Design of transport lines for uniforming beam distribution with octupoles

Y. Y. Guo^{1,2} and G. Xu¹

¹*Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China*

²*University of Chinese Academy of Sciences, Beijing 100049, China*

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The intensity distribution of a typical particle beam is an approximate Gaussian distribution while many applications require the beam to be uniform. Local bright spots on the targets for high-power Gaussian beams create difficult cooling problems and shorten the lifetime of the target. In this paper, we have theoretically studied uniformization of the transverse beam profile using two octupole magnets placed at particular locations in the beam transport line. The details about the requirements on magnet strength and phase advance will be shown. As an example, we take a set of China Spallation Neutron Source parameters to simulate the beam. The simulation result agrees well with theoretical formulas.

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I. INTRODUCTION

Many charged-particle beam applications require illumination of the target with uniform beams. Local bright spots on the target create difficult cooling problems and shorten the lifetime of the target, so we need to adopt a certain approach to uniform the intensity distribution of the Gaussian beam to avoid the negative effect.

The idea that a system of controlled third-order aberrations produced by a pair of octupole magnets is able to substantially cause uniformization of the intensity distribution was first put forward by Philip F. Meads in 1983 [1]. Later, Sherrill and co-workers analytically verified that odd-order magnetic fields are needed to transform a Gaussian distribution into a uniform distribution [2]. In 1991, the Gaussian beam profile of the Brookhaven National Laboratory 200 MeV H^- beam at the Radiation Effect Facility target was transformed into a rectangular profile with almost uniform distribution by placing two octupoles at particular locations along the beam transport line. It proved that a pair of octupoles can smooth the intensity distribution of a beam in the experiment [3]. This theory was also experimentally demonstrated in UNILAC [4]. The extracted beam from the Spallation Neutron Source at Oak Ridge National Laboratory is closed to a uniform beam in the horizontal direction [5]. In addition, Yuri has given the formulas that predict the strengths of the nonlinear magnetic fields required for beam uniformization [6]. In this paper, we have calculated the transverse coordinates of the beam at the target precisely using the transfer matrix method.

With the analysis of constraints, we can get the strength of octupole magnets and relevant parameters in the transport line in theory. Then we will give an application example.

II. BEAM TRANSFORMATION

We adopt the transport system as shown in Fig. 1. There is also a series of quadrupole magnets along the beam line that work together with octupole magnets in order to even the beam distribution.

We assume that the beam distribution is given by

$$I(x_0) = I_0 \exp\left(-\frac{x_0^2}{2\sigma^2}\right). \quad (1)$$

Taking the thin lens approximation, the octupoles deflect the beam in proportion to the cube of the trajectory's displacement at the octupoles:

$$\begin{aligned} x &= x_0, & x' &= x_0' - \frac{1}{6}K(x_0^3 - 3x_0y_0^2), \\ y &= y_0, & y' &= y_0' + \frac{1}{6}K(3y_0x_0^2 - y_0^3), \end{aligned} \quad (2)$$

where K is the integral strength of the octupole.

FIG. 1. A schematic plot of the transport system in the transport line. x_i ($i = 0, 1, 2, 3, 4$) stands for the displacement at different locations.

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According to the beam transport theory, we can get the beam displacement and angle at the target. In order to eliminate the higher than third-order terms, we can make the phase advance $n\pi$ ($n \in \text{integer}$) in both horizontal and vertical planes between the two octupoles [7]. In this condition, the transport matrix between the two octupoles is

$$\begin{bmatrix} a & 0 & 0 & 0 \\ b & \frac{1}{a} & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & \frac{1}{c} \end{bmatrix}.$$

So we can get the displacement and angle at the exit of the second octupole:

$$\begin{aligned} x_3 &= ax_0, & x_3' &= bx_0 - \frac{K_1 + a^4 K_2}{6a} x_0^3 + \frac{1}{a} x_0' + \frac{K_1 + a^2 c^2 K_2}{2a} x_0 y_0^2, \\ y_3 &= cy_0, & y_3' &= dy_0 - \frac{K_1 + c^4 K_2}{6c} y_0^3 + \frac{1}{c} y_0' + \frac{K_1 + a^2 c^2 K_2}{2c} x_0^2 y_0. \end{aligned} \quad (3)$$

In Courant-Snyder parameters, a , b , c , and d can be expressed as

$$a = \pm \sqrt{\frac{\beta_{x2}}{\beta_{x1}}}, \quad b = \pm \frac{\alpha_{x1} - \alpha_{x2}}{\sqrt{\beta_{x1}\beta_{x2}}}, \quad c = \pm \sqrt{\frac{\beta_{y2}}{\beta_{y1}}}, \quad d = \pm \frac{\alpha_{y1} - \alpha_{y2}}{\sqrt{\beta_{y1}\beta_{y2}}}, \quad (4)$$

where $\pm$ are for n even or odd, respectively.

