“Robinson’s sum rule” revisited*

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(Received 31 August 2009; published 2 February 2010)

This discussion revisits two articles on synchrotron radiation damping published in 1958, one by this author and Evgeny K. Tarasov [Zh. Eksp. Teor. Fiz. **34**, 651 (1958); Sov. Phys. JETP **34**, 449 (1958)], and one by Kenneth W. Robinson [Phys. Rev. **111**, 373 (1958)]. The latter is the source of what is known as “Robinson’s sum rule.” Both present the familiar rule, but with very different proofs and calculations of concrete damping decrements. Comparative analysis of these differences reveals serious flaws in Robinson’s proof and calculations.

DOI: 10.1103/PhysRevSTAB.13.024901

PACS numbers: 61.80.Fe, 29.20.-c, 01.65.+g

I. INTRODUCTION AND OVERVIEW

We revisit here two articles on synchrotron radiation damping published in 1958. The first was written by myself with the late Evgeny K. Tarasov and published in the Soviet Union [1]. The second was written by the late Kenneth W. Robinson and appeared four months later in the United States [2].¹ Both works rederived known formulas for decrements in simple isomagnetic rings. Their novelty lay elsewhere. In [1], Orlov-Tarasov derived general formulas for damping decrements in the case of an uncoupled x and y in an arbitrary ring; produced a general sum rule and its proof for a general case by summing the formulas; and then analyzed the formulas with a view to canceling the antidamping of radial oscillations. In [2], Robinson presented the same sum rule, and its proof for a general case, basing the proof on general principles; then, assuming the correctness of rule and proof, he calculated concrete damping decrements for x , y and x , z coupling in order to cancel antidamping. (x is the radial coordinate, y the vertical, and z the longitudinal).

The significant differences between [1] and [2] in theoretical approach and results went unremarked in the literature. The Orlov-Tarasov work was virtually ignored in the West, and Robinson’s became the *locus classicus* of what is generally known as “Robinson’s sum rule.”

The differences between [1] and [2] will be assessed here through an analytic comparison of the two articles. The analysis leaves the Orlov-Tarasov results intact, but reveals very serious errors in Robinson’s proof of the sum rule and his calculations of concrete radiation damping decrements—errors deeper than Tarasov and I had supposed 50 years ago. An overview of the crucial ones will be a useful preliminary.

Robinson’s proof of the sum rule is invalid as a general proof, because it does not hold for nonisomagnetic rings.

*The PRST-AB web site of this article contains a link to Refs. [1,6,10,11] under “Supplemental Material.”

¹The relation of these articles has recently been examined by Talman and Chao [3].

Robinson analyzes determinant D of transfer matrix M for one revolution period of a particle in an accelerator, $D = \exp(\sum \gamma_i)$, where γ_1 , γ_2 , and γ_3 are damping decrements of the longitudinal, radial, and vertical oscillations. $\gamma_i \propto P_\gamma/E$, where P_γ is the synchrotron radiation power and E the particle energy. D is the product of the infinite number of determinants of the elementary matrices for the infinitesimal elements of revolution period T :

$$M = \prod_{n=1}^{\infty} m_n; \quad D = \prod_{n=1}^{\infty} d_n; \quad (1)$$

$$m_n = m(dt_n); \quad \sum_{n=1}^{\infty} dt_n = T.$$

(These equations are mine.) Every m_n is a 6×6 matrix possessing diagonal terms $(m_n)_{kk} = 1 + \xi_k dt$, $k = 1, \dots, 6$, and off-diagonal terms proportional to dt . In d_n we can neglect quadratic terms proportional to $(dt)^2$, so

$$d_n = 1 + \sum_{k=1}^6 \xi_k dt. \quad (2)$$

This idea of Robinson’s is obviously true. It is also obvious that, in general, $D \neq 1 + \sum_k (\int \xi_k dt)$ because its off-diagonal terms can be big at $t = T$. But the off-diagonal terms at $t = T$ that are proportional to P_γ/E remain small, $\int_0^T (P_\gamma/E) dt \ll 1$, as Robinson correctly observes. Without saying why, he concludes that he can neglect the contribution of the off-diagonal terms to D , so equality $D = 1 + \sum (\int \xi_k dt)$ still holds. His formulas make clear the unstated reason: since these off-diagonal terms enter D only quadratically in P_γ/E , they are quadratically small and hence can be neglected. His proof of the sum rule is heavily dependent on the idea of such neglect at every step of the matrix multiplication, from $t = 0$ to $t = T$.

