Experimental study of resonance crossing with a Paul trap

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The effect of resonance crossing on beam stability is studied systematically by employing a novel tabletop experimental tool and a multiparticle simulation code. A large number of ions are confined in a compact linear Paul trap to reproduce the collective beam behavior. We can prove that the ion plasma in the trap is physically equivalent to a charged-particle beam propagating through a strong focusing channel. The plasma confinement force is quickly ramped such that the trap operating point traverses linear and nonlinear resonance stop bands. Assuming a nonscaling fixed field alternating gradient accelerator composed of many identical FODO cells, we measure how much ion losses occur under diverse conditions. It is experimentally and numerically demonstrated that too slow resonance crossing leads to significant ion losses as expected. Particular attention must be paid to the linear coherent resonance excited at a *quarter*-integer tune. When the beam intensity is high, this type of linear stop band can seriously affect the beam quality even for rather fast resonance crossing. A scaling law is given of the emittance growth caused by the quarter-integer resonance crossing.

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I. INTRODUCTION

In most of circular accelerators, transverse betatron tunes are usually fixed at certain optimum values that guarantee beam stability. The machine operating point is, however, not necessarily frozen but actually moves over a wide range in some cases. For instance, when the beam is exposed to a very strong cooling force, the interparticle Coulomb repulsion is gradually strengthened depressing the *effective* tune. As the beam density increases in phase space, the tune shift becomes larger and can eventually reach the magnitude of the initial bare tune at the ultralow-emittance limit. Such a unique situation is expected to occur in advanced cooling experiments if several necessary conditions are satisfied [1,2]. Another good example is nonscaling fixed field alternating gradient (FFAG) accelerators [3]. Considerable theoretical and experimental efforts have recently been devoted to design studies of nonscaling FFAGs for various purposes including hadron therapy, accelerator-driven reactor systems, a muon collider, and a neutrino factory [4–8]. In this type of machines, the bare tunes keep decreasing rapidly while the beam is accelerated by radio-frequency (rf) cavities. It is almost inevitable for the operating point to cross resonance stop bands, some of which may be quite dangerous.

It is known that the natural resonance crossing caused by beam compression in phase space can limit the achievable emittance [9,10]. The cooling process is strongly interrupted once the effective operating point comes to a stop band of the linear (second-order) collective resonance. To avoid crossing such severe stop bands throughout the whole cooling process, the design betatron phase advance per lattice period must be below 90 degrees [10]. On the other hand, in the case of nonscaling FFAGs, the operating point moves due to the reduction of beam focusing forces during acceleration. It is often said that we could cross resonances without serious deterioration of beam quality if the crossing speed is sufficiently fast. This sounds reasonable because the beam must need a certain period of time before it recognizes that the external driving force is surely periodic. The most important question is how fast we have to move the operating point. Numerical simulations and even experimental studies with existing machines have been carried out to answer this practical question [7,11–15].

In the present paper, we address general issues of resonance crossing, applying a novel experimental technique. The tabletop tool called “S-POD (Simulator for Particle Orbit Dynamics)” is used to figure out what happens when the machine operating point traverses stop bands of some resonances. This unique experiment is based on the isomorphism between non-neutral plasmas in a compact trap and charged-particle beams in a linear focusing channel [16]; namely, S-POD can approximately reproduce the rest-frame motion of a beam in the laboratory frame, thus enabling us to obtain fundamental beam-dynamics

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information. Such a handy, inexpensive model experiment could be a powerful means for future beam-physics studies (just like modern computer simulations). At Hiroshima University, several S-POD systems have been developed to explore diverse beam-dynamics issues [17], and we here employ one of them.

The paper is organized as follows. We first give a brief description of the S-POD system in Sec. II for completeness and later convenience. Particle-in-cell (PIC) simulation results are then shown in Sec. III, considering actual experimental conditions. In Sec. IV, we report on the results of systematic S-POD experiments conducted under various choices of primary parameters. Section V is devoted to comparison between experimental and numerical data, which eventually leads to a simple scaling law of emittance growth from the lowest-order coherent resonance crossing. Concluding remarks are finally made in Sec. VI.

II. S-POD

A. Principle

The S-POD system employed for this study is composed mainly of a linear Paul trap (LPT) as displayed in Fig. 1, many DC and AC power sources, a high vacuum system, and a personal computer that controls the whole experimental process, data taking, and saving. LPTs are very popular these days and actually applied to a variety of purposes around the world [18]. They use rf quadrupole fields to confine charged particles transversely while the axial particle confinement is generally achieved by DC potentials on separate electrodes at both ends of the quadrupole section. Our LPT in Fig. 1, designed solely for beam-physics applications, is only 20 cm long and has an aperture of 1 cm in diameter. Owing to the evident

similarity between magnetic and electric quadrupole focusing, the collective motion of a non-neutral plasma in the LPT becomes physically equivalent to that of a charged-particle beam traveling through a linear transport channel at relativistic speed [16]. A group from Princeton Plasma Physics Laboratory also constructed a dedicated LPT system for beam-dynamics studies [19–21].

Each quadrupole rod of our “multisectioned” LPT is divided into five electrically isolated pieces, so that we can form a couple of axial potential wells simultaneously by applying different bias voltages to those pieces [22]. For the present experiments, ions are confined within the axial region of 75 mm long (much greater than the aperture size) in-between short quadrupole sections. The axial potential well then becomes almost flat around the center of the plasma confinement region because the potential barriers formed by the DC bias voltages on the short quadrupole rods do not deeply penetrate into there [23]. According to past experiments [24], the transverse extent of an ion plasma in the LPT is typically a few mm in radius, which means that the plasma actually looks like a long bunch. Note also that there is no possibility of synchrotron resonance because the longitudinal confinement force is static. The axial ion motion is thus unimportant here.

