Analytical expressions of transfer functions for a hard edge dipole magnet using a basic geometrical approach

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In charged particle accelerators, higher order optics studies become important from various points of view, such as dynamic aperture, emittance dilution, beam loss, etc. For some new applications, nonlinear study has become important in single pass optics also. For studying the higher order optics, each magnetic element is represented by a higher order transfer function (map, i.e., a function that relates output coordinates of a trajectory with initial coordinates and momentum deviation). Here in this paper we have provided an alternate method to obtain the analytical formulation of the transfer function for a dipole magnet. This formulation is obtained on the basis of basic geometrical analysis and is exact up to all orders under hard edge approximation. Being an analytical expression, the estimation of higher order effects of dipole magnet can be studied quickly. For checking the correctness of this formulation, we separated terms up to third order, which can be used to obtain the standard transfer matrices up to same order. An example of emittance growth and bunch length modification for a C-chicane-type electron beam bunch compressor is provided using the analytical expression.

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I. INTRODUCTION

In the charged particle accelerators, higher order optics studies become important from various points of view, such as dynamic aperture, beam qualities at the collision point, emittance dilution and beam loss in high intensity proton machines, etc. For some applications, nonlinear studies became important even in single pass optics [1–4]. One of the contributions in nonlinear forces comes through the inherent nonlinear nature of the magnets, i.e., due to curvature in path. Higher order studies, which are related to inherent nonlinearities of a dipole magnet, are done in terms of so-called T , U matrices of the magnetic elements. The first and second order matrix elements (R and T matrix) are derived in Refs. [5,6]. Numerous works have been carried out to obtain the higher order matrix elements for different types of magnetic elements. All these works concentrate on the solution of equation of motion using differential equations or Hamiltonian approach [7–12]. In these approaches, one derives the driving terms for the next desired order of solution and then, using Green's function, the equation is solved or one frames the appropriate Lie operators for the magnets. This approach based on Lie operators is summarized in the paper of Forest [13], in which an accurate higher order transfer map of a dipole magnet with many other magnetic elements are provided. This approach is elegant but mathematically is

complex in nature. Here we want to quote the author himself "...Indeed even if one is aware of the symplectic condition of the map generated by the Hamiltonian H of special relativity, it is not clear to most physicists that the return map from an arbitrary surface of section leads to a canonical symplectic map in some properly selected coordinates. Without going into mathematical details which are obscure to most accelerator physicists," On these approaches a very clear presentation is also provided in the book by Berz [14].

We have carried out an analytical formulation of the transfer function (map) of a dipole magnet with an alternate method, which is derived on the basis of geometry. Here in this paper, we provide a clear representation of this alternate, yet very simple approach with details and examples. Formulation derived here is correct up to all orders under hard edge approximation. First of all, the formulation is obtained for the sector-type dipole magnet where the particle (on design orbit) with correct momentum enters and exits at right angles with pole faces. This is presented for the median plane in Sec. II of this paper. In the next section, the formulation is extended for the particles, which have trajectories at some finite angle with the median plane, i.e., inclusion of coupling. In Sec. IV, the path length dependence of a particle with nonvanishing initial coordinates and momentum deviation is obtained. This exact expression of path length is very useful in quick analysis and design of bunch compressors. Results are compared from numerical computation using MAD. In the subsequent section, for checking the correctness of the obtained analytical expression and for example, expressions are reduced in such a way that these can be expanded up to

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third order in the bending plane (i.e., up to the U matrices). The subsequent section shows an example for computation of emittance growth using this formulation in a simple C -chicane geometry, which is often used for electron beam bunch compressors. After this, the formulation is extended for a general wedge magnet and it is shown that in linear approximations, formulation reduces to the standard edge focusing expressions. An example of computation with an analytical formula and numerical codes for the rectangular magnet is also presented. The conclusion is presented thereafter.

II. ANALYTICAL FORMULATION FOR SECTOR MAGNET IN MEDIAN PLANE

In this section, we have developed a formulation to transfer the coordinates of a charged particle in the median plane on the basis of geometry of a sector-type dipole magnet and with basic nature of motion of the charged particle in the pure dipole field. Figure 1 shows the geometry of the problem. We used two coordinate systems, one is the usual coordinate system x, y, s , whose origin is moving with reference particle, often used in the accelerator theory. The other one is the fixed coordinate system u, v with its origin at the center of the circle made by a reference particle. Further we assume the hard edge model of the magnet, i.e., the effect of fringe field is not included in the formulation. This approach was used by Wolnik [15] to obtain the first order transfer matrix of a dipole magnet. We could obtain the formulation containing all orders by extending this method.