The problem is that this idea is wrong. Every elementary matrix m_n possesses *only one* off-diagonal term (out of 30) proportional to $(P_\gamma/E)dt$. This term arises from the following transformation, taken from Eq. (8) in [1] while keeping only terms proportional to P_γ/E :

$$d\phi = \left(-2\frac{P_\gamma}{E}\dot{\phi} - \frac{2\pi hc\alpha}{L}(2n-1)\frac{P_\gamma}{E}\frac{x}{R} \right) dt + \dots, \quad (3)$$

where ϕ is a phase of the synchrotron oscillations, h the harmonic number, α the momentum compaction factor, L the orbit length, x the radial deviation of the particle, and $n = -(R\partial B/\partial R)/B$. The off-diagonal term enters *D linearly* since, to get $(P_\gamma/E)^2$ in D , matrix M needs another off-diagonal term proportional to P_γ/E . And such a term is absent, all other relevant transformations being purely “diagonal,”

$$d\dot{x} = -\frac{P_\gamma}{E}\dot{x}dt + \dots; \quad d\dot{y} = -\frac{P_\gamma}{E}\dot{y}dt + \dots. \quad (4)$$

Thus, off-diagonal contributions to D can be linear, not quadratic in $(P_\gamma/E)T$. Moreover, on the way from $t = 0$ to $t = T$ they can get some big coefficients from matrix terms independent of P_γ/E . Robinson did not notice this possibility destroying the entire line of his proof. Because he does not correctly take into account the off-diagonal term in (3), he fails to prove his central claim [[2], Eq. (6)] that $D = 1 - 4\langle P_\gamma/E \rangle T$, where $-4\langle P_\gamma/E \rangle dt$ is the sum of the diagonal terms in (3) and (4). (Whether this formula for D is actually true is irrelevant to the validity of his proof.)

Yet another problem with his proof turns on the incorrect idea that one can ignore off-diagonal terms when multiplying an infinite number of elementary matrices. This idea automatically excludes from consideration the physically possible parametric resonances caused by the variation of radiation power along the orbit in a general, nonisomagnetic ring. We can see the possibility of such resonances (correlations between different cofactors) in the off-diagonal term of Eq. (3) above, as well as in the diagonal terms of Eq. (4). The very existence of these resonances conflicts with both Robinson’s sum rule conceived as a “general result” [[2], p. 375] and the basic assumptions of his proof. In short, there is a lack of logical fit between proof and rule.

Robinson’s calculations of decrements are erroneous. First, his calculation of the x, y coupling necessary to cancel antidamping exclusively addresses x, y coupling instead of taking x, y, z coupling into account. This is a conceptual and physical error. For although x, z coupling exists autonomously, x, y coupling can exist only as a part of x, y, z coupling due to the existence of the “natural” x, z coupling in Eq. (3). Moreover, the contribution of this natural coupling can be very big. This holds for any ring.

Second, with respect to nonisomagnetic rings Robinson errs in calculating changes of the x, z coupling necessary to cancel antidamping (in the absence of x, y coupling). Specifically, he fails to notice a correlation—a sort of resonance, again—between radiation power and the ring parameters in nonisomagnetic rings, separately averaging one of those parameters—namely, x/R in (3). This opera-

tion yields a wrong physical picture and equation [(32) in [2]] for damping synchrotron and radial oscillations in the general case of a nonisomagnetic ring:

$$\Delta\alpha_s = (n'\alpha - 1)(P'_\gamma/E), \quad \text{Robinson (32)}$$

$$\Delta\alpha_r = -(n'\alpha + \frac{1}{2})(P'_\gamma/E), \quad \text{Robinson (32),}$$

where indices $s \equiv \phi$ and $r \equiv x$, and $\alpha_{s,r} \equiv \gamma_{s,r}/2$. Equation (32) shows that we can cancel antidamping in principle by varying the n value along the orbit without varying the magnetic field B . However, as [1] proves, this cannot be done [see Eq. (23) below].

II. PROOF OF THE SUM RULE

The Orlov-Tarasov proof of the sum rule does not (as Robinson’s does) use elementary matrices and their multiplication, but equations of motion and successive approximations to solve them. Therefore, the problem of off-diagonal terms simply does not arise in connection with the proof. Our method of calculating the decrement for the normal mode closest to the synchrotron oscillations is the same as that now found in textbooks (for example, [4]). We first assume some phase oscillations, then calculate the response of the radial oscillations to those oscillations, and then include this response as a perturbation in the equation for the phase oscillations. Since we treat the sum of decrements as unknown, we use the same method of successive approximation to calculate the normal mode closest to the radial oscillations: we assume some radial oscillations, calculate the response of the synchrotron oscillations, and then include this response as a perturbation in the equation for the radial oscillations. We note and avoid the possibility of a resonance between a variation of the radiation power and the free radial oscillations in a nonisomagnetic field. As a result, our proof avoids the resonance issue, described above, that undermines Robinson’s proof.