Assuming an axially uniform plasma column confined in a long LPT, we can readily see that the transverse motion of individual particles with rest mass M and charge state q obeys the Hamiltonian [16,23]

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{1}{2}K(\tau)(x^2 - y^2) + \frac{q}{Mc^2}\phi, \quad (1)$$

where c is the speed of light, ϕ is the collective Coulomb potential of the plasma, the linear focusing function $K(\tau)$ is proportional to the rf voltage applied to the quadrupole electrodes, and the independent variable is $\tau = ct$. Since Eq. (1) has the form identical to the well-known betatron Hamiltonian of an intense beam, we can study the effect of resonance crossing in a particular AG lattice by adjusting the waveform of $K(\tau)$ to the lattice design.

Note that the Hamiltonian in Eq. (1) contains no “cross term” that couples the betatron and synchrotron motions. S-POD is, therefore, currently irrelevant to the study of any beam dynamic effects in which momentum dispersion plays an essential role. Noninertial effects are also out of the scope of S-POD experiments; namely, we cannot consider synchrotron radiation and other phenomena caused by particle acceleration [25]. Even with these practical limitations, such compact experiments enable a quick, systematic survey of the beam behavior over a very wide parameter range. S-POD experiments should be particularly useful in exploring self-field-induced collective effects whose high-precision numerical studies are often extremely time consuming.

FIG. 1. The linear Paul trap used for the present experimental study of resonance crossing.

B. Experiment setup

Among a number of possible choices for the rf focusing function $K(\tau)$, we here adopt the simplest sinusoidal waveform to represent a unit AG cell. Since the sinusoidal focusing has a resonance nature very close to that of the FODO lattice, we can approximately model a circular machine if it is composed of identical doublet cells [22]. The nonscaling FFAG “EMMA” consisting of 42 doublets is one such example [4,26]. Bearing EMMA in mind, the rf waveform as illustrated in Fig. 2 is assumed in the following discussion. Every set of 42 consecutive sinusoidal periods corresponds to a single turn around the machine. In addition to this primary focusing wave, we can superimpose one or more perturbation waves of lower frequencies to examine the effect of lattice asymmetry. For instance, the effect of weak error fields, which reduces the strict lattice superperiodicity P from 42 to 1, can be studied by adding another sinusoidal rf field that has the wavelength 42 times longer than that of the primary focusing field (Fig. 2) [27].

$^{40}\text{Ar}^+$ ions produced through the electron bombardment process are used for the experiment. The initial density of the Ar plasma can be controlled by changing the neutral Ar-gas pressure and electron-beam current. We have so far achieved a root-mean-squared (rms) tune depression η up to 0.8 in S-POD, which is more than enough to cover the density range in standard circular accelerators. The rf frequency for the primary focusing wave is fixed at 1 MHz. Then, the rf amplitude necessary for the possible maximum betatron phase advance (180 degrees per single FODO) is about 93 V in the LPT of Fig. 1. In the EMMA FFAG, the betatron tune per unit FODO cell varies from near 0.4 (144 degrees) to below 0.2 (72 degrees) while the beam is accelerated to the output energy of 20 MeV [26,28]. To emulate this rapid tune variation process in S-POD, the rf voltage is linearly ramped over a certain

FIG. 2. Radio-frequency waveform employed for resonance crossing experiments with S-POD. Each sinusoidal period approximately represents a single FODO cell. When we consider a circular machine consisting of 42 FODO cells ($P = 42$), the weak lattice symmetry breakdown due to random gradient errors can be modeled roughly by adding another sinusoidal field ($P = 1$) whose wavelength is 42 times the single FODO period.

range at a certain speed. Provided that the beam is extracted after 10 turns around EMMA, we decrease the rf quadrupole amplitude from 88 V to 54 V within $1 \mu\text{s} \times 42 \text{ cell} \times 10 \text{ turns} = 0.42 \text{ ms}$. As soon as the voltage ramping is completed, we drop the DC potential barrier on one side of the trap to transport the plasma toward a Faraday-cup detector. Although the horizontal and vertical tunes are not the same in the EMMA lattice, we here assume that both tunes are equal through resonance crossing. This assumption simplifies the dynamic behavior of the plasma, allowing us to make a direct comparison between the experiment and one-dimensional (1D) theories [29–31]. When the two tunes are different, stop-band splitting occurs. It is possible to study this two-dimensional (2D) phenomenon with S-POD after minor mechanical modifications. We, however, concentrate upon the simplest case here, leaving the 2D effect as a future issue.

III. PIC SIMULATIONS

For better understanding of experimental observations, self-consistent multiparticle simulations have been performed with the PIC code WARP [32,33]. WARP is equipped with an electromagnetic field solver, so that we can take into account the detailed configurations of conducting electrode boundaries. Since our main concern is the crossing of betatron resonance stop bands, we ignore the longitudinal ion motion to save computing time, assuming axially uniform plasma columns in a long LPT. Similar coasting-beam simulations have been performed previously in Ref. [13]. In WARP simulations here, the space-charge force is self-consistently calculated and applied to macroparticles at least 200 times in every focusing period. We carefully chose the size of a numerical integration step, checking the convergence of output data.

