Modeling two-stage bunch compression with wakefields: Macroscopic properties and microbunching instability

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(Received 21 December 2007; published 23 September 2008)

In a two-stage compression and acceleration system, where each stage compresses a chirped bunch in a magnetic chicane, wakefields affect high-current bunches. The longitudinal wakes affect the macroscopic energy and current profiles of the compressed bunch and cause microbunching at short wavelengths. For macroscopic wavelengths, impedance formulas and tracking simulations show that the wakefields can be dominated by the resistive impedance of coherent edge radiation. For this case, we calculate the minimum initial bunch length that can be compressed without producing an upright tail in phase space and associated current spike. Formulas are also obtained for the jitter in the bunch arrival time downstream of the compressors that results from the bunch-to-bunch variation of current, energy, and chirp. Microbunching may occur at short wavelengths where the longitudinal space-charge wakes dominate or at longer wavelengths dominated by edge radiation. We model this range of wavelengths with frequency-dependent impedance before and after each stage of compression. The growth of current and energy modulations is described by analytic gain formulas that agree with simulations.

DOI: [10.1103/PhysRevSTAB.11.090702](https://doi.org/10.1103/PhysRevSTAB.11.090702)

PACS numbers: 29.27.Bd, 41.60.Cr

I. INTRODUCTION

A two-stage compression and acceleration system, where each stage under compresses a chirped bunch in a magnetic chicane, may be utilized in a free-electron laser (FEL) driver [1]. For a high-current bunch, longitudinal wakefields affect the macroscopic current profile of the compressed bunch and/or cause microbunching [2–10]. The bunch is affected by coherent-radiation wakes, which are approximated by steady-state coherent synchrotron radiation (CSR) within the chicane magnets [11] and coherent edge radiation (CER) [12–15] downstream of the magnets, in addition to the linac geometric and longitudinal-space-charge (LSC) wakes. When the compressions are performed at sufficiently high beam energy, the longitudinal particle distributions are nearly frozen after each chicane. For this case, we model the effect of longitudinal wakes with effective impedances before and after each chicane.

We apply our model to a preliminary two-stage bunch compressor design for the Wisconsin Free Electron Laser (WiFEL) [16], in which the electron-gun current of 50 A is increased by a factor of 20 to 1 kA. At macroscopic wavelengths of $\sim 2\pi$ times the bunch length, the calculated impedance is dominated by CER downstream of the chicane magnets. A drift length of several meters contributes a resistive CER impedance of $\sim Z_0$, where $Z_0 = 120\pi \Omega$ is the impedance of free space [14]. Tracking with the code ELEGANT [17] confirms that the macroscopic effects of coherent-radiation wakes are approximated by resistive

impedance. We use this approximation to obtain formulas for the minimum initial bunch length that can be compressed without producing an upright tail in phase space, and for the jitter in the bunch arrival time resulting from the bunch-to-bunch variation in current, energy and chirp.

The microbunching caused by wakes is considered for a bunch with nonzero emittance and energy spread. To model long wavelengths where CER dominates as well as short wavelengths where LSC dominates, we consider frequency-dependent impedances before and after each chicane. For a small current or energy modulation at the bunch-compressor entrance, the current and energy modulations of the compressed bunch are described by analytic formulas that agree with tracking simulations of a six-dimensional phase space.

For the preliminary WiFEL bunch compressor design, we calculate the minimum compressible initial bunch length, the jitter from bunch-to-bunch variations in current, energy, and chirp, and the microbunching growth. With state-of-the-art control of the linac's superconducting cavities, we plan to obtain acceptable performance by compressing a 200-pC bunch with peak current of 50 A and rms bunch length of 400 μm by a factor of 20, obtaining a peak current of 1 kA.

II. A TWO-STAGE BUNCH COMPRESSOR

For WiFEL [16], we plan to accelerate a bunch from a 4-MeV electron gun that has normalized horizontal emittance ($\beta\gamma\epsilon_x$) and vertical emittance ($\beta\gamma\epsilon_y$) of 1 $\mu\text{m}\text{-rad}$. Here,

β is the velocity divided by the speed of light, γ is the relativistic factor while ε_x and ε_y are the geometric emittances. We first consider radiofrequency (rf) parameters that provide good compression when collective effects are neglected. A schematic is shown in Fig. 1(a), while parameters are given in Table I.

The 4-MeV bunch is accelerated to 251.3 MeV in an injector linac composed of 24 TESLA 1.3-GHz superconducting cavities, with the bunch center at a linac phase 17.8° before the peak. Passage through ten superconducting 3.9-GHz harmonic cavities approximately linearizes the chirp while decelerating the bunch to the energy E_1 of 215 MeV. The chicane BC1, whose transfer matrix element $R_{56}^{(1)}$ equals -87 mm (in the convention where R_{56} is negative for a chicane), compresses the bunch by the ratio $C_1 = 8$ over a length of 12 m. The bunch is then accelerated to the energy E_2 of 485 MeV in the L1 linac, which consists of sixteen 1.3-GHz cavities, with the bunch at a phase 9° after the peak. The chicane BC2, whose transfer matrix element $R_{56}^{(2)}$ is -18 mm, compresses the bunch by the ratio $C_2 = 2.5$ over a length of 10 m. The fully compressed bunch is then accelerated and dechirped in the L2 linac, which consists of 120 1.3-GHz cavities, with the bunch at a phase 50.6° after peak. Its final energy E_f is 1.7 GeV.

In Fig. 1(b), we show lattice functions when this compressor is implemented with FODO focusing in the L1 and L2 linacs, using 5° bending magnets in BC1 and 2.5°

TABLE I. Parameters describing the chicanes BC1 and BC2 and the compression of 200-pC parabolic bunches.

Property	Symbol	Value
BC1 energy	E_1	215 MeV
BC1 compression ratio	C_1	8
BC1 energy-to-position transfer	$R_{56}^{(1)}$	-87 mm
BC2 energy	E_2	485 MeV
BC2 compression ratio	C_2	2.5
BC2 energy-to-position transfer	$R_{56}^{(2)}$	-18 mm
Initial bunch peak current	I_0	50 A
Peak current after BC1	$C_1 I_0$	400 A
Peak current after BC2	$C_1 C_2 I_0$	1 kA
Initial rms bunch length	σ_z	400 μ m
Bunch length after BC1	σ_z/C_1	50 μ m
Bunch length after BC2	$\sigma_z/C_1 C_2$	20 μ m
Normalized transverse emittance	$\beta\gamma\varepsilon_x, \beta\gamma\varepsilon_y$	1 μ m-rad

bending magnets in BC2 whose radius of curvature is 1.83 m. The lattice functions were computed by the MAD-with-acceleration code [18], modified by DESY to include the transverse focusing in a standing wave cavity [19].