Let us once more follow the logic of his proof.

Damping in a six-dimensional phase space is defined by a 6×6 transfer matrix. This matrix is the product of an infinite number of elementary matrices, each corresponding to an infinitesimal step dl (or dt) along the orbit. Matrix terms proportional to the radiation power are proportional to dl . The final determinant is the product of the infinite number of determinants of the elementary matrices. Correspondingly, “In order to determine damping, the determinant of the transfer matrix of the infinitesimal element is evaluated. The only terms in the determinant which will be first order in the length of the element will be due to the diagonal terms of the matrix, and all higher order terms may be neglected” (p. 374). (Indeed, this is true for any single elementary matrix regardless of the physical meaning of its off-diagonal terms.) Robinson observes a little later: “The determinant of the transfer matrix for one

complete period is the product of the transfer matrices of the infinitesimal elements of that period” (p. 374).

Everything is correct up to this point. But then Robinson states: “Since the fractional radiation loss in one period is very small, only first order terms need be considered. . .” (p. 374). His equations (5) and (6) show exactly what this means: when multiplying elementary determinants calculated for even an infinite number of elementary matrices, *we can neglect the off-diagonal terms of those matrices and count only the diagonal ones at any time*, $0 \leq t \leq T$. As we have seen earlier, however, such neglect is forbidden because the off-diagonal terms proportional to the radiation power can enter determinant D for $t = T$ linearly, not quadratically in P_γ/E . Thus, they cannot be neglected.

The following two examples illuminate how Robinson’s idea about off-diagonals is a rather fundamental mathematical and physical error.

Example 1.—This purely mathematical example illustrates how off-diagonals can “penetrate” into the diagonal of matrix M on the way from $t = 0$ to $t = T$. It shows that an infinite number of infinitesimal off-diagonal terms integrated along some length, l_1 , can in fact contribute to the diagonal terms by consequent transformations integrated along another length, l_2 , if those transformations do not commute with transformations along l_1 .

A particle passes over a set of N identical elementary transformations, $N \rightarrow \infty$, with every elementary transformation described by an elementary matrix m_1 in which Vdl symbolizes some value growing proportionally to the passed interval Δl :

$$m_1 = \begin{bmatrix} 1 & Vdl \\ 0 & 1 \end{bmatrix}, \quad Ndl = l. \quad (5)$$

After that, let this particle pass over a set of different elementary transformations,

$$m_2 = \begin{bmatrix} 1 & 0 \\ -d(1/f) & 1 \end{bmatrix}, \quad d(1/f) \equiv [\partial(1/f)/\partial l]dl, \quad (6)$$

where $\int d(1/f)$ is growing with Δl . It is essential here that the growth of $\int Vdl$ is interrupted and that the m_2 matrices do not depend on V . The final matrix is

$$M = (m_2)^N \cdot (m_1)^N = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & Vl \\ 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 1 & Vl \\ -1/f & 1 - Vl/f \end{bmatrix}. \quad (7)$$

We see first that the final matrix contains a diagonal term of the first order in V ; second, this term appears despite the fact that none of the elementary matrices contains it at the diagonal. In this example, the determinant has not changed from 1. However, contrary to Robinson’s assumption, the sum of the diagonal terms *has* changed.

We can easily check the important role of noncommutativity in the process of transforming off-diagonal terms into diagonal terms. If, instead of m_2^N , we use any matrix Q commuting with m_1^N , then the off-diagonal term Vl does not appear at the final diagonal. The elements of Q are $Q_{11} = Q_{22}$, $Q_{21} = 0$.

Example 2.—This mathematico-physical example shows how off-diagonal terms may not only contribute to diagonal terms, but also change the sum of decrements within well-defined areas of resonances created by the radiation power itself. The origin of such resonances is the following.

In a nonisomagnetic ring, the B field met by a particle “oscillates.” Hence $P_\gamma(t)/E$ also oscillates:

$$P_\gamma(t)/E = (P_\gamma/E)_0 + \sum (P_\gamma/E)_k \cos(k\omega_C t + \theta_k), \quad (8)$$

ω_C is revolution frequency. But P_γ/E is a parameter of longitudinal oscillations, Eq. (3), and a friction parameter of both radial and vertical betatron oscillations. So its oscillations produce a parametric resonance if any of the betatron tunes of $x, \dot{x}, y$ equals either integer or half-integer, $\nu = n/2$, and this n is present in the Fourier series (8). (Tune is defined as ratio of frequency to revolution frequency.) Those resonances are absent *in principle* from Robinson’s formulas.

