centroid distribution, it occurs only until $Cs/\Omega \leq 2$. For larger time intervals the slope of the dependence of the amplitude amplification coefficient on x initially decreases (solid circles in Fig. 5) and asymptotically can become even negative (see open circles in Fig. 5).

Typical time dependence of the centroid amplitude on the time also corresponds to its asymptotic decay after passing a maximum value at $Cs = 2\Omega$ (see Fig. 6).

C. Harmonic initial distribution

Let us consider now a special case when initial values of the beam dipole momenta are distributed along the beam according to ($m = 1$, again $\zeta_m \rightarrow \zeta$)

$$\zeta(x, 0) = \cos(x). \quad (51)$$

This initial distribution coincides with that used in Refs. [1], [4], or [7], if we take, for example, in Eq. (21) of Ref. [1] $\theta = \pi/2$ and $\hat{y} = 2$.

A direct substitution of Eq. (51) into Eq. (37) results in the expression in which simplification is not so obvious

FIG. 5. Dependences of the (dimensionless) amplitude $|\zeta|$ on the distance along the beam. The total phase advance of the ion oscillations along the beam is $L\Omega = 20$; solid curves correspond to (1) $t = Cs/\Omega = 0.1$; (2) $t = 0.5$; (3) $t = 1$; (4) $t = 2$; solid circles $t = 3$; open circles $t = 4$.

($t = Cs/\Omega$),

$$\zeta(x, t) = \int_0^x dx' e^{-ix't} I_0(2\sqrt{x'(x-x')}) \Phi(x'), \quad (52)$$

where

$$\Phi(x') = -\sin(x') + \int_0^{x'} dy y \sin(y) \frac{J_1(2\sqrt{x'(x'-y)})}{\sqrt{x'(x'-y)}}, \quad (53)$$

$$= -\sin(x') + x' \int_0^1 dy y \sin(x'y) \frac{J_1(2x'\sqrt{1-y})}{\sqrt{1-y}}. \quad (54)$$

However, we note that according to Eqs. (20)–(23) the amplitude

$$\zeta(x, \omega) = \frac{i[\zeta(x, 0) + F(x, \omega)]}{\omega}$$

obeys the following equation [$\zeta(x, 0) = \zeta_0$]:

$$\frac{d}{dx} \left[\left(1 - \frac{Qx}{\omega} \right) \frac{d\zeta}{dx} \right] + \zeta = \frac{1}{i\omega} \left[\frac{d^2 \zeta_0}{dx^2} + \zeta_0 \right]. \quad (55)$$

If $\zeta(x, 0)$ is defined by Eq. (51), the amplitude $\zeta(x, \omega)$ obeys a homogeneous differential equation

$$\frac{d}{dx} \left[\left(1 - \frac{Qx}{\omega} \right) \frac{d\zeta}{dx} \right] + \zeta = 0, \quad (56)$$

FIG. 6. Dependences of the (dimensionless) amplitude $|\zeta|$ on the time (Cs/Ω). The total phase advance of the ion oscillations along the beam is $L\Omega = 20$, from top to bottom at $Cs/\Omega = 2$, $x = 10$, 5, and 2.

with the border condition

$$\zeta(0, \omega) = \frac{i\zeta_0(0)}{\omega} = \frac{i}{\omega}, \quad \frac{\partial \zeta(x=0, \omega)}{\partial x} = 0. \quad (57)$$

The solution to Eq. (56) can be found easily,

$$\zeta(x, \omega) = -\frac{\pi i}{Q} \{ N_1(2w) J_0(2\sqrt{w(w-x)}) - J_1(2w) N_0(2\sqrt{w(w-x)}) \}, \quad (58)$$

where $w = \omega/Q$. The inverse Fourier transform yields ($t = Cs/\Omega$)

$$\zeta(x, t) = \int_{-\infty}^{\infty} \frac{dw}{2i} e^{-i\omega t} \{ N_1(2w) J_0(2\sqrt{w(w-x)}) - J_1(2w) N_0(2\sqrt{w(w-x)}) \}, \quad (59)$$

or, after reducing the integration contour to the cut $0 < w < x$, we find³

$$\zeta(x, t) = 1 - \int_0^x dw e^{-i\omega t} I_0(2\sqrt{w(x-w)}) J_1(2w). \quad (60)$$

Using the calculations similar to those presented in Appendix B, one can easily prove that $\zeta(x, t)$ from

³Taking here $t = 0$ and using Eq. (51), we can also obtain the following nontrivial integral: $x \int_0^1 dw I_0(2x\sqrt{w(1-w)}) J_1(2xw) = 1 - \cos(x)$.