Figure 2 shows the longitudinal phase space during compression of a parabolic bunch with rms length σ_z of 400 μ m, Gaussian energy spread σ_E of 3 keV, and normalized transverse emittances of 1 μ m-rad. In the phase-space plots, the longitudinal coordinate z is zero at the average electron position, with $z > 0$ at the *tail* of the bunch. The tracking of six-dimensional phase space was performed without collective effects by the ELEGANT code [17].

III. IMPEDANCE

A. Sources of impedance

For high bunch currents, the compression is affected by wakes. Consider a current modulation $\Delta I e^{-i\omega t}$, where ω is the angular frequency, t is the arrival time, c is the speed of light, $k = \omega/c$ is the wave number, and λ is the wavelength. The retarding voltage from a longitudinal wake is $Z(\omega)\Delta I e^{-i\omega t}$, where $Z(\omega)$ is the wake's impedance. We consider four sources of longitudinal impedance.

In SI units, the free space longitudinal space charge (LSC) impedance per unit length, expressed as a function of k , is given for $k > 0$ by [8]

$$Z_{\text{LSC}}(k) = \frac{iZ_0}{\pi k r_b^2} \left[1 - \frac{k r_b}{\gamma} K_1\left(\frac{k r_b}{\gamma}\right) \right], \quad (1)$$

where r_b is the radius of a uniform round electron bunch and K_1 is a modified Bessel function. To approximate the impedance for Gaussian or parabolic transverse distributions with rms dimensions σ_x and σ_y , Eq. (1) may be applied with effective radius $r_b = 0.85(\sigma_x + \sigma_y)$ [8].

FIG. 1. (Color) (a) Schematic of the WiFEL preliminary bunch compressor design, optimized for zero current or trapezoidal bunches. Each linac module contains eight 1.3-GHz superconducting cavities. (b) Lattice functions plotted by the MAD-with-acceleration code.

FIG. 2. Bunch compression without collective effects, for an initial parabolic current distribution with rms length of $400 \mu\text{m}$ and 3-keV Gaussian energy spread, according to ELEGANT simulations of 100 000 particles. In the phase-space plots, the longitudinal coordinate z is zero at the average electron position, with $z > 0$ at the *tail* of the bunch. (a) Initial bunch at 4 MeV. (b) Chirped bunch at the entrance of the BC1 chicane. (c) Bunch after single-stage compression in BC1. (d) Chirped bunch at the entrance of the BC2 chicane. (e) Bunch after full compression in BC2. (f) Dechirped bunch at 1.7 GeV.

For TESLA 9-cell superconducting cavities, the high-frequency limit of the linac geometric impedance per unit length is given for $k > 0$ by [20,21]

$$Z_{\text{linac}}(k) = \frac{iZ_0}{\pi k a^2} \left[1 + (1 + i) \frac{\alpha L_{\text{cell}}}{a} \left(\frac{\pi}{kg} \right)^{1/2} \right]^{-1}, \quad (2)$$

where $a = 35 \text{ mm}$ is the iris radius and $L_{\text{cell}} = 115.4 \text{ mm}$ is the cell period [22], $g = 90 \text{ mm}$ is the distance between irises [23], and $\alpha \approx 0.46$ for $g \approx L_{\text{cell}}$ [21].

For a bending magnet in which the orbit's radius of curvature is ρ , the coherent radiation builds up over a distance of $\sim \rho^{2/3} \lambda^{1/3}$ known as the CSR formation length [24]. Provided that the magnet length L_b exceeds $\rho^{2/3} \lambda^{1/3}$, the coherent radiation becomes “fully formed” and approximated by steady-state CSR. The CSR formation length is comparable to the “overtaking length” $(24/2\pi)^{1/3} \rho^{2/3} \lambda^{1/3}$ over which the radiation overtakes the bunch by a distance $\lambda/2\pi$ [25–27], indicating that

the CSR becomes fully formed when it overtakes the modulation that produces it. Downstream of the magnet, the CSR changes to CER over a distance of $\sim \rho^{2/3} \lambda^{1/3}$. At distance d from the magnet entrance obeying $\rho^{2/3} \lambda^{1/3} < d < \rho^{2/3} \lambda^{1/3} + L_b$, we approximate the impedance by the impedance per unit length of steady-state CSR, given for $k > 0$ by [11]

$$Z_{\text{CSR}}(k) = 1.63 Z_0 (1 + 0.58i) k^{1/3} / (4\pi \rho^{2/3}). \quad (3)$$

For typical chicane magnet parameters [$L_b \sim 0.1$ m, $\rho \sim 1$ m, $k \sim 1/\sigma_z \sim 1/(100 \mu\text{m})$], the integrated CSR impedance over the length L_b is $\sim 0.3 Z_0 (1 + 0.58i)$.

Downstream of the magnet, the wake primarily results from coherent edge radiation. For $k > 0$, the integrated resistive CER impedance in a drift space of length L_d is approximately [14]

$$\int Z_{\text{CER}}(k) = \frac{Z_0}{2\pi} \ln\left(\frac{\min(L_d, \lambda \gamma^2 / 2\pi)}{\rho^{2/3} \lambda^{1/3}}\right), \quad (4)$$

for wavelengths where the right-hand side (rhs) of Eq. (4) is positive. For typical chicane magnet parameters [$\gamma \sim 1000$, $\rho \sim 1$ m, $\lambda/2\pi \sim \sigma_z \sim 100 \mu\text{m}$, $L_d \sim 10$ m], the integrated CER impedance is $\sim Z_0$. Because of the weak logarithmic dependence in Eq. (4), an integrated impedance of $\sim Z_0$ may be expected for a wide range of parameters.

Approximating a chicane magnet's impedance by CSR and CER regions is realistic when the lengths of the CSR and CER regions [L_b and $\min(L_d, \lambda \gamma^2 / 2\pi)$] exceed the length of the transition between them ($\sim \rho^{2/3} \lambda^{1/3}$). Equation (4) neglects any reduction of the CER impedance from the bunch radius r_b . This is valid for $\max[r_b, r_b^2/(\lambda/2\pi)] \ll \rho^{2/3} \lambda^{1/3}$ [14], i.e. $r_b \ll \min(\rho^{2/3} \lambda^{1/3}, \rho^{1/3} \lambda^{2/3})$, which is satisfied by typical chicane magnet parameters [$\rho \sim 1$ m, $\lambda/2\pi \sim \sigma_z \sim 100 \mu\text{m}$, $r_b \sim 100 \mu\text{m}$].

For $k < 0$, the impedances are determined by $Z(k) = Z^*(-k)$. The total impedance is the sum of the contributions given by Eqs. (1)–(4).

