Transverse emittance growth due to rf noise in the high-luminosity LHC crab cavities

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The high-luminosity LHC (HiLumi LHC) upgrade with planned operation from 2025 onward has a goal of achieving a tenfold increase in the number of recorded collisions thanks to a doubling of the intensity per bunch (2.2×10^{11} protons) and a reduction of β^* to 15 cm. Such an increase would significantly expedite new discoveries and exploration. To avoid detrimental effects from long-range beam-beam interactions, the half crossing angle must be increased to 295 microrad. Without bunch crabbing, this large crossing angle and small transverse beam size would result in a luminosity reduction factor of 0.3 (Piwinski angle). Therefore, crab cavities are an important component of the LHC upgrade, and will contribute strongly to achieving an increase in the number of recorded collisions. The proposed crab cavities are electromagnetic devices with a resonance in the radio frequency (rf) region of the spectrum (400.789 MHz). They cause a kick perpendicular to the direction of motion (transverse kick) to restore an effective head-on collision between the particle beams, thereby restoring the geometric factor to 0.8 [K. Oide and K. Yokoya, *Phys. Rev. A* **40**, 315 (1989)]. Noise injected through the rf/low level rf (llrf) system could cause significant transverse emittance growth and limit luminosity lifetime. In this work, a theoretical relationship between the phase and amplitude rf noise spectrum and the transverse emittance growth rate is derived, for a hadron machine assuming zero synchrotron radiation damping and broadband rf noise, excluding infinitely narrow spectral lines. This derivation is for a single beam. Both amplitude and phase noise are investigated. The potential improvement in the presence of the transverse damper is also investigated.

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I. INTRODUCTION

The effect of momentum kicks on the transverse emittance growth has been studied and measured before. Crab cavity tests at KEK [1] have shown the beam sensitivity to crab cavity rf noise [2]. Contrary to the HiLumi LHC though, these tests were conducted with a lepton beam, which has a very short synchrotron radiation damping time, focused on rf noise with a single spectral line, and were dominated by a π -mode instability. Additionally, unlike the HiLumi LHC ($\sigma_z = 7.55$ cm, $\lambda_{\text{rf}} = 75$ cm) the KEK bunch is very short compared to the rf wavelength ($\sigma_z = 0.4$ cm, $\lambda_{\text{rf}} = 59$ cm). Crab cavity noise effects, including simulation results, were also studied in [3] with a focus on the LHC, only for phase noise though. Reference [4] presents the emittance growth caused by transverse dipole kicks in colliding beams in the presence of a transverse damper. Studies have been performed in the Tevatron for the effect of dipole kicks to the transverse emittance growth [5,6] and

the mitigation with a transverse damper [7]. Finally, simulations have been conducted to investigate the beam-beam interaction in the presence of a transverse damper [8]. This work focuses on the crab cavity effect on transverse emittance growth in the HiLumi LHC [9], considers both phase and amplitude noise, incorporates long bunches (applicable to the LHC), and uses the tune distribution in the derivation to model the transverse dynamics.

Tests will be conducted in the Super Proton Synchrotron (SPS) with crab cavity prototypes before installation in the LHC. This work provides the framework for scaling the SPS results to the HiLumi LHC parameters and for estimating the expected behavior. It will drive the specification for the design of the LHC crab cavity low level rf.

Section II presents the methodology used in this work. The transverse emittance dependence on the noise induced momentum kicks is introduced in Sec. III. Then, Sec. IV relates the transverse momentum kicks to the betatron/synchrotron motion and the phase/amplitude noise processes. Sections V and VI provide the theoretical formalism relating the rf noise to transverse emittance growth for phase and amplitude noise, respectively. Section VII presents the expected reduction in emittance growth rates due to the transverse damper. Finally, Sec. VIII validates the above formalism through simulations.

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II. METHODOLOGY

A statistical approach is used to calculate the transverse emittance growth caused by momentum kicks created by the crab cavity rf noise. The statistical approach is appropriate for three reasons.

First, the transverse momentum kicks Δp [10] that will affect the beam are defined by a sequence of random samples (a stochastic process). Therefore, these kicks are not known at every turn. The single particle response is known however and it is easy to track the influence of the momentum kicks on the particle motion over time. The random process Δp is assumed to have a mean of zero, and stationary: at each time its expected value is zero, whereas its statistics are not affected by a shift in the time origin. With these assumptions the random process is fully characterized by its autocorrelation function, or equivalently the power spectral density in the frequency domain, that can be calculated from measurements of the cavity field. $\Delta p(t)$ will be modeled as a continuous-time random process because the measurements of phase and amplitude noise in the cavity belong to the continuous-time domain. The periodic passage of the beam in the cavity will sample the noise spectrum at multiples of the revolution frequency.

Second, since the goal is to track the emittance growth of the whole bunch, it is necessary to obtain the ensemble average of the momentum kick effect over all particles in a bunch. An individual particle ζ is characterized by a set of values generated by the random variable vector $(\hat{x}, \nu_b, \theta, \hat{\phi}, \nu_s, \psi)$, that fully parameterizes its transverse and longitudinal motion. The first three variables are the peak amplitude, tune, and phase at time 0 of the normalized transverse motion (horizontal or vertical) of particle ζ , known as the betatron oscillation. Similarly the last three variables are the peak amplitude (in radians), synchrotron tune, and phase at time 0 of the longitudinal motion of particle ζ (synchrotron oscillation). The ensemble of particles in the bunch are represented by a statistical density function $f(\hat{x}, \nu_b, \theta, \hat{\phi}, \nu_s, \psi)$. In the analysis presented in this work, the density function is independent of time. This is correct for small emittance growth only, which is the case for the HiLumi LHC.

Finally, the same statistical approach is also valid for the action of the LHC transverse damper as it computes the correction kick from an ensemble average of the transverse position over all particles in a bunch.

The crab cavities will act on the horizontal direction for one LHC experiment and on the vertical for the other. For simplicity, only the horizontal emittance growth is considered in this work, but the derivation and conclusions are identical for the two cases. Zero coupling between the horizontal and vertical planes is assumed.

With this statistical approach, the authors deduce the transverse emittance growth rate, as a function of the beam betatron tune distribution and the rf phase and amplitude noise power spectral density.

This statistical analysis is similar to the one used in [11], a study for the superconducting super collider (SSC). In [11] though, the authors first study the emittance growth due to external noise (transverse dipole momentum kicks), then apply the results to the case of quadrupole vibrations and magnetic field fluctuations. Such noise sources generate momentum kicks that are identical for all particles in the bunch. The crab cavity case is different: phase and amplitude noise in the crab cavity are indeed independent of the particle distribution, but the resulting momentum kicks do depend on the longitudinal motion. In this work, the situation during physics is considered: the bucket is a nonaccelerating bucket (180° stable phase) and the phase of the crabbing voltage is adjusted with the synchronous particle (center of the bunch) at the zero crossing. Then, the momentum kick caused by a given amplitude fluctuation will have opposite signs for the head and the tail of the bunch. The longitudinal motion will make a particle move from head to tail at the synchrotron frequency. Therefore, the momentum kicks caused by amplitude noise depend on both the rf noise and the particle's longitudinal motion. Similarly, due to the long LHC bunches, the rf phase noise will not generate a uniform kick along the bunch. The longitudinal tails will see a kick smaller than the kick experienced by the core.

This derivation is for a single beam. Beam-beam effects at the interaction points influence the transverse tune distributions. An effort has been made to use a realistic tune distribution through the appropriate scaling of octupole action.

To reduce the complexity and to aid the reader, the derivations are first presented for the case of short bunch length. Then, the extensions with long bunch length are presented in the Appendix.

III. TRANSVERSE EMITTANCE DEPENDENCE ON MOMENTUM KICKS

The horizontal particle motion of a particle ζ in an accelerator can be described by

$$X = \sqrt{2\beta J} \cos \xi(s)$$

$$X' = -\alpha \sqrt{\frac{2J}{\beta}} \cos \xi(s) - \sqrt{\frac{2J}{\beta}} \sin \xi(s)$$

where X is the particle's deviation from the closed orbit, $X' = dX/ds$ is the divergence, $\beta(s)$ the beta function, $\alpha = -\frac{1}{2} \frac{d\beta}{ds}$, J is the action, $\xi(s) = \int_0^s \frac{ds}{\beta} + \theta$, and s the path length along the orbit.

Setting

$$x = \frac{X}{\sqrt{\beta}} = \sqrt{2J} \cos \xi(s)$$

$$p = \sqrt{\beta} X' + \frac{\alpha}{\sqrt{\beta}} X = \beta \frac{dx}{ds} = -\sqrt{2J} \sin \xi(s)$$

the particle's oscillation follows a circle in a normalized phase-space defined by (x, p) .

In this coordinate system, the transverse emittance ϵ is given by

$$\begin{aligned}\epsilon &= \frac{1}{2} (E[(\mathbf{x} - E[\mathbf{x}])^2] + E[(\mathbf{p} - E[\mathbf{p}])^2]) \\ &= E[(\mathbf{x} - E[\mathbf{x}])^2] = E[(\mathbf{p} - E[\mathbf{p}])^2].\end{aligned}$$

Since the tolerable emittance growth in the HiLumi LHC (few percent per hour) is orders of magnitude slower than filamentation due to the betatron spread (tens of milliseconds), the expected value [12] of $\mathbf{x}$ or $\mathbf{p}$ over all particles in a bunch, will be very small at all times, and thus much smaller than $E[\mathbf{x}^2]$. As a result, the emittance is given by $E[\mathbf{x}^2]$,

$$\epsilon = E[\mathbf{x}^2] = E[\mathbf{p}^2] \quad (1)$$

and the emittance growth depends on the rate of change of $E[\mathbf{x}^2]$. This relationship is confirmed graphically in Sec. VIII by a multiparticle tracking simulation.

The variables (x, p) are observed at the crab cavity location, where the momentum kicks are applied as well. Then, at turn n ,

$$\begin{aligned}\begin{bmatrix} x \\ p \end{bmatrix}_n &= \begin{bmatrix} \cos(2\pi\nu_b) & \sin(2\pi\nu_b) \\ -\sin(2\pi\nu_b) & \cos(2\pi\nu_b) \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}_{n-1} + \begin{bmatrix} 0 \\ \Delta\mathbf{p}(nT_{\text{rev}}) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\pi\nu_b n) & \sin(2\pi\nu_b n) \\ -\sin(2\pi\nu_b n) & \cos(2\pi\nu_b n) \end{bmatrix} \begin{bmatrix} \hat{x} \cos \theta \\ -\hat{x} \sin \theta \end{bmatrix} \\ &\quad + \sum_{k=0}^n \begin{bmatrix} \cos[2\pi\nu_b(n-k)] & \sin[2\pi\nu_b(n-k)] \\ -\sin[2\pi\nu_b(n-k)] & \cos[2\pi\nu_b(n-k)] \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 \\ \Delta\mathbf{p}(kT_{\text{rev}}) \end{bmatrix},\end{aligned}$$

where $\hat{x} = \sqrt{2J}$ is the maximum amplitude of the betatron motion, $\nu_b = \frac{1}{2\pi} \int_0^C \frac{ds}{\beta}$ is the betatron tune of particle ζ , C is the accelerator circumference, and $\Delta\mathbf{p}(kT_{\text{rev}})$ are the noise induced normalized momentum kicks.

Then, the normalized transverse position for particle ζ at turn n is given by

$$\begin{aligned}x_n &= \hat{x} [\cos(2\pi\nu_b n) \cos \theta - \sin(2\pi\nu_b n) \sin \theta] \\ &\quad + \sum_{k=0}^n \Delta\mathbf{p}(kT_{\text{rev}}) \sin[2\pi\nu_b(n-k)] \\ &= \hat{x} \cos(2\pi\nu_b n + \theta) + \tilde{x}_n.\end{aligned} \quad (2)$$

$\tilde{x}_n$ is the noise induced perturbation from the momentum kicks $\Delta\mathbf{p}$ convolved with the particle response. This impulse response assumes zero-damping of the excitation, which is the case for a hadron collider (transverse emittance radiation damping time is 26 h in the LHC at 7 TeV [13]).