The quadrupole rods of the LPT in Fig. 1 have the diameter of 11.5 mm to minimize nonlinear components induced by the nonhyperbolic surface of the electrodes [34]. Low-order nonlinearities are then negligible if the LPT is assembled perfectly. But in reality, we do have finite electrode misalignments that enhance the amplitudes of nonlinear multipole components. In case the rms displacement of the rods from the ideal positions is $50 \mu\text{m}$, the average strength of the third-order nonlinearity (sextupole) becomes 0.34% at the circular aperture of 5 mm in radius [22]. The higher-order nonlinearities are monotonically weakened; 0.19% for the fourth order and 0.13% for the fifth. The nonlinearities are about 3 times enhanced for $100 \mu\text{m}$ rms misalignments. As for the focusing magnets in EMMA, the third- and fourth-order multipole amplitudes about the magnetic center are below 0.1% (typically, on the order of 10^{-4}) of the quadrupole's at the radius of 35 mm, according to the data of rotating coil measurements [35]. Although these numbers seem smaller than those in the LPT, the effect of the nonlinear imperfection fields is expected to be somewhat strengthened in a nonscaling

FFAG because the beam always passes through off-center positions of the quadrupole magnets. If the electron beam in EMMA experiences the same level of net nonlinearity as the ion plasma in the LPT does, slow third-order resonance crossing could be a source of significant emittance growth as demonstrated later.

The experimental conditions outlined in the last section are incorporated in the following multiparticle simulations, except for the initial plasma distribution. Since it is difficult to numerically emulate the plasma production process, we simply start each simulation from a phase-space distribution matched to the sinusoidal focusing lattice. Although several different types of distribution functions (Gaussian, waterbag, and Kapchinsky-Vladimirsky distributions) were tested, the results were qualitatively similar for all cases. Therefore, we only show typical output data based on the initial Gaussian distribution.

A. Low-density regime

Let us first look at the case where the collective Coulomb potential is negligible. EMMA is currently operated in this low-density regime [35]. In this subsection, we simply switch off space-charge interactions in WARP simulations. Root-mean-square emittance evolution during resonance crossing is plotted in Fig. 3 as a function of rf periods (time). The bare tune ν_0 per single rf period is changed from 0.4 to 0.17 in both cases by ramping the amplitude of the rf voltage. The resonance crossing speed in the upper case roughly corresponds to 20-turn (840 FODO periods) acceleration in EMMA while the lower case is twice faster. We naturally observe less emittance growth for quicker resonance crossing. It has been confirmed through separate simulations that the growth rate almost linearly increases as we slow down the bare-tune sweeping. Each picture in Fig. 3 indicates a slight emittance growth near a specific rf period at which the rf amplitude is around 80 V. The bare tune at this quadrupole voltage is $\nu_0 \approx 1/3$, which suggests that the emittance growth is caused by crossing a third-order resonance stop band. The existence of a resonance stop band at $\nu_0 \approx 1/3$ has been numerically clarified in Fig. 4 that shows emittance growth rates when the plasma is stored in the LPT for 0.1 ms (100 rf periods) at fixed tunes. These simulations have assumed 0.1 mm (rms) misalignments of the LPT electrodes that considerably strengthen nonlinear fields. The emittance growth disappears at low density without rod misalignments as demonstrated in Fig. 3 (red line).

B. High density regime

We now turn on the Coulomb potential. Because of the nonlinear nature of space-charge interactions, collective resonances of various orders can be excited even in an ideal linear focusing channel. A 1D Vlasov theory has revealed that the collective mode of m th order becomes unstable when the following condition is fulfilled [29]:

FIG. 3. Multiparticle simulation results without space-charge interactions. The ordinate represents the average of horizontal and vertical rms emittances normalized by its initial value. The rf voltage on the quadrupole electrodes are quickly reduced to move the operating point of the LPT. The initial and final bare tune per single rf period have been set at $\nu_0 = 0.4$ and 0.17 , respectively. In the upper example (a), the ν_0 sweep from 0.4 to 0.17 is completed within 840 rf periods. The sweep speed is twice faster in the lower example (b). Low-order nonlinearities have been intentionally enhanced in these simulations, considering the possible electrode misalignments of 0.1 mm. For reference, simulation results without the misalignments are plotted with red lines, which indicate almost no emittance growth.

$$m(\nu_0 - C_m \Delta\nu) \approx \frac{n}{2}, \quad (2)$$

where $\Delta\nu$ is the tune shift generated by the space-charge repulsion, C_m is the positive constant depending on the mode number m , and n is an integer. In general, the incoherent tune shift $\Delta\nu$ cannot uniquely be determined due to a finite effective tune spread. We thus employ the space-charge-depressed rms tune defined by $\nu = \oint (\epsilon_{\text{rms}} / 2\pi r_{\perp}^2) d\tau$, where ϵ_{rms} is the rms emittance [36] and $r_{\perp}$ is the transverse rms extent of the stationary plasma. The rms tune shift and tune depression can then be calculated from $\Delta\nu = \nu_0 - \nu$ and $\eta = \nu / \nu_0$.

Strictly speaking, there exist not one but many resonances in the vicinity of a particular tune. As readily understood from Eq. (2), resonance conditions for different

FIG. 4. Stop bands in the ultralow density regime (multiparticle simulation results *without* space charge). The rms emittance of an ion plasma after 0.1 ms storage in the LPT was evaluated with the WARP code at many different fixed tunes in the range $0.17 \leq \nu_0 \leq 0.4$. The final emittance has been normalized by the initial value. For direct comparison, the simulation data in Figs. 3(a) and 3(b) (10 times exaggerated) are replotted here with dotted and broken lines. The nonlinear-field enhancement due to electrode misalignments has been taken into account as in Fig. 3.

collective modes become similar whenever the corresponding ratios of n/m are identical [37]. Those resonances with the same n/m ratio overlap each other, which may make it difficult to distinguish the instability of one mode from those of the others. The instability of a highly nonlinear mode is, however, generally quite weak in practice or even invisible due to Landau damping unless the strong external driving force of the same order is present. The amplitudes of the fourth and higher-order multipole fields at the full aperture of the LPT are estimated to be less than 0.5% of the quadrupole amplitude even with the 0.1 mm rms misalignments of the electrodes. Considering this fact, we reasonably expect no strong excitation of highly nonlinear resonances. Figure 4 in the last section actually shows only one clear stop band of the lowest (third-order) nonlinear resonance.