B. Effective impedances

We consider a symmetrical four-magnet chicane whose wakes are dominated by CER in the drift spaces. We represent each impedance within the chicane with effective impedances before and after the chicane. The effective impedances approximate the wake and effect upon compression of the actual impedance when the other impedances in the chicane are neglected. When calculating the effective impedance of the second chicane magnet, we neglect growth in the relative density modulation between the chicane entrance and the second magnet. This is a good approximation in the case of low gain, which describes a typical chicane when the bunch's emittance and energy spread are taken into account [6]. For moderate gain, this is

a reasonable approximation for a chicane (such as the WIFEL design) that has a short drift space after the second magnet, resulting in a small integrated CER wake. For this purpose, a symmetrical three-magnet chicane may be regarded as a four-magnet chicane with a zero drift length between the 2nd and 3rd magnets.

Suppose an electron loses energy ΔE upstream of the BC1 chicane, so that the deflection in each chicane magnet is increased by $\Delta\theta$. Neglecting the small length of the chicane magnets, its path through the chicane is increased to first order in $\Delta\theta$ by the length $2L_1\theta_1\Delta\theta$, where θ_1 is the unperturbed deflection in a chicane magnet, and L_1 is the distance between the 1st and 2nd chicane magnets. If an electron loses energy ΔE between the 1st and 2nd chicane magnets, so that the deflection in the final three chicane magnets is increased by $\Delta\theta$, its path changes by the same length. If it loses energy ΔE between the 2nd and 3rd magnets, so that the deflection in the final two chicane magnets is increased by $\Delta\theta$, its path changes by half as much. Losing energy after the 3rd magnet does not change the path length. Thus, the effect upon bunch compression of an energy loss between the 1st and 2nd chicane magnets is the same as an energy loss upstream of the chicane; the effect of an energy loss between the 2nd and 3rd magnets is the same as an energy loss one-half as large upstream of the chicane, while the effect of an energy loss after the 3rd magnet is the same as that of an energy loss downstream of the chicane.

For a bunch with negligible emittance and energy spread, a current modulation with wave number k_0 upstream of the chicane, $\Delta I e^{-ik_0 z}$, results in an equal modulation between the 1st and 2nd magnets. The effect of the 1st magnet's impedance $Z_{\text{BC1}}(k)$ upon the wake and bunch compression is given by an equal effective impedance upstream of the chicane. Since the modulation is compressed after exiting the 3rd magnet, we represent the impedances of the 3rd and 4th magnets by effective impedances downstream of the chicane equal to $Z_{\text{BC3}}(k)$ and $Z_{\text{BC4}}(k)$, respectively.

Between the 2nd and 3rd magnets of a chicane with low gain, the half-compressed current modulation and its wakefield are $[2C_1/(C_1 + 1)]\Delta I e^{-i[2C_1/(C_1 + 1)]k_0 z}$ and $[2C_1/(C_1 + 1)]Z_{\text{BC1}_2}\{[2C_1/(C_1 + 1)]k_0\} \times \Delta I e^{-i[2C_1/(C_1 + 1)]k_0 z}$. The effect of the wake upon compression is provided by a wake upstream of the chicane that is half as large. We model this wake with an effective impedance upstream of the chicane entrance given by the function $Z_{\text{eff BC1}_2\text{ent}}(k) = [C_1/(C_1 + 1)]Z_{\text{BC1}_2}\{[2C_1/(C_1 + 1)]k\}$, which acts upon the current modulation $\Delta I e^{-ik_0 z}$. Downstream of a low-gain chicane, the wake between the 2nd and 3rd magnets is compressed to equal $C_1 Z_{\text{BC1}_2}\{[2C_1/(C_1 + 1)]k_0\}\Delta I e^{-iC_1 k_0 z}$, while the wake from its upstream effective impedance is compressed by the factor C_1 to give the smaller wake $[C_1^2/(C_1 + 1)]Z_{\text{BC1}_2}\{[2C_1/(C_1 + 1)]k_0\}\Delta I e^{-iC_1 k_0 z}$. By adding an addi-

tional effective impedance downstream of the chicane exit $Z_{\text{eff}_{\text{BC1}_2\text{exit}}}(k) = [1/(C_1 + 1)]Z_{\text{BC1}_2}\{[2/(C_1 + 1)]k\}$, which acts upon the compressed current modulation $C_1\Delta I e^{-iC_1k_0z}$, we approximate the 2nd magnet's wake and compression.

In a bunch with finite emittance, the amplitude of the modulation is reduced within the chicane by longitudinal smearing [4–6,8]. An electron with horizontal displacement x at the chicane entrance is displaced longitudinally by $x\theta_1$ between the 1st and 2nd magnets. For a symmetric distribution of horizontal positions $f_{1\text{ent}}(x)$ at the entrance of the chicane, the modulation amplitude and wake after the 1st magnet are modified by the form factor $\int f_{1\text{ent}}(x) \cos(kx\theta_1) dx$, which equals $\exp[-(k\theta_1\sigma_{x1\text{ent}})^2/2]$ for a Gaussian distribution with rms size $\sigma_{x1\text{ent}}$. The effect of a given wake upon the energy imparted to a longitudinal slice of the incoming bunch is also modified by this form factor. We approximate this emittance effect by modifying the effective impedance of the 1st chicane magnet's CSR and CER by the square of this form factor.

Similarly, an electron with horizontal displacement x at the chicane exit is displaced longitudinally by $-x\theta_1$ between the 3rd and 4th magnets. For a symmetric distribution of horizontal positions $f_{1\text{exit}}(x)$ at the exit of the chicane, the modulation amplitude and wake after the 3rd magnet are modified by the form factor $\int f_{1\text{exit}}(x) \times \cos(kx\theta_1) dx$, which equals $\exp[-(k\theta_1\sigma_{x1\text{exit}})^2/2]$ for a Gaussian distribution with rms size $\sigma_{x1\text{exit}}$. The effect of a given wake upon the energy imparted to a longitudinal slice of the outgoing bunch is also modified by this form factor. We approximate this emittance effect by modifying the effective impedance of the 3rd magnet's CSR and CER by the square of this form factor.

An electron with horizontal angle x' is displaced longitudinally by $-x'L_1\theta_1$ between the 2nd and 3rd magnets. For a symmetric distribution of horizontal angle $f'_1(x')$ at the entrance of the chicane, the modulation amplitude and wake of the 2nd magnet are modified by the form factor $\int f'_1(x') \cos\{[2C_1/(C_1 + 1)]k_0x'L_1\theta_1\} dx'$ for a modulation with wave number k_0 upstream of the chicane. This form factor equals $\exp(-\{[2C_1/(C_1 + 1)]k_0L_1\theta_1\sigma'_{x1}\}^2/2)$ for a Gaussian distribution with rms divergence σ'_{x1} . The effect of a given wake upon the energy imparted to a longitudinal slice of the incoming bunch is also modified by this form factor. We approximate this effect by modifying the effective impedances of the 2nd chicane magnet's CSR and CER located upstream and downstream of the chicane. The effective impedance upstream of the chicane is modified by the square of the form factor $\int f'_1(x') \times \cos\{[2C_1/(C_1 + 1)]kx'L_1\theta_1\} dx'$; where the form factor equals $\exp(-\{[2C_1/(C_1 + 1)]kL_1\theta_1\sigma'_{x1}\}^2/2)$ for a Gaussian distribution with rms divergence σ'_{x1} . The 2nd magnet's effective impedance downstream of the chicane is modified by the square of the form factor $\int f'_1(x') \times$

$\cos\{[2/(C_1 + 1)]kx'L_1\theta_1\} dx'$, in which we have accounted for the fact that the downstream modulation has wave number $k = C_1k_0$. This form factor equals $\exp(-\{[2/(C_1 + 1)]kL_1\theta_1\sigma'_{x1}\}^2/2)$ for a Gaussian distribution with rms divergence σ'_{x1} .

