Arbitrary emittance partitioning between any two dimensions for electron beams

Bruce E. Carlsten,¹ Kip A. Bishofberger,¹ Leanne D. Duffy,¹ Steven J. Russell,¹
Robert D. Ryne,² Nikolai A. Yampolsky,¹ and Alex J. Dragt³

¹*Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

²*Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*

³*University of Maryland, College Park, Maryland 20742, USA*

(Received 7 January 2011; published 26 May 2011)

The flat-beam transform (FBT) for round symmetric beams can be extended using the concept of eigenemittances. By tailoring the initial beam conditions at the cathode, including adding arbitrary correlations between any two dimensions, this extension can be used to provide greater freedom in controlling the beam's final emittances. In principle, this technique can be used to generate extraordinarily transversely bright electron beams. Examples are provided where an equivalent FBT is established between the horizontal and the longitudinal beam dimensions.

DOI: 10.1103/PhysRevSTAB.14.050706

PACS numbers: 41.60.-m, 29.27.-a, 41.85.Lc

I. INTRODUCTION

Currently, free-electron lasers (FELs) are the only devices which can be scaled to generate coherent radiation from microwaves to x rays. The technological limit on generating hard x-ray radiation is mainly determined by the availability of transversely bright electron beams. In order for an FEL to lase optimally, the normalized transverse beam emittance must be small, $\varepsilon_n \leq \beta\gamma\lambda_{x\text{-ray}}/4\pi$, to ensure overlap between the electron and x-ray phase spaces, where β and γ are the beam's velocity normalized to the speed of light c and its relativistic factor, respectively, and $\lambda_{x\text{-ray}}$ is the x-ray wavelength. This condition is easier to satisfy at higher beam energies and this approach was used to design Linac Coherent Light Source [1] which generates 8 keV photons using a 15 GeV electron beam.

The next generation hard x-ray FELs (XFELs), producing photons well above 10 keV, cannot use the same approach due to increased beam energy spread caused by the single-particle synchrotron radiation at high energies [2]. At the end of an undulator, this spread is equal to

$$\frac{\Delta E_{\text{rms}}}{E} = \sqrt{\frac{55}{48\sqrt{3}}} \sqrt{\frac{\hbar e^6 B^3}{4\pi\epsilon_0 m^5 c^6}} \gamma L^{1/2}, \quad (1)$$

where B is the rms undulator field strength, L is the undulator length, and e , m , $\hbar$, and ϵ_0 are the electron charge, electron mass, normalized Planck's constant, and free-space permittivity. FEL performance is degraded when the total beam energy spread is greater than the gain parameter. Estimates for the proposed XFEL for Los Alamos' Matter and Radiation in Extremes (MaRIE)

facility [3] generating 50-keV photons (with a 100-meter long undulator with a 2.4-cm period and a 1-T rms field, driven by a 3.4-kA beam) show that the beam energy is limited to about 20 GeV. At this energy both the induced energy spread and the FEL gain parameter are close to 0.015% and the FEL efficiency rapidly drops at higher energies. This limitation on the beam energy puts a constraint on the normalized transverse beam emittance. Numerical simulations for this case indicate that emittances as high as $0.15 \mu\text{m}$ are marginally acceptable (these transverse emittances are about a factor of 2 more relaxed than the optimal constraint, with some corresponding loss of efficiency), and with performance degrading significantly as the emittance is increased above that. In contrast, the longitudinal emittance for a 150-fs bunch and 0.01% energy spread can be as high as $180 \mu\text{m}$.

Using the scaling law for transverse emittances in current high-brightness photoinjectors, $\varepsilon_n \sim 1 \mu\text{m} (q/nC)^{1/2}$, one can find that the required emittance can be achieved for bunch charges q on the order of 20 pC. A peak current of 3.4 kA implies that such a bunch should be about 6 fs ($2 \mu\text{m}$) long and this ultrashort bunch may be severely impacted by three-dimensional effects of coherent synchrotron radiation (CSR) in the compressing chicane. Alternatively, a 500-pC, 3.4-kA bunch would be 150 fs long which is consistent with demonstrated chicane performance and could produce over an order of magnitude more photons.

A 500-pC electron bunch produced with currently available high-brightness photoinjectors would have normalized beam emittances of $\varepsilon_{x,n}/\varepsilon_{y,n}/\varepsilon_{z,n}$ of $0.7/0.7/1.4 \mu\text{m}$, with a total phase-space volume of $0.7 \mu\text{m}^3$. At the same time, the electron bunch required for a 50 keV XFEL must have normalized beam emittances not exceeding $\varepsilon_{x,n}/\varepsilon_{y,n}/\varepsilon_{z,n}$ of $0.15/0.15/180$, with a total phase-space volume of $4 \mu\text{m}^3$. Therefore, currently available photoinjectors can generate bunches with sufficiently small phase-space

Published by the American Physical Society under the terms of the Creative Commons Attribution 3.0 License. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

volumes, but the partitioning of this phase space into longitudinal and transverse emittances is not correct. This concept is shown notionally in Fig. 1, where we plot the number of photons which can be generated in an FEL using an electron bunch from a currently available photoinjector [black line, using the constraints $\varepsilon_n \leq \beta\gamma\lambda_{x\text{-ray}}/2\pi$ and $\varepsilon_n \sim 1 \mu\text{m}(q/\text{nC})^{1/2}$ to determine the bunch charge, capping the maximum bunch charge at 100 pC] and if the optimal partitioning of the bunch phase-space volume between the longitudinal and the transverse emittances could be achieved (red line). For the red line, a bunch charge of 500 pC is used up to about 120 keV photon energies, because the excess transverse emittance can be repartitioned into the longitudinal dimension, after which the maximum allowable bunch charge also scales inversely as the square of the photon energy. For both lines, an electron-beam energy of 20 GeV and an FEL efficiency (from total electron energy to total photon energy) of $\sim 3 \times 10^{-4}$ are assumed, for ease of comparison.

A novel technique for partitioning the phase-space volume suitable for linear colliders and XFELs was recently proposed [4]. This technique relies on the fact that once the beam is generated, three quantities, known as eigenemittances [5], are conserved through the beam line. This fact is a consequence of the symplectic properties of Hamiltonian systems; preservation of these quantities is only limited by nonlinear effects. A large asymmetry in eigenemittances can be set up by introducing cross-dimensional correlations in the beam as it is formed at the cathode surface [4].

FIG. 1. Rough scaling of photons per pulse versus photon energy for conventional technology (black) and from using eigenemittance partitioning (red). The scaling for conventional technology is dashed beyond 10 keV, due to uncertainty in the CSR model's validity for ultrashort bunches in the compressors.

These cross correlations can be removed downstream at high energy, and the eigenemittances can be recovered as the three beam rms emittances resulting in a transversely very bright beam.

The flat-beam transform (FBT) [6–9] has been described in this context [4]. The FBT employs x - y' and y - x' correlations in a symmetric beam to tailor the beam eigenemittances which are recovered as the transverse beam emittances by using a specific optics algorithm that removes the correlations [6,9]. One of the resulting emittances is decreased and another is increased compared to the uncorrelated beam. The total transverse phase-space volume does not change when the correlations are introduced in a FBT, which makes it a flexible option for significant reduction of one eigenemittance. The FBT scheme was demonstrated experimentally [10], achieving one emittance even less than the initial beam thermal emittance. Another example of the eigenemittance concept, now where there are no initial cross correlations, is the emittance exchanger (EEX) [11,12], which swaps the emittance from one transverse dimension with the axial emittance (and also experimentally demonstrated [13]).

Kim proposed an idea of using a temporally very short electron bunch (and thus very low longitudinal emittance), followed by an FBT and an EEX in succession to achieve the low transverse emittances required for an XFEL [14]. For example, consider the case a 200-fs laser generates a beam with $2.1/2.1/0.15 \mu\text{m}$ intrinsic emittances. The FBT adjusts these numbers to $0.15/30/0.15 \mu\text{m}$, and the EEX swaps $\varepsilon_{y,n}$ and $\varepsilon_{z,n}$, to finally yield $0.15/0.15/30 \mu\text{m}$ emittances, reasonably suitable for an XFEL. However, photoinjector design does not scale linearly enough to produce the needed initial emittances, particularly for such a short axial beam size.

Earlier studies of FBTs were limited to the case of an initially symmetric beam. In this paper we generalize the FBT concept which will allow eigenemittance partitioning of more general and asymmetric initial beams, with three important results. First, we present a simple algorithm to calculate eigenemittances from any set of physically realizable correlations between any two dimensions. Second, we show that the optics elements employed for the symmetric FBT case can also recover the eigenemittances for any arbitrary cross-diagonal beam matrix, with only simple retuning of the optics. Third, we present the beam line optics for recovering eigenemittances as rms beam emittances for a generalized FBT which introduces correlations between either x - y or x - z phase-space planes, for any physically realizable beam. In particular, we study a flat-beam transform with x - z correlations (XZFBT) and demonstrate that, among other configurations, an XZFBT with an elliptical cathode can produce transversely bright electron beam.

Section II briefly discusses the choice of canonical variables and beam units used in our analysis to ensure the following development is symplectically consistent.