In the condition (2), special attention should be paid to the factor $1/2$ on the right-hand side, which is missing in the well-known Sacherer's coherent resonance criterion [30]. The above condition says that there exist twice as many resonance stop bands as Sacherer predicted. Most importantly, the linear resonance ($m = 2$) can occur when the tune is close not only to half integers but also to quarter integers. Since ν_0 crosses $1/4$ in EMMA, this quarter-integer linear resonance might play a crucial role at high beam intensity. The existence of such a linear resonance has been widely known in linear transport theories (while the above criterion applies to all nonlinear modes as well). For instance, a 2D Vlasov theory by Hofmann *et al.* has reached the same conclusion that the strong *second-order* ($m = 2$) resonance can be excited once the bare tune per unit FODO cell exceeds $1/4$, i.e., the phase advance of 90 degrees [38]. The theory even recommends avoiding the *third-order* ($m = 3$) resonance that occurs when the tune per cell goes beyond $1/6$ (60-degree phase advance).

FIG. 5. Stop bands in the high density regime (multiparticle simulation results *with* space charge). The same systematic WARP simulations as in Fig. 4 are carried out with the Coulomb potential switched on. The ordinate represents the rate of rms emittance growth after 0.1 ms plasma storage in the LPT. The initial particle distribution is Gaussian matched to the rf quadrupole focusing force. Three different tune depressions are considered; namely $\eta = 0.99, 0.95$, and 0.9 . In the upper picture, we have assumed no misalignments of the quadrupole rods. The lower picture corresponds to the more realistic case where low-order nonlinearities are considerably enhanced by 0.1-mm (rms) rod misalignments. For easy comparison with later simulation pictures, the abscissa has been reversed.

A careful numerical study in Ref. [39] has confirmed the existence of these second- and third-order coherent resonances. The importance of such coherent parametric resonances is also addressed in connection with cooler storage rings [1,2,10] and next-generation high-power rings [40].

Stop-band distributions at three different tune depressions are depicted in Fig. 5 with and without the enhanced nonlinear rf fields due to electrode misalignments. Unlike the result in Fig. 4, another clear stop band has been created near $\nu_0 \approx 1/4$ as expected from Eq. (2). This linear stop band grows significantly at higher beam density in phase space. Since the tune shift $\Delta\nu$ becomes larger for lower η (stronger space charge), the locations of all stop bands slightly moves toward the high tune side, i.e., leftward in Fig. 5. These observations are consistent to what we found in previous experiments with another S-POD system [22]. The corresponding simulations of resonance crossing at different speeds of bare-tune sweep result in the emittance evolution in Fig. 6. The emittance growth rates are much larger than the results in Fig. 3 where the Coulomb

FIG. 6. Multiparticle simulation results with space-charge interactions. Input parameters here are similar to those adopted in Fig. 3, except that the collective Coulomb potential is now turned on. Two different tune depressions, i.e. $\eta = 0.99$ and 0.9 , are chosen, for example. Red curves are obtained with no electrode misalignments while blue curves take into account rms misalignments of 0.1 mm. For reference, the corresponding stop-band distributions in Fig. 5 are added to each panel with broken lines.

potential is ignored. In particular, the linear resonance at $\nu_0 \approx 1/4$ has seriously affected the plasma quality at high density while no such effect is identified in the simulations without space charge.

Although the large stop band near $\nu_0 \approx 1/4$ can include the effect from the fourth-order resonance as well, the linear instability of the $m = 2$ mode should be most responsible for the observed emittance growth. In fact, if the fourth-order resonance is really dominant there, the third-order resonance near $\nu_0 \approx 1/3$ in Fig. 5 has to be much stronger because, first, it is a lower-order resonance than the fourth-order, second, the sextupole nonlinearity assumed in the simulations is greater than the octupole and, third, we only observe the third-order stop band in Fig. 4 (except for an extremely small emittance growth at $\nu_0 = 1/4$).

C. Ion-loss simulations

The tune depression η is the most important parameter that characterizes the collective behavior of an intense beam. In fact, the resonance condition (2) only contains the tune shift $\Delta\nu$ as a space-charge-related quantity that can uniquely be determined once the choice of η is made. The direct experimental determination of η is extremely difficult not only in the LPT but also in any high-intensity accelerators because it requires a precise survey of phase space. No phase-space monitor is presently available in the S-POD system (while laser-induced fluorescence diagnostics

is under development for a nondestructive measurement of plasma emittance). Despite the lack of detailed information of the ion distribution in phase space, it is still feasible to estimate an approximate magnitude of η by measuring the shifts of stop bands (see Sec. IVA below).

The results of the resonance crossing simulations in Fig. 6 only weakly depend on the initial plasma temperature T_p . Even if we change T_p to some degree, the emittance evolution patterns are not essentially affected as long as the starting value of η is maintained. On the other hand, T_p becomes meaningful when we numerically simulate the actual ion-loss process. In S-POD, a larger emittance growth of a plasma naturally gives rise to more severe particle losses. We can thus judge, through ion-loss measurements, how seriously the plasma is damaged by resonance crossing. The amount of ion losses depends on the mechanical design of the LPT and on the plasma temperature. While T_p is inessential in the emittance evolution, it does change the spatial extent of the plasma at the beginning. This parameter is, therefore, of importance in ion-loss simulations. According to past S-POD experiments based on a similar LPT, the transverse temperature of an equilibrium plasma is typically a few thousand K [24]. We here try three different values of T_p (i.e., 0.1 , 0.2 , and 0.3 eV) to check how much ion losses are expected in S-POD. An example is shown in Fig. 7 where we have assumed the same tune depression, tune ramping speed and range as considered in Fig. 6(d). In this simulation, we