A similar analysis applies to the second chicane BC2, in which θ_2 is the unperturbed deflection, L_2 is the distance between the first and second magnets, and the bunches have rms divergence σ'_{x2} and horizontal sizes $\sigma_{x2\text{ent}}$ and $\sigma_{x2\text{exit}}$ at the entrance and exit.

For Gaussian distributions of horizontal position and angle, we therefore represent the impedances of the first two magnets of BC1 with effective impedances upstream of the chicane entrance equaling

$$Z_{\text{eff}_{\text{BC1}_1}}(k) = Z_{\text{BC1}_1}(k) \exp(-[k\theta_1\sigma_{x1\text{ent}}]^2) \quad (5a)$$

$$Z_{\text{eff}_{\text{BC1}_2\text{ent}}}(k) = \frac{C_1}{C_1 + 1} Z_{\text{BC1}_2} \left(\frac{2C_1k}{C_1 + 1} \right) \times \exp\left(-\left[\frac{2C_1kL_1\theta_1\sigma'_{x1}}{C_1 + 1}\right]^2\right), \quad (5b)$$

where $Z_{\text{BC1}_1}(k)$ and $Z_{\text{BC1}_2}(k)$ are impedances of the first two magnets of BC1 from CSR or CER. For a modulation with initial wavelength λ_0 and initial wave number k_0 , effective impedances of WiFEL are shown in Fig. 3(a). The total effective impedance $Z_{\text{zero}}(k)$, which acts upon the bunch before it is compressed by BC1, is the sum of the effective CSR and CER impedances of the first two magnets of BC1. For initial wavelengths less than 400 μm , the effective impedance $Z_{\text{zero}}(k)$ is suppressed by the bunch's emittance.

Between BC1 and BC2, we have the effective impedances

$$Z_{\text{eff}_{\text{BC1}_2\text{exit}}}(k) = \frac{1}{C_1 + 1} Z_{\text{BC1}_2} \left(\frac{2k}{C_1 + 1} \right) \times \exp\left(-\left[\frac{2kL_1\theta_1\sigma'_{x1}}{C_1 + 1}\right]^2\right), \quad (6a)$$

$$Z_{\text{eff}_{\text{BC1}_3}}(k) = Z_{\text{BC1}_3}(k) \exp(-[k\theta_1\sigma_{x1\text{exit}}]^2) \quad (6b)$$

$$Z_{\text{eff}_{\text{BC1}_4}}(k) = Z_{\text{BC1}_4}(k) \quad (6c)$$

$$Z_{\text{eff}_{\text{BC2}_1}}(k) = Z_{\text{BC2}_1}(k) \exp(-[k\theta_2\sigma_{x2\text{ent}}]^2) \quad (6d)$$

$$Z_{\text{eff}_{\text{BC2}_2\text{ent}}}(k) = \frac{C_2}{C_2 + 1} Z_{\text{BC2}_2} \left(\frac{2C_2k}{C_2 + 1} \right) \times \exp\left(-\left[\frac{2C_2kL_2\theta_2\sigma'_{x2}}{C_2 + 1}\right]^2\right), \quad (6e)$$

where the impedances on the rhs result from CSR or CER. The effective impedance $Z_1(k)$, which acts upon the bunch between BC1 and BC2, is given by summing these effective CSR and CER impedances with the LSC and linac geometric impedances of the L1 linac. For a modulation with initial wavelength λ_0 , the effective impedances experienced between BC1 and BC2 are shown in Fig. 3(b). The LSC impedance and the linac geometric impedance are

FIG. 3. (Color) Effective impedances experienced by a modulation whose wavelength and wave number before compression are λ_0 and k_0 , respectively. (a) Impedance experienced before compression in BC1. (b) Impedance experienced between BC1 and BC2. (c) Impedance experienced after full compression in BC2.

evaluated for a bunch with normalized horizontal emittance and vertical emittance each equaling $1 \mu\text{m-rad}$, accelerated from 215 to 485 MeV at a constant rate over the 16.6-m length of the L1 linac, in which the traditional noncanonical beta functions [28] obey $\langle\beta_x\rangle \approx \langle\beta_y\rangle \approx 25 \text{ m}$. The total effective impedance experienced by the modulation is $Z_1(C_1 k_0)$.

We represent the remaining impedances of BC2 with effective impedances downstream of BC2 from CSR and CER

$$Z_{\text{eff BC2 exit}}(k) = \frac{1}{C_2 + 1} Z_{\text{BC2}} \left(\frac{2k}{C_2 + 1} \right) \times \exp \left(- \left[\frac{2kL_2\theta_2\sigma'_{x2}}{C_2 + 1} \right]^2 \right), \quad (7a)$$

$$Z_{\text{eff BC23}}(k) = Z_{\text{BC23}}(k) \exp(-[k\theta_2\sigma_{x2\text{exit}}]^2) \quad (7b)$$

$$Z_{\text{eff BC24}}(k) = Z_{\text{BC24}}(k). \quad (7c)$$

The total effective impedance $Z_2(k)$ downstream of BC2 is

given by summing the effective CSR and CER impedances with the LSC and linac geometric impedances of the L2 linac. For a modulation with initial wavelength λ_0 , the effective impedances experienced after BC2 are shown in Fig. 3(c). The LSC impedance and the linac geometric impedance are evaluated for a bunch with normalized horizontal and vertical emittances of $1 \mu\text{m-rad}$ that is accelerated from 485 MeV to 1.7 GeV at a constant rate in the L2 linac of length 124.5 m, where $\langle\beta_x\rangle \approx \langle\beta_y\rangle \approx 25 \text{ m}$. The length of BC2's final two magnets is smaller than the steady-state CSR formation length for $\lambda_0 > 3 \text{ mm}$. For $\lambda_0 > 3 \text{ mm}$, where the CSR impedance gives a negligible contribution to the total impedance, the CSR estimate becomes inaccurate; an accurate estimate would require consideration of the wake of a short magnet [25–27]. The total effective impedance experienced after BC2 is $Z_2(C_1 C_2 k_0)$. Since the bunch's longitudinal particle distribution is effectively frozen after the BC2 chicane, the impedance Z_2 affects the bunch's energy distribution, but not its current profile.