To prove the existence of such “radiation resonance” it is sufficient to take the simplest case of vertical betatron oscillations that are separate from x and z oscillations. The equation of motion in this case is

$$\ddot{y} + [P_\gamma(t)/E]\dot{y} + \omega^2(t)y = 0. \quad (9)$$

Using a technique initially developed in [5], let us define the new (complex) coordinates as a, a^* (instead of $y, \dot{y}$),

$$y = a\varphi(t) + a^*\varphi^*(t); \quad \dot{y} = a\dot{\varphi} + a^*\dot{\varphi}^*; \quad (10) \\ \dot{a}\varphi + \dot{a}^*\varphi^* = 0,$$

where $\varphi(t)$ is the Floquet function normalized by condition $\dot{\varphi}\varphi^* - \dot{\varphi}^*\varphi = 2iW$.

$W = \text{Im}\dot{\varphi}\varphi^*$ is a Wronskian. The particle equation of motion in the new variables a, a^* is

$$\dot{a} = -\frac{1}{2iW} \frac{P_\gamma}{E} \varphi^*(a\dot{\varphi} + a^*\dot{\varphi}^*). \quad (11)$$

The two terms on the right side of (11) correspond to two different physical effects. $\text{Re}\dot{\varphi}\varphi^*$ in the term containing $a\dot{\varphi}\varphi^*$, which equals $a(\text{Re}\dot{\varphi}\varphi^* + iW)$, produces a correction of the betatron oscillation tune; iW produces the familiar radiation damping of the vertical oscillations, $da/dt = -[P_\gamma(t)/2E]a$.

If we were to represent the change of variables a, a^* during infinitesimal time dt in matrix form, then this familiar damping would be described by the term $1 - (P_\gamma/2E)dt$ at the diagonal of the matrix. A similar diagonal

term would describe the damping of a^* . The term containing $a^* \dot{\varphi}^* \dot{\varphi}^*$ in (11) would be one of the two off-diagonal terms. The equation for $\dot{y}$ in (4) above does not possess the off-diagonal terms. But such terms appear in the matrix describing the development of a and a^* . (This shows that the definition of off-diagonality has a conditional character.) Now, the determinant of this elementary matrix simply equals the product of the diagonal terms, into which $P_\gamma dt$ enters linearly. The off-diagonal terms enter only as quadratic $(P_\gamma dt)^2$. Further, for one revolution period, Δt , $\int (P_\gamma/E) dt = \langle P_\gamma/E \rangle \Delta t \ll 1$.

From the mathematical viewpoint, these are precisely the conditions under which, according to Robinson (p. 374), we need only count diagonal terms of elementary matrices when multiplying an infinite number of matrices corresponding to a finite period Δt . Robinson concludes from these conditions that the off-diagonal terms create no new diagonal terms during the multiplication, and certainly do not change the determinant. As shown below, neither conclusion is correct in the case of a resonance between radiation power and free oscillations of the particle.

This resonance is connected with the off-diagonal term $a^* \dot{\varphi}^* \dot{\varphi}^*$ in (11). In a strong focusing ring, the betatron Floquet function possesses Fourier modes with tunes $\nu + j$, $j = 0, \pm 1, \pm 2, \dots$. As an example, let us take a case when the main mode of the Floquet function— $\varphi = \varphi_0 \exp(i\nu\omega_C t) + \dots$, φ_0 is real—resonates with the mode $\#n$ in (8), so $\nu \approx n/2$. Having $\dot{\varphi}^* = -i\nu\omega_C \varphi_0 \exp(-i\nu\omega_C t) + \dots$ and keeping only the resonant terms in (11), we get

$$\dot{a} = \left(\frac{\nu\varphi_0^2\omega_C}{2W} \right) \frac{P_{\gamma n}}{2E} a^* \exp[i(n - 2\nu)\omega_C t + \theta_n] + \dots, \quad (12)$$

and, with $g = [(\nu\varphi_0^2\omega_C)/(2W)][P_{\gamma n}/(2E\omega_C)]$, $a = A \exp i\alpha$, $n/2 - \nu = \varepsilon$, $\Phi = 2\varepsilon\omega_C t - 2\alpha + \theta_n$,

$$\frac{dA^2}{dt} = 2g\omega_C A^2 \cos\Phi, \quad (13)$$

$$\frac{d\Phi}{dt} = 2\omega_C(\varepsilon - g \sin\Phi). \quad (14)$$

These are parametric resonance equations, with the integral of motion $H = 2A^2\omega_C(\varepsilon - g \sin\Phi)$. The half width of this resonance equals g . At late times inside the resonance region, $t \rightarrow \infty$, $\sin\Phi \rightarrow \varepsilon/g$, and

$$A \sim \exp[\text{sign}(\cos\Phi)(\sqrt{g^2 - \varepsilon^2})\omega_C t] \quad \text{for } |\varepsilon| < g, \\ t \rightarrow \infty. \quad (15)$$