It is then necessary to take the expected value over all particles in the bunch, that is over the random variable

vector $(\hat{x}, \nu_b, \theta)$. The betatron tune ν_b and the amplitude of the betatron oscillation $\hat{x}$ are correlated through the transverse nonlinearity of the machine (octupole magnets or beam-beam effects). But the phase advance θ at the time when the noise starts is independent of $(\hat{x}, \nu_b)$, and can be assumed uniformly distributed in $[-\pi, \pi]$. The expected value of $E[\mathbf{x}^2]$ over θ for given $(\hat{x}, \nu_b)$, is given by the conditional expectation $E[\mathbf{x}_n^2 | \hat{x}, \nu_b]$ [14]:

$$\begin{aligned}E[\mathbf{x}_n^2 | \hat{x}, \nu_b] &= \hat{x}^2 E[\cos^2(2\pi\nu_b n + \theta)] \\ &\quad + 2\hat{x} E[\tilde{x}_n \cos(2\pi\nu_b n + \theta)] + E[\tilde{x}_n^2 | \hat{x}, \nu_b] \\ &= \frac{\hat{x}^2}{2} E[1 + \cos(4\pi\nu_b n + 2\theta)] \\ &\quad + 2\hat{x} E[\tilde{x}_n \cos(2\pi\nu_b n)] E[\cos(\theta)] \\ &\quad - 2\hat{x} E[\tilde{x}_n \sin(2\pi\nu_b n)] E[\sin(\theta)] + E[\tilde{x}_n^2 | \hat{x}, \nu_b] \\ &= \frac{\hat{x}^2}{2} + E[\tilde{x}_n^2 | \hat{x}, \nu_b].\end{aligned} \quad (3)$$

Therefore, after averaging over the bunch $(\hat{x}, \nu_b)$ distribution, the first term gives the original transverse emittance. As a result, the emittance growth is due to the second term, which corresponds to the contributions from the perturbation kicks. Following Eq. (2),

$$\begin{aligned}E[\tilde{x}_n^2 | \hat{x}, \nu_b] &= E \left\{ \sum_{k=0}^n \sum_{l=0}^n \Delta\mathbf{p}(kT_{\text{rev}}) \Delta\mathbf{p}(lT_{\text{rev}}) \right. \\ &\quad \times \sin[2\pi\nu_b(n-k)] \sin[2\pi\nu_b(n-l)] \left. \right\}.\end{aligned} \quad (4)$$

IV. CRAB CAVITY MOMENTUM KICKS

The synchrotron oscillation of particle ζ is described by

$$\phi_n = \hat{\phi} \cos(2\pi\nu_s n + \psi), \quad (5)$$

where $\hat{\phi}$ is the peak amplitude of the synchrotron oscillation (in radians), ν_s the synchrotron tune, ψ the phase of the synchrotron oscillation at time zero, and n is the turn index.

The change in divergence due to the crab cavity noise momentum kick on a particle is given by

$$\Delta X'_n = \frac{e\Delta V_n}{E_b},$$

where E_b is the particle's energy, $V_n = V_o(1 + \Delta A_n)$ $\sin(\phi_n + \Delta\phi_n)$ is the crab cavity voltage, with ϕ_n the particle's phase with respect to the synchronous particle, V_o is the desired crab cavity voltage, ΔA_n the relative amplitude noise, and $\Delta\phi_n$ the phase noise.

The crab cavity voltage error ΔV_n is given by

$$\begin{aligned}\Delta V_n &= V_o [\sin(\phi_n + \Delta\phi_n) - \sin(\phi_n)] \\ &\quad + V_o \Delta A_n \sin(\phi_n + \Delta\phi_n) \\ &\approx V_o \cos(\phi_n) \Delta\phi_n + V_o \sin(\phi_n) \Delta A_n \\ &\quad + V_o \cos(\phi_n) \Delta\phi_n \Delta A_n.\end{aligned}\quad (6)$$

The last term is negligible. In addition, if the phase and amplitude noise spectra are independent, the two cases can be considered separately. This is a reasonable assumption in the LHC due to the rf noise sources [15].

The normalized momentum kicks due to the crab cavity voltage error ΔV_n are given by

$$\begin{aligned}\Delta p_n &= \sqrt{\beta_{CC}} \Delta X'_n + \frac{\alpha_{CC}}{\sqrt{\beta_{CC}}} \Delta X_n \\ &= \sqrt{\beta_{CC}} \Delta X'_n = \sqrt{\beta_{CC}} \frac{e \Delta V_n}{E_b}\end{aligned}$$

since the crab cavity kicks do not instantaneously change the position ($\Delta X_n = 0$). β_{CC} is the beta function at the crab cavity.

The normalized momentum kick due to phase noise alone and received by particle ζ is then

$$\begin{aligned}\Delta p_{\phi,n} &\approx \sqrt{\beta_{CC}} \frac{e V_o}{E_b} \cos(\phi_n) \Delta\phi_n \\ &= \sqrt{\beta_{CC}} \frac{e V_o}{E_b} \cos[\hat{\phi} \cos(2\pi\nu_s n + \psi)] \Delta\phi_n.\end{aligned}\quad (7)$$

The kick does indeed depend on the synchrotron motion, via $\cos(\phi_n)$. For a short bunch with respect to the rf wavelength, this factor will be close to one for all particles and as a result the bunch experiences a uniform (dipole) kick. For a long bunch this term will reduce the effect for particles in the tails of the longitudinal distribution.

The normalized momentum kick due to amplitude noise alone is

$$\begin{aligned}\Delta p_{A,n} &\approx \sqrt{\beta_{CC}} \frac{e V_o}{E_b} \sin(\phi_n) \Delta A_n \\ &= \sqrt{\beta_{CC}} \frac{e V_o}{E_b} \sin[\hat{\phi} \cos(2\pi\nu_s n + \psi)] \Delta A_n.\end{aligned}\quad (8)$$

The dependence on the synchrotron motion is now via a $\sin(\phi_n)$ factor. This factor is zero for very short bunches. Amplitude noise can be ignored in that case, as experienced at KEK. For long bunches the factor will have a decisive impact on the effect of amplitude noise. The momentum kick depends on the noise random process ΔA_n and on the distribution of the random variables $\hat{\phi}$, ν_s , and ψ which characterize the synchrotron oscillation of the particles in the bunch. Although the random process ΔA_n is independent of the particle longitudinal motion ($\hat{\phi}$, ν_s , ψ), the momentum kicks caused by the crab cavity amplitude noise do depend on it.

V. TRANSVERSE EMITTANCE GROWTH DUE TO PHASE NOISE

In the phase noise case, the momentum kicks follow Eq. (7), and depend on the longitudinal motion and bunch length ($\hat{\phi}$, ν_s , ψ) so that Eq. (4) becomes

$$\begin{aligned}E[\tilde{x}_n^2 | \hat{x}, \nu_b, \hat{\phi}, \nu_s, \psi] &= E[\tilde{x}_n^2 | \nu_b, \hat{\phi}, \nu_s, \psi] \\ &= \beta_{cc} \left(\frac{e V_o}{E_b} \right)^2 \sum_{k=0}^n \sum_{l=0}^n E\{\Delta\phi(kT_{\text{rev}}) \Delta\phi(lT_{\text{rev}}) \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \cos[\hat{\phi} \cos(2\pi\nu_s l + \psi)] \\ &\quad \times \sin[2\pi\nu_b(n-k)] \sin[2\pi\nu_b(n-l)]\} \\ &= \beta_{cc} \left(\frac{e V_o}{E_b} \right)^2 \sum_{k=0}^n \sum_{l=0}^n R_{\Delta\phi}[(k-l)T_{\text{rev}}] \{\cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \cos[\hat{\phi} \cos(2\pi\nu_s l + \psi)]\} \\ &\quad \times \sin[2\pi\nu_b(n-k)] \sin[2\pi\nu_b(n-l)]\end{aligned}\quad (9)$$

by the definition of the autocorrelation function:

$$E\{\Delta\phi[(k+n)T_{\text{rev}}] \Delta\phi(kT_{\text{rev}})\} \triangleq R_{\Delta\phi}[nT_{\text{rev}}].$$

As the bunch length approaches zero, $\hat{\phi}$ is around zero and Eq. (9) becomes

$$\begin{aligned}E[\tilde{x}_n^2 | \nu_b] &= \beta_{cc} \left(\frac{e V_o}{E_b} \right)^2 \sum_{k=0}^n \sum_{l=0}^n R_{\Delta\phi}[(k-l)T_{\text{rev}}] \sin[2\pi\nu_b(n-k)] \sin[2\pi\nu_b(n-l)] \\ &= \frac{\beta_{cc}}{2} \left(\frac{e V_o}{E} \right)^2 \sum_{k=0}^n \sum_{l=0}^n R_{\Delta\phi}[(k-l)T_{\text{rev}}] \{\cos[2\pi\nu_b(k-l)] - \cos[2\pi\nu_b(2n-k-l)]\}\end{aligned}\quad (10)$$

so that the phase noise influence on the transverse emittance does not depend on the longitudinal motion.

Setting $p = k - l$, the set of k, l values is sampled along the diagonals as shown in Fig. 1. The 2-dimensional summation of Eq. (10) is rewritten by first adding all terms on a given diagonal, then summing over all diagonals. As a result, Eq. (10) becomes

$$\begin{aligned} E[\tilde{x}_n^2|\nu_b] &= \frac{\beta_{cc}}{2} \left(\frac{eV_0}{E} \right)^2 \left\{ \sum_{p=0}^n \sum_{l=0}^{n-p} R_{\Delta\phi}[pT_{\text{rev}}] \{ \cos(2\pi\nu_b p) - \cos[2\pi\nu_b(2n - p - 2l)] \} \right. \\ &\quad \left. + \sum_{p=-n}^{-1} \sum_{l=-p}^n R_{\Delta\phi}[pT_{\text{rev}}] \{ \cos(2\pi\nu_b p) - \cos[2\pi\nu_b(2n - p - 2l)] \} \right\} \\ &= \frac{\beta_{cc}}{2} \left(\frac{eV_0}{E} \right)^2 \left\{ \sum_{p=0}^n \sum_{l=0}^{n-p} R_{\Delta\phi}[pT_{\text{rev}}] \{ \cos(2\pi\nu_b p) - \cos(2\pi\nu_b[2n - p - 2l]) \} \right. \\ &\quad \left. + \sum_{p'=1}^n \sum_{l'=0}^{n-p'} R_{\Delta\phi}[p'T_{\text{rev}}] \{ \cos(2\pi\nu_b p') - \cos[2\pi\nu_b(2n - p' - 2l')] \} \right\} \\ &= \frac{\beta_{cc}}{2} \left(\frac{eV_0}{E} \right)^2 \left\{ \sum_{l=0}^n R_{\Delta\phi}[0] \{ 1 - \cos[2\pi\nu_b(2n - 2l)] \} \right. \\ &\quad \left. + 2 \sum_{p=1}^n \sum_{l=0}^{n-p} R_{\Delta\phi}[pT_{\text{rev}}] \{ \cos(2\pi\nu_b p) - \cos(2\pi\nu_b(2n - p - 2l)) \} \right\} \end{aligned} \quad (11)$$

since the autocorrelation function is symmetric and where $p' = -p$ and $l' = l - p'$. Starting with noise injected at time 0, the summation over p will include more and more samples of the autocorrelation function, as n grows. For a noise that does not include infinitely narrow spectral lines, the autocorrelation function will eventually drop to zero after a number of turns (N). N is the number of turns for which NT_{rev} is significantly larger than the noise decoherence time—the time it takes for $R_{\Delta\phi}[t]$ to drop to half of $R_{\Delta\phi}[0]$. N is independent of the beam parameters. So, for $n > N$, $R_{\Delta\phi}[nT_{\text{rev}}]$ can be ignored, and

$$\begin{aligned} E[\tilde{x}_n^2|\nu_b] &= \frac{\beta_{cc}}{2} \left(\frac{eV_0}{E} \right)^2 \left\{ R_{\Delta\phi}[0] \sum_{l=0}^n 1 + 2 \sum_{p=1}^N R_{\Delta\phi}[pT_{\text{rev}}] \cos(2\pi\nu_b p) \sum_{l=0}^{n-p} 1 \right\} \\ &\quad - \frac{\beta_{cc}}{2} \left(\frac{eV_0}{E} \right)^2 \left\{ R_{\Delta\phi}[0] \sum_{l=0}^n \cos[2\pi\nu_b(2n - 2l)] + 2 \sum_{p=1}^N R_{\Delta\phi}[pT_{\text{rev}}] \sum_{l=0}^{n-p} \cos[2\pi\nu_b(2n - p - 2l)] \right\} \\ &= \beta_{cc} \left(\frac{eV_0}{E_b} \right)^2 \left\{ \frac{R_{\Delta\phi}[0]}{2} (n+1) + \sum_{p=1}^N R_{\Delta\phi}[pT_{\text{rev}}] (n-p+1) \cos[2\pi p\nu_b] \right\} \\ &\quad - \beta_{cc} \left(\frac{eV_0}{E_b} \right)^2 \left\{ \frac{R_{\Delta\phi}[0]}{2} \sum_{k=0}^n \cos[4\pi\nu_b k] + \sum_{p=1}^N R_{\Delta\phi}[pT_{\text{rev}}] \sum_{k=p}^n \cos[2\pi(2k-p)\nu_b] \right\}, \end{aligned} \quad (12)$$

where $k = n - l$. The expression in the first curly brackets is linear in n and will therefore lead to a linear emittance growth (identical growth at every turn). The expression in the second curly brackets is oscillatory at twice the betatron tune. Emittance growth has therefore three characteristics:

- (i) A first transient, lasting for a time (N turns) proportional to the decoherence time of the rf noise process. This will always be much shorter than the timescale of interest for luminosity lifetime in physics (except for infinitely narrow [16] spectral lines, not considered in this paper).
- (ii) An oscillation at twice the betatron tune. With a tune close to $1/3$, this will create a fast (but small amplitude) ripple from turn to turn
- (iii) A linear growth, the significant trend during the long physics fills.