The impedances before and after BC1 are dominated by the resistive impedance of CER for initial wavelengths exceeding 1 mm. Since a bunch with initial rms bunch length σ_z is primarily composed of wavelengths exceeding $2\pi\sigma_z$, the impedances before and after BC1 are dominated by CER for $2\pi\sigma_z > 1 \text{ mm}$, i.e. $\sigma_z > 160 \mu\text{m}$. For short wavelengths, the LSC impedance dominates.

Since we are planning for WIFEL initial bunch lengths of $\sim 400 \mu\text{m}$, we expect that macroscopic wake effects before and after BC1 will be dominated by the resistive impedance of CER.

IV. MACROSCOPIC MODELING WITH RESISTIVE IMPEDANCE

A. Tracking simulations

To study the approximation of wakes by resistive effective impedances, we performed tracking simulations. Numerical noise was reduced by sampling the initial phase space with Halton sequences [7,29]. To isolate the effect of coherent radiation, LSC and linac geometric wakes are not included in these initial simulations.

We performed longitudinal tracking with a modified version of the code LITRACK [30], in which impedances $Z_{\text{zero}}(k)$, $Z_1(k)$, and $Z_2(k)$ (calculated without LSC and linac impedances) are approximated by resistances R_0 , R_1 , and R_2 . To model initial bunch lengths in the range of $400 \mu\text{m}$ to 1 mm , we averaged the values of $|Z_{\text{zero}}(k_0)|$, $|Z_1(C_1 k_0)|$, and $|Z_2(C_1 C_2 k_0)|$ for initial wavelengths of $2\pi \times 400 \mu\text{m}$ and $2\pi \times 1 \text{ mm}$ (i.e., 2.5 and 6.28 mm), obtaining resistances R_0 , R_1 , and R_2 of 327, 1121, and 756 Ω , respectively. Since a drift length of several meters has resistive CER impedance of $\sim Z_0$, and the WIFEL chicanes have much shorter drift lengths after their second magnets, $R_0 \sim Z_0$, $R_1 \sim 3Z_0$, and $R_2 \sim 2Z_0$. These approximate longitudinal simulations are compared with

tracking by the ELEGANT code [17], which tracks a six-dimensional phase space.

Tracking without wakes was performed for a trapezoidal bunch of 100 000 particles with flat-top of 2 mm, peak current of 50 A, Gaussian energy spread of 3 keV, and normalized transverse emittances of 1 $\mu\text{m-rad}$. At the head and tail, the bunch current drops linearly to zero over a distance of 1 mm. Figure 4(a) shows the longitudinal phase space at 1.7 GeV according to the longitudinal tracking code LITRACK and the ELEGANT code. To ensure that the bunch entering the BC1 compressor is the same with both codes, we begin tracking at 215 MeV, where the bunch's energy profile is the sum of the 4-MeV electron-gun energy with the rf fields of the injector linac and harmonic cavities.

The output phase spaces agree within $\Delta E/E = 0.01\%$. The bunch is compressed by a factor of 20 to a peak current of 1 kA; the compressed bunch's flat-top current is constant within 5%.

Figure 4(b) shows the fully compressed trapezoidal bunch at 1.7 GeV, tracked by LITRACK with resistive impedances R_0 , R_1 , and R_2 . The compressed bunch forms three line segments in phase space, in which the middle line segment is horizontal.

Figure 4(b) also shows an ELEGANT simulation where longitudinal coherent-radiation wakes are applied in all of the chicane bending magnets and drift regions downstream of them. The longitudinal coherent-radiation wakes in the magnets and downstream drift regions are computed by ELEGANT in the $\gamma \rightarrow \infty$ limit, according to the method of Ref. [27]. Transient effects at the beginning of the magnets and drift regions are included. We expect this method to provide an accurate calculation of CER for wavelengths where $r_b \ll \min(\rho^{2/3}\lambda^{1/3}, \rho^{1/3}\lambda^{2/3})$, provided that the drift region length L_d does not greatly exceed the edge-radiation formation length of $\lambda\gamma^2/2\pi$ [14]. These requirements are satisfied for wavelengths where the calculated impedance is dominated by CER. Shielding of the wakes by the vacuum chamber is not modeled.

Figure 4(c) shows LITRACK and ELEGANT tracking of a Gaussian bunch with peak current of 50 A and rms bunch length of 1 mm, while Fig. 4(d) shows tracking of a parabolic bunch with peak current of 50 A and rms bunch length of 1 mm. In all cases, an initial Gaussian energy

spread of 3 keV and normalized transverse emittances of 1 $\mu\text{m-rad}$ are modeled.

According to ELEGANT tracking, the CSR and CER wakes increase the horizontal projected and slice emittances at 1.7 GeV by $\sim 30\%$ and $\sim 10\%$, respectively, while shifting the bunch centroid horizontally by $\sim 100 \mu\text{m}$ in position and $\sim 5 \mu\text{rad}$ in angle. These shifts, which are comparable to the bunch's rms size and divergence, can be compensated by steering magnets.

The similarity of the longitudinal tracking with LITRACK and ELEGANT confirms that the macroscopic effects of coherent radiation can be approximated by resistive effective impedances. When the geometric wakes of the L1 and L2 linacs are also included in the tracking, the only significant difference is a linear chirp of the bunch at 1.7 GeV. This is consistent with Fig. 3, in which CER is the dominant impedance for wavelengths comparable to 2π times the bunch length.

B. Upright bunches in phase space

In Appendix A, we consider the passage of a cold bunch through a two-stage compressor with resistive impedances R_0 , R_1 , and R_2 . The longitudinal coordinates of an electron after each chicane, z_1 and z_2 , are related to the initial coordinate z . Since the longitudinal particle distribution is effectively frozen after BC2, the relation $z_2(z)$ applies to WIFEL when the impedances $Z_{\text{zero}}(k)$ and $Z_1(k)$ are approximately resistive: for $\sigma_z > 160 \mu\text{m}$.

For a bunch with initial current magnitude $I(z)$, whose initial energy profile differs from a linear chirp by $\Delta E(z)$, the longitudinal coordinate after BC1 obeys

$$z_1(z) = z/C_1 + (R_{56}^{(1)}/E_1)[\Delta E(z) - eR_0I(z)], \quad (8)$$

with derivative with respect to z of

$$z_1'(z) = C_1^{-1}\{1 + C_1(R_{56}^{(1)}/E_1)[\Delta E'(z) - eR_0I'(z)]\}, \quad (9)$$

where $e > 0$ is the magnitude of the electron charge.