FIG. 7. Results of ion-loss simulations with the WARP code. The input parameters are identical to those assumed in Fig. 6(d), except for the initial plasma temperature T_p . The rms misalignments of the electrodes have been set at 0.1 mm. The ordinate of the lower panel represents the number of ions surviving, divided by its initial value. We here consider $T_p = 0.1, 0.2$, and 0.3 eV, recalling previous experimental data [24]. To simulate ion losses in actual experiments, macroparticles that have gone across the electrode boundaries are just removed. The observed T_p dependence of emittance evolution is caused largely by the macroparticle losses.

have simply removed particles once they hit the quadrupole rods. Not surprisingly, many particles are lost when the LPT operating point traverses the regions of large emittance growth. The loss rate is enhanced, as expected, at higher plasma temperature. We have also confirmed that almost no ion losses occur over a wide range of the crossing speed u when the tune depression is reduced to 0.99 (see Fig. 14 in the next section).

IV. EXPERIMENTAL RESULTS

A. Stop-band measurements

Before proceeding to resonance crossing experiments, we experimentally locate stop bands in tune space. A large number of $^{40}\text{Ar}^+$ ions, typically from 10^5 to 10^7 , produced by the electron bombardment process are confined for a certain period in the LPT and then released by dropping the axial DC potential barrier on the Faraday-cup side. In Fig. 8, we show the stop-band distributions measured at three different values of N_{in} (the number of ions initially confined in the LPT), i.e., $N_{\text{in}} \approx 10^5, 10^6$, and 10^7 . The blue and red lines indicate, respectively, the ion numbers after 1 and 10 ms storage in the LPT at fixed bare tunes. Since the operating rf frequency is 1 MHz, 10 ms corresponds to 10 000 focusing periods. Results of the immediate extraction (no storage) after plasma generation are plotted in black for reference, which show no sharp ion losses anywhere. Each line in this picture is made of 600

FIG. 8. Stop-band distributions measured by the S-POD system. The ordinate is the number of Ar ions surviving after a short storage in the LPT at a certain bare tune ν_0 . The storage periods considered in these experiments are 0 ms (immediate extraction; black), 1 ms (blue), and 10 ms (red). The noises in the low-density experiments ($N_{\text{in}} \approx 10^5$) are due simply to the limited signal-to-noise ratio of the Faraday cup.

independent data points obtained with different rf amplitudes at approximately the same N_{in} . S-POD automatically executes the necessary experimental procedure without human intervention to retune parameters. It takes our current control system less than 90 minutes to complete these 600 measurements; a single measurement (including the plasma production, storage and extraction), data transfer to the personal computer, and saving take roughly 8 seconds.

The experimental observations in Fig. 8 are consistent to the numerical results given in the last section. At low plasma density, remarkable ion losses occur only at $\nu_0 \approx 1/3$ because of the third-order resonance. As expected, the linear stop band appears near $\nu_0 \approx 1/4$ at higher density and the particle loss rate there eventually goes beyond that from the third-order resonance. All stop bands slightly shift to the right as the plasma becomes denser. We also notice another weak stop band near $\nu_0 \approx 1/6$, which originates from a *third-order* resonance ($m = 3, n = 1$) predicted by Eq. (2) [29] and by previous theoretical studies [31,38,39]. Figure 8 is strong experimental evidence for the validity of these coherent resonance theories.

Although the direct evaluation of the tune depression η is practically difficult as explained in the last subsection, we can deduce its probable magnitude by comparing the experimental observations and WARP simulation data [22]. Results of S-POD experiments starting with various N_{in} values are plotted in Fig. 9. Careful comparison with WARP simulations in the lower panel enables us to make a rough estimate of η . At $N_{\text{in}} \approx 10^7$, the estimated tune depression is 0.85–0.9 while the space-charge-induced tune shift is almost negligible at $N_{\text{in}} \approx 10^5$. We can confirm, through

FIG. 9. Comparison between the stop-band distributions obtained through S-POD experiments and WARP simulations. The plasma storage periods are fixed at 10 msec in the experiments and at 0.1 msec in the simulations. The initial ion number N_{in} is chosen in the range $10^5 \leq N_{in} \leq 10^7$. The weak stop band near $\nu_0 = 1/6$ is 30 times enlarged in the lower panel. The scale of the ordinate is arbitrary.

systematic ion-loss simulations, that the present η estimation is reasonable (see Sec. V).

B. Particle loss rate vs resonance crossing speed

Figure 10 demonstrates how the number of confined ions after a bare-tune sweep varies depending on resonance crossing speed. Similarly to the WARP simulations, the operating tune ν_0 per single sinusoidal focusing period is swept from 0.4 to 0.17. The abscissa stands for the dimensionless speed u defined by the ratio of δ (tune variation width) to n_{rf} (number of rf focusing periods for the bare tune to go across the width δ). The tune variation width $\delta = 0.4 - 0.17 = 0.23$ in this experiment as well as the numerical data given in Sec. III (Figs. 3 and 6). In case a beam is extracted from EMMA in 10 turns, n_{rf} is equal to 10 times 42 periods, which yields $u = 0.23/(10 \times 42) \approx 5.48 \times 10^{-4}$. The vertical broken line in Fig. 10 indicates this speed for reference. Right after ν_0 reaches the final value 0.17, the plasma is extracted from the LPT toward the Faraday cup. Significant ion losses have occurred even for rather fast resonance crossing when the plasma is initially dense. About a half of ions are lost at $u = 2 \times 10^{-4}$ in the highest-density case (green circles). By contrast, only little particle losses are observed in the other two cases at the same crossing speed.