The fixed coordinate axes u and v are chosen in such a way that the entry edge of the sector-type dipole magnet coincides with the v axis. Arc AA_1 (marked by C_1 in the

FIG. 1. Geometry of design trajectory and a deviated particle trajectory in the median plane of a sector dipole magnet with fixed coordinate system.

Fig. 1) defines the design trajectory and describes the moving curved coordinate system. The coordinates of a particle are defined with respect to the design trajectory. The arc of the design trajectory has the radius of curvature ρ in the median plane with a bending angle of θ . The center of circle O made by this trajectory is at the origin of coordinate axes. We choose another particle, which has its entry point at B in the magnet. The initial coordinates of this particle are defined by a displacement of x_i and an angle of α_i from the design trajectory with a momentum offset of δ ($\delta = \Delta p/p$). This particular particle exits from the magnet at point B_1 . In this section, we are considering the particle trajectory, which completely lies in the median plane, i.e., the particle does not have any vertical angle with the median plane. This particle under the ideal dipole field will make a circular path with the center of the circle at O' and a radius of curvature ρ_1 (marked by C_2 in Fig. 1). At the exit from the magnet, this particle has a displacement of x_e and an angle of α_e from the design trajectory. We will obtain these coordinates as a function of x_i, α_i , and δ .

Let O' have the coordinates u_c, v_c in a fixed coordinate system. These coordinates are expressed in terms of the particle's initial conditions and radius of curvature as follows:

$$u_c = \rho_1 \sin \alpha_i, \quad (1)$$

$$v_c = (x_i + \rho) - \rho_1 \cos \alpha_i. \quad (2)$$

Here $\rho_1 = \rho(1 + \delta)$. The coordinates of the point B_1 (u_1, v_1), from where the displaced particle exits from the magnet, is given by

$$u_1 = (\rho + x_e) \sin \theta, \quad (3)$$

$$v_1 = (\rho + x_e) \cos \theta. \quad (4)$$

From Eq. (3), we get x_e , i.e.,

$$x_e = \frac{u_1}{\sin \theta} - \rho. \quad (5)$$

The angle of this particle at exit with design trajectory can be obtained from the difference in slopes of circular arcs C_1 and C_2 at points A_1 and B_1 , respectively.

The slopes of the arcs C_1 and C_2 at the exit points are defined as

$$\theta_{c1} = \tan^{-1}\left(-\frac{1}{m}\right) \quad \text{and} \quad \theta_{c2} = \tan^{-1}\left(-\frac{u_1 - u_c}{mu_1 - v_c}\right).$$

Here $m = \cot \theta$ (slope of the magnetic edge in fixed coordinate system).

Thus, the exit angle will be

$$\alpha_e = \tan^{-1}\left(-\frac{u_1 - u_c}{mu_1 - v_c}\right) - \tan^{-1}\left(-\frac{1}{m}\right). \quad (6)$$

Coordinates x_e and α_e both depend on u_1 ; relating u_1 as a function of initial conditions (x_i, α_i, δ) will give the

relation between the coordinates at the exit and entry of the particle in the magnet in the moving coordinate system. Point B_1 in Fig. 1 is also the point of intersection between the circular arc made by this particle and the exit edge of the magnet. This information can be used to obtain u_1 as a function of initial coordinates, as follows:

$$\text{circle } C_2: (u - u_c)^2 + (v - v_c)^2 = \rho_1^2, \quad (7)$$

$$\text{edge } L_1: v = mu. \quad (8)$$

The coordinate for the point of intersection u_1 is given below, obtained from the above two equations:

$$u_1 = \frac{u_c + mv_c}{1 + m^2} \pm \frac{\rho(1 + \delta)}{\sqrt{1 + m^2}} \sqrt{(1 + P)(1 - Q)}. \quad (9)$$

Here

$$P = \frac{(u_c + mv_c)^2}{(1 + m^2)} \frac{1}{\rho_1^2} \frac{1}{(1 - Q)}, \quad (10)$$

$$Q = \frac{u_c^2 + v_c^2}{\rho_1^2}. \quad (11)$$

For the solution of u_1 [in Eq. (9)], a positive sign is used for the bending angle smaller than π while a negative sign is used for the bending angle equal to or greater than π . In a practical situation, generally a positive sign is used and hence in this paper also, a positive sign has been used. Putting Eqs. (1) and (2) in Eqs. (10) and (11), one can obtain the value of intersection point u_1 [from Eq. (9)] as a function of initial conditions. Using the value of u_1 , the displacement and angle of the particle at the exit of the dipole magnet can be obtained [from Eqs. (5) and (6)]. In deriving these formulations no approximation has been made except that the magnet is hard edge.