The point is that this exponent in the matrix representation for a , a^* will be among both the diagonal and off-diagonal terms of the transformation matrix; therefore, the determinant will obviously not equal 1. The sign of $\cos\Phi$ depends on the initial conditions. Both plus and minus signs are

possible only if $\sin\Phi = \varepsilon/g$ at the injection. In all other cases, A in (15) is growing exponentially. (Formula (15) shows only the imaginary part of y 's eigenfrequency.) Since $2W \sim \nu\varphi_0^2\omega_C$, the resonance part of the damping decrement is of the same order of magnitude as a usual radiation damping decrement.

What is at stake here for Robinson's proof is its validity as a proof of the sum rule *as he has characterized it*: “a general result for any type of electron accelerator if the average electron energy is constant” (p. 375). Robinson's apparently innocuous view on off-diagonal terms is like a Trojan horse with respect to his proof. It leads him to ignore, in principle, the integrated effects of an infinite number of off-diagonal terms of different elementary matrices. In particular, he ignores parametric resonances, which are demonstrably possible and relevant to nonisomagnetic rings. This disqualifies his proof as a proof for a general result.

The Orlov-Tarasov proof narrowly avoids the above problem because our sum rule carries the implicit qualification: one has to avoid betatron resonances. This qualification is implied by our explicit choice of noninteger betatron tunes (p. 451), which is made so that we can “drop oscillating functions and so throw out all the terms containing a^* ” (p. 451). [The betatron-tune issue arises because we use a transformation similar to (10) above for the radial oscillations. So Eq. (16) in [1] is like Eq. (11) above, except that it has more terms due to the x, z coupling.] We see the possibility of resonances arising from the appearance of a^* in the equation for $\dot{a}$. Our mistake is to mention only the restricted areas around integer tunes, while the half integers are equally important.

III. FORMULATION OF THE SUM RULE

The very first formulation of the sum rule was presented (without a proof) by Robinson in an unpublished Cambridge Electron Accelerator technical report of 1956 [6]: the sum of damping decrements equals $-\langle 4P_\gamma(t)/E \rangle$. Consistent with [2], the formulation makes no mention of possible resonances. In retrospect, it is clear that the best formulation of the sum rule will take the explicit form.

The sum of the radiation damping decrements equals $-\langle 4P_\gamma(t)/E \rangle$ for any type of magnetic lattice outside the borders of parametric radiation resonances defined by this lattice.

Since this is what Tarasov and I had in mind, the validity of our proof is unaffected by it. We would nevertheless have done well to say explicitly what we meant. In any event, the explicit formulation entails no big change from the established one on the practical side, since the radiation resonances are numerically narrow and naturally avoided together with the usual betatron resonances. Moreover, it accurately reflects how the sum rule has been interpreted in the context of actual practice.

IV. x, y COUPLING

Tarasov and I regarded radial-vertical coupling as an ineffective way to cancel antidamping. So we did not calculate decrements in cases of radial-vertical coupling in [1] and paid little attention to such calculations in Robinson’s 1956 technical report [6], which he replicated in his article [2]. Thus, we failed to see that his calculations were in fact incorrect.

The problem of x, y coupling (to cancel antidamping) is that it cannot be analyzed separately from unavoidable, naturally existing x, z coupling (to produce antidamping). This means that in the case of x, y coupling, one must solve the x, y, z problem. However, Robinson treats problems of z, x and x, y coupling as separate ones.

To get a feel for the scale of this error, let us compare two decrements of longitudinal damping in the simplest cases of two different weak-focusing isomagnetic rings, one with and one without x, y coupling.