Eq. (60) obeys Eqs. (20) [$m = 1$],

$$\frac{1}{i} \frac{\partial \zeta}{\partial t} = \frac{F}{iQ} = G, \quad G = -\frac{\partial}{\partial x} \left(x \frac{\partial \zeta}{\partial x} + \frac{\partial G}{\partial x} \right). \quad (61)$$

Typical dependencies of the amplitude of oscillations ($|\zeta|$) on time (see Fig. 7) in this case differ from that calculated for the two previously discussed cases. After initial bump, which again occurs when $Cs = 2\Omega$, the amplitude decreases towards a constant value $|\zeta| \approx 1$.

Due to similarity in initial distribution of bunch centroids along the beam, this example enables a numerical comparison of the obtained here solutions with that previously obtained in Refs. [1] or [7]. As was already discussed, these papers predict wrong behavior of the excited amplitudes at small ($Cs/\Omega \rightarrow 0$) and large ($Cs/\Omega \geq 2$) time intervals. We may expect, however, some coincidence in the asymptotic region $x\sqrt{Cs/\Omega} > 1$. To test such a possibility we fit $|\zeta(x, t)|$ calculated using Eq. (60) to the functions which were calculated in Refs. [1,7],

$$\zeta_t = A_f |\cos(x)| I_0(x\sqrt{Cs/[\Omega t_0]}), \quad x\sqrt{Cs/\Omega} > 1. \quad (62)$$

Here, A_f and t_0 are the fitting parameters and $I_0(x)$ is the Bessel function of an imaginary argument.⁴ The fitting parameters were selected inspecting dependencies on $x\sqrt{Cs/\Omega}$ of the relative deviation of the excited oscillation amplitude ($|\zeta(x, t)| - |\cos x|$) calculated using Eq. (60) from that calculated using ζ_t ,

FIG. 7. Dependences of the (dimensionless) amplitude $|\zeta|$ on the time (Cs/Ω). From top to bottom at $Cs/\Omega = 2$, $x = 10$, 5, and 2.

⁴Using equations of this subsection, a reader can easily test other possibilities.

$$\Delta_t = 2 \frac{||\zeta(x, t)| - \zeta_t|}{|\zeta(x, t)| + \zeta_t - 2|\cos(x)|}. \quad (63)$$

Good fitting results (Figs. 8 and 9) were obtained for $A_f \approx 1$. As seen from Fig. 8, for the point $x = 10$, and if $t_0 = 1.5$, the relative deviation (Δ_t) is less than 30%, provided that

$$2.5 \leq x\sqrt{\frac{Cs}{\Omega}} \leq 10, \quad x = 10.$$

The value of Δ_t exceeds 10% within the enclosed interval (Fig.8)

$$3.5 \leq x\sqrt{\frac{Cs}{\Omega}} \leq 9, \quad x = 10.$$

Another example is shown in Fig. 9 for the point $x = 3$. Again, the best fitting was obtained for $A_f \approx 1$. As is seen, the border of the 30% accuracy shrinks approximately to the interval

$$2.5 \leq x\sqrt{\frac{Cs}{\Omega}} \leq 5, \quad x = 3,$$

the value of the best fitting parameter has changed to $t_0 \approx 3$.

Presented examples show that Eq. (62) provides very poor approximations to the exact solutions even in the considered quasiasymptotic regions of the parameter $x\sqrt{Cs/\Omega}$. Due to different asymptotic behaviors in time and in the cases of the good fittings, exact solutions $\zeta(x, t)$ and non-BNS damped functions ζ_t from Eq. (62)

FIG. 8. Dependences of the relative deviation (Δ_t) of the excited oscillation amplitude from that for the non-BNS damped solution (ζ_t) on $x\sqrt{Cs/\Omega}$. Solid line: $t_0 = 1.5$; solid circles: $t_0 = 1.3$; open circles: $t_0 = 1.7$; $x = 10$.