III. PARTICLE TRAJECTORY AT SOME ANGLE WITH MEDIAN PLANE

If the particle trajectory is inclined with respect to the median plane, then the horizontal coordinates will have dependence on vertical coordinates. Thus, the motion is coupled and here also the geometrical method can be employed for the calculation of coupling terms. The geometry of the problem is shown in Fig. 2. A particle has

FIG. 2. Effect of initial nonzero vertical angle β_i on the path of particle inside a sector dipole magnet.

an arc AA_1 in the median plane. If this particle enters at some vertical angle β_i with the median plane (particle's initial coordinates are x_i , α_i , y_i , and β_i with a momentum offset δ) it will trace out a different arc AA_2 (this arc is a part of a helix, traced out by the particle in the inclined magnetic field) and will exit with a vertical displacement y_e . Now the total path inside the dipole magnet will be longer; hence, x_e and α_e will be modified. The radius of curvature will also be changed for this trajectory due to its inclination with respect to the magnetic field. Let the radius of curvature for such an inclined trajectory be ρ_1' . The exit vertical angle β_e will be the same as the vertical angle at entrance (β_i), but at the exit the vertical displacement y_e will be a function of β_i and δ . The arc AA_2 has a circular projection AA_1' on the median plane with a radius of curvature ρ_1' . The effective bending angle in the median plane will be different from the design bending angle, leading to coupling of vertical motion with the horizontal motion. The effective magnetic field for bending the inclined trajectory is $B \cos \beta_i$ and thus the radius of curvature ρ_1' becomes $\rho_1 / \cos \beta_i$. The center of curvature has the coordinate point O' . The coordinates of this center are given by

$$u'_c = \rho_1' \sin \alpha_i = \frac{\rho_1 \sin \alpha_i}{\cos \beta_i}, \quad (12)$$

$$v'_c = (x_i + \rho) - \rho_1' \cos \alpha_i = (x_i + \rho) - \frac{\rho_1 \cos \alpha_i}{\cos \beta_i}. \quad (13)$$

In Eqs. (9)–(11), replacing the u_c , v_c from the above values of u'_c and v'_c and using ρ_1' in place of ρ_1 , we get the u_1 for the particle with nonzero initial vertical angle. Using this value of u_1 (and also u'_c and v'_c) in Eqs. (5) and (6), we can get the exit horizontal displacement and angle for the inclined particle, exactly similar to the particle with zero initial vertical angle.

In order to obtain the results of vertical coordinates at exit with coupling, the path length dependence on initial horizontal coordinates in the median plane is required. This has been derived in the next section. Knowing the path length dependence on initial conditions is also useful for designing a bunch compressor.

IV. LONGITUDINAL AND VERTICAL TRANSFER FUNCTIONS WITH EXAMPLE OF A TYPICAL SECTOR MAGNET

In design and analysis of bunch compressors, it is very important to know how the path length depends on momentum as well as on transverse coordinates. For momentum dependence, these terms are quantified in the form of matrix elements R_{56} , T_{566} , U_{5666} , etc. Again following the geometrical approach, the dependence of path length on momentum and transverse coordinates can be obtained, correct up to all orders. The path length can be obtained by integrating the circular arc from initial coordinates to final coordinates in the fixed u - v coordinate system. In the median plane it is given by

$$\begin{aligned}
S &= \int_{u=0}^{u=u_1} \sqrt{\left(1 + \left\{\frac{dv}{du}\right\}^2\right)} du = \int_0^{u_1} \frac{\rho_1}{v - v_c} du \\
&= \rho_1 \int_0^{u_1} \frac{1}{\sqrt{\rho_1^2 - (u - u_c)^2}} du, \\
S &= \rho_1 \left[\sin^{-1}\left(\frac{u_1 - u_c}{\rho_1}\right) - \sin^{-1}\left(\frac{-u_c}{\rho_1}\right) \right]. \quad (14)
\end{aligned}$$

For the particle entering at vertical angle β_i , ρ_1 , u_c , v_c will be replaced by ρ'_1 , u'_c , v'_c respectively. The difference of this term with respect to the length of the design arc, i.e., $S_0 = \rho\theta$, gives the path length difference due to nonvanishing initial coordinates. This is one of the most important relations for bunch compressor designs. For inclined trajectory, the path length becomes $S' = S/\cos\beta_i$. Using the value of S , the final vertical coordinate at the exit is given by (see Fig. 2)

$$y_e = y_i + S \tan\beta_i. \quad (15)$$

In the above equation, S depends on x_i and α_i (through u_1 and u_c), hence provides the coupling terms for vertical coordinate. These exact expressions are purely algebraic in nature, i.e., no integration or differentiation is involved. The complete process of computation can be summarized as shown in Fig. 3.