Here we make $\varphi_{x1} = \pi$, $\varphi_{y1} = \pi$. On the target, we have

$$\begin{aligned} x_4 &= -x_0' \sqrt{\beta_{x1}\beta_{x3}} \sin\varphi_{x2} - \frac{x_0 \sqrt{\beta_{x3}} (\cos\varphi_{x2} + \alpha_{x1} \sin\varphi_{x2})}{\sqrt{\beta_{x1}}} + \frac{x_0^3 (K_1 \beta_{x1}^2 + K_2 \beta_{x2}^2) \sqrt{\beta_{x3}} \sin\varphi_{x2}}{6\beta_{x1}^{3/2}} \\ &\quad - \frac{x_0 y_0^2 \sqrt{\beta_{x3}} (K_1 \beta_{x1} \beta_{y1} + K_2 \beta_{x2} \beta_{y2}) \sin\varphi_{x2}}{2\sqrt{\beta_{x1}} \beta_{y1}}, \\ x_4' &= \frac{x_0' \sqrt{\beta_{x1}} (-\cos\varphi_{x2} + \alpha_{x3} \sin\varphi_{x2})}{\sqrt{\beta_{x3}}} - \frac{x_0 [(\alpha_{x1} - \alpha_{x3}) \cos\varphi_{x2} - (1 + \alpha_{x1} \alpha_{x3}) \sin\varphi_{x2}]}{\sqrt{\beta_{x1}\beta_{x3}}} + x_0^3 \left[\frac{K_2 \beta_{x2}^2 (\cos\varphi_{x2} - \alpha_{x3} \sin\varphi_{x2})}{6\beta_{x1}^{3/2} \sqrt{\beta_{x3}}} \right. \\ &\quad \left. - \frac{K_1 \sqrt{\beta_{x1}} (-\cos\varphi_{x2} + \alpha_{x3} \sin\varphi_{x2})}{6\sqrt{\beta_{x3}}} \right] + x_0 y_0^2 \left[-\frac{K_1 \sqrt{\beta_{x1}} (\cos\varphi_{x2} - \alpha_{x3} \sin\varphi_{x2})}{2\sqrt{\beta_{x3}}} + \frac{K_2 \beta_{x2} \beta_{y2} (-\cos\varphi_{x2} + \alpha_{x3} \sin\varphi_{x2})}{2\sqrt{\beta_{x1}} \sqrt{\beta_{x3}} \beta_{y1}} \right], \\ y_4 &= -\frac{y_0 \sqrt{\beta_{y3}} (\cos\varphi_{y2} + \alpha_{y1} \sin\varphi_{y2})}{\sqrt{\beta_{y1}}} - y_0' \sqrt{\beta_{y1}\beta_{y3}} \sin\varphi_{y2} - \frac{x_0^2 y_0 \sqrt{\beta_{y3}} (K_1 \beta_{x1} \beta_{y1} + K_2 \beta_{x2} \beta_{y2}) \sin\varphi_{y2}}{2\sqrt{\beta_{y1}} \beta_{x1}} \\ &\quad + \frac{y_0^3 (K_1 \beta_{y1}^2 + K_2 \beta_{y2}^2) \sqrt{\beta_{y3}} \sin\varphi_{y2}}{6\beta_{y1}^{3/2}}, \\ y_4' &= \frac{y_0' \sqrt{\beta_{y1}} (-\cos\varphi_{y2} + \alpha_{y3} \sin\varphi_{y2})}{\sqrt{\beta_{y3}}} - \frac{y_0 [(\alpha_{y1} - \alpha_{y3}) \cos\varphi_{y2} - (1 + \alpha_{y1} \alpha_{y3}) \sin\varphi_{y2}]}{\sqrt{\beta_{y1}\beta_{y3}}} + y_0^3 \left[\frac{K_2 \beta_{y2}^2 (\cos\varphi_{y2} - \alpha_{y3} \sin\varphi_{y2})}{6\beta_{y1}^{3/2} \sqrt{\beta_{y3}}} \right. \\ &\quad \left. - \frac{K_1 \sqrt{\beta_{y1}} (-\cos\varphi_{y2} + \alpha_{y3} \sin\varphi_{y2})}{6\sqrt{\beta_{y3}}} \right] + x_0^2 y_0 \left[-\frac{K_1 \sqrt{\beta_{y1}} (\cos\varphi_{y2} - \alpha_{y3} \sin\varphi_{y2})}{2\sqrt{\beta_{y3}}} + \frac{K_2 \beta_{x2} \beta_{y2} (-\cos\varphi_{y2} + \alpha_{y3} \sin\varphi_{y2})}{2\sqrt{\beta_{y1}} \sqrt{\beta_{y3}} \beta_{x1}} \right], \end{aligned} \quad (5)$$