FIG. 10. Number of remaining ions after resonance crossing vs speed of bare-tune sweep. The bare tune per single rf period is changed from 0.4 to 0.17 at various ramping speeds of the rf quadrupole amplitude. The resonance crossing speed (abscissa) is defined as $u = \delta/n_{rf}$ where δ is the tune variation width (0.23 in all three cases) and n_{rf} is the number of rf focusing periods spent for the tune sweep over δ . Three different initial plasma densities are considered; namely, $N_{in} \approx 10^5$ (blue square), 10^6 (red triangle), and 10^7 (green circle). For reference, the crossing speed corresponding to the 10-turn extraction from EMMA is indicated by a vertical broken line.

The natural decay rate of an ion plasma confined in the LPT depends on its density and the operating tune. In general, the plasma lifetime becomes shorter as the density becomes higher and/or the LPT operating point is closer to the Mathieu stability boundary. We, however, understand from Fig. 8 that the natural ion losses within 10 ms are not prominent even at high density in the tune range considered here. Although we recognize slight decreases of the ion number on the high tune side, the ion losses of resonance origin are clearly much more severe. This fact allows us to conclude that the observed ion losses in Fig. 10 largely come from crossing resonances.

C. Single resonance crossing

It is interesting and also practically meaningful to ask which resonance most seriously affects the beam quality when it is crossed at a particular speed. Figure 11 answers this question. The three panels summarize the experimental observations obtained for different tune-sweep ranges. The widths of ν_0 variation are equal in all three cases, while the initial and final tunes are chosen to be (a) (0.4, 0.32), (b) (0.31, 0.23), and (c) (0.22, 0.14) such that the LPT operating point traverses only one of the three stop bands identified in Fig. 8. The experiments are carried out at different initial plasma densities. Green circles correspond to 10^7 ions at the beginning, red triangles to 10^6 ions, and blue squares to 10^5 ions. We see that at low starting density, only the nonlinear stop band near $\nu_0 \approx 1/3$ gives rise to noticeable particle losses as anticipated from the results in Fig. 8. The other two stop bands have no effect unless u is very small. We must, however, recall that the third-order resonance has been enhanced in the LPT by the electrode

FIG. 11. Number of remaining ions after the LPT operating point crosses a particular resonance at different speeds. The ranges of the bare-tune sweep are chosen as follows: (initial ν_0 , final ν_0) = (0.4, 0.32), (0.31, 0.23), and (0.22, 0.14). Following the experiments in Fig. 8, we have taken three different initial densities into consideration, namely, $N_{\text{in}} = 10^5$ (blue square), 10^6 (red triangle), and 10^7 (green circle).

misalignments. Since the sextupole nonlinearity is generally weaker in most accelerators, the particle loss rate near $\nu_0 \approx 1/3$ should be much reduced as numerically verified in Sec. III or even become negligible in practice.

At high density, the second-order stop band near $\nu_0 \approx 1/4$ starts to play a major role. In fact, Fig. 11 as well as Fig. 8 points out that the particle loss rate near $\nu_0 \approx 1/4$ eventually exceeds what we have at $\nu_0 \approx 1/3$. Since this linear resonance is intrinsic and independent of lattice errors, it may cause trouble in general nonscaling FFAGs whenever the beam is space-charge dominant. As mentioned above, the quarter-integer linear stop band actually limits the beam emittance achievable by cooling [10,40]. For more information, we have measured the time evolution of ion numbers, putting the LPT operating point in the middle of a stop band. The results are shown in Fig. 12 where we have chosen two initial ion numbers, i.e., $N_{\text{in}} \approx 10^6$ in the upper panel and 10^7 in the lower. The bare tune ν_0 is fixed at proper values at which either the second- or third-order resonance is most active. The black dotted line in each panel is added just for the purpose of demonstrating that no particle losses occur out of a stop band

FIG. 12. Time evolution of the confined ion number at fixed tunes where either the second-order resonance (red) or the third-order resonance (blue) is activated. The operating tunes are different in the upper and lower cases because of the density dependence of the tune shift (see Fig. 8). The black line in each panel is obtained at a resonance-free tune ($\nu_0 = 0.31$). The initial numbers of Ar^+ ions in the LPT are (a) 10^6 and (b) 10^7 .

($\nu_0 \approx 0.31$). In the case of relatively low density, the plasma decays faster within the third-order stop band. At higher density, the decay rate at the linear resonance goes beyond that from the third-order resonance despite the fact that the sextupole field has been strengthened here by inevitable electrode misalignments.

D. Effect of lattice imperfection

Similar experiments have been carried out adding the long perturbation wave as illustrated in Fig. 2. Typical results of resonance crossing and stop-band measurements are given in Fig. 13. The amplitude of the error-induced wave is fixed at 1% of the primary focusing wave amplitude. The linear and nonlinear stop bands appear to be slightly widened by the additional rf field. This is due to the sidebands generated by the perturbation wave. At a lower ion number ($N_{\text{in}} \approx 10^6$), we actually recognize two small stop bands growing beside the original major resonance at $\nu_0 \approx 1/3$. As the ion density increases, these stop bands become wider eventually overlapping each other. The appearance of these error-induced sidebands has been confirmed in WARP simulations as well. According to separate experiments of single resonance crossing, the stop-band