An example of computation using the derived analytical formula is presented here, in which beam is tracked through a sector dipole magnet. The sector dipole magnet is chosen with a bending angle of 30.75° and a bending radius of 0.96 m. These parameters of the magnet are close to the parameters of one of the magnets used in the CTF3 (CERN) transfer line-2 bunch compressor. The results are compared with numerical computation using computer code MAD [16]. For this computation, a total of 3000 particles are distributed (Gaussian distribution) in 6D phase space. The study has been carried out to obtain the emittance growth and change in bunch length as a function of initial transverse and longitudinal emittance. In the computations of bunch length, a difference between analytical result and result obtained from computer code has been observed with an increase in initial horizontal emittance. This difference occurs due to effects of transverse geometrical terms on bunch length. Results of the above computations have been presented in Fig. 4, in which rms bunch length is plotted with the initial horizontal emittance

FIG. 3. Steps for computing final coordinates of the particles from initial coordinates using analytical expression.

FIG. 4. Variation in bunch length (on momentum particles) with change in horizontal emittance.

for monochromatic beam of correct energy. For transverse emittance growth, results from the analytical expression and numerical computation are almost same.

With a variation in initial bunch length also, there is difference in the final bunch length in both computations (again particles with on momentum). This difference increases up to $\sim 10\%$ in a single magnet for very short bunch length ($\sim$ sub-ps range) but for longer initial bunch lengths, both results are almost equal. This is shown in Fig. 5.

Including initial momentum spread (Gaussian distributed) also, there is no appreciable change between the results of transverse emittance using numerical computation and analytical formulation. The changes are observed in longitudinal phase space (hence, bunch length) with increasing initial momentum spread. Figure 6 shows the difference between longitudinal phase space obtained using formula and computer code (8.3 ps is the bunch length after the linac in CTF3, hence this bunch length is chosen in the example).

FIG. 5. Variation in difference in final bunch length obtained from analytical expression and MAD as a function of initial bunch length.

FIG. 6. Longitudinal phase space after passing from the magnet (obtained from MAD and formula).

V. FORMULA VALIDATION WITH EXPLICIT EXPRESSIONS UP TO THIRD ORDER

A. Linear terms

Expansion up to first order of the terms in Eqs. (1), (2), and (9) to Eq. (11) generates the usual R matrix of the sector dipole magnet and can be shown as follows.

In first order u_c and v_c can be written as

$$u_c \approx \rho \alpha_i \quad \text{and} \quad v_c \approx x_i - \rho \delta_i.$$

As u_c and v_c contains minimum first order terms (i.e., no zero order or constant term), P and Q contain second order or higher terms (see Sec. II). Thus up to first order, u_1 can be written as

$$u_1 \approx \frac{\rho}{\sqrt{1+m^2}} + \frac{m}{1+m^2} x_i + \frac{\rho}{1+m^2} \alpha_i - \frac{m\rho}{1+m^2} \delta + \frac{\rho}{\sqrt{1+m^2}} \delta.$$

Using Eq. (5), x_e can be obtained:

$$x_e \approx \frac{m}{\sqrt{1+m^2}} x_i + \frac{\rho}{\sqrt{1+m^2}} \alpha_i - \frac{m\rho}{\sqrt{1+m^2}} \delta + \rho \delta.$$

Here $m = \cot\theta$. Using this relation in the above equation, finally one obtains

$$x_e \approx \cos\theta x_i + \rho \sin\theta \alpha_i + \rho(1 - \cos\theta) \delta. \quad (16)$$

From Eq. (6), we get expression using trigonometric identity

$$\alpha_e = \tan^{-1} \left\{ \left[-\frac{1}{m} \left(1 - \frac{u_c}{u_1} \right) \left(1 - \frac{v_c}{mu_1} \right)^{-1} + \frac{1}{m} \right] \times \left[1 + \frac{1}{m^2} \left(1 - \frac{u_c}{u_1} \right) \left(1 - \frac{v_c}{mu_1} \right)^{-1} \right]^{-1} \right\}.$$

From the above expression, the following expression can be obtained after a little algebraic manipulation:

$$\alpha_e = \tan^{-1} \left[-\frac{mZ}{1+m^2} \left(1 + \frac{Z}{1+m^2} \right)^{-1} \right]. \quad (17)$$

Here $Z = \frac{v_c}{m} - u_c)u_1^{-1} + \frac{v_c}{m} (\frac{v_c}{m} - u_c)u_1^{-2} + (\frac{v_c}{m})^2 \times (\frac{v_c}{m} - u_c)u_1^{-3} + \dots$.