The equations of motion of a relativistic particle, $v = c$, are [1]

$$\ddot{\phi} + 2\Gamma\dot{\phi} + \Omega^2\phi + k\Gamma(2n - 1)x/R = 0, \quad (16)$$

$$\ddot{x} + \Gamma\dot{x} + c^2(1 - n)x/R^2 + (c^2/kR)\dot{\phi} - c^2sy/R^2 = 0, \quad (17)$$

$$\ddot{y} + \Gamma\dot{y} + c^2ny/R^2 - c^2sx/R^2 = 0. \quad (18)$$

In the weak-focusing isomagnetic ring, all coefficients in (16)–(18) are constants. Here I use the synchrotron phase ϕ instead of z ; $\Gamma = P_\gamma/E$; $k = 2\pi h c \alpha/L$, where h is the rf harmonic number, L is the orbit length, and α is the momentum compaction factor; $n = -R(\partial H_y/\partial x)/H$, and $s = R(\partial H_y/\partial y)/H$ (a skew quadrupole field). Applying the normal mode technique to (16)–(18), we can easily get the frequency of the normal mode closest to the synchrotron oscillations: $\omega_s = \Omega + i\gamma_\phi$, where γ_ϕ corresponds to the damping, $\phi \sim \phi_0 \exp(-\gamma_\phi t)$. With $\Omega^2/(c/R)^2 \ll n, (1 - n)$,

$$\gamma_\phi = \Gamma \left[1 - \frac{n(2n - 1)}{2[n(1 - n) - s^2]} \right]. \quad (19)$$

In the absence of x, y coupling, $s = 0$, $\gamma_\phi = (3 - 4n)/2(1 - n)$ [7]. We see that a sufficiently strong coupling of the cross-sectional x, y betatron oscillations, $s \sim n$, dramatically changes the damping of the longitudinal z oscillations. Robinson is therefore wrong to suppose that x, y coupling “has the desirable characteristic of not reducing the damping rate of the synchronous oscillation” (p. 378). In addition, $(\gamma_x + \gamma_y)$ is different from Robinson’s calculated results, since γ_ϕ is changed by the x, y coupling, while the sum of decrements remains constant.

V. x, z COUPLING

Both [1] and [2] consider using x, z coupling to cancel antidamping without any x, y coupling. Robinson offers qualitative advice on “redesigning the magnetic structure such as adding quadrupole lenses or making the field strength different in focusing and defocusing sectors” (p. 377). This advice already appears in his 1956 technical report [6], which we reference in [1] and later [8] because it is broadly correct and consistent with our results.

However, the advice is followed by formulas and statements that contain a serious error about how to reduce antidamping. In [2] Robinson says that antidamping may be reduced by adding “a magnet with a large n value (n'), such that the radiation loss decreases with increasing radius” (p. 377). It is clear from this statement and Eqs. (31)–(33) which follow it—as well as from his numerical calculations in [6] (pp. 8–9)—that he expects reduction of antidamping even if the magnetic field of additional magnets with $n' \gg n$ is the same as the field of magnets having the usual field and n values. [Robinson’s Eq. (32) is reproduced in Sec. I above.] But as [1] proves, adding such magnets has practically zero effect: “[I]ntroduction of additional magnets with arbitrarily large n but with the same field as in the other magnets does not change the dependence of $\langle \rho^{-2} \rangle$ on the energy oscillations” (p. 453), and hence does not change the damping. (Here I have corrected the published English translation, where the Russian word for oscillations, *kolebaniya*, is imprecisely and confusingly rendered as “fluctuations.”) Further, “To change the damping, it is necessary to vary the field” (p. 453).

In order to keep the beam dynamics unchanged, Robinson combines magnets with positive and negative fields having the same n' . The total effect of his system is not zero only because of that, and therefore rather accidentally. But since only half of his magnets affect damping, their number should be doubled to achieve the reduction of antidamping he aims for.²

The source of Robinson’s error here is neglect of a correlation (a resonance) affecting the synchrotron oscillations. It is between radiation $P_\gamma = P_\gamma(z)$; the field gradient represented by the n value, $n = n(z)$; and the momentum dispersion function, which is the ψ function in [1], $\psi = \psi(z)$. This correlation, up to a constant factor K , is none other than parameter ϑ of a well-known modern textbook [[4], Eq. (8.25)]:

²The reader is referred to [8] for a more correct system of magnets similar to Robinson’s, which Tarasov and I helped design and calculate. See [9] for calculations we also made for rms amplitudes arising from the combination of radiation quantum fluctuations and classical radiation damping. Although submitted in 1957, these articles were published only in 1959 and 1958, respectively. So far as I know, Robinson took note of neither.

$$K\vartheta = -\langle P_\gamma(t)[2n(t) - 1]\psi(t)/R(t)E \rangle, \quad (20)$$

where $\langle \cdots \rangle$ means the average over time; $-K\vartheta$ corresponds to the left part of Eq. (27) in [1]. (In [1] we do not use the notation ϑ .) This parameter is also visible in Eq. (16) above, when it is rewritten for an arbitrary ring. In the arbitrary ring, all parameters depend on time as they are met by the rotating particle. Robinson takes the factor $\psi(t)/R(t)$ out of ϑ and averages it over time separately from other factors, getting $\langle \psi(t)/R(t) \rangle = \alpha$, momentum compaction factor. This is an obvious error, but its significance is not obvious without the analysis of $K\vartheta$ we make in Eqs. (22) and (27) of [1], which is as follows.