In Sec. III, theoretical definitions are presented for initial beam correlations, and Sec. IV expands the eigenemittance concept beyond previous work [4] to include all possible correlations and a simple set of formulas to calculate the eigenemittances. After a review of x - y FBTs in Sec. V, explicit configurations for XZFBTs, including the necessary correlation-removing optics, is offered in Sec. VI. A final section discusses practical limitations due to photoinjector nonlinearities and compares correlations coupling all three dimensions, and includes a numerical example for the proposed MaRIE XFEL injector, another for the JLAB 500-kV DC gun, and one using an energy-attenuating foil at higher energy.

II. DEFINITION OF CONSISTENT BEAM VARIABLES

A key concept in this paper is that each eigenemittance is not tied to a specific dimension; while conserved through linear symplectic transformations, they can be ultimately expressed in any dimension depending on the specific beam optics. The invariance under linear symplectic transformations demands that our approach is traceable to canonical variables. In this section, we show that the beam coordinates we use in the following sections are directly traceable to canonical coordinates, and identify that the only approximation used for the beam coordinates, which we will alternatively call the beam trajectory, is the paraxial approximation.

In our analysis we use the arclength s (or the longitudinal coordinate, z , in rectilinear systems) as the independent variable. In Cartesian coordinates, the 6-vector of canonical coordinates and momenta is given by $\mathbf{s}_{\text{can}}^T = (x, p_x, y, p_y, t, p_t)$, where x and y denote transverse Cartesian coordinates, p_x and p_y denote transverse canonical momenta, $\vec{p} = \vec{p}_{\text{mech}} + q\vec{A}$ (where $\vec{p}_{\text{mech}}$ is the

mechanical momenta, q is the charge, and $\vec{A}$ is the vector potential), t is the arrival time at a location s in the beam line, and p_t is the negative of the total energy, $-\gamma mc^2$. Beam dynamics codes often use variables that are dimensionless deviations from a reference trajectory. To avoid notational clutter, we will write the dimensionless deviations as (x, p_x, y, p_y, t, p_t) , derived from the dimensional variables, the design trajectory $\mathbf{s}_{\text{can},0}^T$, and the scaling quantities l, δ , and ω , as will be described in a future paper [15]:

$$\begin{aligned} x &\leftarrow (x - x_0)/l, & p_x &\leftarrow (p_x - p_{x0})/\delta, \\ y &\leftarrow (y - y_0)/l, & p_y &\leftarrow (p_y - p_{y0})/\delta, \\ t &\leftarrow \omega(t - t_0), & p_t &\leftarrow (p_t - p_{t0})/(\omega l \delta). \end{aligned} \quad (2)$$

Because we will use unnormalized quantities we set $\delta = \beta_0 \gamma_0 mc$. Additionally, we use $l = 1$ m and $\omega l/c = -1$. With these choices, and making use of the definitions of p_x , p_y , and p_t , the 6-vector of canonical coordinates and momenta is therefore $\mathbf{s}_{\text{can}}^T = [x, (\gamma \beta_x / \gamma_0 \beta_0), y, (\gamma \beta_y / \gamma_0 \beta_0), c \Delta t, (\Delta \gamma / \gamma_0 \beta_0)]$ in drift regions where the vector potential vanishes, and where now we have identified Δt as a time variable which is positive for the head of the bunch and negative for the tail.

For the purposes of this paper, we transform these canonical variables into the more traditional (noncanonical) formulation used in linac design. Let $x' = dx/dz$ and $y' = dy/dz$. It follows that the unnormalized beam vector can be written in the paraxial approximation as $\mathbf{s}^T = [x, x', y, y', c \Delta t, \Delta(\gamma \beta) / \gamma_0]$, where we have used $\Delta \gamma = \beta \Delta(\gamma \beta) = \gamma \beta \frac{\Delta(\gamma \beta)}{\gamma}$ and kept both the transverse and longitudinal momentum deviations to first order in the small quantities. When these conditions (lowest order paraxial expansion in a drift space) are satisfied, we can construct the following unnormalized beam matrix,

$$\sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle & \langle x y \rangle & \langle x y' \rangle & \langle x(c \Delta t) \rangle & \left\langle x \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle \\ \langle x x' \rangle & \langle x'^2 \rangle & \langle x' y \rangle & \langle x' y' \rangle & \langle x'(c \Delta t) \rangle & \left\langle x' \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle \\ \langle x y \rangle & \langle x' y \rangle & \langle y^2 \rangle & \langle y y' \rangle & \langle y(c \Delta t) \rangle & \left\langle y \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle \\ \langle x y' \rangle & \langle x' y' \rangle & \langle y y' \rangle & \langle y'^2 \rangle & \langle y'(c \Delta t) \rangle & \left\langle y' \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle \\ \langle x(c \Delta t) \rangle & \langle x'(c \Delta t) \rangle & \langle y(c \Delta t) \rangle & \langle y'(c \Delta t) \rangle & \langle (c \Delta t)^2 \rangle & \left\langle (c \Delta t) \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle \\ \left\langle x \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle & \left\langle x' \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle & \left\langle y \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle & \left\langle y' \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle & \left\langle (c \Delta t) \frac{\Delta(\beta \gamma)}{\gamma_0} \right\rangle & \left\langle \left(\frac{\Delta(\beta \gamma)}{\gamma_0} \right)^2 \right\rangle \end{pmatrix}, \quad (3)$$

where the brackets indicate ensemble averages over the entire electron bunch distribution. In the absence of coupling among the phase planes, the rms unnormalized geometrical emittances in this limit follow immediately

by inspection of the 2×2 determinants on the diagonal [where we use the subscript “ z ” to denote quantities associated with the longitudinal (temporal) phase space]:

$$\begin{aligned}\varepsilon_x &= \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} & \varepsilon_y &= \sqrt{\langle y^2 \rangle \langle y'^2 \rangle - \langle yy' \rangle^2} \\ \varepsilon_z &= \sqrt{\langle (c\Delta t)^2 \rangle \left\langle \left(\frac{\Delta(\beta\gamma)}{\gamma_0} \right)^2 \right\rangle - \left\langle (c\Delta t) \frac{\Delta(\beta\gamma)}{\gamma_0} \right\rangle^2}.\end{aligned}\quad (4)$$

In summary, we have used the definition of canonical variables to motivate the unnormalized beam matrix in traditional variables, where $c(\Delta t)$ is the appropriate longitudinal coordinate and $\Delta(\beta\gamma)/\gamma_0$ is the appropriate longitudinal momentum deviation term. [Equivalently, we could have used $\Delta\gamma/(\gamma_0\beta_0)$ but chose this alternative form to simplify the coordinate when the beam is at very low energies near the cathode.]

It is worth recalling that the beam trajectory parameters are not canonical because they do not include the vector potential, so these expressions (in terms of identifying eigenemittances) are only valid in drift regions where the vector potential vanishes. Thus, this formalism is accurate when one is considering the transfer matrix of an optics element from the field-free region before the element to another after the element, as we do in the following analysis. Likewise, they can be used to describe correlations in the beam from magnetic field present at the cathode only after the beam has entered a field-free region.

III. BEAM CORRELATIONS AND TRANSFORMS

In general, the beam matrix transforms through any linear optics from position 1 to position 2 by

$$\sigma_2 = R\sigma_1 R^T, \quad (5)$$

where the transform matrix R transforms an unnormalized particle vector $\vec{\xi}_2 = R\vec{\xi}_1$ and $\vec{\xi}^T = [x, x', y, y', (c\Delta t), \Delta(\beta\gamma)/\gamma_0]$ as shown above. All electrodynamic motion of charged particles in the electromagnetic field is symplectic, so all transfer matrices R obey

$$J_6 = R^T J_6 R, \quad (6)$$

where now

$$J_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \quad (7)$$

Following Kim [8], we use the form of nonsymplectic transformations on an initially diagonal beam matrix to represent initial beam correlations, specifically once the beam is in a field-free region. We start with general correlations, followed by specific symplectic transformations which establish x - y and x - z correlations. In the Appendix, we provide specific x - y transformations and x - z transformations that are needed in subsequent sections.

A. Initial beam correlations

The purpose of this section is to define initial beam correlations in a convenient way, and to establish the most general kind of beam matrices we need to consider for general FBTs. The uncorrelated diagonal beam matrix, σ_0 , defined by Eq. (3), is

$$\sigma_0 = \begin{pmatrix} \sigma_x^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{x'}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{y'}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{z'}^2 \end{pmatrix}, \quad (8)$$

where we have introduced the notation $\sigma_x^2 = \langle x^2 \rangle$, $\sigma_y^2 = \langle y^2 \rangle$, $\sigma_z^2 = \langle (c\Delta t)^2 \rangle$, $\sigma_{x'}^2 = \langle x'^2 \rangle$, $\sigma_{y'}^2 = \langle y'^2 \rangle$, and $\sigma_{z'}^2 = \langle (\frac{\Delta(\beta\gamma)}{\gamma_0})^2 \rangle$. Following Kim [8], suppose that the correlated beam matrix can be established by the following transform [analogous to Eq. (5)]:

$$\sigma_{\text{corr}} = (I + C)\sigma_0(I + C)^T, \quad (9)$$

where C is any specifically chosen correlation matrix and $\det(I + C) = 1$. In this paper, we focus on two-dimensional couplings, so at this point we will mostly restrict our analysis to 4×4 beam and transfer matrices, where we consider alternatively x - y or x - z phase spaces.