Thus keeping only linear terms, gives

$$\begin{aligned} \alpha_e &\approx -\frac{m}{1+m^2}Z \\ &\approx -\frac{1}{\rho\sqrt{1+m^2}}x_i + \frac{m}{\sqrt{1+m^2}}\alpha_i + \frac{1}{\sqrt{1+m^2}}\delta. \end{aligned}$$

This gives

$$\alpha_e \approx -\frac{\sin\theta}{\rho}x_i + \cos\theta\alpha_i + \sin\theta\delta. \quad (18)$$

In linear approximation

$$S = \rho\theta \quad \text{and} \quad y_e = y_i + S\beta_i, \quad (19)$$

$$\beta_e = \beta_i, \quad (20)$$

Eqs. (16) and (18)–(20) give linear transformation, i.e., the respective elements of the R matrix for the sector dipole magnet.

B. Explicit higher order terms

For an example, in this section we deduce expressions for obtaining x_e and α_e up to third order from the derived formulation. The linear, second order, and third order terms in the expansion are the elements of well-known R , T , and U matrices, respectively, which are mentioned in Refs. [11,12,17]. Although, there is no need to separate the terms of different orders as the derived formulation is correct up to all orders, however for checking the correctness of the formulation, we have carried out the separation of different orders for transfer functions.

These can be derived easily from this geometrical formulation by expanding the terms up to higher order terms in Eqs. (1), (2), and (9) to Eq. (13).

The P and Q (see Sec. II) will also be expanded up to third order. Variations of P and Q for different bending radius and bending angles are shown in Figs. 7 and 8. In practical situations, $P < 1$ and $Q < 1$, therefore expansion of Eq. (10) and (11) can be carried out easily.

These expansions lead to the following results:

$$\begin{aligned} Q &= \frac{1}{\rho^2}x_i^2 + \alpha_i^2 - \frac{2}{\rho}x_i\delta + \delta^2 + \frac{1}{\rho}x_i\alpha^2 - \frac{2}{\rho^2}x_i^2\delta \\ &\quad + \frac{4}{\rho}x_i\delta^2 - \alpha_i^2\delta - 2\delta^3, \\ P &= \frac{1}{M} \left\{ \frac{m^2x_i^2}{\rho^2} + \frac{2mx_i\alpha_i}{\rho} + \alpha_i^2 - \frac{2m^2x_i\delta}{\rho} - 2m\alpha_i\delta \right. \\ &\quad + m^2\delta^2 + \frac{m^2x_i\alpha_i^2}{\rho} + m\alpha_i^3 - \frac{2mx_i\alpha_i\delta}{\rho} - \frac{2m^2x_i^2\delta}{\rho^2} \\ &\quad \left. + \frac{4m^2x_i\delta^2}{\rho} - m^2\alpha_i^2\delta + 2m\alpha_i\delta^2 - 2m^2\delta^3 \right\}. \end{aligned}$$

Here $m = \cot\theta'$ and $M = 1 + m^2$. Now Eq. (9) (Sec. II) can be computed explicitly up to third order. The first part of this expression is as follows:

$$\begin{aligned} \frac{u_c + mv_c}{1+m^2} &= \frac{1}{M} \left\{ mx_i + \rho\alpha_i - m\rho\delta + \frac{m\rho}{2}\alpha_i^2 \right. \\ &\quad \left. + \rho\alpha_i\delta - \frac{\rho}{6}\alpha_i^3 + \frac{m\rho}{2}\alpha_i^2\delta \right\}. \end{aligned}$$

The second part of expression is

$$\begin{aligned} &\frac{\rho(1+\delta)}{\sqrt{1+m^2}}(1+P)^{1/2}(1-Q)^{1/2} \\ &= \frac{1}{M^{3/2}} \left\{ M\rho + M\rho\delta - \frac{x_i^2}{2\rho} + mx_i\alpha_i - \frac{m^2\rho\alpha_i^2}{2} \right. \\ &\quad + x_i\delta - m\rho\alpha_i\delta - \frac{m\rho\delta^2}{2} - \frac{x_i\alpha_i^2}{2} + \frac{m\rho\alpha_i^3}{2} \\ &\quad \left. + \frac{x_i^2\delta}{2\rho} + (1-2m^2)x_i\delta^2 + \rho(1-m^2)\alpha_i^2\delta + \frac{\rho}{2}\delta^3 \right\}. \end{aligned}$$

FIG. 7. Variation in P and Q with bending radius for fixed bending angle of 30.75° . The x_i and α_i are taken as 100 mm and 20 mrad, respectively.