When the rhs of Eq. (9) is zero, an upright bunch occurs after BC1. Thus, part of the bunch is upright after BC1 when

$$(-C_1R_{56}^{(1)}/E_1)[\Delta E'(z) - eR_0I'(z)] = 1. \quad (10)$$

When none of the bunch is upright after BC1, the longitudinal coordinate after BC2 is

$$z_2(z) = \frac{1}{C_1C_2}\left\{z + \left(\frac{C_1R_{56}^{(1)}}{E_1} + \frac{C_1^2C_2R_{56}^{(2)}}{E_2}\right)[\Delta E(z) - eR_0I(z)] - \frac{C_1^2C_2(R_{56}^{(2)}/E_2)eR_1I(z)}{1 + C_1(R_{56}^{(1)}/E_1)[\Delta E'(z) - eR_0I'(z)]}\right\} \quad (11)$$

with derivative

$$\begin{aligned} z_2'(z) = \frac{1}{C_1C_2}\left\{1 + \left(\frac{C_1R_{56}^{(1)}}{E_1} + \frac{C_1^2C_2R_{56}^{(2)}}{E_2}\right)[\Delta E'(z) - eR_0I'(z)] - \frac{C_1^2C_2(R_{56}^{(2)}/E_2)eR_1I'(z)}{1 + C_1(R_{56}^{(1)}/E_1)[\Delta E'(z) - eR_0I'(z)]}\right. \\ \left. + \frac{C_1^3C_2(R_{56}^{(1)}/E_1)(R_{56}^{(2)}/E_2)eR_1I(z)[\Delta E''(z) - eR_0I''(z)]}{\{1 + C_1(R_{56}^{(1)}/E_1)[\Delta E'(z) - eR_0I'(z)]\}^2}\right\}. \quad (12) \end{aligned}$$

FIG. 4. Phase space after compression and acceleration to 1.7 GeV, for bunches with peak current of 50 A before compression. The bunch tail is at positive z . LITRACK (with resistive impedance) results are on the left, while ELEGANT results with coherent radiation are on the right. (a) Trapezoidal bunch with 2-mm flattop, 1-mm head and tail, tracked without collective effects. (b) Trapezoidal bunch with 2-mm flattop, 1-mm head and tail. (c) Gaussian bunch with $\sigma_z = 1$ mm. (d) Parabolic bunch with $\sigma_z = 1$ mm.

When $z'_2(z) = 0$, part of the bunch is upright in phase space after BC2.

When $|(C_1 R_{56}^{(1)}/E_1)[\Delta E'(z) - eR_0 I'(z)]| \ll 1$ for all z , upright bunch behavior after BC1 is safely avoided, and Eq. (12) may be approximated as

$$z'_2(z) \approx \frac{1}{C_1 C_2} \left\{ 1 + \left(\frac{C_1 R_{56}^{(1)}}{E_1} + \frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) \times [\Delta E'(z) - eR_0 I'(z)] - \frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} eR_1 I'(z) \right\}, \quad (13)$$

provided that $I(z)$ and $\Delta E(z)$ do not vary on a length scale much shorter than the bunch.

For a bunch with a linear chirp $[\Delta E(z) = 0]$, Eq. (13) predicts upright bunch behavior for

$$I'(z) = \left[\left(\frac{C_1 R_{56}^{(1)}}{E_1} + \frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) eR_0 + \left(\frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) eR_1 \right]^{-1}. \quad (14)$$

When $\Delta E(z) = eR_0 I(z)$, the effect of the resistance R_0 is perfectly compensated to achieve linear compression in BC1. In this idealized case, upright bunch behavior is predicted by Eq. (13) for

$$I'(z) = E_2 / (C_1^2 C_2 R_{56}^{(2)} eR_1). \quad (15)$$

C. Minimum initial bunch length

Consider a bunch with peak current magnitude I_0 that is compressed to a peak current $C_1 C_2 I_0$. In this section we show that the compressed bunch may have an upright tail region in phase space and an associated current spike if the initial uncompressed bunch is too short. We consider the case where $|(C_1 R_{56}^{(1)}/E_1)[\Delta E'(z) - eR_0 I'(z)]| \ll 1$, so that upright bunch behavior after BC1 is safely avoided.

For a linearly chirped trapezoidal initial bunch, in which the current at the head and tail drop linearly to zero over a distance Δz , Eq. (14) indicates that the tail of the compressed bunch will be upright in phase space after BC2 when

$$\Delta z = \Delta z_{\text{crit}} = \left| \left(\frac{C_1 R_{56}^{(1)}}{E_1} + \frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) eR_0 + \left(\frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) eR_1 \right| I_0. \quad (16)$$

For $I_0 = 50$ A, $\Delta z_{\text{crit}} = 480 \mu\text{m}$ is predicted for WifEL. The upright bunch tail predicted for $\Delta z = \Delta z_{\text{crit}}$ occurs in the LITRACK and ELEGANT simulations of Fig. 5(a) performed for normalized transverse emittances of $1 \mu\text{m-rad}$.

If the effect of R_0 could be perfectly compensated to give linear compression in BC1, Eq. (15) predicts an up-

right tail would occur for

$$\Delta z = \Delta z_{\text{crit}} = |C_1^2 C_2 R_{56}^{(2)} eR_1 I_0 / E_2|. \quad (17)$$

For perfect compensation, which is impractical for the tails of a trapezoidal bunch, Δz_{crit} becomes $330 \mu\text{m}$.

For a Gaussian initial bunch with peak current I_0 , $I(z) = I_0 e^{-z^2/2\sigma_z^2}$. An upright tail region is predicted for a linear chirp by Eq. (14) when

$$\sigma_z \leq \sigma_{z_{\text{crit}}} = e^{-1/2} \left| \left(\frac{C_1 R_{56}^{(1)}}{E_1} + \frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) eR_0 + \left(\frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) eR_1 \right| I_0. \quad (18)$$

For $I_0 = 50$ A, $\sigma_{z_{\text{crit}}}$ equals $290 \mu\text{m}$. Gaussian bunches tracked in Fig. 5(b) show upright bunch tails for $\sigma_z = \sigma_{z_{\text{crit}}}$.

If the effect of R_0 could be perfectly compensated to give linear compression in BC1, Eq. (15) predicts an upright tail for

$$\sigma_z \leq \sigma_{z_{\text{crit}}} = e^{-1/2} |C_1^2 C_2 R_{56}^{(2)} eR_1 I_0 / E_2|, \quad (19)$$

which equals $200 \mu\text{m}$.

For a parabolic initial bunch with peak current I_0 , $I(z) = I_0 [1 - z^2/(5\sigma_z^2)]$. An upright tail region is predicted for a linear chirp by Eq. (14) when

$$\sigma_z \leq \sigma_{z_{\text{crit}}} = \frac{2}{\sqrt{5}} \left| \left(\frac{C_1 R_{56}^{(1)}}{E_1} + \frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) eR_0 + \left(\frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) eR_1 \right| I_0. \quad (20)$$

For $I_0 = 50$ A, $\sigma_{z_{\text{crit}}}$ equals $430 \mu\text{m}$. The parabolic bunches tracked in Fig. 5(c) show upright bunch tails and current spikes for $\sigma_z = \sigma_{z_{\text{crit}}}$.