Entries in the correlation matrix establish beam correlations, but the addition of single correlations does not change the determinant of the beam matrix. They also define which beam element depends on which other one. For example, consider the two x - y correlation matrices

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & c_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (10)$$

and

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_{14} & 0 & 0 & 0 \end{pmatrix}. \quad (11)$$

These correlations lead to these correlated beam matrices:

$$\sigma_{C1} = \begin{pmatrix} \sigma_x^2 + c_{14}^2 \sigma_{y'}^2 & 0 & 0 & c_{14} \sigma_{y'}^2 \\ 0 & \sigma_{x'}^2 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & 0 \\ c_{14} \sigma_{y'}^2 & 0 & 0 & \sigma_{y'}^2 \end{pmatrix} \quad (12)$$

and

$$\sigma_{C2} = \begin{pmatrix} \sigma_x^2 & 0 & 0 & c_{41}\sigma_x^2 \\ 0 & \sigma_{x'}^2 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & 0 \\ c_{41}\sigma_x^2 & 0 & 0 & \sigma_{y'}^2 + c_{41}^2\sigma_x^2 \end{pmatrix}. \quad (13)$$

In σ_{C1} , the horizontal beam size σ_x^2 is increased through a $\sigma_{y'}^2$ contribution, while in σ_{C2} , the $\sigma_{y'}^2$ term is increased through a σ_x^2 contribution. It is easy to show, however, that both beam matrices have determinants equal to the uncorrelated beam matrix.

Correlations can be stacked in a multiplicative manner, so that the final correlation matrix can be given as a product of n individual correlation matrices:

$$I + C_{\text{total}} = \prod_{i=1}^n (I + C_i). \quad (14)$$

If the correlations are decoupled, the correlation coefficients from the C_i matrices reappear in C_{total} . For example, an axial magnetic field on the cathode is the first step of an x - y FBT, which creates x - y' and x' - y correlations once the beam enters a field-free region. The total correlation matrix (within the two dimensions) becomes

$$\begin{aligned} I + C_{\text{total}} &= \left[I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 \end{pmatrix} \right] \left[I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \\ &= \left[I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 \end{pmatrix} \right]. \end{aligned} \quad (15)$$

The beam matrix, at a waist outside of the axial field, can then be written:

$$\sigma_{\text{axial field}} = \begin{pmatrix} \sigma_x^2 & 0 & 0 & -a\sigma_x^2 \\ 0 & \sigma_{x'}^2 + a^2\sigma_y^2 & a\sigma_y^2 & 0 \\ 0 & a\sigma_y^2 & \sigma_y^2 & 0 \\ -a\sigma_x^2 & 0 & 0 & \sigma_{y'}^2 + a^2\sigma_x^2 \end{pmatrix}, \quad (16)$$

where now $a = \frac{e}{2\gamma\beta mc} B_{\text{cath}} (R_{\text{cath}}/R_{\text{beam}})^2$ and $R_{\text{cath}}/R_{\text{beam}}$ is the ratio of the beam size at the cathode to the beam size at the waist. These correlations are antisymmetric (opposite signs), and the angular momentum can be represented by $\mathbf{L} = |\langle xy' - yx' \rangle|/2 = |a|(\sigma_x^2 + \sigma_y^2)/2$. Also, the correlations in Eq. (15) were written as if y' and x' are functions of x and y , not vice versa.

For more general correlations, coupling occurs. Consider a reasonably general family of two-dimensional correlation given by x' being a function of y , y' being a function of x , and either x or y being a function of the other dimension (it is hard to imagine a practical system when any variable depends on either x' or y' , especially one of these on the other). These three correlations lead to a significantly more complex beam matrix. We will consider both the cases x depends on y and the reverse.

B. y , x' , and y' depend on x , y , and x , respectively

We shall transform an initial particle vector (x_i, x'_i, y_i, y'_i) to a final particle vector (x_f, x'_f, y_f, y'_f) such that $y_f = y_i + dx_i$, $y'_f = y'_i + bx_i$, and $x'_f = x'_i + ey_f$. After combining the first two, adding the third yields

$$\begin{aligned} I + C_{\text{total}} &= \left[I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \end{pmatrix} \right] \left[I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \\ &= \left[I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ ed & 0 & e & 0 \\ d & 0 & 0 & 0 \\ b & 0 & 0 & 0 \end{pmatrix} \right]. \end{aligned} \quad (17)$$

An initially diagonal 4×4 beam matrix

$$\sigma_0 = \begin{pmatrix} \sigma_x^2 & 0 & 0 & 0 \\ 0 & \sigma_{x'}^2 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & 0 \\ 0 & 0 & 0 & \sigma_{y'}^2 \end{pmatrix} \quad (18)$$

becomes

$$\sigma_{C: y(x)} = \begin{pmatrix} \sigma_x^2 & ed\sigma_x^2 & d\sigma_x^2 & b\sigma_x^2 \\ ed\sigma_x^2 & \sigma_{x'}^2 + e^2d^2\sigma_x^2 + e^2\sigma_y^2 & ed^2\sigma_x^2 + e\sigma_y^2 & edb\sigma_x^2 \\ d\sigma_x^2 & ed^2\sigma_x^2 + e\sigma_y^2 & \sigma_y^2 + d^2\sigma_x^2 & db\sigma_x^2 \\ b\sigma_x^2 & edb\sigma_x^2 & db\sigma_x^2 & \sigma_{y'}^2 + b^2\sigma_x^2 \end{pmatrix}. \quad (19)$$

C. x , x' , and y' depend on y , y , and x , respectively

We repeat the same procedure, but change the first correlation to $x_f = x_i + dy_i$. The final beam matrix now becomes

$$\sigma_{C: x(y)} = \begin{pmatrix} \sigma_x^2 + d^2 \sigma_y^2 & ed\sigma_y^2 & d\sigma_y^2 & bd^2 \sigma_y^2 + b\sigma_x^2 \\ ed\sigma_y^2 & \sigma_{x'}^2 + e^2 \sigma_y^2 & e\sigma_y^2 & edb\sigma_y^2 \\ d\sigma_y^2 & e\sigma_y^2 & \sigma_y^2 & db\sigma_y^2 \\ bd^2 \sigma_y^2 + b\sigma_x^2 & edb\sigma_y^2 & db\sigma_y^2 & \sigma_{y'}^2 + b^2 d^2 \sigma_y^2 + b^2 \sigma_x^2 \end{pmatrix} \quad (20)$$

which is also quite complicated. More interestingly, every term in $\sigma_{C: x(y)}$ is different than the equivalent term in $\sigma_{C: y(x)}$, even for the special case of $d = 1$. In general, we use correlations between x and z too, so in the following we generalize the notation where σ_x , $\sigma_{x'}$, σ_y , and $\sigma_{y'}$ are replaced by σ_1 , σ_2 , σ_3 , and σ_4 , respectively.

None of the correlated beam matrices are irreducible in terms of symplectic transformations; they can always be transformed to a diagonal beam matrix with the eigenemittances appearing on the diagonal (and likewise back to any form with an appropriate grouping of correlations). The form of the beam matrices in Eqs. (19) and (20) can be considered completely general and irreducible using uncoupled transfer matrices (where the XY and YX submatrices are zero).

A collection of symplectic transfer matrices needed for recovering the beam eigenemittances are supplied in the Appendix and which will be used in the following three sections. The key observations in the Appendix are that the x - z counterparts of x - y optics elements are a chicane for a drift, an rf cavity for a quadrupole, and a transversely deflecting rf cavity for a skew quadrupole.

IV. EIGENEMITTANCES

The power of the eigenemittance concept is the following: by establishing a desirable set of eigenemittances when the beam is generated (by incorporating clever beam correlations), they will be the observed emittances after all correlations are removed. A corollary to this statement is that the eigenemittances are unaffected by the specific transport before these correlations are removed, including acceleration (assuming irresolvable higher-order issues remain small). The demonstration of the FBT transform at the A0 photoinjector at Fermilab has shown that, at least in certain cases, these higher-order concerns can be kept small [10].

For any beam matrix, the eigenemittances are solvable through several different techniques. Three different methods are presented here: the first is the most general and provides an explicit representation of the required transfer matrix, while the last is the most straightforward and uses matrix multiplication and the quadratic formula to determine the eigenemittances.

A. Dragt's method

The eigenemittances and the corresponding transfer matrix M which diagonalizes the beam matrix σ can be found using the following Dragt's algorithm implemented in the MARYLIE code [16]. First, we evaluate the matrix

$A = e^{\lambda J_6 \sigma}$ using its Taylor series, where λ is some small scalar making the norm of the matrix A close to unity. Then the matrix A is symplectic having all the eigenvalues on the unit circle. We construct the transfer matrix M from the eigenvalues of A which transforms A into the normal form, $N = M^{-1}AM$ [17]. The resulting matrix M transforms the original beam matrix into diagonal form $\sigma_{\text{diag}} = M\sigma M^T$ [17]. The matrix M represents the transfer matrix of the entire “unraveling” process, and, in principle, can be decomposed into a series of required beam-optics components.