FIG. 9. Same as in Fig. 8, but $x = 3$. Solid line: $t_0 = 3$; solid circles: $t_0 = 2$; open circles: $t_0 = 4$.

can intersect in a couple of points (open circles and solid lines in Figs. 8 and 9). Near these intersection points the value of Δ_t decreases.

V. CONCLUSION

Within the framework of the assumptions about the betatron mode decoupling and in the smoothed focusing approximation, equations describing the fast ion instability in the linear approximation in the electron bunch offsets are solved exactly. The solutions are expressed in terms of the Fourier integral over the finite tune interval in the beam spectrum. If ζ is the electron beam centroid position in the phase space, the amplitudes in this Fourier integral are proportional to the initial distribution of the value $\partial\zeta/\partial x$ along the beam. The time dependencies of the obtained solutions is strongly determined by the shape of this distribution and by the longitudinal bunch position in the beam. The instability of the beam oscillations manifests itself in amplifications of the oscillation amplitudes along the beam and in initial growths of the oscillation amplitude in time. However, the amplitudes of these unstable oscillations are asymptotically ($s > 2\Omega/C$) limited by the natural BNS damping due to an increase in the number of the produced ions along the beam and the proportional increase in the particle betatron tune shifts.

The amplification coefficient of the oscillation amplitudes along the beam depends on both time, on the longitudinal position inside the beam, and on the shape of the function $\partial\zeta(x, 0)/\partial x$. It can be close to an exponential for large x .

Dependencies of the amplitudes of unstable oscillations on time are not universal, but are strongly determined by the beam parameters. We did not find initial distributions of the beam dipole momenta when their subsequent dependences on time follow Eq. (1) with a reasonable accuracy.⁵ For large x , the position of the first maximum in the amplitude dependence on the time approaches the universal value

$$s_0 \approx \frac{2\Omega}{C} = \frac{2^{3/2}\gamma k}{n_A\sigma_i} \sqrt{\left(\frac{m}{m_p}\right) \frac{L\sigma_x\sigma_y}{AN_b r_e}}. \quad (64)$$

Here, m_p is the proton mass. This value can be used for rough estimations of the rise times of the instability in these conditions. Except for a possibility of an exponential amplification of the oscillation amplitudes along the beam, the predictions containing the obtained exact solutions do not agree with results of previous calculations (see, for example, Refs. [1,4,7]).

Note that the growth rate ($1/s_0$) in Eq. (64) agrees surprisingly well with that obtained in Ref. [3] for the problem where the instability occurs due to interaction of the beam with uncaptured ions. However, in the case of uncaptured ions the betatron tune shift along the train saturates. That suppresses the BNS damping for the long trains and results in the quasiexponential asymptotical increase of the oscillation amplitude in time as it is described in Eq. (1). This coincidence can be due to the fact that the beam breakup instability occurs due to the resonance excitation of the tail bunches in the beam by the foregoing ones. If the damping occurs due to a nonresonant process, initially the excitation of oscillations prevails so that the oscillation amplitudes grow. During this stage the growth rate is determined by the interactions of electrons via produced ions, which yields $1/s_0$. In the case of the BNS damping, the growth of the amplitudes is limited by the detuning of the betatron frequencies of the bunch oscillations along the beam.

⁵After simple but tedious calculations the reader can find out that the functions with such asymptotes obey the equation, which can be obtained from Eq. (2), if we omit in its right-hand side the term describing the modulation of the betatron tune along the beam ($-2Ck\tau y$). Such an equation can describe the fast ion instability in a special case, when the tune shift due to ions is exactly compensated, for example, due to interactions of bunches with surrounding electrodes [$\Delta k_{ext}(\tau) = -C\tau$]. The singularities of the Fourier amplitudes of the solutions of this equation in the plane of the complex variable w are reduced to essential singularities. This alteration in the character of the singularities of the Fourier amplitudes of the solution results in the asymptote described in Eq. (1). A more realistic case, when the ratio $\Delta k_{ext}/C\tau$ is an arbitrary number, demands too complicated analytic calculation and should be studied separately.