Figure 2 shows a numerical computation of $E[\tilde{x}_n^2|\nu_b]$ for a low-pass noise spectrum (-3 dB at $0.004 f_{\text{rev}}$). The autocorrelation function $R[nT_{\text{rev}}]$ is a decaying exponential with a time constant of 4.4 ms (≈ 50 turns). After a transient of about 50 turns corresponding to the noise decoherence time, the growth, smoothed over a time period much longer than the betatron period, is linear. Superimposed to this linear growth, there is a small fluctuation at twice the betatron tune. The betatron tune was set to 0.025 so that the

FIG. 1. Graphic of summation index change.

FIG. 2. $E[\tilde{x}_n^2|\nu_b]$ over the first 200 turns (blue dots) computed from Eq. (10), compared to the linear trend of Eq. (13).

betatron period is much longer than the sampling period of one turn and thus clearly visible on the plot. The LHC operates with a noninteger tune close to $1/3$ (a betatron period of about 3 turns). Therefore, the emittance growth oscillation period will be close to 1.5 turns.

From Eq. (12), the linear component of the emittance growth is therefore

$$E[\tilde{x}_n^2|\nu_b] \approx \beta_{cc} \left(\frac{eV_o}{E_b} \right)^2 \left[\frac{(n+1)}{2} R_{\Delta\phi}[0] + \sum_{p=1}^{\infty} (n-p+1) R_{\Delta\phi}[pT_{\text{rev}}] \cos[2\pi p\nu_b] \right] \quad (13)$$

where N was replaced by infinity as the autocorrelation function is assumed to be zero for $n > N$.

The increase between two turns is then

$$\begin{aligned} E[\tilde{x}_n^2|\nu_b] - E[\tilde{x}_{n-1}^2|\nu_b] &= \beta_{cc} \left(\frac{eV_o}{E_b} \right)^2 \left\{ \frac{1}{2} R_{\Delta\phi}[0] + \sum_{p=1}^{\infty} R_{\Delta\phi}(pT_{\text{rev}}) \cos[2\pi p\nu_b] \right\} \\ &= \frac{\beta_{cc}}{2} \left(\frac{eV_o}{E_b} \right)^2 \sum_{k=-\infty}^{\infty} R_{\Delta\phi}[kT_{\text{rev}}] e^{-j2\pi k\nu_b}. \end{aligned} \quad (14)$$

The autocorrelation function is the inverse Fourier transform of the power spectral density (PSD) $S_{\Delta\phi}(f)$:

$$R_{\Delta\phi}[t] = \int_{-\infty}^{\infty} S_{\Delta\phi}(f) e^{j2\pi f t} df.$$

With this definition, Eq. (14) becomes

$$\begin{aligned} E[\tilde{x}_n^2|\nu_b] - E[\tilde{x}_{n-1}^2|\nu_b] &= \frac{\beta_{cc}}{2} \left(\frac{eV_o}{E_b} \right)^2 \int_{-\infty}^{\infty} S_{\Delta\phi}(f) \left[\sum_{k=-\infty}^{\infty} e^{j2\pi k(fT_{\text{rev}} - \nu_b)} \right] df. \end{aligned} \quad (15)$$

Using the identity

$$\sum_{k=-\infty}^{\infty} e^{j2\pi k(fT_{\text{rev}} - \nu_b)} = f_{\text{rev}} \sum_{k=-\infty}^{\infty} \delta[f - kf_{\text{rev}} - \nu_b f_{\text{rev}}].$$

Equation (15) becomes

$$\begin{aligned} E[\tilde{x}_n^2|\nu_b] - E[\tilde{x}_{n-1}^2|\nu_b] &= \frac{\beta_{cc}}{2} \left(\frac{eV_o}{E_b} \right)^2 f_{\text{rev}} \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k + \nu_b)f_{\text{rev}}]. \end{aligned} \quad (16)$$

Finally, since the PSD is even symmetric, the spectrum can be sampled at both the positive and negative betatron sidebands with a reduction of the scaling factor by $1/2$. Then, Eq. (16) can be written as

$$\begin{aligned} E[\tilde{x}_n^2|\nu_b] - E[\tilde{x}_{n-1}^2|\nu_b] &= \beta_{cc} \left(\frac{eV_o}{2E_b} \right)^2 f_{\text{rev}} \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k \pm \nu_b)f_{\text{rev}}]. \end{aligned} \quad (17)$$

Equation (17) shows that, for particles with tune ν_b , and after a transient corresponding to the noise decoherence time, the growth, smoothed over a time long compared to the betatron period, is linear with a rate equal to the sum of the noise power spectral density on all betatron bands. Small fluctuation at twice the betatron tune will be superimposed to this linear growth, as was shown in Eq. (12) and Fig. 2.

This derivation is for particles at tune ν_b . Equation (17) should be averaged over the tune distribution of the bunch to get the increase in the variance of the transverse position due to phase noise. The probability density function of the betatron tune ν_b over all particles is given by $\rho(\nu_b)$, with mean $\bar{\nu}_b$ and standard deviation σ_{ν_b} .

$$\begin{aligned} E[\tilde{x}_n^2] - E[\tilde{x}_{n-1}^2] &= \beta_{cc} \left(\frac{eV_o}{2E_b} \right)^2 f_{\text{rev}} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k \pm \nu_b)f_{\text{rev}}] \rho(\nu_b) d\nu_b. \end{aligned} \quad (18)$$

Physically, Eq. (18) implies that the emittance growth depends on the frequency-domain overlap between the noise spectrum and the betatron tune distribution. Noise outside this overlap has no effect on transverse emittance growth. Additionally, the growth rate depends linearly on the power spectral density. Furthermore, the periodicity of the beam aliases the noise spectrum to the band from DC to f_{rev} , or equivalently, the noise spectrum is sampled by the tune distribution around each revolution harmonic. As shown in Eq. (17), particles may be affected differently if the noise PSD varies significantly within the tune spread. In that case the emittance growth will depend on the actual betatron tune distribution. This is the case in the presence of the transverse damper, as shown in Sec. VII.

If the betatron tune spread σ_{ν_b} is sufficiently narrow that $S_{\Delta\phi}(\nu_b f_{\text{rev}})$ is constant within the betatron spread—which is the case in the HiLumi LHC—the effect of noise is independent of the actual tune distribution and Eq. (18) becomes

$$E[\tilde{x}_n^2] - E[\tilde{x}_{n-1}^2] = \beta_{cc} \left(\frac{eV_o}{2E_b} \right)^2 f_{\text{rev}} \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k \pm \bar{\nu}_b)f_{\text{rev}}].$$

Finally, since $\epsilon = E[x^2] = E[p^2]$ in our coordinate system,

$$\begin{aligned} \frac{d\epsilon_x}{dt} &\approx \frac{\Delta E[x_n^2]}{T_{\text{rev}}} = \frac{\Delta E[\tilde{x}_n^2]}{T_{\text{rev}}} \\ &= \beta_{cc} \left(\frac{eV_o f_{\text{rev}}}{2E_b} \right)^2 \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k \pm \bar{\nu}_b)f_{\text{rev}}]. \end{aligned} \quad (19)$$

This result agrees with previous work in [6,11], among others.

The above derivation assumes a short bunch length ($\hat{\phi} \approx 0$). Appendix A shows how this derivation can be adapted for a bunch of any length when $\hat{\phi}$ follows a distribution $f_{\hat{\phi}}(\hat{\phi})$. For a two-dimensional Gaussian longitudinal distribution in phase space, $\hat{\phi}$ follows a Rayleigh distribution, and Eq. (19) becomes

$$\begin{aligned} \frac{d\epsilon_x}{dt} &= \beta_{cc} \left(\frac{eV_o f_{\text{rev}}}{2E_b} \right)^2 C_{\Delta\phi}(\sigma_{\phi}) \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k \pm \bar{\nu}_b)f_{\text{rev}}] \\ C_{\Delta\phi}(\sigma_{\phi}) &= e^{-\sigma_{\phi}^2} \left[I_0(\sigma_{\phi}^2) + 2 \sum_{l=1}^{\infty} I_{2l}(\sigma_{\phi}^2) \right] \end{aligned} \quad (20)$$

where $I_{2n}[x]$ is the modified Bessel function of the first kind and σ_{ϕ} the rms longitudinal bunch line density (in radians at the crab cavity frequency). Figure 3 shows the term $C_{\Delta\phi}(\sigma_{\phi})$, the correction term due to bunch length. As the bunch length increases, the effect of phase noise on transverse emittance growth is reduced: particles undergoing large synchrotron oscillations see a smaller effect due to the difference between the reference and shifted sine waves when they cross the cavity at the peak of their longitudinal synchrotron oscillation. In fact, particles crossing the cavity at a $\pm\pi/2$ phase offset see no kick from phase noise.

FIG. 3. Growth rate dependence on bunch length $C_{\Delta\phi}(\sigma_{\phi})$, phase noise. The LHC nominal bunch length is shown for reference.

The case when the longitudinal bunch centroid follows sinusoidal motion is also of interest. This centroid motion is equivalent to a modulation of the betatron tune by the centroid motion tune ν_c . The total noise power sampled is unchanged for a white noise spectrum. Additionally, the centroid motion leads to an apparent bunch lengthening, and thus to a small reduction of phase noise effects and a small increase of amplitude noise effects. Since though the longitudinal bunch centroid motion in the LHC is less than 1° and the 4σ bunch length is 145° , these effects will be negligible. A full mathematical description of the bunch centroid motion has been developed and associated simulations have been conducted to confirm the above statements. They are beyond the scope of this work and thus are not presented here.

VI. TRANSVERSE EMITTANCE GROWTH DUE TO AMPLITUDE NOISE

The main difference from the phase noise derivation is the momentum kick dependence on the particle statistics. Using Eq. (8), Eq. (4) for the amplitude noise case becomes

$$\begin{aligned} E[\tilde{x}_n^2 | \nu_b, \hat{\phi}, \nu_s, \psi] &= \sum_{k=0}^n \sum_{l=0}^n E\{\Delta p_A(kT_{\text{rev}}) \Delta p_A(lT_{\text{rev}}) \sin[2\pi\nu_b(n-k)] \sin[2\pi\nu_b(n-l)]\} \\ &= \beta_{cc} \left(\frac{eV_o}{E_b} \right)^2 \sum_{k=0}^n \sum_{l=0}^n E\{\Delta A(kT_{\text{rev}}) \Delta A(lT_{\text{rev}}) \\ &\quad \times \sin[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \sin[\hat{\phi} \cos(2\pi\nu_s l + \psi)] \sin[2\pi\nu_b(n-k)] \sin[2\pi\nu_b(n-l)]\}. \end{aligned} \quad (21)$$

It is thus straightforward to show (Appendix B) that—if the noise PSD is constant within the betatron tune spread—the emittance growth rate due to amplitude noise is given by

$$\begin{aligned} \frac{d\epsilon_x}{dt} &= \beta_{cc} \left(\frac{eV_o f_{\text{rev}}}{2E_b} \right)^2 C_{\Delta A}(\sigma_\phi) \\ &\times \sum_{k=-\infty}^{\infty} S_{\Delta A}[(k \pm \bar{\nu}_b \pm \bar{\nu}_s) f_{\text{rev}}] \\ C_{\Delta A}(\sigma_\phi) &= e^{-\sigma_\phi^2} \sum_{l=0}^{\infty} I_{2l+1}[\sigma_\phi^2]. \end{aligned} \quad (22)$$

Similarly to Eq. (20), Eq. (22) implies that the particle beam is only sensitive to the part of the noise spectrum overlapping with the tune distribution, but *shifted* up or down by the synchrotron tune. The beam will react strongly if the amplitude noise is on the synchrotron bands; amplitude noise kicks the head and tail of the bunch in opposite transverse directions. As a particle moves from the head to the tail at the synchrotron frequency, amplitude noise at $\nu_b \pm \nu_s$ will result in kicking a given particle at the ν_b frequency, and thus in a resonant response.

The growth rate also depends linearly on the power spectral density again and the noise spectrum is sampled around each revolution harmonic. Unlike the phase noise case though, the transverse emittance growth rate in the presence of amplitude noise increases with the second moment of the longitudinal line density, following the correction term in Fig. 4. This is expected since the amplitude noise is zero at the center of the bunch and increases toward the head and tail of the bunch. The very small factor for short bunch length explains why amplitude noise was of no concern at KEK.