Let the radius and field of the usual bending magnets be R_0 and B_0 , which will be the constants in the following equations. Then

$$\begin{aligned} P_\gamma(t)R(t) &= P_{\gamma 0}R_0 + [P_\gamma(t)R(t) - P_{\gamma 0}R_0] \\ &= P_{\gamma 0}R_0[1 + (B - B_0)/B_0], \end{aligned} \quad (21)$$

where $P_{\gamma 0}$ is the constant equal to the power of radiation in the usual magnet. From this,

$$\begin{aligned} K\vartheta &= \left\langle -\frac{P_\gamma(t)}{E} \frac{\psi(t)}{R(t)} - 2P_{\gamma 0}R_0 \frac{[B(t) - B_0]}{B_0} \right. \\ &\quad \times \left. \frac{[n(t) - 1]}{R^2(t)} \psi(t) - 2P_{\gamma 0}R_0 \frac{[n(t) - 1]}{R^2(t)} \psi(t) \right\rangle. \end{aligned} \quad (22)$$

From the equation for the dispersion function, we get $\langle (n - 1)\psi/R^2 \rangle = -\langle 1/R \rangle$. Therefore,

$$\begin{aligned} K\vartheta &= \left\langle 2\frac{P_{\gamma 0}R_0}{ER(t)} - \frac{P_\gamma(t)\psi(t)}{ER(t)} \right. \\ &\quad \left. - 2P_{\gamma 0}R_0 \frac{[B(t) - B_0][n(t) - 1]\psi(t)}{B_0R^2(t)} \right\rangle. \end{aligned} \quad (23)$$

This formula shows that, contrary to Robinson's expectation, his additional magnets with the same field, $B = B_0$, as the usual magnets produce almost zero effect in a strong focusing ring, regardless of their n . No matter what $n = n(t)$ is, we get nothing but the isomagnetic ring formula if we keep $B(t) = B_0$. Only additional magnets with a negative field help cancel the antidamping of radial oscillations in his system of magnets.

Our analysis of [1] and [2] is of more than purely theoretical interest. It highlights points that deserve a place in the standard treatment of damping, especially in textbooks and handbooks. For example, it would be good to mention that parametric resonances may violate the sum rule in well-defined areas. Likewise, when stating that the special variation of the magnetic field gradient along the orbit can cancel antidamping, it would be good to note the need for simultaneous variation of the magnetic field. Finally, concrete formulas derived for cases of varying fields (e.g., fringe fields) could benefit from being consis-

tent with the fact—a consequence of Eq. (22) in [1]—that damping at an edge of a wedge magnet depends not only on the angle and dispersion function, but also on the correlation between them and the magnetic field development.

ACKNOWLEDGMENTS

I thank Sidney Orlov for invaluable discussions and editorial help, and the MIT Libraries Document Service and Cornell University Library for providing copies of Robinson's technical reports.

APPENDIX: SOME HISTORICAL BACKGROUND

Robinson's article was his first published statement of the sum rule and proof. He had already formulated the rule in a 1956 Cambridge Electron Accelerator technical report [6], as noted above, and his proof in a CEA report of 1957 [10]. These and a CEA report of 1958 [11] constitute a kind of draft of his 1958 article.

Tarasov and I wrote no preliminary reports. After work on the 7 GeV proton synchrotron design at ITEP in Moscow, we had turned to electron synchrotrons in September of 1956 when I became responsible for the theoretical part of the 5 GeV electron synchrotron design in Yerevan, Armenia. Tarasov continued his work in Moscow while participating in the Yerevan design. We arrived at synchrotron radiation problems in 1957. Literature on the subject was already to hand—papers of Iwanenko and Pomeranchuk (my former boss at ITEP) [12], Sokolov and Ternov [13], Kolomenski and Lebedev [14], Schwinger [15], and Sands [7]. We analyzed damping decrements in an arbitrary ring, quantum excitation of amplitudes, and ways to cancel antidamping, and submitted the corresponding articles for publication [1,8,9]—starting with [1] in September. It was a hectic year.

Robinson's unpublished 1956 report [6] first came to our attention at the rushed galley-reading stage of [1], when a colleague showed up with it from a conference abroad.³ Skimming it in light of our own results, we saw that Robinson's sum rule looked just like ours, and thought his suggestion to use different magnetic fields in different magnets sounded good. But some parts looked very wrong—particularly, his equation $\rho_r = (n'\alpha + 1/2)P'_\gamma/E$ for the radial decrement. Unlike ours [see (22) in [1] or (23) above], his equation did not contain a magnetic field. However, it was too late to add any discussion to our article. So we simply tacked onto the end of it a vague, positive reference to Robinson and his report, with his suggestion about magnetic fields in mind. When Robinson's own article appeared, we noted the same wrong equation [(32) in [2]] that had struck us earlier; we found no reference to our article.