where K_1 and K_2 are the integral strength of the first and the second octupole, β_{xi} , β_{yi} , α_{xi} , α_{yi} ($i = 1, 2, 3$) are the horizontal and vertical Courant-Snyder parameters, $i = 1, 2, 3$ stand for location at the first octupole, the second octupole, and the target. $\varphi_{x1}, \varphi_{y1}$ are the horizontal and vertical phase advance between the two octupoles, $\varphi_{x2}, \varphi_{y2}$

are the horizontal and vertical phase advance between the second octupole and the target, x_0, y_0, x_0', y_0' stand for the displacement and angle just before the first octupole, and x_4, y_4, x_4', y_4' stand for the displacement and angle on the target.

When the phase advances are $n\pi$ ($n \in \text{integer}$) in both the horizontal and the vertical planes between the

two octupoles, there are not any terms higher than 3 in the expression of the displacement and angle on the target. In the linear collider an octupole doublet can be used for tail folding, the phase advance between octupoles is close to

zero, and the fifth-order terms are used to generate the aberration [8].

In order to eliminate the coupling term, we make $K_1\beta_{x1}\beta_{y1} + K_2\beta_{x2}\beta_{y2} = 0$, or $K_1 = -\frac{K_2\beta_{x2}\beta_{y2}}{\beta_{x1}\beta_{y1}}$, then we have

$$\begin{aligned}
 x_4 &= -x_0' \sqrt{\beta_{x1}\beta_{x3}} \sin\varphi_{x2} + \frac{K_2 x_0^3 \beta_{x2} \sqrt{\beta_{x3}} (\beta_{x2}\beta_{y1} - \beta_{x1}\beta_{y2}) \sin\varphi_{x2}}{6\beta_{x1}^{3/2} \beta_{y1}} - \frac{x_0 \sqrt{\beta_{x3}} (\cos\varphi_{x2} + \alpha_{x1} \sin\varphi_{x2})}{\sqrt{\beta_{x1}}}, \\
 x_4' &= \frac{x_0' \sqrt{\beta_{x1}} (-\cos\varphi_{x2} + \alpha_{x3} \sin\varphi_{x2})}{\sqrt{\beta_{x3}}} - \frac{x_0 [(\alpha_{x1} - \alpha_{x3}) \cos\varphi_{x2} - (1 + \alpha_{x1} \alpha_{x3}) \sin\varphi_{x2}]}{\sqrt{\beta_{x1}\beta_{x3}}} \\
 &\quad - x_0^3 \left[\frac{K_2 \beta_{x2} (\beta_{x1}\beta_{y2} - \beta_{x2}\beta_{y1}) (\cos\varphi_{x2} - \alpha_{x3} \sin\varphi_{x2})}{6\beta_{x1}^{3/2} \sqrt{\beta_{x3}} \beta_{y1}} \right], \\
 y_4 &= -y_0' \sqrt{\beta_{y1}\beta_{y3}} \sin\varphi_{y2} - \frac{K_2 y_0^3 \beta_{y2} \sqrt{\beta_{y3}} (\beta_{x2}\beta_{y1} - \beta_{x1}\beta_{y2}) \sin\varphi_{y2}}{6\beta_{y1}^{3/2} \beta_{x1}} - \frac{y_0 \sqrt{\beta_{y3}} (\cos\varphi_{y2} + \alpha_{y1} \sin\varphi_{y2})}{\sqrt{\beta_{y1}}}, \\
 y_4' &= \frac{y_0' \sqrt{\beta_{y1}} (-\cos\varphi_{y2} + \alpha_{y3} \sin\varphi_{y2})}{\sqrt{\beta_{y3}}} - \frac{y_0 [(\alpha_{y1} - \alpha_{y3}) \cos\varphi_{y2} - (1 + \alpha_{y1} \alpha_{y3}) \sin\varphi_{y2}]}{\sqrt{\beta_{y1}\beta_{y3}}} \\
 &\quad - \frac{K_2 y_0^3 \beta_{y2} (\beta_{x1}\beta_{y2} - \beta_{x2}\beta_{y1}) (-\cos\varphi_{y2} + \alpha_{y3} \sin\varphi_{y2})}{6\beta_{y1}^{3/2} \beta_{x1} \sqrt{\beta_{y3}}}.
 \end{aligned} \tag{6}$$