In summary, Eqs. (20) and (22) differ by a factor of 2 (due to the sampling at two synchrotron sidebands for the latter case), as well as an opposite dependance on bunch length; phase noise effects reduce with bunch length, whereas amplitude noise effects increase. For the HiLumi LHC nominal bunch length, and for the same

FIG. 4. Growth rate dependence on bunch length $C_{\Delta A}(\sigma_\phi)$, amplitude noise. The LHC nominal bunch length is shown for reference.

noise spectrum, the growth rate due to phase noise is 2.65 times higher than the growth rate due to amplitude noise.

VII. GROWTH RATE REDUCTION DUE TO TRANSVERSE DAMPER

The LHC transverse damper [17] is a bunch-by-bunch system: it measures the mean transverse position $E[x]$ of each bunch individually, and generates a momentum kick, proportional to this measurement, but of opposite sign and with a 90° phase shift. As a result, the transverse damper could possibly partly mitigate the effect of the momentum kicks caused by the crab cavity phase and amplitude noise. Its efficiency in mitigating the effect of noise in a collider has been demonstrated in the Tevatron [7].

A. Beam transfer function

As the transverse damper measures the particle position averaged over the entire bunch, a derivation of the beam transfer function (BTF) H_{BTF} is necessary for this analysis. The continuous-time BTF has been presented in several publications, for example in [18,19]. Since the work presented here is in discrete-time, the corresponding BTF will be derived and will be shown to converge to the classic notation.

The bunch response is given by the expected value of the impulse response over the bunch:

$$h_n = E[\sin(2\pi\nu_b n)] = \int_{-\infty}^{\infty} \sin(2\pi\nu_b n) \rho(\nu_b) d\nu_b.$$

The discrete-time BTF, relating the momentum kicks to the average bunch displacement, is the z -transform of the bunch response evaluated on the unit circle. Note that the summation index starts at $n = 0$ because the response is causal.

$$\begin{aligned} H_{\text{BTF}}(z) &\triangleq \sum_{n=0}^{\infty} h_n z^{-n} = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \sin(2\pi\nu_b n) z^{-n} \rho(\nu_b) d\nu_b \\ H_{\text{BTF}}(e^{j2\pi\Omega}) &= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{e^{j2\pi n(\nu_b - \Omega)} - e^{-j2\pi n(\nu_b + \Omega)}}{2j} \right\} \\ &\times \rho(\nu_b) d\nu_b \end{aligned}$$

where the normalized frequency $\Omega = f/f_{\text{rev}}$ ranges between $-1/2$ and $1/2$.

It is shown in Appendix C that the infinite sum inside the integration converges to

$$\begin{aligned} &\sum_{n=0}^{\infty} \left\{ \frac{e^{j2\pi n(\nu_b - \Omega)} - e^{-j2\pi n(\nu_b + \Omega)}}{2j} \right\} \\ &= \frac{1}{4j} \{ \delta(\nu_b - \Omega) - \delta(\nu_b + \Omega) \} \\ &+ \frac{1}{4\pi} \text{P.V.} \left\{ \frac{1}{(\nu_b - \Omega)} + \frac{1}{(\nu_b + \Omega)} \right\} \end{aligned}$$

and the bunch transfer function becomes

$$H_{\text{BTF}}(e^{j2\pi\Omega}) = \frac{1}{4j} \{ \rho(\Omega) - \rho(-\Omega) \} \\ + \frac{1}{4\pi} \text{P.V.} \left\{ \int_{-\infty}^{\infty} \frac{\rho(\nu_b)}{(\nu_b - \Omega)} d\nu_b \right. \\ \left. + \int_{-\infty}^{\infty} \frac{\rho(\nu_b)}{(\nu_b + \Omega)} d\nu_b \right\}.$$

The last two integrals are not well behaved, but their Cauchy principal values do converge if the tune distribution is continuous. As the tune-spread is very small compared to the mean tune, the first integral will peak for frequency in the tune region, and the second integral for the negative (image) frequencies. Note that the above BTF is Hermitian, as it should. $\rho(\nu_b)$ is zero for negative ν_b . As a result, H_{BTF} for positive Ω can be approximated by

$$H_{\text{BTF}}(e^{j2\pi\Omega}) = \frac{1}{4j} \rho(\Omega) + \frac{1}{4\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\rho(\nu_b)}{(\nu_b - \Omega)} d\nu_b.$$

It is then convenient to scale and shift the tune distribution as in [19], using the functions $g(u)$, $f(u)$:

$$u = \frac{\nu_{b0} - \Omega}{\Delta\nu_b}, \\ g(u) = \pi\Delta\nu_b \rho(\nu_{b0} - u\Delta\nu_b), \\ f(u) = \Delta\nu_b \text{P.V.} \int_{-\infty}^{\infty} \frac{\rho(\nu_b)}{(\nu_b - \nu_{b0} + u\Delta\nu_b)} d\nu_b.$$

Then, the beam transfer function becomes

$$H_{\text{BTF}}(e^{j2\pi\Omega}) = \frac{1}{4\pi j \Delta\nu_b} g(u) + \frac{1}{4\pi \Delta\nu_b} f(u). \quad (23)$$

This definition is identical to Eqs. (5.26–27) in [19], except for the use of betatron tune in place of betatron frequency. Thus, $f(u)$ and $g(u)$ are scaled versions of the real and imaginary parts of the BTF. They only depend on the shape of the tune distribution. As shown in [19], for each type of tune distribution (normal, parabolic, exponential, etc.), the values $\Delta\nu_b$ and ν_{b0} can be chosen so that $f(u)$ and $g(u)$ depend only on u and no longer depend on $\Delta\nu_b$, ν_{b0} . In this work, $\Delta\nu_b = \sigma_{\nu_b}$.

B. Effective momentum kicks

The main rf noise source is the rf demodulator [15]. The resulting noise will be in the feedback regulation bandwidth which will span 100 kHz based on the specifications of the crab cavity llrf. This low frequency excitation will excite low-order modes only. For the HiLumi LHC operational scenario, the transverse damper imperfections (1 MHz bandwidth, 4.5 turns delay in processing) have no effect

on its low frequency response. Additionally, it will be shown in Sec. VII C that for the LHC a long bunch length is the main limitation in damper efficiency. Therefore, for the purpose of this analysis, an ideal transverse damper is considered where the pickup and kicker are just 90° apart. The damper response is then

$$H'(e^{j2\pi\Omega}) = jG \text{sgn}(\Omega). \quad (24)$$

With this model, the total momentum kick Δp_n received by a particle at turn n is the sum of the kick caused by the crab cavity plus the correction applied by the transverse damper kicker. The correction term is common for all particles in the bunch, whereas the noise term varies for a long bunch.

The signal measured by the damper pickup is the ensemble average (taken over all particles in the bunch) of the transverse position of the particles [Eq. (2)], under the influence of the noise induced momentum kicks Δp [Eqs. (7) and (8)]. For a short bunch and considering phase noise,

$$E[\tilde{x}_n] = E \left\{ \sum_{k=0}^n \Delta p(kT_{\text{rev}}) \sin[2\pi\nu_b(n-k)] \right\} \\ = \sum_{k=0}^n \sqrt{\beta_{CC}} \frac{eV_o}{E_b} \Delta\phi(kT_{\text{rev}}) E\{\sin[2\pi\nu_b(n-k)]\} \\ = \sqrt{\beta_{CC}} \frac{eV_o}{E_b} \sum_{k=0}^n \Delta\phi(kT_{\text{rev}}) h_{n-k} \quad (25)$$

since the kicks are common for all particles for a given turn and the expected value corresponds to the ensemble average over the bunch. The final term in this expression is the convolution of the bunch response with the momentum kicks, and in the frequency domain is obtained by filtering the momentum kicks with the BTF, as shown on Fig. 5.

It is then possible to estimate the reduction of the transverse emittance growth due to the damper by

FIG. 5. Momentum kicks and damper corrections.

FIG. 6. Transfer function between Δp_n and $\Delta p'_n$.

calculating how it modifies the PSD of the total momentum kicks Δp_n applied to the particles, resulting in an effective momentum kick $\Delta p'_n$, as shown in Fig. 5.

$\Delta p'_n$ can be represented as a filtered version of Δp_n , as shown in Fig. 6. When a random process $\Delta p'_n$ is generated by filtering another random process Δp_n through transfer function $K(e^{j2\pi\Omega})$, its PSD is the product of the PSD of Δp_n and the square modulus of $K(e^{j2\pi\Omega})$. In this case the transfer function $K(e^{j2\pi\Omega})$ is the closed-loop response of the feedback loop shown on Fig. 5. It is thus possible to calculate the noise PSD experienced by the particles:

$$S_{\Delta p'}(f) = \frac{1}{|1 + H'(e^{j2\pi\Omega})H_{BTF}(e^{j2\pi\Omega})|^2} S_{\Delta p}(f) = R_d(\Omega) S_{\Delta p}(f), \quad (26)$$

where $\Omega = f/f_{\text{rev}}$ and $R_d(\Omega)$ is the noise reduction factor due to the damper.

C. Correction of phase noise

Equation (7) shows that the momentum kicks due to phase noise are almost equal for particles in the core of the bunch [$\cos(\phi_n)$ dependence], so the transverse damper can reduce the effect of phase noise on transverse emittance growth. For short bunches, the momentum kicks due to phase noise are independent of the particle motion.

Then, for short bunches the phase noise PSD in Eq. (17) is reduced according to Eq. (26) leading to

$$E[\tilde{x}_n^2|\nu_b] - E[\tilde{x}_{n-1}^2|\nu_b] = \beta_{cc} \left(\frac{eV_o}{2E_b} \right)^2 f_{\text{rev}} \times \sum_{k=-\infty}^{\infty} R_d(\nu_b) S_{\Delta\phi}[(k \pm \nu_b)f_{\text{rev}}] R_d(\nu_b) = \frac{1}{|1 + jGH_{BTF}(e^{j2\pi\nu_b})|^2} \quad (27)$$

since $\text{sgn}(\nu_b) = 1$. The phase noise PSD reduction $R_d(\nu_b)$ is a function of the beam transfer function and consequently of the tune distribution. The damper has a significant effect for tunes corresponding to many particles (typically the core of bunch) but no effect for tunes weakly populated (bunch tails). As a result, the tail population will naturally deplete since it will experience a higher effective noise PSD. Figure 7 shows the noise PSD reduction $R_d(\nu_b)$

FIG. 7. Noise PSD reduction versus tune for a Gaussian tune distribution ($\sigma_{\nu_b} = 0.003$).

versus tune for various damper gain values. It is evident that the reduction is much greater at the core of the bunch.

Then, integrating Eq. (27) over the tune distribution,

$$E[\tilde{x}_n^2] - E[\tilde{x}_{n-1}^2] = \beta_{cc} \left(\frac{eV_o}{2E_b} \right)^2 f_{\text{rev}} \int_{-\infty}^{\infty} \times \sum_{k=-\infty}^{\infty} \frac{S_{\Delta\phi}[(k \pm \nu_b)f_{\text{rev}}]}{|1 + jGH_{BTF}(e^{j2\pi\nu_b})|^2} \rho(\nu_b) d\nu_b.$$

Assuming that the phase noise PSD does not change within the betatron tune spread,

$$E[\tilde{x}_n^2] - E[\tilde{x}_{n-1}^2] = \beta_{cc} \left(\frac{eV_o}{2E_b} \right)^2 f_{\text{rev}} \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k \pm \bar{\nu}_b)f_{\text{rev}}] \times \left\{ \int_{-\infty}^{\infty} \frac{\rho(\nu_b)}{|1 + jGH_{BTF}(e^{j2\pi\nu_b})|^2} d\nu_b \right\}.$$

The term within the curly brackets is $\bar{R}_d$, the expected value of the noise reduction factor, and it represents the growth rate reduction due to the action of the transverse damper. Using Eq. (23) it can be expressed as

$$\begin{aligned} \bar{R}_d &= \int_{-\infty}^{\infty} \frac{\rho(\nu_b)}{|1 + jGH_{BTF}(e^{j2\pi\nu_b})|^2} d\nu_b \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(u)}{[1 + \frac{G}{4\pi\sigma_{\nu_b}} g(u)]^2 + [\frac{G}{4\pi\sigma_{\nu_b}} f(u)]^2} du \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(u)}{[1 + \alpha g(u)]^2 + [\alpha f(u)]^2} du \\ \alpha &= \frac{G}{4\pi\sigma_{\nu_b}} \end{aligned} \quad (28)$$

so for a given tune distribution, the damper action on the full bunch emittance growth depends on α only. The

transverse damper will mitigate the noise if the damping time ($2T_{\text{rev}}/G$) is smaller than the decoherence time $T_{\text{rev}}/(2\pi\sigma_{\nu_b})$. Actually α is exactly equal to the ratio of these time constants.