FIG. 8. Variation in P and Q with bending angle for fixed bending radius of 0.96 m. The x_i and α_i are taken as 100 mm and 20 mrad, respectively.

Addition of these two parts gives u_1 . Using u_1 in Eq. (5), x_e can be obtained explicitly up to third order in initial coordinates, which produces elements of the well-known matrices up to third order.

Similarly, using expansion of Z (and its higher order, i.e., Z^2 , Z^3) in Eq. (17) provides matrix elements for α_e up to third order. Using Eq. (17), the final angle α_e can be obtained from the following expression:

$$\alpha_e = -\frac{m}{1+m^2}Z + \frac{m}{(1+m^2)^2}Z^2 - \frac{m}{(1+m^2)^3}\left(1 + \frac{m^2}{3}\right)Z^3. \quad (21)$$

From the above equation, an explicit form of well-known matrix elements can be obtained.

VI. EXAMPLE OF A C-CHICANE BUNCH COMPRESSOR

The results in the previous section show a difference between the bunch length (obtained by analytical formulation and numerical computation) and this difference is larger compared to a difference in transverse emittance. Therefore, in this section, we present a specific example for computation of electron beam emittance growth and change in bunch length using the analytical expressions of sector-type dipole magnets in a C -chicane-type bunch compressor (without including the effects of coherent

synchrotron radiation). The computation is compared with the results of emittance growth and change in bunch length with results of numerical computation using MAD. The given example indicates that the analytical formula differs much in longitudinal plane. In an actual bunch compressor other effects (like errors and coherent synchrotron radiation) also cause emittance growth and, in general, may be more significant than the presented effect in the example. The layout of the C -chicane geometry for an achromatic tunable R_{56} optics is shown in Fig. 9. In this figure, rectangles represent quadrupoles.

The bunch compressor is tuned to $R_{56} = +0.25$ m (MAD sign convention) and a chirped beam (for attaining minimum bunch length) is tracked through the chicane optics. The optical parameters for this bunch compressor are given in Table I. The initial beam in transverse phase space (horizontal and vertical) is shown in Fig. 10, which is generated by 3000 randomly Gaussian distributed particles (initial longitudinal distribution will be different in each case due to different chirping; in chirping only linear effects, i.e., R_{65} and R_{66} are considered). The phase space distortion is obtained using only numerical computation in one case and in another case numerical computation with dipole replaced by analytical expression. The results after tracking are shown for the initial bunch length of

FIG. 9. Geometry of C -chicane bunch compressor.

TABLE I. Optical parameters of bunch compressor.

Parameters	Values
Beam energy	300 MeV
Horizontal emittance	10 nm-rad
Vertical emittance	10 nm-rad
Initial β_H	7.922 m
Initial β_V	3.271 m
Initial α_H	0.000
Initial α_V	0.000

FIG. 10. Initial beam distribution in horizontal and vertical phase space.

1 and 8.3 ps (bunch length from linac of CTF3) in Fig. 11. In transverse phase space, the analytical expression and computer model yield almost the same results [even for larger momentum spread numerical computation does not differ much from analytical results, which is depicted in Figs. 11(e) and 11(f)], but in the case of longitudinal phase space, there is a considerable difference in results from the numerical computation and the analytical expression. This mismatch increases with initial bunch length. For observing this behavior we increased the initial bunch length to 80 ps, which generates a large initial momentum spread due to chirping. The mismatch in the longitudinal plane is much pronounced in this case [Fig. 11(c)]. In case of results obtained using MAD, the quadratic term (T_{566}) brings an almost symmetric curve in phase space (due to neglecting the fourth and higher order terms), while using an analytical expression the effect of this term seems to be canceled out up to some extent in this particular example due to higher order (odd) terms in momentum offset. Keeping the initial emittance the same in all these cases brings the same pure geometric effects on the bunch length. Therefore as pointed out in the example of a single magnet, the difference in bunch length comes from geometric effects, so in this case, higher order geometric-chromatic mixed effects are also increasing (due to increase in momentum spread) and generate a difference from numerical computation. At very large momentum spread, the higher order terms produce effects opposite to second order term for negative momentum offset, which is clearly observed in the tracking simulations. Therefore using an analytical expression, effects of higher order terms are taken care of (although order of numerical computation can be increased and lead to longer computation time). The higher order tracking results obtained using computer code ELEGANT (for dipole magnet exact Hamiltonian with integration order = 4) [18] are closer to analytical results. Figure 11(d) shows such a result for the largest bunch length 80 ps (i.e., with

higher momentum spread also). In this tracking, in one case all four dipoles are replaced by an analytical formula at the place of the ELEGANT map and in other cases ELEGANT is used for all the elements. At shorter bunch length (hence, smaller momentum spread), this agreement is almost complete. Thus, higher order numerical computation converges towards the analytical expression.