FIG. 13. Experimental results with the perturbation wave on. The error-induced sinusoidal voltage in Fig. 2 is applied to the quadrupole electrodes in addition to the primary focusing rf potential. (a) Stop-band distribution at relatively high initial ion number ($N_{in} \approx 10^6$ and 10^7). The plasma storage period in each measurement at a fixed tune is 1 ms. The third-order stop band at $N_{in} \approx 10^6$ is enlarged with more data points to elucidate the existence of sidebands rising. The two sidebands eventually get together with the large central stop band at higher density, creating a single widened instability region in the vicinity of $\nu_0 \approx 1/3$. For reference, the corresponding results without the perturbation (blue line in Fig. 8) are plotted with broken lines. (b) Number of remaining ions after resonance crossing vs speed of bare-tune sweep. The experimental conditions are identical to those considered in Fig. 10, except for the excitation of the error-induced wave. The results in Fig. 10 are replotted with black dots.

widening raises the ion-loss rate particularly at the third-order resonance when the initial plasma density is high. Nevertheless, the experimental observations in the lower panel look analogous to the previous results (black dots) in Fig. 10. This fact can be understood by considering the effect of the third-order resonance crossing on the linear resonance crossing. Since ion losses at the third-order stop band are enhanced with the error-induced wave on and the plasma density is more reduced, the effect from the second-order resonance crossing is somewhat weakened compared to the case without the perturbation. As a result, the net ion losses become comparable to those in Fig. 10 after the two stop bands are crossed consecutively.

Needless to say, random lattice errors produce not only the first Fourier harmonic but also many others that have shorter wavelengths. In such general cases with various imperfection waves, narrow but visible stop bands may be created additionally, leading to more particle losses. The excitations of resonance sidebands similar to those in Fig. 13 are also expected depending on the original lattice symmetry and imperfection wavelength. This practical issue of error-induced effects will be discussed in detail in a future paper.

V. SCALING LAW

The measurement data of Fig. 10 are replotted in Fig. 14 in comparison with the ion-loss simulations. The ordinate represents the ratio of ion numbers before and after the full tune ramping from $\nu_0 = 0.4$ to 0.17 . The initial plasma temperature has been set at $T_p = 0.3$ eV in all WARP simulations while four different tune depressions are assumed at the beginning. These results suggest that the effective tune depression η is initially near 0.85 at $N_{in} \approx 10^7$ and changed to about 0.95 at $N_{in} \approx 10^6$. The emittance growth rates calculated from these simulations at $u = 5.48 \times 10^{-4}$ (corresponding to 10-turn extraction from EMMA) are 1.0 for $\eta = 0.99$, 1.5 for $\eta = 0.95$, 3.4 for $\eta = 0.90$ with 5% ion loss, and 3.9 for $\eta = 0.85$ with 23% ion loss. We thus expect that only little deterioration of beam quality will take place in EMMA even with rather strong nonlinearity if the space-charge-induced tune shift is less than a few % of the bare tune.

Since the coherent resonance at $\nu_0 = 1/4$ intrinsically exists in any high-intensity machines even without imperfection fields, it is practically important to figure out how the plasma gets heated at this resonance depending on the crossing speed and initial particle density. The upper panel

FIG. 14. Comparison of measurement data with the corresponding ion-loss simulations. The experimental results in Fig. 10 are replotted with filled circles ($N_{in} \approx 10^7$) and triangles ($N_{in} \approx 10^6$). The ordinate (Ion-Loss Rate) stands for the final ion number divided by the initial. The initial plasma temperature is chosen to be 0.3 eV in all simulations that consider four different values of η , i.e., $\eta = 0.85$ (circle), 0.9 (diamond), 0.95 (triangle), and 0.99 (square).

FIG. 15. Comparison of measurement data with the corresponding ion-loss simulations. Experimental results of the quarter-integer resonance crossing with various initial ion numbers are plotted in the upper panel (a) as a function of the crossing speed u . The tune-sweep range is the same as in Fig. 11(b); namely, from 0.31 to 0.23. The corresponding WARP simulations assuming the initial plasma temperature of 0.3 eV are shown in the lower panel (b).

in Fig. 15 shows the experimental data obtained with various N_{in} 's. Similarly to the previous experiments of single resonance crossing [Fig. 11(b)], the operating tune ν_0 is ramped from 0.31 to 0.23 to see the effect from the quarter-integer stop band. The corresponding WARP simulation results with several different values of η are given in the lower panel, which agree fairly well with the experimental observations. Although the current S-POD system has no emittance monitor, the good agreement between the experimental and numerical results justifies using the simulation data to draw an approximate scaling law of possible emittance growth after the intrinsic linear resonance crossing. Figure 16 shows some examples of the numerically evaluated final emittance at $\nu_0 = 0.23$ normalized by the initial value at $\nu_0 = 0.31$. Since the emittance behavior during resonance crossing is insensitive to the initial plasma temperature unless too many particles escape from the LPT acceptance (see Fig. 7), a lower plasma temperature (0.1 eV) has been assumed here to lighten particle losses. Even then, a considerable amount of ions are lost when a large emittance growth occurs. The open squares and triangles in Fig. 16 indicate numerical data with finite particle losses. We clearly see the linear dependence of the emittance ratio $\kappa_{1/4}$ on the tune depression η in the region where particle losses are negligible. For more information, results of about 1500 systematic WARP simulations with various u and η are summarized in Fig. 17.

FIG. 16. Examples of WARP simulation data for quarter-integer resonance crossing. The emittance growth rate $\kappa_{1/4}$ after crossing the stop band at $\nu_0 \approx 1/4$ is plotted at four different values of u as a function of η . The initial plasma temperature T_p has been set at 0.1 eV in all simulations here to minimize particle losses. $\kappa_{1/4}$ is insensitive to T_p as long as particle losses during resonance crossing are negligible. Open symbols represent the occurrence of finite particle losses. The solid straight lines are fitting results based on the simple function in Eq. (3).

The white line in the picture indicates the border below which finite particle losses are detected during the resonance crossing. Naturally, the emittance growth becomes more severe for a slower crossing speed and/or higher particle density.