B. Eigenvalue equation

Alternatively, the eigenemittances ε_{eig} of the beam matrix σ can be found through solving the eigenvalue problem [17],

$$\det(J_6 \sigma - i \varepsilon_{\text{eig}} I) = 0, \quad (21)$$

where I is the unit matrix, although the corresponding transfer matrix which diagonalizes the beam matrix remains unknown. Equations (5) and (6) in Ref. [4] are a special transformation where the correlations are of the form

$$s = (I + C)s_0, \quad C = \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}, \quad (22)$$

where A and B are 2×2 matrices and we are now considering either the x - y or x - z phase spaces. Note this is not of the most general form, Eqs. (19) and (20), but in the form of what we define as a nonsymmetric cross-diagonal FBT. The transform in Eq. (22) results in the subsequent beam matrix σ related to the intrinsic beam matrix σ_0 as in Eq. (9). Reference [4] shows that the constraint $\det(AB) = \text{Tr}(AB) = 0$ must hold, and that the eigenemittances of the modified beam matrix σ can be found by solving the 4×4 characteristic equation $\det(J_4 \sigma - i \varepsilon_{\text{eig}} I) = 0$. The specific form of C yields the following characteristic equation:

$$\varepsilon_{\text{eig}}^4 - \varepsilon_{\text{eig}}^2 [\varepsilon_{12}^2 + \varepsilon_{34}^2 + Q + (\varepsilon_{12}^2 \det B^2 + \varepsilon_{34}^2 \det A^2)] + \varepsilon_{12}^2 \varepsilon_{34}^2 = 0, \quad (23)$$

where $Q = \sigma_1^2 \sigma_3^2 (a_{21}^2 + b_{21}^2) + \sigma_1^2 \sigma_4^2 (a_{11}^2 + b_{22}^2) + \sigma_2^2 \sigma_3^2 (a_{22}^2 + b_{11}^2) + \sigma_2^2 \sigma_4^2 (a_{12}^2 + b_{12}^2)$, $\varepsilon_{12}^2 = \sigma_1^2 \sigma_2^2$ and $\varepsilon_{34}^2 = \sigma_3^2 \sigma_4^2$ are the uncorrelated beam emittances (since σ_0 has no correlations), and a_{ij} and b_{ij} are the elements of the matrices A and B , respectively.

The biquadratic equation (23) defines two eigenemittances which differ from the intrinsic emittances ε_{12} and ε_{34} of the initially diagonal beam matrix σ_0 . There are several important properties caused by the cross-correlation matrix C . First, the product of two eigenemittances remains the same, which directly follows from the preservation of the phase-space density, $\det(I + C) = 1$. Also, the sum of the two eigenemittances grows with the magnitude of any correlation since $Q > 0$ and $\det A^2, \det B^2 \geq 0$. This property states that the maximum-to-minimum eigenemittance ratio can only grow when cross correlations in the beam matrix are introduced. As a consequence, these schemes cannot be used for creating a bunch with equal emittances. At the same time, these schemes can be used for reducing the smallest eigenemittance which can be beneficial for producing electron bunches with ultrahigh transverse brightness.

C. Application of conserved moments

Using the two conservation properties of four-dimensional symplectic systems, the conservation of the determinant and of the trace invariant $-\frac{1}{2} \text{Tr}(J\sigma J\sigma)$ [18], Kim [8] found for the special case where $\sigma_1 = \sigma_3$ and the initial uncoupled emittances are equal $\varepsilon_{12} = \varepsilon_{34} = \varepsilon_0$, and there is an axial magnetic field at the cathode as in Eq. (16), that the eigenemittances are given by an expression equivalent to $\varepsilon_{\text{eig},\pm}^2 = \varepsilon_0^2 + 2a^2\sigma_1^4 \pm \sqrt{4a^2\sigma_1^4\varepsilon_0^2 + 4a^4\sigma_1^8}$, which can be derived from Eq. (23) and is used in the next section.

The more general cases, defined in Eqs. (19) and (20), are studied next. The nonzero $\langle xx' \rangle$ and $\langle yy' \rangle$ correlations are inconvenient, so we can eliminate them by first transforming through focusing elements defined by

$$R_{\text{diag}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\sigma_{C,12}/\sigma_{C,11} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\sigma_{C,34}/\sigma_{C,33} & 1 \end{pmatrix}, \quad (24)$$

where the matrix element values are from the correlated beam matrices in Eqs. (19) and (20). Those correlated beam matrices are transformed to a matrix of the form

$$\sigma_{\text{beam}} = \begin{pmatrix} \bar{\sigma}_1^2 & 0 & D & B \\ 0 & \bar{\sigma}_2^2 & E & F \\ D & E & \bar{\sigma}_3^2 & 0 \\ B & F & 0 & \bar{\sigma}_4^2 \end{pmatrix}. \quad (25)$$

Conservation of the determinant and the trace invariant leads to eigenemittances satisfying

$$\varepsilon_{\text{eig},\pm}^2 = U \pm V, \quad (26)$$

where

$$U = \frac{1}{2}(\bar{\sigma}_1^2\bar{\sigma}_2^2 + \bar{\sigma}_3^2\bar{\sigma}_4^2 - 2BE + 2FD) \quad (27)$$

and

$$V^2 = \frac{1}{4}(\bar{\sigma}_1^2\bar{\sigma}_2^2 + \bar{\sigma}_3^2\bar{\sigma}_4^2 - 2BE + 2FD)^2 - (\bar{\sigma}_1^2\bar{\sigma}_2^2\bar{\sigma}_3^2\bar{\sigma}_4^2 - F^2\bar{\sigma}_1^2\bar{\sigma}_3^2 - E^2\bar{\sigma}_1^2\bar{\sigma}_4^2 - D^2\bar{\sigma}_2^2\bar{\sigma}_4^2 - B^2\bar{\sigma}_2^2\bar{\sigma}_3^2 + D^2F^2 + E^2B^2 - 2EBDF). \quad (28)$$

It is convenient to use these equations to solve for the eigenemittances numerically given correlations of the forms shown in Eqs. (19) and (20). Several nominal numerical results are plotted in Fig. 2. In these plots, the correlations b , e , and f are the most general set of correlations, defined by Eq. (17). The correlations b and e correspond to the cross-diagonal correlations as in a standard FBT, and are equal in magnitude but opposite in sign for the case of an axial magnetic field and a symmetric FBT. If they are equal in magnitude and sign, they form a symplectic correlation instead of a nonsymplectic correlation (for example, a skew quadrupole if in x - y or a transversely deflecting rf cavity in x - z). The correlation d refers to a direct correlation between the coordinates (a rotated ellipse either in x - y or in x - z).

In Figs. 2(a) and 2(b), the original uncorrelated diagonal beam-matrix elements $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ [as used in Eqs. (19) and (20)] are varied, keeping the product of the emittances $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$, and for $b = 1, e = -1$ [Fig. 2(a)] and $b = 1, e = 1$ [Fig. 2(b)]. The four cases in either plot correspond to different initial conditions: (i) $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$ (solid line); (ii) $\sigma_1 = \sqrt{0.5}, \sigma_3 = \sqrt{2}, \sigma_2 = \sigma_4 = 1$ (dot-dashed line); (iii) $\sigma_1 = \sigma_2 = \sqrt{0.5}, \sigma_3 = \sigma_4 = \sqrt{2}$ (dashed line); and (iv) $\sigma_1 = 0.5, \sigma_3 = 2, \sigma_2 = \sigma_4 = 1$ (short dashed line). These combinations were picked to consider an initially symmetric beam (i), a beam with an initial emittance ratio of 2 (ii), and two beam with initial emittance ratios of 4 (iii) and (iv). These plots show that the sign of d is not important, a fact proven by expanding B, D, E , and F in terms of d . Increasing the magnitude of d increases the eigenemittance ratio, as previously discussed. Although the $b = e = 0$ case is not plotted, it is identical to Fig. 2(b), showing that symplectic correlations like $b = 1, e = 1$ can be ignored. The effect of the nonsymplectic correlation $b = 1, e = -1$ by comparing the two plots at $d = 0$ is very apparent. For large d , the initial b and e do not matter much. Also, a small σ_1 [case (ii)] or large σ_2 [case (iv)] suppresses the effect of the correlation d , as one would expect. A final note is that Fig. 2(b), where the only effective correlation is d , is equivalent to simply rotating the coordinate system.

Figure 2(c) represents the asymmetric FBT case, where $b = 1, e = -1, d = 0$, and the initial beam shape is varied. For both curves, the divergences $\sigma_2 = \sigma_4 = 1$. For the solid line, $\sigma_3 = 1$ and σ_1 is decreased, causing the change in the initial emittance ratio. For the dot-dashed line, σ_1 is decreased while σ_3 is increased, keeping the product $\sigma_1\sigma_3 = 1$. In other words, the solid line

FIG. 2. Numerical modeling of the final emittance ratio for a variety of beam-matrix elements: (a) varying d while setting $b = 1$, $e = -1$; (b) varying d while setting $b = 1$, $e = 1$; (c) final emittance ratio for a nonsymmetric FBT, varying the initial emittance ratio.

corresponds to the case that a round initial beam is truncated to an ellipse, losing charge and the product of the initial emittance decreases. The dot-dashed line corresponds to an initial beam shape that is squished in one dimension but grows in the other, keeping the total charge and product of the initial emittances constant. For the purposes of maximizing the x-ray flux from an XFEL,

the dot-dashed curve is more relevant, but the solid curve shows an unexpected decrease in the ratio of final emittances as the ratio of initial emittances is increased.

At this point, we have accomplished the first of the main results of this paper—to provide a simple algorithm to calculate the eigenemittances from arbitrary correlations between any two dimensions.