Recall that $g(u) = \pi\Delta\nu_b\rho(\nu_{b0} - u\Delta\nu_b)$ is a scaled version of the betatron tune distribution. It is positive valued, and it integrates to π . The reduction factor will therefore also depend on the actual bunch tune distribution and will be smaller than 1 for all distributions/gains, if the damper gain is positive (negative values represent anti-damping). The functions $f(u)$ and $g(u)$ are provided for various distributions in [19]. Then, it is possible to calculate the correction factor $\bar{R}_d$ as a function of α as shown in Fig. 8. All curves correspond to the short bunch length approximation. The Gaussian distribution corresponds to a case with low octupole field and high chromaticity. The correction factor in the case of strong octupole fields (exponential tune distribution) is also shown. All the curves asymptotically approach $1/\alpha^2$ when the damping time becomes much smaller than the betatron decoherence time ($\alpha \gg 1$). This approximation was derived in [11,20] for dipole kicks. In Sec. VIII the correction factor is estimated via simulations and compared to the values predicted from Eq. (28).

Note that the damper gives the same kick to all particles, while the crab cavity phase noise gives smaller kicks to the bunch tails. Figure 5 is therefore correct for short bunches only. In this case, both the crab cavity phase noise and the damper give the same kick to all particles and the subtraction is correct. For long bunches, the damper action will be reduced, since particles at $\phi = \pm\pi/2$ are not affected by phase noise, but they will receive the damper kicks intended to reduce the motion of the bunch core, resulting in excitation. The complete derivation for the case of long bunch length is presented in Appendix D. The

growth rate reduction due to the damper for any bunch length is given by

$$\bar{R}_d = \frac{1}{\pi} \int_{-\infty}^{\infty} g(u) \times \left\{ 1 - \frac{e^{-\sigma_\phi^2}}{C_{\Delta\phi}(\sigma_\phi)} \frac{\alpha^2 [g(u)^2 + f(u)^2] + 2\alpha g(u)}{[1 + \alpha g(u)]^2 + [\alpha f(u)]^2} \right\} du. \quad (29)$$

Simulation results in Sec. VIII indeed show reduced damper effectiveness for long bunches. It is easy to show that Eq. (29) reduces to Eq. (28) for short bunches ($\sigma_\phi \rightarrow 0$).

Figure 9 shows the noise reduction factor as a function of α for different values of the bunch length. As the bunch length is increased, the damper is less efficient in two ways. First, the measurement (depending on the BTF) is indeed smaller as particles away from the longitudinal bunch core experience a smaller kick. This reduces the damper gain. Second, particles away from the core require a smaller correction, but the damper kick is uniform. As a result, the damper is inefficient in the longitudinal tails. Unlike the short bunch case, the damper correction does not asymptotically approach $1/\alpha^2$, but rather it saturates at a value that increases with bunch length.

D. Correction of amplitude noise

The synchronous phase in the HiLumi LHC is practically π . Therefore, the longitudinal distribution is symmetric (nonaccelerating bucket). As a result, the amplitude noise will cause symmetric transverse head-tail oscillations of the bunch and the mean position measured by the damper pickup will be zero *at every turn*. The bunch-by-bunch

FIG. 8. Damper correction factor as a function of α and tune distribution.

FIG. 9. Damper correction factor as a function of α for different values of σ_ϕ .

transverse damper *cannot* reduce the effect of crab cavity amplitude noise.

It is worth noting that even though the position averaged over all particles in a bunch has zero mean at all time, its variance is not zero. Statistically the transverse damper will measure an average position and apply a correction in the correct direction. It will therefore “cool” the beam. The variance of the position averaged over N particles in the bunch is inversely proportional to the number of particles. The stochastic cooling rate is therefore observable in simulations with small (less than 10^5) particles per bunch but is negligible in the real LHC (1.2×10^{11} p/bunch). This effect has been observed in past simulations [8] and was investigated during this work as well.

VIII. VALIDATION THROUGH HEADTAIL SIMULATIONS

HEADTAIL is a software package developed at CERN for simulation of multiparticle beam dynamics with collective effects [21,22]. The code includes various beam and machine parameters and computes the evolution of individual particles within a bunch over an adjustable number of turns. The bunch is normally distributed in six-dimensional space (x, p_x, y, p_y, z, p_z) and at every turn the phase advance of each particle is computed in each phase-space. It is thus possible to include nonlinear elements, such as the action of the octupoles, the rf voltage, the betatron, and synchrotron tune spreads, and more. The momentum kicks caused by the crab cavity noise are added to p_x at every turn following Eqs. (7) and (8).

The validity of the theoretical formalism presented in this work was tested through a series of HEADTAIL simulations where amplitude or phase noise was injected in the transverse plane. Results from the x direction are presented in this section for simplicity, but the results can be generalized to the y direction as well. Equations (20) and (22) show that the transverse emittance growth rate depends on the noise PSD, the bunch length, and the tune. These dependencies are tested in Secs. VIII C, VIII D, and VIII E. Then, the damper correction [Eqs. (28) and (29)] that additionally depends on the tune distribution and the damper gain, is evaluated in Sec. VIII F.

The normalized transverse emittance $\epsilon_n = \gamma_r \beta_r \epsilon_x$ is used in this section, where γ_r, β_r are the relativistic gamma and beta factors. This is done since the normalized emittance is quoted more often in literature.

A. Emittance relationship to $E[x^2]$

Figure 10 shows a representative simulation result. The noise PSD was increased compared to the expected values to achieve reasonable simulation times ($\approx 0.3\%$ per second). Even in this unrealistic high noise simulation, filamentation is much faster than the emittance growth. As a result, $E[x^2]$ is four orders of magnitude larger

FIG. 10. Normalized emittance and $\gamma_r \beta_r E[x^2]$ from simulation.

than $E[x^2]$ in this simulation and its growth rate closely tracks the emittance growth as expected, since $E[x^2]$ is negligible. In the LHC, this difference will be even more pronounced, since the transverse filamentation time is less than a second, several orders of magnitude smaller than the target emittance growth time of 5% per hour. This confirms Eq. (1): $E[x^2]$ represents the emittance. Figure 11 shows the transverse distribution in the beginning of the simulation and after 10^5 turns. It is again evident that the changes in $E[x]$ are imperceptible, whereas there is a small change in $E[x^2]$ proportional to the emittance growth.

B. Parameters

The parameters shown in Table I have been used to simulate HiLumi LHC conditions. The original six-dimensional phase space distribution is created randomly based on a six-dimensional Gaussian distribution. 10^5 to 3×10^5 particles

FIG. 11. Transverse position distribution at turns 0 and 10^5 .

TABLE I. HiLumi LHC parameters.

f_{rev} (Hz)	ν_x	f_{rf} (MHz)	V_o (MV)	V_{rf} (MV)	β_{CC} (m)	ϵ_x (pm rad)	ϵ_n ($\mu\text{m rad}$)	E_b (TeV)	σ_ϕ (rad)
11245	62.31	400.789	3	16	4000	335	2.5	7	0.6325

are used in these simulations. The crab cavity voltage is 3 MV. The rf accelerating voltage is 16 MV. The accelerating and crab cavities are at the same frequency.

A single crab cavity is modeled for computational simplicity. The HiLumi LHC will have 8 cavities per ring and per plane, so any estimations using the theoretical formalism presented in this work should be scaled by 8 since the noise sources are uncorrelated and the emittance growth rate is proportional to the noise PSD. The major noise source (rf demodulator [15]) will be uncorrelated as each crab cavity has its own demodulator and feedback regulation. The aliased rf demodulator noise will be flat in the 0 to f_{rev} band. Therefore, a white noise spectrum is a realistic model for the LHC case. Most of the simulations are conducted with such a spectrum.

The tune distribution has a big effect on the transverse emittance growth in the presence of the damper. During physics in the HiLumi LHC, the tune distribution will be dominated by the head-on beam-beam (HOBB) effects [23], with smaller, but not insignificant, chromaticity contributions. The HOBB effects can be modeled in single-beam HEADTAIL simulations through appropriate choice of octupole current, leading to a similar tune distribution. Figure 12 shows a realistic representation of the expected tune distribution for the HiLumi LHC. This distribution $\rho_{\text{sim}}(\nu_b)$ is used in the simulations presented in this work with a rms tune spread of 0.003. Figure 12 additionally shows the two extreme cases for reference (strong octupoles/no chromaticity resulting in an

exponential tune distribution, and high chromaticity/low octupole field, resulting in a normal tune distribution).

C. Growth rate dependence on noise PSD

Simulations were then performed with phase or amplitude noise to show the validity of Eqs. (20) and (22). White noise of varying PSD was injected. Filtered noise centered around the betatron frequency was also used to show the emittance growth rate dependence on PSD, not on total noise power.

Table II shows that the estimated normalized transverse emittance growth rate very closely resembles the values predicted by Eqs. (20) and (22). More importantly, the growth rate scales with the PSD around the betatron frequency, but not with the total noise power, as expected. Even though the total power is significantly lower when narrowband filtered noise is injected, the growth rate still scales with the PSD. It should be noted that the PSD and total power refer to their aliased versions over a band from DC to f_{rev} .

Figure 13 shows the emittance growth in the simulation for the first three cases of Table II with the predicted trend lines, showing very good agreement.

Figure 14 plots the transverse emittance growth rate versus the applied phase or amplitude noise PSD at $\bar{\nu}_b$. The resulting slope is within 2% of the expected proportionality factor from Eqs. (20) and (22). The phase noise slope is 2.65 times higher than the amplitude noise case, as estimated in Sec. VI.

Multiple simulation runs were conducted for a phase noise PSD of $9.0\text{e-}22 \text{ rad}^2/\text{Hz}$ and 10^5 simulated turns. The initial six-dimensional bunch distribution and the noise random process were randomly regenerated for each run. Figure 15 shows the resulting emittance growth. The small

FIG. 12. Betatron tune distributions $\rho(\nu_b)$ with the same 0.003 rms tune spread.

TABLE II. Simulation and estimated normalized transverse emittance growth rates [$\sigma_\phi = 0.6325 \text{ rad}$, $\rho_{\text{sim}}(\nu_b)$ transverse tune distribution].

Type	PSD at $\bar{\nu}_b$ ($10^{-10}/\text{Hz}$)	Total Power (10^{-6})	Calculated $\frac{d\epsilon_n}{dt} = \gamma_r \beta_r \frac{d\epsilon_x}{dt}$ (nm/s)	Simulation $\frac{d\epsilon_n}{dt}$ (nm/s)
$\Delta\phi$ (white)	0.14 rad^2	0.15 rad^2	3.4	3.5
$\Delta\phi$ (white)	0.40 rad^2	0.46 rad^2	10.1	10.3
$\Delta\phi$ (white)	0.72 rad^2	0.81 rad^2	16.6	16.5
$\Delta\phi$ (white)	1.2 rad^2	1.4 rad^2	30.4	30.0
$\Delta\phi$ (white)	5.5 rad^2	6.2 rad^2	131	130
$\Delta\phi$ (filtered)	0.14 rad^2	0.030 rad^2	3.4	3.4
ΔA (white)	0.33	0.36	3.1	3.1
ΔA (white)	1.0	1.1	9.4	9.3
ΔA (white)	2.9	3.3	27.5	27.2
ΔA (filtered)	0.32	0.07	3.0	3.1

FIG. 13. Simulation results with different phase noise PSD (solid lines). Theoretically estimated trend line (dashed lines).

spread is expected due to the statistical origin of the emittance growth. The slope of the emittance growth for each run was extracted via a linear fit. Then, the mean $\mu = 10.3 \text{ nm/s}$ and standard deviation $\sigma = 0.4 \text{ nm/s}$ of these slopes were computed over all runs. The theoretically estimated value for these settings is 10.1 nm/s . 10^5 to 3×10^5 turns were used in the simulations presented in this paper to reduce the uncertainty in the results.

D. Growth rate dependence on bunch length

Simulations were performed to validate the dependence of the transverse emittance growth rate on bunch length for phase or amplitude noise. Figures 16 and 17 show the dependence of the growth rate on bunch length for the phase and amplitude noise case, respectively. This

FIG. 14. Normalized transverse emittance growth rate with noise PSD at $\bar{\nu}_b$. Error bars represent $\pm\sigma$ over ten trials.

FIG. 15. Multiple simulation runs for the same phase noise PSD (solid lines). Theoretically estimated trend line (dashed line).

correction corresponds to the terms $C_{\Delta\phi}(\sigma_\phi)$, $C_{\Delta A}(\sigma_\phi)$ in Eqs. (20) and (22). The simulation results match very well with the expected growth rates.