VII. FORMULATION FOR WEDGE MAGNET

The formulation for a general wedge magnet can also be obtained using geometrical analysis and is derived in this section. Generally, the transfer matrix of a wedge magnet is obtained considering an equivalent sector-type dipole magnet. We also follow a similar approach. The geometry of the problem is shown in Fig. 12. Two red lines show the entry (L_1) and exit edge (L_2) of a wedge magnet. The coordinate axis " ν " coincides with the equivalent sector magnet entry edge and the blue line OB' shows the exit edge of this sector magnet. The equivalent sector magnet and the wedge magnet have the same design trajectory (i.e., circle marked with C_1 , i.e., arc AA'). The entry edge of the wedge magnet is inclined at an angle " e_i " with the ν axis. The particle in the median plane, with " x_i " and " α_i " as the initial coordinates for an equivalent sector magnet, have the initial coordinates " x_{ri} " and " α_{ri} " at the edge of a wedge magnet. After these coordinates, the particle follows the circular path decided by the magnetic field of the wedge magnet. The exit coordinates from the wedge magnet edge for this particle are " x_{re} " and " α_{re} " in the usual accelerator coordinate system, which are " x_e " and " α_e " for the equivalent sector magnet. The exit edge of the wedge magnet is inclined at an angle of e_e with the exit edge of the equivalent sector magnet. Usually the wedge is represented by a thin lens (even in higher orders), therefore there is no difference between x_i and x_{ri} and similarly between x_{re} and x_e . In this case, these are different. The actual particle coordinate at the entry edge x_{ri} is somewhat less than the x_i due to curvature in

FIG. 11. Phase space distribution of beam after bunch compressor (analytical and numerical model).

the design orbit. Therefore for obtaining the transfer function for a wedge magnet includes two steps, i.e., first obtain the actual coordinates (i.e., x_{ie} and α_{ie}) at the entry edge from the coordinates x_i and α_i (in usual magnet map it is

represented by a thin wedge lens). The second step is to obtain the output coordinates x_{re} and α_{re} from these coordinates, which is very similar to the procedure for a sector magnet, outlined in Secs. II and III.

FIG. 12. Geometry of design trajectory and a deviated particle trajectory in the median plane of a wedge dipole magnet with fixed coordinate system and equivalent sector magnet.

Let a particle enter the edge at point B_r . The distance of this point from the design orbit along the edge is given by

$$L_i = \frac{x_i \cos \alpha_i}{\cos(\alpha_i + e_i)}, \quad (22)$$

$$x_{ri} = -\rho + \sqrt{\rho^2 + L_i^2 + 2\rho L_i \cos e_i},$$

$$\psi_i = \sin^{-1} \left[\frac{L_i}{x_{ri} + \rho} \sin e_i \right], \quad \alpha_{ri} = \alpha_i + \psi_i. \quad (23)$$

Now the deviated particle traces an arc of the circle up to B_r' with the center of the circle O' and radius of ρ_1 [Eq. (7) of Sec. III]. The coordinates at the exit edge of the wedge magnet are the common point of the circle C_2 and edge L_2 . The coordinates of the center of the circle

$$u_c = (x_{ri} + \rho) \sin \psi_i + \rho_1 \sin(\alpha_i - \psi_i), \quad (24)$$

$$v_c = (x_{ri} + \rho) \cos \psi_i - \rho_1 \cos(\alpha_i - \psi_i). \quad (25)$$

The x_{re} is the distance between B_r' and F , which is given by

$$x_{re} = -\rho + \sqrt{\rho^2 + L_e^2 + 2\rho L_e \cos e_e}. \quad (26)$$

Here

$$L_e = \sqrt{(u_{B_r'} - \rho \sin \theta)^2 + (v_{B_r'} - \rho \cos \theta)^2}. \quad (27)$$

Here $u_{B_r'}$ and $v_{B_r'}$ are the common point between the exit edge of the magnet and the trajectory of the deviated particle and is given by

$$u_{B_r'} = \frac{1}{(1 + m_e^2)} [\{u_c - m_e(c_e - v_c)\} + \sqrt{\{u_c - m_e(c_e - v_c)\}^2 - (1 + m_e^2)\{(c_e - v_c)^2 + u_c^2 - \rho_1^2\}}], \quad (28)$$

$$v_{B_r'} = m_e u_{B_r'} + c_e. \quad (29)$$

Here $m_e = \cot(\theta - e_e)$ and $c_e = \rho(\cos \theta - m_e \sin \theta)$.