V. CONVENTIONAL x - y FBT

In a FBT, an axial magnetic field is applied at the location of the beam cathode, giving the beam canonical angular momentum and generating the previously discussed eigenemittances. When the beam exits the solenoid, the beam begins to rotate (shear), with b and e type correlations. This rotation leads to a large apparent emittance growth, added in quadrature to the beam's intrinsic transverse emittances. Three skew quadrupoles then eliminate the rotation, thereby removing the apparent emittance contribution. However, the b and e correlations successfully split the eigenemittances as described in the previous section, so once all correlations are removed with three skew quadrupoles, one emittance is lower than an equivalent-but-unmagnetized beam would be, while the other emittance is larger (making the product the same). A schematic of this approach is shown in Fig. 3.

After the beam exits the magnetic pole piece, the beam matrix is (assuming the beam is at a waist), from Eq. (16),

$$\sigma_{\text{FBT}} = \begin{pmatrix} \sigma_x^2 & 0 & 0 & -a\sigma_x^2 \\ 0 & \sigma_{x'}^2 + a^2\sigma_x^2 & a\sigma_x^2 & 0 \\ 0 & a\sigma_x^2 & \sigma_x^2 & 0 \\ -a\sigma_x^2 & 0 & 0 & \sigma_{x'}^2 + a^2\sigma_x^2 \end{pmatrix}, \quad (29)$$

where, for a standard FBT, we assume $\sigma_x = \sigma_y$ and $\sigma_{x'} = \sigma_{y'}$. The “intrinsic emittance,” defined as $\varepsilon_0 = \sigma_x \sigma_{x'}$, is that emittance that the beam would possess if it were generated without any correlations, that is, without any axial magnetic field. With the field, the observed emittance at the exit of the solenoid is $\varepsilon_{\text{beam}} = \sqrt{\varepsilon_0^2 + \mathbf{L}^2}$, where as before, $\mathbf{L} = (e|B_{\text{cath}}|R_{\text{cath}}^2)/(8\gamma\beta cm) = |a|\sigma_x^2 = \frac{1}{2}|\langle xy' - yx' \rangle|$. After the skew quadrupoles, the eigenemittances

$$\varepsilon_{\text{eig},-} = \frac{\varepsilon_0^2}{2\mathbf{L}} \quad \varepsilon_{\text{eig},+} = 2\mathbf{L} \quad (30)$$

are recovered in the limit $\mathbf{L}^2 \gg \varepsilon_0^2$.

Note that since the $\sigma_{XY} = \begin{pmatrix} \langle xy \rangle & \langle xy' \rangle \\ \langle x'y \rangle & \langle x'y' \rangle \end{pmatrix}$ submatrix is constrained by the conservation of canonical angular momentum and $\langle xy \rangle = \langle x'y' \rangle = 0$ for an axisymmetric beam, the matrix σ_{XY} cannot change until the first skew quadrupole. Any effect on the beam matrix due to axisymmetric nonlinearities in the electron diode (or photoinjector for a more practical case) can only occur in the σ_{XX} and σ_{YY} submatrices. Because of this, the beam emittances before the first skew quadrupole can evolve arbitrarily, but an effective intrinsic emittance ε_0 consistent with Eq. (30) can always be defined by $\varepsilon_0 = \sqrt{\varepsilon_{\text{beam}}^2 - \mathbf{L}^2}$. Thus, the FBT formulas always apply. If there is no emittance growth in the electron diode, the intrinsic emittance refers to the actual intrinsic thermal emittance during the production of the electron beam. If nonlinearities are present, the intrinsic emittance more closely represents the emittance that grows under the presence of the nonlinear fields, for the case of no applied axial magnetic field. However, the axial field will modify the emittance growth, and, although Eq. (30) is valid, it may be hard to predict what ε_0 actually is for practical designs. Despite that complexity, in [10] the lower measured eigenemittance value was smaller than even the beam's thermal emittance from the cathode, a significant verification of this technique.

Earlier work [7] provided the needed symplectic transfer matrices to recover the eigenemittances. A specific class of optics solutions for the symplectic flat-beam transformation was found in [9] based on [6], for the condition the beam is at a waist at the location of the first skew quadrupole. In [9], the coordinates of a particle with zero intrinsic transverse divergence is followed through the skew quadrupoles and drifts, and the quadrupole strengths and drift lengths are found by setting the final horizontal particle position and divergence to both vanish at the end of the final skew quadrupole, which, although do not exactly recover the eigenemittances (similar to the solution in [6]), lead to a nearly exact recovery, where the increase in the product of the final emittances goes as $\varepsilon_0^2/\mathbf{L}^2$.

FIG. 3. Conventional x - y FBT configuration—initial beam correlations are established at the cathode with an applied axial magnetic field, altering the beam's eigenemittances, and these new eigenemittances are recovered through three skew quadrupoles.

Let us assume that the first two skew quadrupoles are separated by a distance M and the third quadrupole is a distance L after the second one. We use a normalized magnetic field parameter $a = \frac{e}{2\gamma\beta mc} B_{\text{cath}}(r_{0,\text{edge}}/r_{\text{edge}})^2$ and normalized quadrupole strengths of

$$b = \frac{e}{\gamma\beta mc} \int B'_{\text{quad},1} dl \quad c = -\frac{e}{\gamma\beta mc} \int B'_{\text{quad},2} dl$$

$$d = \frac{e}{\gamma\beta mc} \int B'_{\text{quad},3} dl, \quad (31)$$

for each of the three quadrupoles (and where b , d , and e no longer refer to correlations). Using $e = b - a$ and $f = b + a$ we find three unique constraints on the quadrupole and magnetic field strengths [9]:

$$1 = MLfc \quad Mc = e(L + M)$$

$$c = d + e(1 - MLdc) \quad (32)$$

and six parameters we can chose to satisfy them (L , M , a , b , c , and d). Following [9], a convenient simplification can be made if we assume the quadrupole separations are equal ($M = L$) and if we constrain the focal length of the middle quadrupole to be twice the quadrupole separation ($Lc = 1/2$). These constraints determine all the other parameters, which are now $c = \frac{1}{2L}$, $d = \frac{4}{7}c$, $e = \frac{1}{2}c$, and $f = 4c$, which in turn also give $a = -\frac{7}{4}c$ and $b = \frac{9}{4}c$. These solutions are the actual field and field gradient values for the axial field at the cathode and in the quadrupoles.

Written this way, these results are trivial to generalize for any arbitrary initial beam as long as it is both at horizontal and vertical waists and the correlations are only between x - y' and y - x' , as for an asymmetric FBT. In that case, if we let $\langle x'y' \rangle = u\sigma_y^2$ and $\langle xy' \rangle = v\sigma_x^2$, the same equations hold, but now with $e = b + u$ and $f = b + v$. In general, it may not be possible to pick the correlations u and v , so they need to be independent variables. Because there are only three constraints, b can also be picked as some convenient value, such that both e and f are positive, and we can let $M = L$ again for convenience. In that case, L , c , and d are found to be

$$L = \frac{1}{\sqrt{2fe}}, \quad c = 2e, \quad \text{and} \quad d = \frac{e}{1 - 2L^2e^2}. \quad (33)$$

It is important to note that the standard FBT optics can be used for a beam if u and v are negatives of each other (where $u = -v = a$ as before), or consequently, for a beam of any size or ellipticity if there is an axial magnetic field at the cathode. Additionally, using Eqs. (26)–(28), we find that

$$\varepsilon_{\text{eig},\pm}^2 = \varepsilon_0^2 \left(\frac{1 + R^2}{2} \right) + 2a^2 R^2 \sigma_x^4 \pm \left[\varepsilon_0^4 \left(\frac{1 - R^2}{2} \right)^2 + 2a^2 R^2 \sigma_x^4 \varepsilon_0^2 (1 + R^2) + 4a^4 R^4 \sigma_x^8 \right]^{1/2} \quad (34)$$

are the eigenemittances of an initially elliptical beam with axial field on the cathode, and where the uncorrelated emittances are ε_0 and $R\varepsilon_0$, the horizontal beam size at the cathode is σ_x , and the vertical beam size at the cathode is $\sigma_y = R\sigma_x$. Note in the limit that the angular momentum term dominates ($|a|R\sigma_x^2 \gg \varepsilon_0$), the eigenemittances are then $\varepsilon_{\text{eig},+} = 2|a|R\sigma_x^2$ and $\varepsilon_{\text{eig},-} = \varepsilon_0^2/2|a|\sigma_x^2$, which reduce to the usual FBT case, Eq. (30), when $R = 1$. It is interesting to note, in this limit and for the case of a beam where the horizontal size is increased and the vertical size is decreased (such that $R \leq 1$, $\sigma_x = \sigma_t/\sqrt{R}$, $\sigma_y = \sqrt{R}\sigma_t$, and the product of the uncorrelated emittances stays constant, $\varepsilon_x = \varepsilon_t/\sqrt{R}$ and $\varepsilon_y = \sqrt{R}\varepsilon_t$), that the eigenemittances are $\varepsilon_{\text{eig},+} = 2|a|\sigma_t^2$ and $\varepsilon_{\text{eig},-} = \varepsilon_t^2/(2|a|\sigma_t^2)$, which are independent of the aspect ratio R . Note that the ratio of the eigenemittances is given by $\varepsilon_{\text{eig},+}/\varepsilon_{\text{eig},-} = 4a^2\sigma_t^4/\varepsilon_t^2$.