E. Growth rate dependence on tune distribution

According to Eq. (20), the transverse emittance growth rate is independent of tune distribution for a noise spectrum that is flat within the tune distribution. Simulations were performed with a white phase noise source to test the validity of this statement. Two parameters were varied. First, the tune spread of $\rho_{\text{sim}}(\nu_b)$ was increased. Then, the chromaticity was increased. As shown in Fig. 12, an increase in chromaticity not only increases the tune spread, but also results in a more symmetric distribution. Table III shows the corresponding results. No statistically significant

FIG. 16. Growth rate dependence on bunch length $C_{\Delta\phi}(\sigma_\phi)$, phase noise.

FIG. 17. Growth rate dependence on bunch length $C_{\Delta A}(\sigma_{\phi})$, amplitude noise.

change in the growth rate is observed with the changes of tune spread or tune distribution.

F. Growth rate dependence on the transverse damper

Figure 18 shows the simulation results with the expected correction factor for the tune distribution used in the simulations as a function of α , for three different bunch lengths. The parameter α [defined in Eq. (28)], was varied by adjusting the damper gain. The tune spread was also varied with similar results (not shown here). There is a close agreement between the data and the theoretical expectation. The $\sigma_{\phi} = 0.63$ rad case corresponds to the nominal LHC bunch (1 ns $4\sigma_t$). The α value for the planned HiLumi LHC damper gain and tune spread is marked as well.

The ideal transverse damper used in the simulations, acts as stochastic cooler. Since the stochastic cooling rate is inversely proportional to the number of particles ([8,24,25]), an appropriate number of particles was used in these simulations to keep the stochastic cooling rate at

FIG. 18. Damper correction factor as a function of α [Eq. (29)] for ρ_{sim} .

least an order of magnitude smaller in amplitude than the noise induced growth rate.

Similar tests were conducted with amplitude noise in the presence of the transverse damper. When the number of particles is large enough to minimize the stochastic cooling effect, the growth rates were identical with or without the damper, as expected.

IX. CONCLUSIONS AND FUTURE DIRECTIONS

A theoretical formalism has been presented relating the crab cavity phase and amplitude noise with the bunch transverse emittance growth, including the dependence on bunch length and the tune distribution. The effect of the transverse damper was also investigated. The formalism was validated through multiparticle simulations. This formalism is essential for estimating the expected transverse emittance growth in the HiLumi LHC and providing the rf feedback designers with specifications for the crab cavity low level rf system. These estimates and specifications will be presented in a subsequent publication.

Future directions of this work also include studies of the effect of different tune distributions on the achieved growth rates in the presence of damper, in particular for the expected distribution due to HOB effects. Past literature has shown that in the case of dipole kicks or very short bunches, part of the excitation noise is converted in a coherent π -mode oscillation ([2,4]). This oscillation should be strongly damped by the transverse damper in the case of phase noise—which results in an ensemble displacement of the bunch—but not with amplitude noise. A quantitative estimate of this effect is being studied. Early results from simulations with the complete beam-beam interaction [26] produce very similar transverse emittance growth rates as this single-beam work. Validation of the single-beam model

TABLE III. Simulated normalized transverse emittance growth rates with tune distribution. RMS tune changed by appropriate adjustment of octupole current and chromaticity. White noise with 1.4×10^{-6} rad² total power.

RMS Tune spread (Hz)	Chromaticity	Simulation rate (nm/s)
32	10	30.0
62	10	30.3
92	10	31.3
122	10	32.4
152	10	31.1
42	20	30.8
51	30	30.2
61	40	29.8

will continue with more extensive simulations including the complete beam interaction.

The effect of higher chromaticity settings will also be investigated in case the HiLumi LHC operational settings are adjusted. Results from HEADTAIL simulations though indicate that emittance growth rates with high chromaticity are lower than values estimated by the formalism presented in this work. This will be confirmed by an extension of this formalism.

This work has focused on the emittance growth rate. It would be interesting to additionally study the change of the transverse distribution as a function of damper gain, chromaticity, and tune distribution, since the damper correction is not uniform through the bunch. It acts on the frequencies corresponding to the motion of the core of the bunch. The luminosity would be less affected if the emittance growth is mostly due to an increased population of the tails, rather than in the case of a uniform growth through the distribution. The luminosity reduction will be quantified as a function of the resulting transverse distribution. Additionally, if the transverse tails have different frequencies (the case with HOBBS), phase noise could be selectively injected at these frequencies to deplete the tails. Such a procedure would be very beneficial since it would strongly reduce the transverse losses following a crab cavity trip [27].

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APPENDIX A: PHASE NOISE WITH LONG BUNCH LENGTH

In the case of long bunch length, the term within the curly brackets in Eq. (9) depends on the longitudinal parameters $(\hat{\phi}, \nu_s, \psi)$. Using the frequency modulation formula

$$\cos[a \cos(b)] = J_0(a) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(a) \cos(2nb)$$

where $J_i(a)$ are Bessel functions of the first kind,

$$\begin{aligned} & \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \cos[\hat{\phi} \cos(2\pi\nu_s l + \psi)] \\ &= \left\{ J_0[\hat{\phi}] + 2 \sum_{r=1}^{\infty} (-1)^r J_{2r}[\hat{\phi}] \cos[2r(2\pi\nu_s k + \psi)] \right\} \left\{ J_0[\hat{\phi}] + 2 \sum_{q=1}^{\infty} (-1)^q J_{2q}[\hat{\phi}] \cos[2q(2\pi\nu_s l + \psi)] \right\} \\ &= J_0^2[\hat{\phi}] + 2J_0[\hat{\phi}] \sum_{r=1}^{\infty} (-1)^r J_{2r}[\hat{\phi}] \cos[2r(2\pi\nu_s k + \psi)] + 2J_0[\hat{\phi}] \sum_{q=1}^{\infty} (-1)^q J_{2q}[\hat{\phi}] \cos[2q(2\pi\nu_s l + \psi)] \\ &\quad + 4 \sum_{r=1}^{\infty} \sum_{q=1}^{\infty} (-1)^{r+q} J_{2r}[\hat{\phi}] J_{2q}[\hat{\phi}] \cos[2r(2\pi\nu_s k + \psi)] \cos[2q(2\pi\nu_s l + \psi)] \\ &= J_0^2[\hat{\phi}] + 4J_0[\hat{\phi}] \sum_{r=1}^{\infty} (-1)^r J_{2r}[\hat{\phi}] \cos[2r(2\pi\nu_s k + \psi)] \\ &\quad + 2 \sum_{r=1}^{\infty} \sum_{q=1}^{\infty} (-1)^{r+q} J_{2r}[\hat{\phi}] J_{2q}[\hat{\phi}] \cos[4\pi\nu_s(rk + ql) + (2r + 2q)\psi] \\ &\quad + 2 \sum_{r=1}^{\infty} \sum_{q=1}^{\infty} (-1)^{r+q} J_{2r}[\hat{\phi}] J_{2q}[\hat{\phi}] \cos[4\pi\nu_s(rk - ql) + (2r - 2q)\psi]. \end{aligned} \tag{A1}$$

Since the bucket is nonaccelerating, the initial phase of the synchrotron oscillation ψ is independent of (ϕ, ν_s) and can be assumed uniformly distributed on $[-\pi, \pi]$. After averaging over this random variable, all cosine functions that contain this phase will sum to zero. The last summation will keep terms with $r = q$ only. As a result, the expected value of this term with respect to ψ is given by

$$\begin{aligned} & E\{\cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \cos[\hat{\phi} \cos(2\pi\nu_s l + \psi)] | \hat{\phi}, \nu_s\} \\ &= J_0^2[\hat{\phi}] + 2 \sum_{q=1}^{\infty} J_{2q}^2[\hat{\phi}] \cos[4\pi\nu_s q(k - l)]. \end{aligned} \tag{A2}$$

The first term leads to a scaling that will decrease with bunch length. In the subsequent terms, the autocorrelation

function is modulated by a cosine at even multiples of the synchrotron frequency.

Then, Eq. (9) is given by

$$E[\tilde{x}_n^2|\nu_b, \hat{\phi}, \nu_s] = \beta_{cc} \left(\frac{eV_o}{E_b} \right)^2 \sum_{k=0}^n \sum_{l=0}^n R_{\Delta\phi}[(k-l)T_{\text{rev}}] \left\{ J_o^2[\hat{\phi}] + 2 \sum_{q=1}^{\infty} J_{2q}^2[\hat{\phi}] \cos[4\pi\nu_s q(k-l)] \right\} \\ \times \sin[2\pi\nu_b(n-k)] \sin[2\pi\nu_b(n-l)] \quad (\text{A3})$$

for long bunch length. This equation differs from Eq. (10) only in the term in curly brackets. Following the same index changes as in Sec. V and similarly disregarding the initial transient and the oscillations at twice the betatron tune, the linear increase between two turns can be reduced to

$$E[\tilde{x}_n^2|\nu_b, \hat{\phi}, \nu_s] - E[\tilde{x}_{n-1}^2|\nu_b, \hat{\phi}, \nu_s] = \frac{\beta_{cc}}{2} \left(\frac{eV_o}{E_b} \right)^2 \left\{ \begin{aligned} & (J_o[\hat{\phi}])^2 \sum_{k=-\infty}^{\infty} R_{\Delta\phi}[kT_{\text{rev}}] e^{-j2\pi k\nu_b} + \\ & 2 \sum_{p=1}^{\infty} (J_{2p}[\hat{\phi}])^2 \sum_{k=-\infty}^{\infty} R_{\Delta\phi}[kT_{\text{rev}}] \cos(4\pi p k \nu_s) e^{-j2\pi k\nu_b} \end{aligned} \right\} \\ = \frac{\beta_{cc}}{2} \left(\frac{eV_o}{E_b} \right)^2 f_{\text{rev}} \left\{ \begin{aligned} & (J_o[\hat{\phi}])^2 \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k+\nu_b)f_{\text{rev}}] + \\ & \sum_{p=1}^{\infty} (J_{2p}[\hat{\phi}])^2 \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k+\nu_b+2p\nu_s)f_{\text{rev}}] + \\ & \sum_{p=1}^{\infty} (J_{2p}[\hat{\phi}])^2 \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k+\nu_b-2p\nu_s)f_{\text{rev}}] \end{aligned} \right\}. \quad (\text{A4})$$

The finite bunch length reduces the effect of the noise PSD at the exact betatron frequency and adds a small contribution from the spectral density on the even order synchrotron sidebands. These are scaled by squared Bessel functions of even orders. The synchrotron frequency (≈ 20 Hz) is much smaller than the betatron frequency (≈ 3400 Hz) in the LHC. Assuming that the noise PSD is constant over the synchrotron sidebands,

$$S_{\Delta\phi}[(k+\nu_b \pm 2p\nu_s)f_{\text{rev}}] \approx S_{\Delta\phi}[(k+\nu_b)f_{\text{rev}}].$$

With this approximation, Eq. (A4) becomes

$$E[\tilde{x}_n^2|\nu_b, \hat{\phi}] - E[\tilde{x}_{n-1}^2|\nu_b, \hat{\phi}] \\ = \frac{\beta_{cc}}{2} \left(\frac{eV_o}{E_b} \right)^2 f_{\text{rev}} \left[(J_o[\hat{\phi}])^2 + 2 \sum_{l=1}^{\infty} (J_{2l}[\hat{\phi}])^2 \right] \\ \times \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k+\nu_b)f_{\text{rev}}]$$

which is now independent of ν_s .

Assuming that the amplitude of synchrotron motion is independent of betatron tune and considering a longitudinal two-dimensional Gaussian distribution, with longitudinal line density standard deviation σ_ϕ , $\hat{\phi}$ then follows a Rayleigh distribution with density function

$$f_{\hat{\phi}}(\hat{\phi}) = \frac{\hat{\phi}}{\sigma_\phi^2} e^{-\frac{\hat{\phi}^2}{2\sigma_\phi^2}}$$

for $\hat{\phi}$ in $[0, \infty]$ so that

$$E[\hat{\phi}^2] = 2\sigma_\phi^2.$$

This assumption is valid in the LHC, where the betatron tune distribution comes from the beam-beam interaction (in physics) and σ_ϕ is one tenth of the rf wavelength, so that the bunch does not fill the bucket. It is also correct for a tune spread dominated by the octupole field. It is not correct though in the case of high chromaticity, which leads to a strong coupling between ν_b and $\hat{\phi}$ as both would depend on the particle momentum.