FIG. 17. Contour plot of the emittance growth rate on the η - u plane. The final-to-initial emittance ratio $\kappa_{1/4}$ after the quarter-integer resonance crossing is evaluated from systematic WARP simulations and color-coded in η - u parameter space. The initial plasma temperature at $\nu_0 = 0.31$ has been set at 0.1 eV in all simulations. The white solid line indicates the border below which finite particle losses occur due to a large emittance growth before ν_0 reaches the final value 0.23.

FIG. 18. Fitting constant estimation. The parameters η_0 and a in Eq. (3) are evaluated from WARP simulation data as a function of u . The solid lines correspond to the simple fitting functions defined by Eq. (4).

Each solid line in Fig. 16 is a simple linear fitting based on the function

$$\kappa_{1/4} - 1 = a(\eta_0 - \eta), \quad (3)$$

where the fitting constants η_0 and a both depend on the resonance crossing speed u . A similar scaling law has previously been found by Lee [13]. Note that space-charge-induced instabilities are self-limited; namely, the emittance growth is sooner or later stopped because it reduces the plasma density, thus weakening the space-charge interactions. The fitting with Eq. (3) is, therefore, only valid when $\kappa_{1/4}$ is not too large. The parameter η_0 roughly represents the threshold tune depression at which the emittance starts to grow. For the examples in Fig. 16, we have $(a, \eta_0) = (121.9, 0.977)$ for $u = 8 \times 10^{-5}$, $(a, \eta_0) = (73.1, 0.964)$ for $u = 2 \times 10^{-4}$, $(a, \eta_0) = (51.0, 0.949)$ for $u = 4 \times 10^{-4}$, and $(a, \eta_0) = (38.7, 0.925)$ for $u = 8 \times 10^{-4}$. Figure 18 illustrates the u dependence of the two fitting constants. A quick estimate of the critical tune depression η_0 and the slope a can be made by approximating these curves with proper functions of u . The solid line in each panel is a simple trial fitting function; we have here adopted

$$\eta_0 \approx -g_1 \ln u + g_2 \quad \text{and} \quad a \approx h_1 u^{-h_2} \quad (4)$$

with the fitting constants $(g_1, g_2) = (0.028, 0.723)$ and $(h_1, h_2) = (0.896, 0.519)$. We are now able to plot $\kappa_{1/4}$ as a function of the crossing speed u by using the scaling law

FIG. 19. WARP simulation data of quarter-integer resonance crossing. The u dependence of the emittance growth rate $\kappa_{1/4}$ is plotted for four different initial values of η . The initial plasma temperature T_p has been set at 0.1 eV in all simulations here to minimize particle losses. Similarly to the examples in Fig. 16, data points with finite particle losses are indicated by open symbols. The solid lines are obtained from the scaling law in Eq. (3) together with Eq. (4).

in Eq. (3) together with Eq. (4). The solid lines in Fig. 19 show four examples with different values of η while symbols represents WARP simulation results. We observe reasonable agreement between the scaling law and PIC simulation results.

VI. CONCLUDING REMARKS

We have investigated fundamental features of resonance crossing by means of the very compact experimental system “S-POD” developed at Hiroshima University [16,17,22]. Since S-POD provides us with an experimental environment physically equivalent to AG beam transport channels, diverse beam-physics subjects can be studied easily and systematically without the use of large-scale machines. Non-neutral plasmas are produced in the LPT and then confined by rf quadrupole fields under specific conditions. To emulate a typical resonance crossing process in nonscaling FFAG accelerators, the operating tune of the LPT is changed over wide ranges at various speeds and at various plasma densities. Self-consistent multi-particle simulations are also performed for comparison with experimental observations. We have confirmed both experimentally and numerically that the particle loss rate becomes higher for slower resonance crossing.

When a machine holds very high lattice symmetry, three instability regions are identified as shown in Figs. 5 and 8. We conclude that these instabilities are due mainly to the lowest- and second lowest-order resonances rather than to higher-order nonlinear resonances; namely, one of them at $\nu_0 \approx 1/4$ is the linear ($m = 2$) stop band and the other two at $\nu_0 \approx 1/6$ and $1/3$ are the third order ($m = 3$) while

there could exist weak higher-order stop bands overlapping at these tunes. Although this interpretation is contradictory to that in Ref. [13], the coherent resonance criterion in Eq. (2) [29] and other previous works on space-charge-dominated beam transport [1,10,31,38–40] support our conclusion. In addition, the present experimental and numerical results suggest that crossing the nonlinear stop band at $\nu_0 \approx 1/6$ is almost harmless [see Figs. 6 and 11(c)]. Another third-order resonance at $\nu_0 \approx 1/3$ is stronger, but at low particle density, the emittance growth from this resonance crossing should be negligible in the absence of excess nonlinearity due to electrode misalignments (see Fig. 6). The linear stop band at $\nu_0 \approx 1/4$, generated by the lowest-order resonance, is most dangerous and unavoidable at high density unless the tune variation range is sufficiently narrower than 0.25 per single focusing period. As the plasma becomes denser, this stop band grows causing serious emittance growth even at rather high resonance crossing speed [see, e.g., Fig. 11(b)].

Employing 2D WARP simulation data that are in good agreement with the experimental observations, we have derived a simple scaling law of the emittance growth caused by the intrinsic resonance crossing at $\nu_0 \approx 1/4$. As shown in Eq. (3) and in Fig. 16, the emittance ratio $\kappa_{1/4}$ before and after the quarter-integer resonance crossing has approximate linear dependence on the tune depression η . This conclusion is consistent to the previous finding in Ref. [13]. The two fitting parameters can roughly be estimated from Eq. (4) as a function of the resonance crossing speed u .

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