Here we have accomplished the second goal of this paper—we have shown that the optics scheme used for a standard FBT can be used to recover the eigenemittances for any cross-diagonal beam matrix, where an initial beam has arbitrary nonzero moments $\langle x'y' \rangle$ and $\langle xy' \rangle$, but $\langle xy \rangle = \langle x'y' \rangle = 0$, and the beam is at a horizontal and a vertical waist, only possibly requiring retuning the first skew quadrupole. We have also provided an exact expression of the eigenemittances of an elliptical beam in an axial magnetic field.

VI. XZFBT DESIGNS

In this section we provide an algorithm to find the optics needed for nearly exactly recovering the eigenemittances from any general two-dimensional correlations, in a more general sense than the algorithm in the previous section. We also provide some examples of photoinjector and other geometries that can generate these types of correlations.

A. Optics scheme for recovering eigenemittances for general two-dimensional correlations

We find it convenient to have the beam correlations in the form

$$R_{\text{ideal correl}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \lambda & 0 & 1 & 0 \\ \zeta & 0 & 0 & 1 \end{pmatrix} \quad (35)$$

(or its equivalent in terms of x as a function of y or z instead of y or z as a function of x) to simplify the algebra for diagonalizing the beam matrix. We will use the same procedure employed in the previous section to determine parameters for three additional skew quadrupoles (or transverse rf cavities) to nearly exactly diagonalize the beam matrix. There are an infinite number of actual optics schemes that can do this; this will be only

one representation. For significant initial correlations, the increase in the product of the emittances will be tiny, as in [6,9].

It is trivial to generate the correlation matrix in Eq. (35) starting with one of the form shown in Eq. (19), by using a coupling transfer matrix of the form shown in Eq. (A8) or Eq. (A11), where a in those matrices equals $-e$ from Eq. (19). Alternatively, the correlation matrix might have the slightly different form

$$R_{\text{correl}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta & 1 & \alpha & 0 \\ \lambda & 0 & 1 & 0 \\ \delta & 0 & 0 & 1 \end{pmatrix} \quad (36)$$

which is equivalent to the form in Eq. (17) under simple, uncoupled, symplectic transformations. This correlation matrix transforms into the one shown in Eq. (35) after first transforming the beam through a coupling matrix of the form shown in Eq. (A8) or Eq. (A11) where now $a = -\alpha$ and then through a horizontal focusing element where the inverse focal length is $f = -\beta + \alpha\lambda$.

Now we consider a particle with initial vector $(x, x', z, z') = (x_0, 0, z_0 + \lambda x_0, \zeta x_0)$ using the beam correlation shown in Eq. (35). Again, we are assuming a particle with no intrinsic “divergence” because the FBT constraint will require that this particle is mapped to a final horizontal position and divergence that are zero. If we let this particle pass through three transverse rf cavities of normalized strengths b , c , and d in that order, with a drift of length L between the first two and M between the last two (to keep all the notation the same as in the previous optics cases), the final horizontal position and divergence are

$$\begin{aligned} x_f &= x_0 \{1 + Lb\lambda + Mb\lambda + Mc[\lambda + L(b + \zeta)]\} \\ &\quad + z_0 [bL + M(b + c)] \\ x'_f &= x_0 \{b\lambda + c[\lambda + L(b + \zeta)] + d[\lambda + (L + M)(b + \zeta)] \\ &\quad + cM(1 + bL\zeta)\} + z_0 [b + c + d(1 + bcLM)]. \end{aligned} \quad (37)$$

Setting the four coefficients to zero leads to only three unique constraints (as before, one can be derived from the other four), which can be written as

$$\begin{aligned} 0 &= 1 + cLM(b + \zeta) & 0 &= bL + bM + cM \\ 0 &= (b + c + d) + bcdLM. \end{aligned} \quad (38)$$

Surprisingly, λ has dropped out from these equations, so the optics are independent of the x - z coupling (or x - y coupling for that case). We have five parameters to solve these three equations, with ζ given. As before, it is convenient to separate the rf cavities equally ($L = M$), which reduces the three constraints to

$$L^2 = \frac{1}{2b(b + \zeta)} \quad c = -2b \quad d = \frac{b(b + \zeta)}{\zeta}. \quad (39)$$

Picking $b = \zeta$ again for convenience (our last free parameter), the optics for a general XZFBT are fully defined by $b = \zeta$, $c = -2\zeta$, $d = 2\zeta$, and $L = M = 1/2\zeta$, where the rf field strength is found from these normalized coefficients using Eq. (A11).

In the special case $\zeta = 0$, L cannot equal M , but any other relation between L and M works. For example, we can choose $M = 2L$, in which case the solution is given by $c = -3b/2$, $L = 1/b\sqrt{3}$, and $d = b/4$, where b can be chosen as any convenient value.

B. Examples of correlations for XZFBTs

First, it is useful to consider what an x - z correlation means. If the coordinates were x and y , the eigenemittances are trivial—it is just the beam emittances for a nonrotated beam in a rotated coordinate system, $(\tilde{x}_1, \tilde{x}_2)$. But this is not true for an initially correlated x - z beam because the x and z velocities are not isotropic—we cannot just rotate an x - z correlated beam to identify and recover the eigenemittances.

In [4], we considered creating the needed x - z correlations with a drive laser pulse hitting a vertical cathode surface at some angle or if the laser pulse acquires a time delay depending on its transverse coordinate. In order to determine the nonsymplectic correlations introduced in the beam matrix we consider two laser pulses having the same total spot size and the same duration at a given transverse position as illustrated in Fig. 4. The coordinates inside the pulses are related as

$$x = x_0, \quad c\Delta t = c\Delta t_0 + x_0 \tan\theta, \quad (40)$$

where x_0 and Δt_0 correspond to the coordinates of the nontilted pulse and θ is the tilt angle of the laser pulse.

To condition the beam matrix to recover the beam eigenemittances, we assume either an identity transform through the photoinjector or one of the form shown in Eq. (A15), and use the procedure described at the start of this section. Note for this special case (just a correlation between x and z), the optics do not depend on the amplitude of the correlations at all. The optics configuration for

FIG. 4. The correspondence of the photon positions between the normal and tilted incident laser pulses.

this case would look very similar to Fig. 3, where the cathode is illuminated by a tilted cathode and there would be four transversely deflecting rf cavities uniformly spaced (instead of three skew quadrupoles as in Fig. 3).

Alternative initial beam correlations that can be used for an XZFBT include recessing the cathode at an angle with an applied vector potential from an external wiggler field and using a photocathode with a work function that changes linearly across the cathode surface.

It is also possible to consider a two-stage FBT/XZFBT hybrid, where correlations are imposed on the beam at higher energy, using a nonsymplectic beam line element. A round beam at the cathode with a standard FBT can be used to transfer emittance from one transverse dimension to the other. Having the beam generate an energy correlation with transverse position in a downstream element could impose an x - z' coupling that in turns moves the emittance from that second transverse plane with the larger subsequent emittance to the longitudinal plane. The nonsymplectic element may also increase the beam's energy spread and emittance, leading to a complex performance tradeoff. If the initial transverse emittance and energy spread are labeled by *initial*, the induced transverse emittance growth (added in quadrature) and energy spread are labeled by *ind*, and the rms induced energy slew correlated with transverse position is $(\frac{\Delta\gamma}{\gamma})_{\text{slew}}$, the eigenemittances after such an element become [19]

$$\begin{aligned}\varepsilon_- &= \frac{[(\frac{\Delta\gamma}{\gamma})_{\text{ind}}^2 + (\frac{\Delta\gamma}{\gamma})_{\text{initial}}^2]^{1/2}}{(\frac{\Delta\gamma}{\gamma})_{\text{slew}}} (\varepsilon_{x,\text{ind}}^2 + \varepsilon_{x,\text{initial}}^2)^{1/2} \\ \varepsilon_+ &= \gamma \left(\frac{\Delta\gamma}{\gamma} \right)_{\text{slew}} \sigma_{z'}.\end{aligned}\quad (41)$$

Too large an induced energy spread or emittance growth limits the ability to perform this kind of two-stage FBT/XZFBT hybrid. Using collisional ionization or Bremsstrahlung radiation to generate the correlated energy slew in a wedge-shaped foil was first discussed in 1970 [20] and with something similar to eigenemittances calculated in 1983 [21]. Although at first glance it appears that the induced energy spread is too large to lead to a significant improvement in emittances, a small percentage of particles dominates the induced energy spread, and if one is willing to sacrifice a significant portion of the beam (say start with 1 nC and end with 250 pC by using intercepting optics after the foil), significant improvements are possible [19]. An alternative approach is to use a transversely tapered wiggler field where the electrons' incoherent synchrotron radiation leads to the needed transverse energy-spread slew. Nominally, a T-scale field is needed for a 1-m long element at a beam energy of 1 GeV.

At this point, we have completed the third main point of this paper. We have described geometries that can lead to correlations between the horizontal and longitudinal

directions and we have provided a prescription for using at most four transversely deflecting cavities to recover the eigenemittances, and, specifically, to minimize the horizontal emittance at the expense of the longitudinal emittance.