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APPENDIX A: CALCULATION OF THE INTEGRAL IN EQ. (34)

We calculate the integral

$$\int_{x'}^x \frac{I_0(2\sqrt{w(x-w)})J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}} dw = \frac{1}{x'} [I_0(2\sqrt{x'(x-x')}) - 1]. \quad (\text{A1})$$

Using

$$\frac{d}{dx'} J_0(2\sqrt{w(w-x')}) = J_1 \frac{\sqrt{w}}{\sqrt{w-x'}} = w \frac{J_1}{\sqrt{w(w-x')}},$$

we write

$$\int_{x'}^x \frac{I_0(2\sqrt{w(x-w)})J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}} dw = \frac{1}{x'} I_0(2\sqrt{x'(x-x')}) + \Phi(x, x'), \quad (\text{A2})$$

where

$$\Phi(x, x') = \frac{d}{dx'} \int_{x'}^x \frac{dw}{w} I_0(2\sqrt{w(x-w)})J_0(2\sqrt{w(w-x')}). \quad (\text{A3})$$

The function $\Phi(x, x')$ can be calculated using the following expansion [6]:

$$I_0(2\sqrt{w(x-w)})J_0(2\sqrt{w(w-x')}) = \sum_{k=0}^{\infty} \frac{w^k(x-w)^k}{(k!)^2} F\left(-k, -k, 1, -\frac{w-x'}{x-w}\right). \quad (\text{A4})$$

Here, $F(-k, -k, 1, z)$ is the hypergeometric function

$$F(-k, -k, 1, z) = 1 + \frac{k^2}{1} z + \frac{[k(k-1)]^2}{(2!)^2} z^2 + \frac{[k(k-1)(k-2)]^2}{(3!)^2} z^3 + \dots + z^k \quad (\text{A5})$$

$$= (k!)^2 \sum_{m=0}^k \frac{z^m}{[m!(k-m)!]^2}. \quad (\text{A6})$$

So, we can write

$$I_0(2\sqrt{w(x-w)})J_0(2\sqrt{w(w-x')}) = \sum_{k=0}^{\infty} w^k(x-w)^k \sum_{m=0}^k \frac{(-1)^m}{[m!(k-m)!]^2} \frac{(w-x')^m}{(x-w)^m},$$

while

$$\begin{aligned} \Phi &= \frac{d}{dx'} \sum_{k=0}^{\infty} \int_{x'}^x w^{k-1}(x-w)^k \left(\sum_{m=0}^k \frac{(-1)^m}{[m!(k-m)!]^2} \frac{(w-x')^m}{(x-w)^m} \right) dw \\ &= -\frac{1}{x'} + \frac{d}{dx'} \sum_{k=1}^{\infty} \int_{x'}^x w^{k-1}(x-w)^k \left(\sum_{m=0}^k \frac{(-1)^m}{[m!(k-m)!]^2} \frac{(w-x')^m}{(x-w)^m} \right) dw = -\frac{1}{x'} + \sum_{k=1}^{\infty} \frac{d\Phi_k}{dx'}, \end{aligned} \quad (\text{A7})$$

where

$$\Phi_k = \int_{x'}^x w^{k-1}(x-w)^k \left(\sum_{m=0}^k \frac{(-1)^m}{[m!(k-m)!]^2} \frac{(w-x')^m}{(x-w)^m} \right) dw \quad (\text{A8})$$

$$= \sum_{m=0}^k \frac{(-1)^m}{[m!(k-m)!]^2} \int_{x'}^x w^{k-1}(x-w)^{k-m}(w-x')^m dw. \quad (\text{A9})$$

Calculating the integrals directly, we find that for all $k \geq 1$, $\Phi_k = 0$, so that

$$\Phi = -\frac{1}{x'}.$$
 (A10)

Substituting this expression into Eq. (A2) we arrive at Eq. (A1).

$$\frac{\partial \zeta}{\partial u} = F, \quad \frac{\partial^2 F}{\partial x^2} + F = iQ \frac{\partial}{\partial x} \left(x \frac{\partial \zeta}{\partial x} \right). \quad (B1)$$

Since $t = Qu/2$, we write

$$\frac{\partial \zeta}{\partial u} = \frac{\partial \zeta}{\partial t} \frac{Q}{2}, \quad \frac{1}{2i} \frac{\partial \zeta}{\partial t} = \frac{F}{iQ} = G.$$

We expect that

$$G = \frac{\partial}{\partial x} \left(x \frac{\partial \zeta}{\partial x} - \frac{\partial G}{\partial x} \right), \quad (B2)$$