The angle of the deviated particle with respect to the design orbit at the exit edge is given by

$$\alpha_{re} = \tan^{-1} \left(-\frac{u_{B_r'} - u_c}{v_{B_r'} - v_c} \right) - \tan^{-1} \left(-\frac{1}{m} \right). \quad (30)$$

Thus, Eqs. (26) and (30) are the desired transfer function in the bending plane.

Then there will be an additional optional step if one wants to obtain again the coordinates x_e and α_e from the x_{re} and α_{re} . This transformation is also provided below:

$$\psi_e = \sin^{-1} \left[\frac{L_e}{x_{re} + \rho} \sin e_e \right] \quad \alpha_e = \alpha_{re} - \psi_e, \quad (31)$$

$$x_e = \frac{(x_{re} + \rho) \cos \alpha_e}{\cos \alpha_{re}} - \rho. \quad (32)$$

Thus, the path length of the deviated particle inside the magnet is given by

$$S = \rho_1 \left[\sin^{-1} \left(\frac{u_{B_r'} - u_c}{\rho_1} \right) - \sin^{-1} \left(\frac{u_{B_r} - u_c}{\rho_1} \right) \right]. \quad (33)$$

Here

$$u_{B_r} = \frac{1}{(1 + m_i^2)} [\{u_c - m_i(\rho - v_c)\} + \sqrt{\{u_c - m_i(\rho - v_c)\}^2 - (1 + m_i^2)\{(\rho - v_c)^2 + u_c^2 - \rho_1^2\}}]$$

FIG. 13. Variation in difference in final horizontal and vertical emittance obtained from analytical expression and MAD as a function of initial horizontal emittance.

and m_i is the slope of the entry edge from the u axis. The difference between S and $\rho\theta$ gives the path length difference for the deviated particle. If there is some finite initial vertical angle of the particle with the median plane, then the radius of curvature will be increased by $1/\cos\beta_i$ (Sec. III). The vertical coordinates can be obtained similar to the sector magnet using S as follows:

$$y_{re} = y_{ri} + S \tan\beta_{ri}, \quad (34)$$

$$\beta_{re} = \beta_{ri}. \quad (35)$$

It is easy to see that the transformation from x_i, α_i to x_{ri}, α_{ri} (and similarly from x_{re}, α_{re} to x_e, α_e) becomes usual thin (de)focusing lens with strength of $\tan(e_i)/\rho$ on retaining linear terms only. Also the formulas reduce to the sector magnet formulas in the case of $e_i = e_e = 0$.

An example of emittance change after beam transportation through a rectangular magnet is shown in Fig. 13 as a function of input horizontal emittance (vertical emittance is kept constant and very small, i.e., 1 nm-rad) using analytical expression and MAD. The parameters of the magnet are the same as of the example in Sec. IV and assuming a monochromatic beam. As pointed out earlier in the section that analytical expression brings a difference between x_{ri} and x_i (similarly x_{re} and x_e), the emittance growth is less than the growth indicated by MAD in the horizontal plane while in the vertical plane it is more. In the vertical plane due to coupling (higher order), the analytical expression changes vertical emittance (although change is very small) due to an increase in input horizontal emittance. The difference in bunch length with momentum deviation for the rectangular magnet

using analytical expression and MAD is not much pronounced.

VIII. CONCLUSION

Using an alternate method and following the geometrical approach, we have obtained the analytical expression of transfer function for pure dipole magnet using hard edge approximation. The formulation obtained using a simple approach is correct up to all orders and provides a way to estimate the effect of magnets on beam analytically. Although accurate descriptions are available for such problem in the literature using other approaches, this simple approach adds another method to our knowledge. The example discussed above shows the effects of higher orders on bunch length, mainly on phase space distribution, as obtained by application of the analytical expression. Therefore the formulation may be very useful in quick estimation of change in bunch length and emittance growth in bunch compressors or other transfer line design where higher order contributions are playing an important role. Though numerically computed results are close to the results obtained analytically, it is always better to obtain accurate values using analytical results. The expression reduces to usual R , T , and U matrices when explicit expansion is carried out up to first, second, and third order, respectively.

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