³We never knew of any other Robinson reports. I learned of [10,11] only in September 2008.

So we continued on our path and Robinson continued on his, never discussing our very different theoretical approaches and results. The Soviet Union then—the height of the Cold War—was still a paranoid, totalitarian society. Tarasov and I were definitely not among the select group of scientists permitted free contacts with foreigners and trips abroad. After a 1956 prodemocracy speech at ITEP, I had been fired from ITEP and narrowly avoided arrest followed by a ticket to the gulag. The authorities were still opening my correspondence when we wrote our article. Tarasov was guilty by association, ordered by the KGB to break off all scientific and personal relations with me. Although he ignored the order, he probably shared my caution about contacts with foreigners at this time. It would have been a big risk to try and initiate discussion with Robinson, and ludicrous to trawl for other unpublished reports he might have written. Nor would it have made sense to take such a risk merely to satisfy our curiosity about his misguided conclusions, since we were convinced ours were correct. (Our Soviet colleagues were likewise convinced. They later nominated me for the prestigious State Prize of 1973 for my work on synchrotron radiation. The authorities then nullified the nomination—politics again.)

As for Robinson, it is reasonable to assume that he knew our article, if only in the English translation that appeared in Sov. Phys. JETP [1] a couple of months after his own article was published. JETP was a well-known journal and Robinson’s article even has a reference to it. So one can imagine, for the following reasons, that his silence reflected not ignorance of our article but a mix of caution and conviction similar to our own.

In 1988, a CERN physicist told me that our formula for the radial damping decrement [(22) in [1]], the best result in our article, looked “very weird.” Many scientists in the West, it seems, did not understand our formulas. Perhaps Robinson had been among them. Convinced that he was right and being, by all accounts, a very nice man, he would not have risked compromising Tarasov and myself by initiating contact—especially not to discuss weird results. One can also imagine him feeling that unfathomable results were best passed over silently in print and elsewhere. Western physicists apparently followed his lead. By 1979, Robinson had died and I was in the gulag. All possibility of discussion between us was gone. Thus, his 1958 article remained the first and last word on the subject and some of its errors made their way into the literature.

Our notation probably did not promote understanding of our results. It was common in the USSR at the time, but likely to puzzle readers unfamiliar with it. Some belated elucidation may be in order here. As the text of [1] explains between Eqs. (6) and (7), $\bar{P}_\gamma$ is the (frequency) average of

the radiation power. $\bar{P}_{\gamma s}$ is the radiation power at the equilibrium orbit; it equals zero in straight sections. $\langle P_{\gamma s} \rangle$ is the radiation power averaged along the equilibrium orbit. $n = -[\rho_s(\partial H_s/\partial r)]/H_s$ goes to $-\infty$ in a straight section. However, $n/\rho_s^2 = -(\partial H_s/\partial r)/(H_s\rho_s)$ does not go to infinity. When a particle moves from a magnet to a straight section, it meets $H_s \rightarrow 0$, $\rho_s \rightarrow \infty$, $H_s\rho_s = \text{constant}$, and $n/\rho_s^2 = -(\partial H_s/\partial r)/(\text{rigidity})$, a well-defined physical quantity.

For any x and any x_0 , $x = x_0 + (x - x_0)$. In Eq. (13) in [1], $x = \bar{P}_{\gamma s}\rho_s$ and $x_0 = P_{\gamma 0}\rho_0$. The latter can be chosen arbitrarily. For a given equilibrium energy of a particle, E_s , $P_{\gamma 0} \propto H_0^2 E_s^2$. Hence, H_0 is defined: $P_{\gamma 0}\rho_0 \propto H_0 E_s^3$. $P_{\gamma 0}$ is the same at any azimuth.

Analogously, $\bar{P}_{\gamma s}\rho_s \propto H_s E_s^3$, which explains our Eq. (21). Thus, the choice of $H_0 \propto P_{\gamma 0}\rho_0$, with E_s given, is arbitrary. But if $H_s = H_{sM}$ inside most magnets, then the best choice of H_0 is of course $H_0 = H_{sM}$. It should be noted that the straight sections—where $H_s = 0$ by definition—are not passive elements. They correspond to the simplest variation of the magnetic field: $(H_s - H_0) = -H_0 \neq 0$.

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