Here, we test that obtained solutions obey Eq. (20). As an example we make the calculations for the mode $m = -1$. In this case, Eq. (20) reads

where [see Eq. (37)]

$$G = \frac{1}{2i} \frac{\partial \zeta}{\partial t} = \int_0^x dx' x' \frac{d\zeta_0(x')}{dx'} e^{2ix't} I_0(2\sqrt{x'(x-x')}) - \int_0^x dw w e^{2iwt} I_0(2\sqrt{w(x-w)}) \int_0^w dx' x' \frac{d\zeta_0(x')}{dx'} \frac{J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}}. \quad (B3)$$

Direct calculations result in

$$\begin{aligned} \frac{\partial G}{\partial x} = & x \frac{d\zeta_0(x)}{dx} e^{2ixt} + \int_0^x dx' x' \frac{d\zeta_0(x')}{dx'} e^{2ix't} \frac{\partial}{\partial x} I_0(2\sqrt{x'(x-x')}) - x e^{2ixt} \int_0^x dx' x' \frac{d\zeta_0(x')}{dx'} \frac{J_1(2\sqrt{x(x-x')})}{\sqrt{x(x-x')}} \\ & - \int_0^x dw w e^{2iwt} \frac{\partial}{\partial x} I_0(2\sqrt{w(x-w)}) \int_0^w dx' x' \frac{d\zeta_0(x')}{dx'} \frac{J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}}, \end{aligned} \quad (B4)$$

and

$$\begin{aligned} x \frac{\partial \zeta}{\partial x} = & x \frac{d\zeta_0(x)}{dx} e^{2ixt} + \int_0^x dx' \frac{d\zeta_0(x')}{dx'} e^{2ix't} x \frac{\partial}{\partial x} I_0(2\sqrt{x'(x-x')}) - x e^{2ixt} \int_0^x dx' x' \frac{d\zeta_0(x')}{dx'} \frac{J_1(2\sqrt{x(x-x')})}{\sqrt{x(x-x')}} \\ & - \int_0^x dw e^{2iwt} x \frac{\partial}{\partial x} I_0(2\sqrt{w(x-w)}) \int_0^w dx' x' \frac{d\zeta_0(x')}{dx'} \frac{J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}}. \end{aligned} \quad (B5)$$

Subtracting these expressions we obtain

$$\begin{aligned} x \frac{\partial \zeta}{\partial x} - \frac{\partial G}{\partial x} = & \int_0^x dx' \frac{d\zeta_0(x')}{dx'} e^{2ix't} (x-x') \frac{\partial}{\partial x} I_0(2\sqrt{x'(x-x')}) \\ & - \int_0^x dw e^{2iwt} (x-w) \frac{\partial}{\partial x} I_0(2\sqrt{w(x-w)}) \int_0^w dx' x' \frac{d\zeta_0(x')}{dx'} \frac{J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}}. \end{aligned} \quad (B6)$$

Substituting this expression into Eq. (B2), we note that the derivative over the upper limit yields exactly zero, while the remaining differentiation results in

$$\begin{aligned} G = & \int_0^x dx' \frac{d\zeta_0(x')}{dx'} e^{2ix't} \frac{\partial}{\partial x} \left[(x-x') \frac{\partial}{\partial x} I_0(2\sqrt{x'(x-x')}) \right] - \int_0^x dw e^{2iwt} \frac{\partial}{\partial x} \left[(x-w) \frac{\partial}{\partial x} I_0(2\sqrt{w(x-w)}) \right] \\ & \times \int_0^w dx' x' \frac{d\zeta_0(x')}{dx'} \frac{J_1(2\sqrt{w(w-x')})}{\sqrt{w(w-x')}}. \end{aligned} \quad (B7)$$

Using the Bessel equations

$$\frac{\partial}{\partial x} \left[(x-x') \frac{\partial}{\partial x} I_0(2\sqrt{x'(x-x')}) \right] = I_0(2\sqrt{x'(x-x')}),$$

and

$$\frac{\partial}{\partial x} \left[(x-w) \frac{\partial}{\partial x} I_0(2\sqrt{w(x-w)}) \right] = I_0(2\sqrt{w(x-w)}),$$

we arrive at Eq. (B3).

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