III. DISTRIBUTION OF CUBIC FUNCTION $g = x - kx^3$

Assuming that the probability distribution of x is Gaussian distribution of average 0 and variance 1: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}$. Now let us consider the probability distribution of g , where $g = x - kx^3$. The inverse function of $g = x - kx^3$ has three branches: $x_1 = f_1(k, g)$, $x_2 = f_2(k, g)$, $x_3 = f_3(k, g)$. These branches are shown in Fig. 2: Figure 2 shows the sketch of inverse functions of $g = x - kx^3$. The branch points of the curve are $\pm\sqrt{\frac{4}{27k}}$. So the probability distribution of g is divided into three parts:

$$\begin{aligned}
 f_{o1}(k, g) &= -f(x_1) \frac{\partial x_1}{\partial g}, \quad g < -\sqrt{\frac{4}{27k}}, \\
 f_{o2}(k, g) &= -f(x_1) \frac{\partial x_1}{\partial g} - f(x_3) \frac{\partial x_3}{\partial g} + f(x_2) \frac{\partial x_2}{\partial g}, \quad -\sqrt{\frac{4}{27k}} \leq g \leq \sqrt{\frac{4}{27k}}, \\
 f_{o3}(k, g) &= -f(x_3) \frac{\partial x_3}{\partial g}, \quad g > \sqrt{\frac{4}{27k}}.
 \end{aligned} \tag{7}$$

We define $f_o(k, g) = f_{o2}(k, 0)$, $-\sqrt{\frac{4}{27k}} \leq g \leq \sqrt{\frac{4}{27k}}$.

When k takes different values, the probability distribution of g will be quite different. Figure 3 shows the probability distribution curve of g when k takes the value 0.14, 0.095, 0.032, respectively. The black dashed curve is $f(x)$; the black curve is $f_o(k, g)$ when $k = 0.095$. The green curve is the probability distribution of g when $k = 0.14$, $f_{o1}(0.14, g)$, and $f_{o3}(0.14, g)$ are very close to zero at this time; the middle part of $f_{o2}(0.14, g)$ is uniform, but it has a great value on both sides. The blue curve is the probability distribution of g when $k = 0.032$; the distribution is more similar to Gaussian distribution. The red curve is the probability distribution of g when $k = 0.095$. $f_{o1}(0.095, g)$ and $f_{o3}(0.095, g)$ are also very close to zero at this time. In the range of $-\sqrt{\frac{4}{27k}}$ to $\sqrt{\frac{4}{27k}}$, the areas

FIG. 2. The sketch of inverse functions of $g = x - kx^3$. The dashed curve is $f_1(k, g)$, the solid curve is $f_2(k, g)$, and the dot-dashed one is $f_3(k, g)$. The branch points of the curve are $\pm\sqrt{\frac{4}{27k}}$.