VII. DISCUSSION

The previous results provide options to provide ultrahigh bright transverse emittances in addition to Kim's original suggestion to use a FBT and an EEX with a very short round bunch [14]. We can use the results in Sec. V to design arbitrary x - y FBTs where the initial beam is not round or may have unequal transverse emittances, knowing that initial nonsymplectic correlations can never bring the emittances closer. Equivalently, we can also design an equivalent FBT between one of the transverse dimensions and the longitudinal dimension. In that case, consider a highly elliptical photoinjector geometry, where there is about a 22:1 ratio between the horizontal and vertical sizes (which ensure the vertical emittance already meets the nominal tiny transverse emittance constraint). With initial emittances $\varepsilon_{x,n}/\varepsilon_{y,n}/\varepsilon_{z,n}$ of 3.3/0.15/1.4 μm , an XZFBT can be designed to reach a final emittance of 0.15/0.15/30 μm , using the notional correlations in the previous section. Because the eigenemittances are conserved, we only need to start with the right correlations at the cathode. Typical initial normalized values would be $\sigma_x = 10$ mm and $\beta\gamma\sigma_{x'} = 0.35$ mrad for the horizontal direction, and $\sigma_y = 1$ mm and $\beta\gamma\sigma_{y'} = 1.4$ mrad for the longitudinal direction. Using Eqs. (17) and (26)–(28), this XZFBT would require $d = 2.5$ for an initial x - z correlation and $e = 3.5$ μm^{-1} for an initial x' - z correlation, both of which are achievable. There is a concern that nonlinearities in a photoinjector may be problematic with such a large aspect ratio, particularly since an XZFBT does not have the conservation properties of an x - y FBT. Recent FEL simulations [22] indicate that final transverse emittance asymmetries as high as 4:1 may only decrease the x-ray flux by as little as 15%, which will reduce the needed initial beam asymmetry. In this case, with a cathode aspect ratio of 5.3:1 (where the initial x emittance is increased from 0.7 to 1.61 μm and the initial y emittance is decreased from 0.7 to 0.3 μm), a laser tilt can provide the needed x - z coupling to produce an emittance split of 0.3/0.075/30 μm . If a cathode with a 1-mm radius can produce the 0.7 μm transverse emittances, with a 3.3 ps laser pulse, then with a cathode of 2.3-mm rms horizontal size and 0.43-mm rms vertical size and with a drive laser angle 83° off normal incidence will establish these eigenemittances. The ability to recover the eigenemittances established in the manner described above depends on the ability to minimize nonlinear effects from diluting the linear correlations in phase space. Emittance compensation has demonstrated that this can likely be done if the photoinjector is designed carefully. The resulting bunch

length from tilting the drive laser would be about 20 ps, and may require a lower frequency or DC photoinjector to maintain linearity.

For a second example, let us consider what emittance partitioning can be done at a conventional DC photoinjector, such as the gun test stand at the Jefferson National Accelerator Laboratory. Nominal parameters are a 5 mm radius cathode producing a beam with a normalized transverse emittance of $5 \mu\text{m}$ and a bunch length of 15 ps with a $15 \mu\text{m}$ longitudinal emittance. The drive laser illuminates the cathode at a 45° angle and the phase front is typically corrected to remove any horizontal-longitudinal coupling. However, if the drive laser tilt is not corrected, $d = 1$ from Eq. (17), and the horizontal and longitudinal emittances are converted to about 3.5 and $21.5 \mu\text{m}$, respectively. With stronger coupling ($d = 2$ which corresponds to an illumination angle of 63.4 degrees), the converted emittances are 2.2 and $33.8 \mu\text{m}$ respectively, which again indicates that improvements of factors of least 2 in emittance seem to be straightforward.

As a final example, let us consider a two-stage FBT/ZXFBS hybrid as described in the previous section, with the foil nominally at 100 MeV. This configuration has the advantage that it is less susceptible to nonlinear fields in a photoinjector due to the conservation properties supported by the conventional FBT. A round cathode with an applied axial magnetic field can be used to modify the intrinsic emittances $\varepsilon_{x,n}/\varepsilon_{y,n}/\varepsilon_{z,n}$ of $0.7/0.7/1.4 \mu\text{m}$ to $3.3/0.15/1.4 \mu\text{m}$ with a conventional FBT. A wedge-shaped foil could provide a ramped energy attenuation as a function of horizontal position. If the foil is designed so that it provides 50 keV more attenuation at one horizontal end of the bunch compared to the other (for a 1-ps long beam with $100\text{-}\mu\text{m}$ horizontal size), an XZFBT optics section would transfer the excess horizontal emittance to the longitudinal plane, with a final emittance partitioning of $0.15/0.15/30 \mu\text{m}$, again using Eqs. (17) and (26)–(28), where $b = 5$. This technique fails if the foil generates too excessive an energy spread, but simulations indicate that as long as the induced rms “transverse slice” energy spread is no more than 10%–20% of the energy slew, transverse emittances as low as $0.25 \mu\text{m}$ can be achieved with this configuration.

More generally to minimize the effects of photoinjector nonlinearities, three-dimensional correlations may be needed, where round cathodes can be conveniently used [23]. Just as one correlation between different dimensions can reduce one eigenemittance at the expense of another, two correlations, if sufficiently unrelated, can reduce two eigenemittances at the expense of the third. The two need to be “related” according to the table shown in Fig. 5(a) [23], where the variable in the row is correlated as a function of the variable in the column. In this table, two correlations with the same color must be chosen (but not immediately next to each other). For example, an x - y

FIG. 5. (a) Table of acceptable correlations coupling all three dimensions that lead to two tiny eigenemittances and one large eigenemittance. Acceptable pairs of correlations are one from each of the two groups of the same color, with the black areas as excluded, where the row index is a function of the column index. (b) Eigenemittance evolution as a function of x - y and x' - z correlations [using the yellow sections from part (a)]. When both correlations vanish, the eigenemittances are the initial uncorrelated emittances of $1/1/1 \mu\text{m}$.

correlation (row x , column y_0 , where x is a function of y) and a p_x - z_0 correlation (where p_x is a function of z) are both yellow; thus we expect the two eigenemittances to be lower than the equivalent uncorrelated beam emittances.

The plot in Fig. 5(b) shows the effect of these correlations on the eigenemittance values. The origin, hidden in the back, represents a totally uncorrelated beam (and the three eigenemittances are degenerate with eigenvalues $\lambda_i = 1$). As the x - y and p_x - z correlations are increased to a normalized value of 10, at the closest corner, one eigenemittance has increased, while the other two have decreased substantially. Not all pairs of correlations reduce two lower eigenemittances, and, as seen in Fig. 5(a), the two final correlated values must be conjugates of each other [23].

Some improvement can be trivially made—keeping a round cathode and using a slight x - z correlation with an XZFBT to just drop one of the transverse dimensions, as in the DC gun example above, would certainly improve

XFEL performance. Detailed nonlinear simulations of the dynamics in an actual photoinjector need to be made next to evaluate the performance of these different options, including likely field errors and misalignments. It is likely that some two-dimensional FBT configurations or full three-dimensional correlations will be more resistant to nonlinearities and provide a path to significantly lower transverse emittances than others.

APPENDIX: SYMPLECTIC BEAM TRANSFORMS

Using the appropriate beam units for proper treatment of symplectic transformations that were found in Sec. II, the following is a list of common optics elements and their transfer matrices needed for arbitrary FBT optics configurations. Of particular importance, we identify the x - z equivalents of standard x - y optics elements. In particular, a chicane acts like a longitudinal drift, an rf cavity provides longitudinal focusing when the bunch passes the cavity at the zero phase crossing, and a transversely deflecting rf cavity provides x - z coupling in a manner analogous to the x - y coupling from a skew quadrupole.

1. Drifts in x , y , and z

The full six-dimensional transfer matrix for a drift is given by

$$R_{\text{drift}} = \begin{pmatrix} 1 & D & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & D & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{D}{(\gamma\beta)^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A1})$$

Note that there is very little longitudinal “drift,” and it vanishes at high energy. The longitudinal equivalent for a drift is a chicane, with transfer matrix

$$R_{\text{chicane}} = \begin{pmatrix} 1 & 2L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2\varepsilon \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{A2})$$

where

$$L = S_1 \frac{1}{\cos^3 \theta_0} + 2 \frac{D}{\cos \theta_0} + \frac{S_2}{2}, \quad (\text{A3})$$

$$\varepsilon = S_1 \frac{\sin^2 \theta_0}{\cos^3 \theta_0} + 2 \frac{D}{\sin \theta_0} \left(\frac{\sin \theta_0}{\cos \theta_0} - \theta_0 \right). \quad (\text{A4})$$

the dipoles have width D , S_1 is the distance between dipoles in each dogleg, S_2 is the distance between the doglegs, and the dipoles have nominal bend angle θ_0 . Combinations of quadrupoles, drifts, and chicanes can produce arbitrary drifts in all three dimensions simultaneously.

2. Focusing in x , y , and z

For the x - y dimension, an ideal (thin lens) quadrupole is

$$R_{\text{quad}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -f & 1 \end{pmatrix}, \quad (\text{A5})$$

where f is the inverse focal length. Combinations of quadrupoles and drifts can be made to provide this idealized matrix:

$$R_{\text{focus}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & g & 1 \end{pmatrix}, \quad (\text{A6})$$

where f and g are arbitrary. The z analogy of a quadrupole is an rf cavity passed at zero crossing (known as a slew cavity), with transfer matrix

$$R_{\text{rf cav}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b & 1 \end{pmatrix}, \quad (\text{A7})$$

where $b = 2eV_{\text{gap}} \sin(\omega d/2\beta c)/(\gamma m c^2 d)$, and V_{gap} , ω , and d are the cavity's gap voltage, frequency, and length, respectively.