If the effect of R_0 could be perfectly compensated to give linear compression in BC1, Eq. (15) predicts an upright tail for

$$\sigma_z \leq \sigma_{z_{\text{crit}}} = (2/\sqrt{5}) |C_1^2 C_2 R_{56}^{(2)} eR_1 I_0 / E_2|, \quad (21)$$

which equals $300 \mu\text{m}$. With perfect compensation of R_0 , the bunch length may be decreased 30% without obtaining an upright tail after BC2.

D. Wake compensation

In WifEL, we plan to compress a 200-pC parabolic bunch with peak current of 50 A, rms bunch length of $400 \mu\text{m}$, and normalized transverse emittances of $1 \mu\text{m-rad}$. Figure 5(c) indicates that the compressed bunch

FIG. 5. Phase space at 1.7 GeV showing upright bunch tails for bunches with peak current of 50 A and Gaussian energy spread of 3 keV before compression. LITRACK (with resistive impedance) results are on the left; ELEGANT results with coherent radiation are on the right. (a) Trapezoidal bunch with 2-mm flattop, 480- μm head and tail. (b) Gaussian bunch with $\sigma_z = 290 \mu\text{m}$. (c) Parabolic bunch with $\sigma_z = 430 \mu\text{m}$.

will have an upright tail if wake effects are not compensated. For a parabolic bunch, the resistive impedance R_0 can be compensated by reducing the harmonic cavity voltage so that the rf curvature provides a nonlinear (quadratic) chirp at the entrance of BC1, in addition to the approximately linear chirp.

We adjusted the rf parameters by longitudinal tracking with a modified version of LITRACK that applies resistive impedances R_0 , R_1 , and R_2 . The chicane parameters and energies were not changed. With the quick turnaround time of LITRACK, acceptable compensation is quickly found by trial and error. A LITRACK simulation of the compressed bunch at 1.7 GeV is shown in Fig. 6(a). According to

LITRACK, the adjusted rf parameters provide good compression in which the energy profile has a minimum when the bunch current peaks at 1 kA. This behavior is nearly reproduced in Fig. 6(a) when 6D tracking from 215 MeV to 1.7 GeV is performed with coherent radiation by the ELEGANT code, using the rf parameters obtained from the LITRACK simulations.

When the geometric wakes of the L1 and L2 linacs are also included in the LITRACK model, adjusting the phases of the L1 and L2 linacs by trial and error gives the compressed bunch shown in Fig. 6(b). Figure 6(b) also shows tracking by ELEGANT in which coherent radiation and the linac geometric wakes of L1 and L2 are included. The

FIG. 6. Compensation of wake effects for an initial parabolic bunch with peak current of 50 A, $\sigma_z = 400 \mu\text{m}$, and Gaussian energy spread of 3 keV. The left-hand plots show the longitudinal phase space at 1.7 GeV according to LITRACK (with resistive impedance approximation of chicane wakes). The middle plots show the longitudinal phase space at 1.7 GeV according to ELEGANT. The right-hand plots show the current distribution at 1.7 GeV according to ELEGANT. The bunch tail is at positive z . (a) CSR and CER wakes. (b) CSR, CER, linac L1, and linac L2 wakes. (c) CSR, CER, linac L1, linac L2, injector linac, and harmonic cavity wakes.

ELEGANT tracking confirms that good compression is provided by the rf parameters obtained from LITRACK. In the ELEGANT tracking, the bunch current peaks at 870 A in Figs. 6(a) and 6(b). The peak current may be increased to 1 kA by fine-tuning, in which the injector linac phase is increased by 0.15° .

In Fig. 6(c), we also include the geometric wakes of the injector linac and harmonic cavities, calculated for a bunch whose current distribution is frozen between 4 and 215 MeV. Using LITRACK to adjust rf parameters by trial and error, we obtained the compressed bunch phase space shown in Fig. 6(c), in which the energy and current are

nearly constant for the central portion of the bunch. The wakes of the injector linac and harmonic cavities cause current spikes at the head and tail of the compressed bunch. Figure 6(c) also shows tracking by ELEGANT that includes the injector linac and harmonic cavity geometric wakes, coherent radiation and linac geometric wakes. The ELEGANT tracking confirms that good compression is provided by the rf parameters obtained from LITRACK.

In Figs. 6(a)–6(c), the harmonic cavity voltage was in the range of 8–20 MeV, the injector linac phase was between -21° and -18.95° , while the L1 phase was between -5° and 13° . A large L2 phase between 40°

and 50° dechirps the bunch before its energy reaches 1.7 GeV.

Figure 6 shows that the resistive-wake approximation of CER is useful in compensating wakes by trial and error. By simulating and compensating resistive wakes with a modified version of LITRACK, we obtained the bunch compressor parameters of Table I that can compress a 200-pC parabolic bunch by a factor of 20 to obtain a current of 1 kA. Using the resistive-wake approximation of CER may also im-

prove the compensation of wakes by reverse tracking of the longitudinal phase space, in comparison with reverse-tracking that neglects CSR and CER [31].

For the case where coherent radiation and the geometric wakes of the L1 and L2 linacs are included in ELEGANT tracking, Fig. 7 shows details of the compression process. Figure 8 shows details when tracking includes coherent radiation and the geometric wakes of the injector linac, harmonic cavities, L1 and L2 linacs.

FIG. 7. Bunch compression for an initial parabolic current distribution with rms length of $400\ \mu\text{m}$ and charge of 200 pC. The longitudinal phase space evolution is given by ELEGANT simulations of 100 000 particles that include the coherent-radiation wakes of both chicanes in addition to the geometric wakes of the L1 and L2 linacs. (a) Initial bunch at 4 MeV. (b) Chirped bunch at the entrance of the BC1 chicane. (c) Bunch after single-stage compression in BC1. (d) Chirped bunch at the entrance of the BC2 chicane. (e) Bunch after full compression in BC2. (f) Dechirped bunch at 1.7 GeV.