FIG. 3. The probability distribution of g when $k = 0.14, 0.095, 0.032$, respectively. The black dashed curve is $f(x)$, the black curve is $fo(k, g)$ when $k = 0.095$. The green curve is the probability distribution of g when $k = 0.14$, the blue curve is the probability distribution of g when $k = 0.032$ and the red one is the probability distribution of g when $k = 0.095$. The probabilities are very small when $g < -\sqrt{\frac{4}{27k}}$ and $g > \sqrt{\frac{4}{27k}}$. There logarithmic functions are shown in the box when $g < -\sqrt{\frac{4}{27k}}$, the green curve, red curve, and the blue one are corresponding to $k = 0.14, 0.095, 0.032$, respectively. The possibilities when $g > \sqrt{\frac{4}{27k}}$ are symmetrical with that when $g < -\sqrt{\frac{4}{27k}}$.

of shaded regions which are above and below the intersection of $fo(0.095, g)$ and $f(0.095, g)$ are equal to each other. We think this is the most uniform distribution.

IV. APPLICATION EXAMPLE

From the previous section, if the intensity distribution of x_0 is a Gaussian distribution of variance σ , then the distribution of $x = x_0 - \frac{k}{\sigma^2} x_0^3$ is most uniform when $k = 0.095$. Take $\alpha_{x1} = 0, \alpha_{y1} = 0$, then the ratio of the third-order coefficient to the first-order coefficient in the expression of x_4 is

$$\frac{K_2 \beta_{x2} (\beta_{x1} \beta_{y2} - \beta_{x2} \beta_{y1}) \sin \varphi_{x2}}{6 \beta_{x1} \beta_{y1} \cos \varphi_{x2}} = -\frac{0.095}{\beta_{x1} \varepsilon}. \quad (8)$$

The ratio of the third-order coefficient to the first-order coefficient in the expression of y_4 is similar:

$$\frac{K_2 \beta_{y2} (\beta_{x2} \beta_{y1} - \beta_{x1} \beta_{y2}) \sin \varphi_{y2}}{6 \beta_{x1} \beta_{y1} \cos \varphi_{y2}} = -\frac{0.095}{\beta_{y1} \varepsilon}, \quad (9)$$

where ε is rms emittance.

Then we have

$$\begin{aligned} K_2 &= -\frac{0.57 \beta_{y1} \cos \varphi_{x2}}{\beta_{x2} (\beta_{x1} \beta_{y2} - \beta_{x2} \beta_{y1}) \sin \varphi_{x2} \varepsilon} \\ &= -\frac{0.57 \beta_{x1} \cos \varphi_{y2}}{\beta_{y2} (\beta_{x2} \beta_{y1} - \beta_{x1} \beta_{y2}) \sin \varphi_{y2} \varepsilon}. \end{aligned} \quad (10)$$

So

$$\frac{\beta_{y1} \cot \varphi_{x2}}{\beta_{x2}} = -\frac{\beta_{x1} \cot \varphi_{y2}}{\beta_{y2}}. \quad (11)$$

TABLE I. The CSNS RTBT parameters.

Parameter	Value	
Beam energy (GeV)	E	1.6
Beam average current (μA)	I	63
Beam peak current (A)	$I_{\max}$	3.6
Acceptance ($\pi mm mrad$)	A	350
Emittance ($\pi mm mrad$, rms)	ε	9
Magnetic rigidity (Tm)	$B\rho$	7.867 07
Target size (mm^2)	$a \times b$	120×40
Thickness of V C and installation gap (mm) ^a	d	5
Pipe size (mm)	D	250
Field strength on pole of octupole (T)	B	0.7
Octupole length (m)	l	0.6

^aV C stands for the vacuum chamber.

We can make $\beta_{x1} = \beta_{y2}, \beta_{y1} = \beta_{x2}, \cot \varphi_{x2} = -\cot \varphi_{y2}$ or $\varphi_{x2} = \pi - \varphi, \varphi_{y2} = \pi + \varphi$, and φ is a very small value, then $K_1 = -K_2$.

Here we take a set of China Spallation Neutron Source (CSNS) parameters to simulate the beam. The related CSNS ring to target beam transport (RTBT) parameters are shown in Table I.