3. Coupling between two dimensions

A quadrupole rotated 45 degrees has horizontal focusing proportional to a particle's vertical position and vertical focusing proportional to a particle's horizontal position. For a skew quadrupole with field gradient B' , the transfer matrix is given by

$$R_{\text{skew}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{A8})$$

where $a = \frac{e}{\gamma\beta mc} \int B' dl$ and the integral is taken over the particle's path in the quadrupole.

Note that a skew-quadrupole transfer matrix transforms the original beam matrix into

$$\sigma_2 = R_{\text{skew}} \sigma_0 R_{\text{skew}}^T$$

$$= \begin{pmatrix} \sigma_x^2 & 0 & 0 & a\sigma_x^2 & 0 & 0 \\ 0 & \sigma_{x'}^2 + a^2\sigma_y^2 & a\sigma_y^2 & 0 & 0 & 0 \\ 0 & a\sigma_y^2 & \sigma_y^2 & 0 & 0 & 0 \\ a\sigma_x^2 & 0 & 0 & \sigma_{y'}^2 + a^2\sigma_x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{z'}^2 \end{pmatrix}. \quad (\text{A9})$$

Note that, although this matrix looks similar to the beam matrix originating from an axial field on the cathode [Eq. (16)], the signs are different.

A transversely deflecting rf cavity is the x - z analog of a skew quadrupole. Consider an rf cavity with transverse center $(x, y) = (0, 0)$ of horizontal width $2w$ and vertical width $2b$ with fields

$$\begin{aligned} E_z &= A \sin\left(\frac{x\pi}{b}\right) \cos\left(\frac{y\pi}{2w}\right) e^{j\omega t} \\ B_y &= j \frac{A\pi}{b\omega} \cos\left(\frac{x\pi}{b}\right) \cos\left(\frac{y\pi}{2w}\right) e^{j\omega t} \\ B_x &= j \frac{A\pi}{2w\omega} \sin\left(\frac{x\pi}{b}\right) \sin\left(\frac{y\pi}{2w}\right) e^{j\omega t}. \end{aligned} \quad (\text{A10})$$

These fields lead to a transfer matrix of

$$R_{\text{rf}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & a & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ a & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{A11})$$

where a particle at time $t = 0$ passes the center of the cavity when the magnetic field vanishes, and $a = eAd \frac{\pi}{b} \frac{1}{m\gamma\beta c}$.

4. Modifying correlations with symplectic transformations

In the goal of being able to construct arbitrary FBTs, two useful transformations are presented. The first has the ability to “fix” antidiagonal terms in the beam matrix, and the second can fix diagonal terms. Applicable to any two dimensions, we start with the beam matrix:

$$\sigma_{C0} = \begin{pmatrix} \sigma_1^2 & 0 & 0 & b\sigma_1^2 \\ 0 & \sigma_1^2 & a\sigma_3^2 & 0 \\ 0 & a\sigma_3^2 & \sigma_3^2 & 0 \\ b\sigma_1^2 & 0 & 0 & \sigma_4^2 \end{pmatrix}. \quad (\text{A12})$$

A symplectic coupling matrix of the form (skew quadrupole or transversely deflecting rf cavity)

$$R_{\text{coupling}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{pmatrix} \quad (\text{A13})$$

transforms the initial correlated beam matrix into

$$\sigma_{C: \text{coupled}} = \begin{pmatrix} \sigma_1^2 & 0 & 0 & (b+c)\sigma_1^2 \\ 0 & \sigma_2^2 + 2ac\sigma_3^2 + c^2\sigma_3^2 & (a+c)\sigma_3^2 & 0 \\ 0 & (a+c)\sigma_3^2 & \sigma_3^2 & 0 \\ (b+c)\sigma_1^2 & 0 & 0 & \sigma_4^2 + 2abc\sigma_1^2 + c^2\sigma_1^2 \end{pmatrix} \quad (\text{A14})$$

and we see we can somewhat arbitrarily adjust the anti-diagonal elements (e.g., make certain elements on the cross diagonal vanish or become negative values of each other). Also, a transfer matrix of the form

$$R_{\text{magnify}} = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & 1/M & 0 & 0 \\ 0 & 0 & N & 0 \\ 0 & 0 & 0 & 1/N \end{pmatrix} \quad (\text{A15})$$

magnifies/demagnifies the 12 and 34 submatrices, with only equivalent changes in the antidiagonal elements:

$$\begin{aligned} \sigma_{\text{magnified}} &= R_{\text{magnify}} \begin{pmatrix} \sigma_1^2 & 0 & 0 & b\sigma_1^2 \\ 0 & \sigma_1^2 & a\sigma_3^2 & 0 \\ 0 & a\sigma_3^2 & \sigma_3^2 & 0 \\ b\sigma_1^2 & 0 & 0 & \sigma_4^2 \end{pmatrix} R_{\text{magnify}}^T \\ &= \begin{pmatrix} M^2\sigma_1^2 & 0 & 0 & Mb\sigma_1^2/N \\ 0 & \sigma_1^2/M^2 & Na\sigma_3^2/M & 0 \\ 0 & Na\sigma_3^2/M & N^2\sigma_3^2 & 0 \\ Mb\sigma_1^2/N & 0 & 0 & \sigma_4^2/N^2 \end{pmatrix}. \end{aligned} \quad (\text{A16})$$

These tools may make some optics cases easier to realize and can help visualize the effect of the optics used in the main sections of this paper.

-
- [1] P. Emma, Nat. Photon. **4**, 641 (2010) [<http://lcls.slac.stanford.edu>].
- [2] J. Rossbach, E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov, Nucl. Instrum. Methods Phys. Res., Sect. A **393**, 152 (1997).
- [3] <http://marie.lanl.gov/>.
- [4] N. Yampolsky, B. Carlsten, K. Bishofberger, S. Russell, R. Ryne, and A. Dragt, “Controlling Electron-beam Emittance Partitioning for Future X-ray Light Sources” (unpublished).
- [5] For example, A. Dragt *et al.*, Phys. Rev. A **45**, 2572 (1992); A. Dragt *et al.*, Annu. Rev. Nucl. Part. Sci. **38**, 455 (1988).
- [6] R. Brinkmann, Ya. Derbenev, and K. Floettmann, Phys. Rev. ST Accel. Beams **4**, 053501 (2001).
- [7] A. Burov, S. Nagaitsev, and Y. Derbenev, Phys. Rev. E **66**, 016503 (2002).
- [8] K.-J. Kim, Phys. Rev. ST Accel. Beams **6**, 104002 (2003).
- [9] B. E. Carlsten and K. A. Bishofberger, New J. Phys. **8**, 286 (2006).
- [10] P. Piot, Y.-E. Sun, and K.-J. Kim, Phys. Rev. ST Accel. Beams **9**, 031001 (2006).
- [11] M. Cornacchia and P. Emma, Phys. Rev. ST Accel. Beams **5**, 084001 (2002).
- [12] P. Emma, Z. Huang, K.-J. Kim, and P. Piot, Phys. Rev. ST Accel. Beams **9**, 100702 (2006).
- [13] A. Johnson, J. Ruan, H. Edwards, T. Koeth, A. Lumpkin, P. Piot, J. Santucci, Y.-E. Sun, and R. Thurman-Keup, in Proceedings of the 2010 International Particle Accelerator Conference IPAC10, Kyoto, Japan, 2010 [<http://accelconf.web.cern.ch/accelconf/IPAC10>].
- [14] K.-J. Kim, in 13th Advanced Accelerator Conference Workshop, Santa Cruz, 2008, AIP Conf. Proc. No. 1086, edited by C.B. Schroeder, W. Leemans, and E. Esarey (American Institute of Physics, Melville, NY, 2009).
- [15] R. D. Ryne *et al.* (unpublished).
- [16] Alex J. Dragt *et al.*, MARYLIE 3.0 Users Manual, <http://www.physics.umd.edu/dsat/> (2003).
- [17] Alex J. Dragt, Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics, <http://www.physics.umd.edu/dsat/> (2010).
- [18] See W. Lysenko and M. Overly, in Proceedings of the Workshop on Linear Accelerators and Beam Optics Codes, San Diego, CA, 1988, edited by C. R. Eiminhez, AIP Conf. Proc. No. 177 (American Institute of Physics, New York, 1988), p. 323; A. J. Dragt, F. Neri, and G. Rangarajan, Phys. Rev. A **45**, 2572 (1992).
- [19] B. E. Carlsten, K. A. Bishofberger, L. D. Duffy, M. Holloway, R. McCrady, and N. A. Yampolsky, “Two-stage Generation of Beam Correlations in Electron Beams to Establish Low Transverse Beam Emittances” (unpublished).
- [20] C. Bovet, LBL Internal Report No. ERAN 89, 1970.
- [21] J. M. Peterson, IEEE Trans. Nucl. Sci. **30**, 2403 (1983).
- [22] Q. R. Marksteiner (private communication).
- [23] L. D. Duffy, K. A. Bishofberger, B. E. Carlsten, Q. R. Marksteiner, and S. J. Russell, “Exploring Minimal Correlation Scenarios to Produce Transversely Bright Electron Beams using the Eigenemittance Concept” (unpublished).