Using this distribution, the expected value of the square Bessel functions over $\hat{\phi}$ is given by

$$E[(J_n[\hat{\phi}])^2] = \int_0^\infty (J_n[x])^2 \frac{x}{\sigma_\phi^2} e^{-\frac{x^2}{2\sigma_\phi^2}} dx = e^{-\sigma_\phi^2} I_n[\sigma_\phi^2],$$

where $I_n[x]$ is the modified Bessel function of the first kind. Averaging over all particles with a given betatron tune, the growth rate becomes

$$E[\tilde{x}_n^2|\nu_b] - E[\tilde{x}_{n-1}^2|\nu_b] \\ = \frac{\beta_{cc}}{2} \left(\frac{eV_o}{E_b} \right)^2 f_{\text{rev}} \left\{ e^{-\sigma_\phi^2} \left[I_o[\sigma_\phi^2] + 2 \sum_{l=1}^{\infty} I_{2l}[\sigma_\phi^2] \right] \right\} \\ \times \sum_{k=-\infty}^{\infty} S_{\Delta\phi}[(k+\nu_b)f_{\text{rev}}].$$

Comparing this expression with Eq. (16) it is evident that the transverse emittance growth rate is reduced with increased bunch length by a factor $C_{\Delta\phi}(\sigma_\phi)$ given by

TABLE IV. Contribution of synchrotron sideband harmonics to correction factor for $\sigma_\phi = 0.6325$ rads.

0	$\pm 2\nu_s$	$\pm 4\nu_s$	$\pm 6\nu_s$	$\pm 8\nu_s$	$\pm 10\nu_s$	$\pm 12\nu_s$	$\pm 14\nu_s$	$\pm 16\nu_s$
$I_0[\sigma_\phi^2]$	$I_2[\sigma_\phi^2]$	$I_4[\sigma_\phi^2]$	$I_6[\sigma_\phi^2]$	$I_8[\sigma_\phi^2]$	$I_{10}[\sigma_\phi^2]$	$I_{12}[\sigma_\phi^2]$	$I_{14}[\sigma_\phi^2]$	$I_{16}[\sigma_\phi^2]$
1.04	0.02	6.7e-5	9e-8	6e-11	3e-14	9e-18	2e-21	3e-25

$$C_{\Delta\phi}(\sigma_\phi) = e^{-\sigma_\phi^2} \left[I_0[\sigma_\phi^2] + 2 \sum_{l=1}^{\infty} I_{2l}[\sigma_\phi^2] \right]. \quad (\text{A5})$$

Table IV lists the weighting of the synchrotron sidebands harmonics in the resulting growth rate, for the nominal HiLumi LHC rms bunch length of 0.6325 rads. The contribution of the higher harmonics is negligible.

APPENDIX B: AMPLITUDE NOISE WITH LONG BUNCH LENGTH

Using the frequency modulation formula

$$\sin[a \cos(b)] = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(a) \cos[(2n+1)b]$$

the longitudinal motion dependent terms in Eq. (21) become

$$\begin{aligned} & \sin[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \sin[\hat{\phi} \cos(2\pi\nu_s l + \psi)] \\ &= \left\{ 2 \sum_{p=0}^{\infty} (-1)^p J_{2p+1}[\hat{\phi}] \cos[(2p+1)(2\pi\nu_s k + \psi)] \right\} \left\{ 2 \sum_{q=0}^{\infty} (-1)^q J_{2q+1}[\hat{\phi}] \cos[(2q+1)(2\pi\nu_s l + \psi)] \right\} \\ &= 4 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} J_{2p+1}[\hat{\phi}] J_{2q+1}[\hat{\phi}] \cos[(2p+1)(2\pi\nu_s k + \psi)] \cos[(2q+1)(2\pi\nu_s l + \psi)] \\ &= 2 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} J_{2p+1}[\hat{\phi}] J_{2q+1}[\hat{\phi}] \{ \cos[2\pi\nu_s [(2p+1)k - (2q+1)l] + 2(p-q)\psi] \\ &\quad + \cos[2\pi\nu_s [(2p+1)k + (2q+1)l] + (2p+2q+2)\psi] \}. \end{aligned}$$

After averaging over the uniform ψ distribution, the first term will keep terms with $p = q$ only and the second term is zero since p and q are non-negative. As a result, the expected value with respect to ψ is given by

$$E\{\sin[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \sin[\hat{\phi} \cos(2\pi\nu_s l + \psi)] | \hat{\phi}, \nu_s\} = 2 \sum_{p=0}^{\infty} (J_{2p+1}[\hat{\phi}])^2 \cos[2\pi\nu_s (2p+1)(k-l)].$$

The scaling factor containing the squared Bessel functions of odd orders will decrease with bunch length.

Then, Eq. (21) is given by

$$\begin{aligned} E[\tilde{\mathbf{x}}_n^2 | \nu_b, \hat{\phi}, \nu_s] &= \beta_{cc} \left(\frac{eV_o}{E_b} \right)^2 \sum_{k=0}^n \sum_{l=0}^n R_{\Delta A}[(k-l)T_{\text{rev}}] \left\{ 2 \sum_{p=0}^{\infty} (J_{2p+1}[\hat{\phi}])^2 \cos[2\pi\nu_s (2p+1)(k-l)] \right\} \\ &\quad \times \sin[2\pi\nu_b(n-k)] \sin[2\pi\nu_b(n-l)]. \end{aligned}$$

The autocorrelation function is modulated by a cosine at odd multiples of the synchrotron frequency. Following the same index changes as in Sec. V and similarly disregarding the initial transient and the oscillations at twice the betatron tune, the linear increase between two turns can be reduced to

$$\begin{aligned}
E[\tilde{x}_n^2|\nu_b, \hat{\phi}, \nu_s] - E[\tilde{x}_{n-1}^2|\nu_b, \hat{\phi}, \nu_s] &= \frac{\beta_{cc}}{2} \left(\frac{eV_o}{E_b} \right)^2 2 \sum_{p=0}^{\infty} (J_{2p+1}[\hat{\phi}])^2 \sum_{k=-\infty}^{\infty} R_{\Delta A}[kT_{\text{rev}}] \cos[2\pi\nu_s(2p+1)k] e^{-j2\pi k\nu_b} \\
&= \frac{\beta_{cc}}{2} \left(\frac{eV_o}{E_b} \right)^2 f_{\text{rev}} \left\{ \frac{\sum_{p=0}^{\infty} (J_{2p+1}[\hat{\phi}])^2 \sum_{k=-\infty}^{\infty} S_{\Delta A}[(k + \nu_b + (2p+1)\nu_s)f_{\text{rev}}]}{\sum_{p=0}^{\infty} (J_{2p+1}[\hat{\phi}])^2 \sum_{k=-\infty}^{\infty} S_{\Delta A}[(k + \nu_b - (2p+1)\nu_s)f_{\text{rev}}]} \right\}. \quad (\text{B1})
\end{aligned}$$

The only difference between Eqs. (A4) and (B1) is the index of the Bessel functions (odd in the amplitude case) and that the PSD is now sampled on the odd synchrotron sidebands of the betatron tune lines. Assuming that the noise PSD does not change over the small range covered by the synchrotron sidebands (the synchrotron frequency is about ≈ 20 Hz) and after averaging over ν_b and $\hat{\phi}$ as in Appendix A, the emittance growth rate in the case of

amplitude noise is given by Eq. (22). The additional factor of $1/2$ in Eq. (22) is due to the summation over positive and negative betatron bands.

APPENDIX C: INFINITE SUM DERIVATION

To simplify the derivation in Sec. VII A, the following infinite sum has to be evaluated:

$$\begin{aligned}
\sum_{n=0}^N e^{j2\pi n\theta} &= \frac{1 - e^{j2\pi(N+1)\theta}}{1 - e^{j2\pi\theta}} = \frac{e^{j\pi(N+1)\theta} e^{-j\pi(N+1)\theta} - e^{j\pi(N+1)\theta}}{e^{j\pi\theta} e^{-j\pi\theta} - e^{j\pi\theta}} = e^{j\pi N\theta} \frac{\sin[\pi(N+1)\theta]}{\sin(\pi\theta)} \\
&= \frac{\cos(\pi N\theta) \sin[\pi(N+1)\theta]}{\sin(\pi\theta)} + j \frac{\sin(\pi N\theta) \sin[\pi(N+1)\theta]}{\sin(\pi\theta)} \\
&= \frac{1}{2} \frac{\sin(\pi\theta) + \sin[\pi(2N+1)\theta]}{\sin(\pi\theta)} + \frac{j}{2} \frac{\cos(\pi\theta) - \cos[\pi(2N+1)\theta]}{\sin(\pi\theta)} \\
&= \frac{1}{2} + \frac{1}{2} \frac{\sin[\pi(2N+1)\theta]}{\sin(\pi\theta)} + \frac{j}{2} \frac{\cos(\pi\theta) - \cos[\pi(2N+1)\theta]}{\sin(\pi\theta)}
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{n=0}^N e^{j2\pi n(\nu-\Omega)} &= \frac{1}{2} + \frac{1}{2} \frac{\sin[\pi(2N+1)(\nu-\Omega)]}{\sin[\pi(\nu-\Omega)]} + \frac{j}{2} \frac{\cos[\pi(\nu-\Omega)] - \cos[\pi(2N+1)(\nu-\Omega)]}{\sin[\pi(\nu-\Omega)]} \\
\sum_{n=0}^N e^{-j2\pi n(\nu+\Omega)} &= \frac{1}{2} + \frac{1}{2} \frac{\sin[\pi(2N+1)(\nu+\Omega)]}{\sin[\pi(\nu+\Omega)]} - \frac{j}{2} \frac{\cos[\pi(\nu+\Omega)] - \cos[\pi(2N+1)(\nu+\Omega)]}{\sin[\pi(\nu+\Omega)]}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^N e^{j2\pi n(\nu-\Omega)} - \sum_{n=0}^N e^{-j2\pi n(\nu+\Omega)} \\
&= \frac{1}{2} \left\{ \frac{\sin[\pi(2N+1)(\nu-\Omega)]}{\sin[\pi(\nu-\Omega)]} - \frac{\sin[\pi(2N+1)(\nu+\Omega)]}{\sin[\pi(\nu+\Omega)]} \right\} \\
&\quad + \frac{j}{2} \left\{ \frac{\cos[\pi(\nu-\Omega)] - \cos[\pi(2N+1)(\nu-\Omega)]}{\sin[\pi(\nu-\Omega)]} + \frac{\cos[\pi(\nu+\Omega)] - \cos[\pi(2N+1)(\nu+\Omega)]}{\sin[\pi(\nu+\Omega)]} \right\}.
\end{aligned}$$

Following [19],

$$\lim_{N \rightarrow \infty} \frac{\sin(\pi N\theta)}{\sin(\pi\theta)} = \delta(\theta) \quad \lim_{N \rightarrow \infty} \frac{\cos(\pi\theta) - \cos(\pi N\theta)}{\sin(\pi\theta)} = \text{P.V.} \frac{1}{\pi\theta}.$$

As a result,

$$\lim_{N \rightarrow \infty} \left\{ \sum_{n=0}^N e^{j2\pi n(\nu - \Omega)} - \sum_{n=0}^N e^{-j2\pi n(\nu + \Omega)} \right\} = \frac{1}{2} \{ \delta(\nu - \Omega) - \delta(\nu + \Omega) \} + \frac{j}{2\pi} \text{P.V.} \left\{ \frac{1}{(\nu - \Omega)} + \frac{1}{(\nu + \Omega)} \right\}.$$

It should be noted that the above expression is a distribution and is only used as an argument within an integral in this work.

APPENDIX D: DAMPER ACTION WITH PHASE NOISE AND LONG BUNCH LENGTH

The perturbed transverse displacement for a given particle in the presence of the damper is

$$\tilde{x}_n = \sum_{k=0}^n \sin[2\pi\nu_b(n-k)] \left(\sqrt{\beta_{cc}} \frac{eV_o}{E_b} \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \Delta\phi_k - h'_k * E[\tilde{x}_k | \Delta\Phi_k] \right),$$

where $*$ denotes a convolution, $\Delta\Phi_n = \{\Delta\phi_n, \Delta\phi_{n-1}, \dots, \Delta\phi_0\}$ is the series of past phase noise samples, and h'_n is the impulse response of the damper action on turn n , as shown in Fig. 5. Therefore, the ensemble average over the bunch is given by

$$\begin{aligned} E[\tilde{x}_n | \Delta\Phi_n] &= \sqrt{\beta_{cc}} \frac{eV_o}{E_b} \sum_{k=0}^n E\{\sin[2\pi\nu_b(n-k)] \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)]\} \Delta\phi_k - \sum_{k=0}^n E\{\sin[2\pi\nu_b(n-k)]\} (h'_k * E[\tilde{x}_k | \Delta\Phi_k]) \\ &= \sqrt{\beta_{cc}} \frac{eV_o}{E_b} \sum_{k=0}^n E\{\sin[2\pi\nu_b(n-k)] \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)]\} \Delta\phi_k - h_n * h'_n * E[\tilde{x}_n | \Delta\Phi_n], \end{aligned} \quad (\text{D1})$$

where h_n is the impulse response of the BTf (Sec. VII A).