The integral strength of the octupole must satisfy the following relation:

$$|K_1| = \frac{6Bl}{B\rho(\sqrt{A\beta_{x1}} + d)^3}. \quad (12)$$

So

$$\begin{aligned} K_2 &= -\frac{0.57 \beta_{y1} \cot \varphi_{x2}}{\beta_{x2} (\beta_{x1} \beta_{y2} - \beta_{x2} \beta_{y1}) \varepsilon} \\ &= \frac{0.32}{(\sqrt{\beta_{x1}} \cdot 350 \times 10^{-6} + 0.005)^3}. \end{aligned} \quad (13)$$

Because $\beta_{x2} \beta_{y1} \ll \beta_{x1} \beta_{y2}$, we omit $\beta_{x2} \beta_{y1}$ in Eq. (13), so we can get a relational expression of φ_{x2} and β_{x1} :

$$\varphi_{x2} = \text{Arc cot} \left(-\frac{5057.72 \beta_{x1}^2}{(\sqrt{350 \beta_{x1}} + 5)^3} \right). \quad (14)$$

FIG. 4. The relational graph of φ and β_{x1} .

FIG. 5. Real-space scaled intensity distribution at the target in the horizontal direction (the upper plot) and the vertical direction (the bottom plot) when $\varphi = \frac{1}{180}\pi$. The dashed curve stands for the initial Gaussian distribution; the solid curve stands for the uniformed beam distribution.

FIG. 6. Real-space scaled intensity distribution at the target in the horizontal direction (the upper plot) and the vertical direction (the bottom plot) when $\varphi = \frac{13.466}{180}\pi$. The dashed curve stands for the initial Gaussian distribution, the solid curve stands for the uniformed beam distribution, the slope of the solid curve is the impact of the x_0' and y_0' terms in the expression of x_4 and y_4 .

That is,

$$\varphi = \pi - \varphi_{x2} = \pi - \text{Arccot} \cot \left(-\frac{5057.72\beta_{x1}^2}{(\sqrt{350}\beta_{x1} + 5)^3} \right). \quad (15)$$

The expression is shown in Fig. 4.

Because of the limit of the pipe size and the target size, here we take $\beta_{x1} = \beta_{y2} = 40$, $\beta_{x2} = \beta_{y1} = 7$, $\beta_{x3} = 153$, $\beta_{y3} = 17$, then we can get $\varphi = \frac{13.466}{180}\pi$.

Now we will discuss the effect of φ on the beam distribution individually. When we take $\varphi = \frac{1}{180}\pi$, the expressions of x_4 and y_4 are as follows:

$$\begin{aligned} x_4 &= 1.95546x_0 - 516.025x_0^3 - 1.36531x_0', \\ y_4 &= 1.55815y_0 - 2349.59y_0^3 + 0.190383y_0'. \end{aligned} \quad (16)$$

The x_0' and y_0' terms are very small; the distribution of beam before and after uniforming is shown in Fig. 5.

Figure 5 shows the real-space scaled intensity distribution at the target in the horizontal direction and the vertical direction when $\varphi = \frac{1}{180}\pi$. The beam is uniform with almost rectangular distribution after uniforming, but the

TABLE II. The lattice parameters.

Label	Element	Length (m)	Strength
O1	Octupole	0	-170.621
DOCT	Drift	0.3	
D1	Drift	4.172	
Q1	Quadrupole	0.5	0.243
D2	Drift	4.084	
Q2	Quadrupole	0.5	-0.332
D3	Drift	9.991	
Q3	Quadrupole	0.5	0.412
D4	Drift	0.759	
Q4	Quadrupole	0.5	-0.412
D3	Drift	9.991	
Q5	Quadrupole	0.5	0.332
D2	Drift	4.084	
Q6	Quadrupole	0.5	-0.243
D1	Drift	4.172	
DOCT	Drift	0.3	
O2	Octupole	0	170.621
DOCT	Drift	0.3	
D5	Drift	3.647	
Q7	Quadrupole	0.5	-0.215
D6	Drift	2.455	
Q8	Quadrupole	0.5	0.641
D7	Drift	0.788	
Q9	Quadrupole	0.5	-0.379
D8	Drift	4.649	
Q10	Quadrupole	0.5	0.513
D9	Drift	7.339	
Q11	Quadrupole	0.5	-0.329
D10	Drift	12.462

FIG. 7. Beam envelop.

particles do not completely cover the target and hit the target in the middle part of the region. In addition, the strength of the octupole is unavailable in this case. As we increase the value of φ gradually, the area that the particles hit on the target becomes larger, and both sides of the rectangle start to become a slope. The reason is the impact of x_0' and y_0' terms.