The first term can be evaluated as follows,

$$\begin{aligned} &E\{\sin[2\pi\nu_b(n-k)] \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)]\} \Delta\phi_k \\ &= E\left[\sin[2\pi\nu_b(n-k)] \right. \\ &\quad \times \left. \left\{ J_0[\hat{\phi}] + 2 \sum_{p=1}^{\infty} (-1)^p J_{2p}[\hat{\phi}] \cos[2p(2\pi\nu_s k + \psi)] \right\} \right] \Delta\phi_k. \end{aligned}$$

Assuming that ψ is uniformly distributed and independent of the other random variables, all terms average to zero except for $p = 0$. Then,

$$\begin{aligned} &E\{\sin[2\pi\nu_b(n-k)] \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)]\} \Delta\phi_k \\ &= E\{J_0[\hat{\phi}] \sin[2\pi\nu_b(n-k)]\} \Delta\phi_k \\ &= E\{J_0[\hat{\phi}]\} E\{\sin[2\pi\nu_b(n-k)]\} \Delta\phi_k \end{aligned}$$

further assuming that ν_b is independent of $\hat{\phi}$. This assumption is valid for the expected HiLumi LHC parameters, but does not hold for a case of high chromaticity.

Using the Rayleigh distribution as in Appendix A,

$$E\{J_0[\hat{\phi}]\} = \int_0^{\infty} J_0[x] \frac{x}{\sigma_{\hat{\phi}}^2} e^{-\frac{x^2}{2\sigma_{\hat{\phi}}^2}} dx = e^{-\frac{\sigma_{\hat{\phi}}^2}{2}} \quad (\text{D2})$$

and thus

$$\begin{aligned} &E\{\sin[2\pi\nu_b(n-k)] \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)]\} \Delta\phi_k \\ &= e^{-\frac{\sigma_{\hat{\phi}}^2}{2}} h_{n-k} \Delta\phi_k. \end{aligned}$$

Substituting this expression in Eq. (D1),

$$\begin{aligned} E[\tilde{x}_n | \Delta\Phi_n] &= \left[\sqrt{\beta_{cc}} \frac{eV_o}{E_b} e^{-\frac{\sigma_{\hat{\phi}}^2}{2}} h_n * \Delta\phi_n \right] - h_n * h'_n \\ &\quad * E[\tilde{x}_n | \Delta\Phi_n]. \end{aligned}$$

This recursion between the phase noise and the average transverse displacement is shown graphically in Fig. 19.

FIG. 19. Block diagram showing the transfer function between $E[\tilde{x}_n | \Delta\Phi_n]$ and phase noise in the presence of damper.

Then, using control theory, the average transverse displacement can be expressed as a filtered version of $\Delta\phi_n$

$$E[\tilde{x}_n|\Delta\Phi_n] = h_n'' * \Delta\phi_n,$$

where h_n'' is the impulse response of this filter, with a z-transform $H''(z)$ given by

$$H''(z) = \sqrt{\beta_{cc}} \frac{eV_o}{E_b} e^{-\frac{\sigma_\phi^2}{2}} \frac{H_{\text{BTF}}(z)}{1 + H'(z)H_{\text{BTF}}(z)}, \quad (\text{D3})$$

where $H'(z)$ is the damper response defined in Eq. (24). Note that for $G = 0$ and $\sigma_\phi \rightarrow 0$, $H''(z) = \sqrt{\beta_{cc}} \frac{eV_o}{E_b} H_{\text{BTF}}(z)$ as expected.

As a result,

$$\tilde{x}_n = \sum_{k=0}^n \sin[2\pi\nu_b(n-k)] \left(\sqrt{\beta_{cc}} \frac{eV_o}{E_b} \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \Delta\phi_k - h_k' * h_k'' * \Delta\phi_k \right).$$

Then,

$$\tilde{x}_n^2 = \sum_{k=0}^n \sum_{r=0}^n \left[\begin{aligned} &\sin[2\pi\nu_b(n-r)] \sin[2\pi\nu_b(n-k)] \\ &\times \left\{ \sqrt{\beta_{cc}} \frac{eV_o}{E_b} \cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)] \Delta\phi_k - (h_k' * h_k'' * \Delta\phi_k) \right\} \\ &\times \left\{ \sqrt{\beta_{cc}} \frac{eV_o}{E_b} \cos[\hat{\phi} \cos(2\pi\nu_s r + \psi)] \Delta\phi_r - (h_r' * h_r'' * \Delta\phi_r) \right\} \end{aligned} \right].$$

Using

$$\cos[\hat{\phi} \cos(2\pi\nu_s k + \psi)] = J_0[\hat{\phi}] + 2 \sum_{r=1}^{\infty} (-1)^r J_{2r}[\hat{\phi}] \cos[2r(2\pi\nu_s k + \psi)]$$

and averaging over the uniformly distributed ψ

$$E[\tilde{x}_n^2|\Delta\Phi_n, \nu_b, \hat{\phi}, \nu_s] = \sum_{k=0}^n \sum_{r=0}^n \left[\begin{aligned} &\sin[2\pi\nu_b(n-r)] \sin[2\pi\nu_b(n-k)] \\ &\times \left\{ \begin{aligned} &\beta_{cc} \left(\frac{eV_o}{E_b} \right)^2 \left\{ J_0^2[\hat{\phi}] + 2 \sum_{p=1}^{\infty} J_{2p}^2[\hat{\phi}] \cos[4p(2\pi\nu_s(k-r))] \right\} \Delta\phi_k \Delta\phi_r \\ &- \sqrt{\beta_{cc}} \frac{eV_o}{E_b} J_0[\hat{\phi}] \Delta\phi_r (h_k' * h_k'' * \Delta\phi_k) \\ &- \sqrt{\beta_{cc}} \frac{eV_o}{E_b} J_0[\hat{\phi}] \Delta\phi_k (h_r' * h_r'' * \Delta\phi_r) \\ &+ (h_k' * h_k'' * \Delta\phi_k) (h_r' * h_r'' * \Delta\phi_r) \end{aligned} \right\} \end{aligned} \right].$$

The expected value above includes phase noise dependent autocorrelation and cross-correlation functions:

$$\begin{aligned} E[\Delta\phi_k \Delta\phi_r] &= R_{\Delta\phi}[(k-r)T_{\text{rev}}], \\ E[\Delta\phi_r (h_k' * h_k'' * \Delta\phi_k)] &= R_{\Delta\phi}[(k-r)T_{\text{rev}}] * h_{k-r}' * h_{k-r}'', \\ E[(h_k' * h_k'' * \Delta\phi_k) (h_r' * h_r'' * \Delta\phi_r)] &= R_{\Delta\phi}[(k-r)T_{\text{rev}}] * h_{k-r}' * h_{r-k}' * h_{k-r}'' * h_{r-k}''. \end{aligned}$$

As a result, using the assumption that the noise PSD is flat over the synchrotron sidebands and after averaging over $\hat{\phi}$ assuming a Rayleigh distribution (as in Appendix A),

$$E[\tilde{\mathbf{x}}_n^2 | \nu_b] = \beta_{cc} \left(\frac{eV_o}{E_b} \right)^2 \sum_{k=0}^n \sum_{r=0}^n \times \left[\begin{aligned} & \sin[2\pi\nu_b(n-r)] \sin[2\pi\nu_b(n-k)] \\ & \left\{ \begin{aligned} & C_{\Delta\phi}(\sigma_\phi) R_{\Delta\phi}[(k-r)T_{\text{rev}}] \\ & - \frac{E_b}{\sqrt{\beta_{cc}eV_o}} e^{-\frac{\sigma_\phi^2}{2}} R_{\Delta\phi}[(k-r)T_{\text{rev}}] * h'_{k-r} * h''_{k-r} \\ & - \frac{E_b}{\sqrt{\beta_{cc}eV_o}} e^{-\frac{\sigma_\phi^2}{2}} R_{\Delta\phi}[(r-k)T_{\text{rev}}] * h'_{r-k} * h''_{r-k} \\ & + \left(\frac{E_b}{\sqrt{\beta_{cc}eV_o}} \right)^2 R_{\Delta\phi}[(k-r)T_{\text{rev}}] * h'_{k-r} * h'_{r-k} * h''_{k-r} * h''_{r-k} \end{aligned} \right\} \end{aligned} \right].$$

Comparing to Eq. (10), it is clear that the correction due to the damper is given by the scaling and filtering of $R_{\Delta\phi}[(k-r)T_{\text{rev}}]$ in the curly brackets, which is equal to $C_{\Delta\phi}(\sigma_\phi)$ without the damper ($h'_n = 0$). This is the term $R_d(\nu_b)S_{\Delta\phi}(f)$ from Sec. VII. Following the analysis of Sec. V, the term $R_{\Delta\phi}[kT_{\text{rev}}]$ in Eq. (14) is substituted by

$$\begin{aligned} & C_{\Delta\phi}(\sigma_\phi) R_{\Delta\phi}[kT_{\text{rev}}] - \frac{E_b}{\sqrt{\beta_{cc}eV_o}} e^{-\frac{\sigma_\phi^2}{2}} R_{\Delta\phi}[kT_{\text{rev}}] * h'_k * h''_k - \frac{E_b}{\sqrt{\beta_{cc}eV_o}} e^{-\frac{\sigma_\phi^2}{2}} R_{\Delta\phi}[-kT_{\text{rev}}] * h'_{-k} * h''_{-k} \\ & + \left(\frac{E_b}{\sqrt{\beta_{cc}eV_o}} \right)^2 R_{\Delta\phi}[kT_{\text{rev}}] * h'_k * h'_{-k} * h''_k * h''_{-k}. \end{aligned}$$

Taking the Fourier transform of this term and since the PSD is even symmetric,

$$\begin{aligned} C_{\Delta\phi}(\sigma_\phi) R_d(\Omega) S_{\Delta\phi}(f) &= C_{\Delta\phi}(\sigma_\phi) S_{\Delta\phi}(f) - \frac{E_b}{\sqrt{\beta_{cc}eV_o}} e^{-\frac{\sigma_\phi^2}{2}} S_{\Delta\phi}(f) H'[e^{j2\pi\Omega}] H''[e^{j2\pi\Omega}] \\ & - \frac{E_b}{\sqrt{\beta_{cc}eV_o}} e^{-\frac{\sigma_\phi^2}{2}} S_{\Delta\phi}(f) \bar{H}'[e^{j2\pi\Omega}] \bar{H}''[e^{j2\pi\Omega}] + \left(\frac{E_b}{\sqrt{\beta_{cc}eV_o}} \right)^2 S_{\Delta\phi}(f) |H'[e^{j2\pi\Omega}]|^2 |H''[e^{j2\pi\Omega}]|^2, \end{aligned}$$

where $\Omega = f/f_{\text{rev}}$ and the factor $C_{\Delta\phi}(\sigma_\phi)$ is introduced on the left-hand side since it corresponds to the growth rate reduction due to the long bunch length in the absence of damper [Eq. (A5)]. Then, the phase noise PSD reduction due to the damper $R_d(\Omega)$ is given by

$$\begin{aligned} R_d(\Omega) &= \frac{1}{C_{\Delta\phi}(\sigma_\phi)} \left[C_{\Delta\phi}(\sigma_\phi) - \frac{E_b}{\sqrt{\beta_{cc}eV_o}} e^{-\frac{\sigma_\phi^2}{2}} H'[e^{j2\pi\Omega}] H''[e^{j2\pi\Omega}] \right. \\ & \left. - \frac{E_b}{\sqrt{\beta_{cc}eV_o}} e^{-\frac{\sigma_\phi^2}{2}} \bar{H}'[e^{j2\pi\Omega}] \bar{H}''[e^{j2\pi\Omega}] + \left(\frac{E_b}{\sqrt{\beta_{cc}eV_o}} \right)^2 |H'[e^{j2\pi\Omega}]|^2 |H''[e^{j2\pi\Omega}]|^2 \right]. \end{aligned}$$

Then, for positive frequencies

$$R_d(\Omega) = 1 - \frac{e^{-\frac{\sigma_\phi^2}{2}}}{C_{\Delta\phi}(\sigma_\phi)} \frac{G^2 |H_{\text{BTF}}[e^{j2\pi\Omega}]|^2 + jG(H_{\text{BTF}}[e^{j2\pi\Omega}] - \bar{H}_{\text{BTF}}[e^{j2\pi\Omega}])}{|1 + jGH_{\text{BTF}}[e^{j2\pi\Omega}]|^2}$$

using Eqs. (24) and (D3). For $G \rightarrow 0$ this term goes to 1.

Finally, integrating over the tune distribution and using the beam transfer function from Eq. (23), the expected value of the noise reduction factor is given by

$$\begin{aligned} \bar{R}_d &= \frac{1}{\pi} \int_{-\infty}^{\infty} g(u) \left\{ 1 - \frac{e^{-\frac{\sigma_\phi^2}{2}}}{C_{\Delta\phi}(\sigma_\phi)} \right. \\ & \left. \times \frac{\alpha^2 [g^2(u) + f^2(u)] + 2\alpha g(u)}{[1 + \alpha g(u)]^2 + [\alpha f(u)]^2} \right\} du. \end{aligned}$$

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