When we take $\varphi = \frac{13.466}{180}\pi$, the expressions of x_4 and y_4 are as follows:

$$\begin{aligned} x_4 &= 1.90204x_0 - 501.92929x_0^3 - 18.20830x_0', \\ y_4 &= 1.51559y_0 - 2285.41048y_0^3 + 2.53902y_0'. \end{aligned} \quad (17)$$

Figure 6 shows the initial distribution and the uniform distribution after optimizing in the horizontal direction and the vertical direction when $\varphi = \frac{13.466}{180}\pi$. The beam just can cover the target and the strength of the octupole is acceptable.

In this condition, we have done a particle-tracking simulation with MAD8 to compare the particle coordinates with what we get from the theory above. The lattice parameters are listed in Table II where we have taken the thin lens approximation and the strength of the octupole in Table II is the integral strength. The beam envelope graph is shown in Fig. 7.

In Table III, we give two sets of initial values x_0, x_0', y_0, y_0' arbitrarily and compare the particle coordinates through the four elements in Fig. 1 from the theoretical calculations above with the simulated results with MAD8 by thin and thick octupoles. From Table III, we can see that the particle coordinates from the theoretical calculations and simulation results in thin lens approximation are identical. On the other hand, although the results with thick octupoles are not exactly equal, they are very close.

TABLE III. The particle coordinates comparison of theory and simulation with MAD8.

	x_0 [mm]	x_0' [0.001]	y_0 [mm]	y_0' [0.001]	x_0 [mm]	x_0' [0.001]	y_0 [mm]	y_0' [0.001]
	20.000	0.400	8.000	1.000	35.500	1.000	15.400	2.250
	x_1 [mm]	x_1' [0.001]	y_1 [mm]	y_1' [0.001]	x_1 [mm]	x_1' [0.001]	y_1 [mm]	y_1' [0.001]
Theory	20.000	0.518	8.000	0.742	35.500	1.554	15.400	0.698
MAD8	20.000	0.518	8.000	0.742	35.500	1.554	15.400	0.698
(thin lens)								
MAD8	20.275	0.515	8.523	0.730	36.266	1.545	16.285	0.622
	x_2 [mm]	x_2' [0.001]	y_2 [mm]	y_2' [0.001]	x_2 [mm]	x_2' [0.001]	y_2 [mm]	y_2' [0.001]
Theory	-8.367	-1.239	-19.124	-0.310	-14.851	-3.715	-36.813	-0.292
MAD8	-8.367	-1.239	-19.124	-0.310	-14.851	-3.715	-36.813	-0.292
(thin lens)								
MAD8	-7.614	-1.231	-19.142	-0.305	-12.567	-3.693	-37.879	-0.260
	x_3 [mm]	x_3' [0.001]	y_3 [mm]	y_3' [0.001]	x_3 [mm]	x_3' [0.001]	y_3 [mm]	y_3' [0.001]
Theory	-8.367	-1.483	-19.124	-0.226	-14.851	-5.339	-36.813	0.434
MAD8	-8.367	-1.483	-19.124	-0.226	-14.851	-5.339	-36.813	0.434
(thin lens)								
MAD8	-8.420	-1.469	-19.294	-0.209	-15.228	-5.308	-37.725	0.648
	x_4 [mm]	x_4' [0.001]	y_4 [mm]	y_4' [0.001]	x_4 [mm]	x_4' [0.001]	y_4 [mm]	y_4' [0.001]
Theory	26.736	2.032	13.494	2.663	26.836	2.886	20.704	2.855
MAD8	26.736	2.032	13.494	2.663	26.836	2.886	20.704	2.855
(thin lens)								
MAD8	27.091	2.051	13.503	2.653	28.781	3.003	19.985	2.417

V. SUMMARY

In order to avoid the negative influence of the Gaussian beam to the target, we place an octupole system in the beam transport line. In the thin lens approximation, when the phase advances are $n\pi$ ($n \in \text{integer}$) in both horizontal and vertical planes between the two octupoles, the terms higher than 3 in the expressions of displacement and angle on the target will disappear, and if $K_1 = -\frac{K_2\beta_{x2}\beta_{y2}}{\beta_{x1}\beta_{y1}}$, the coupling term will disappear, too. Moreover, the closer to π the phase advances in the horizontal and the vertical plane between the second octupole and the target, the more uniform the distribution on the target, but the higher the strength of the octupole magnet.

We take the parameters of CSNS to simulate the beam with MAD8. The simulation results agree well with the theoretical formulas.

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