E. Jitter

In WiFEL, compressed bunches are seeded with a laser to generate the FEL output. Consequently, the allowable jitter in the bunch arrival time at the FEL with respect to the reference timing is smaller than the compressed bunch duration. Equation (11) may be used to estimate the jitter resulting from variations in current, energy, and chirp at the entrance of the two-stage compressor. We consider the case where $|(C_1 R_{56}^{(1)}/E_1)[\Delta E'(z) - eR_0 I'(z)]| \ll 1$, so that up-right bunch behavior after BC1 is safely avoided. In this

case, Eq. (11) gives $\partial z_2(z)/\partial I(z) \approx (-C_1 C_2)^{-1}[(C_1 R_{56}^{(1)}/E_1 + C_1^2 C_2 R_{56}^{(2)}/E_2)eR_0 + C_1^2 C_2 R_{56}^{(2)}eR_1/E_2]$. Thus, the arrival-time jitter δt is related to the bunch-to-bunch variation in the initial current profile δI by

$$c\delta t \approx \left| (C_1 C_2)^{-1} \left[\left(\frac{C_1 R_{56}^{(1)}}{E_1} + \frac{C_1^2 C_2 R_{56}^{(2)}}{E_2} \right) eR_0 + \frac{C_1^2 C_2 R_{56}^{(2)} eR_1}{E_2} \right] \right| \delta I. \quad (22)$$

FIG. 8. Bunch compression for an initial parabolic current distribution with rms length of $400 \mu\text{m}$ and charge of 200 pC . The longitudinal phase space evolution is given by ELEGANT simulations of 100 000 particles that include the coherent-radiation wakes of both chicane in addition to the geometric wakes of the injector, harmonic cavities, L1 and L2 linacs. (a) Initial bunch at 4 MeV. (b) Chirped bunch at the entrance of the BC1 chicane. (c) Bunch after single-stage compression in BC1. (d) Chirped bunch at the entrance of the BC2 chicane. (e) Bunch after full compression in BC2. (f) Dechirped bunch at 1.7 GeV.

From Eq. (11), $\partial z_2(z)/\partial \Delta E(z) = (C_1 C_2)^{-1} \times (C_1 R_{56}^{(1)}/E_1 + C_1^2 C_2 R_{56}^{(2)}/E_2)$, so that the arrival-time jitter is related to the bunch-to-bunch energy variation δE at the entrance of BC1 by

$$c\delta t = |(C_1 C_2)^{-1} (C_1 R_{56}^{(1)}/E_1 + C_1^2 C_2 R_{56}^{(2)}/E_2)| \delta E. \quad (23)$$

The bunch arrival time is insensitive to δE when $C_1 R_{56}^{(1)}/E_1 + C_1^2 C_2 R_{56}^{(2)}/E_2 = 0$, which requires overcompression in one of the stages. [In this case, the total compression is also insensitive to chirp, since $\partial(C_1 C_2)/\partial h_1 = -C_1 C_2 E_1 (C_1 R_{56}^{(1)}/E_1 + C_1^2 C_2 R_{56}^{(2)}/E_2)$, where h_1 is the relative energy chirp.] However, overcompression may cause undesirable large wakes within a chicane magnet, splitting the bunch longitudinally into two bunches [15].

From Eq. (11), $\partial z_2(z)/\partial \Delta E'(z) \approx C_1^2 R_{56}^{(1)} R_{56}^{(2)} e R_1 I(z)/E_1 E_2$. For peak current I_0 , the arrival-time jitter is related to the bunch-to-bunch chirp variation $\delta E'$ by

$$c\delta t \approx |C_1^2 R_{56}^{(1)} R_{56}^{(2)} e R_1 I_0 / (E_1 E_2)| \delta E'. \quad (24)$$

For WifEL [16], jitter less than 15 fs rms is desired. Evaluating Eqs. (22)–(24) with resistances R_0 and R_1 of 327 and 1121 Ω , we find that this requires rms errors $\delta I < 9.3$ A, $\delta E < 9.8$ keV, and $\delta E' < 83$ MeV/m, in approximate agreement with ELEGANT simulations. The requirement $\delta I < 9.3$ A is satisfied if bunches are created at a photocathode with creation-time variation $\ll \sigma_z/c = 1.3$ ps and peak current variation < 9.3 A. Under the assumption of uncorrelated errors in the injector linac cavities, the expected error of N cavities is $N^{1/2}$ times the error of a single cavity. For this case, a cavity's amplitude and phase variation must be less than 0.02% and 0.03° rms to achieve $\delta E < 9.8$ keV, in which case the requirement $\delta E' < 83$ MeV/m is also satisfied. The required amplitude and phase stability are at the current state of the art in control of superconducting cavities [32]. For acceleration from 485 MeV to 1.7 GeV with linac phase of 40°–50°, a 15-fs arrival-time jitter after BC2 contributes energy jitter up to 180 keV in the 1.7 GeV bunches, which is smaller than the allowable energy jitter of ~ 500 keV.

V. MICROBUNCHING

When a high-current bunch is compressed in magnetic chicanes, longitudinal modulations of the bunch current and energy may grow [2–10]. To prevent large modulations in the compressed bunch, growth may be suppressed by increasing the energy spread of the uncompressed bunch with a laser heater [7,8]. Microbunching growth may be calculated by modeling the effect of CSR, LSC, and/or linac wakes upon the bunch [4–6,8–10]. However, we expect that CER and LSC are the dominant wakes for the WifEL compressor.

In Appendix B, we model CER, LSC, CSR, and linac geometric wakes with frequency-dependent effective im-

pedances, for the case where $Z_{\text{zero}}(k) = 0$. At the entrance of the first chicane, we consider a bunch in which each slice has the symmetric normalized energy distribution $f(\delta)$, where δ is the energy deviation from an ideal linear chirp. This describes an uncompressed bunch with slice energy spread from the electron gun or a matched laser heater. We follow the bunch's evolution through both stages of compression, including the rotated phase space at the entrance of the second chicane BC2. We obtain analytical formulas for the growth of a small single-wavelength modulation of the bunch current or energy of the form $\Delta I e^{-i\omega_0 t}$ or $\Delta E e^{-i\omega_0 t}$. The growth of a sum of small modulations is given by linear superposition.

The frequency-dependent effective impedances downstream of the two chicanes are denoted $Z_1(k)$ and $Z_2(k)$. For an initial modulation with wave number $k_0 = \omega_0/c$, let $Z_1 \equiv Z_1(C_1 k_0)$ and $Z_2 \equiv Z_2(C_1 C_2 k_0)$.

Each longitudinal slice of the uncompressed bunch (i.e., value of z) has a distribution of energies and an average energy $\varepsilon(z)$. Similarly, each slice of the compressed bunch has a distribution of energies and an average energy. For a bunch entering BC1 at energy E_1 , longitudinal modulations of the bunch current I and average energy E obey, for single-stage compression with $Z_{\text{zero}}(k_0) = 0$,

$$T_{II} \equiv \frac{\Delta I/I_{\text{out}}}{\Delta I/I_{\text{in}}} = F_1, \quad (25)$$

$$T_{EI} \equiv \frac{\Delta E/E_{\text{out}}}{\Delta E/E_{\text{in}}} = \frac{-F_1 C_1 e Z_1 I_0}{E_{\text{out}}} + \frac{i G_1 C_1}{E_{\text{out}}}, \quad (26)$$

$$T_{IE} \equiv \frac{\Delta I/I_{\text{out}}}{\Delta E/E_{\text{in}}} = i F_1 C_1 k_0 R_{56}^{(1)}, \quad (27)$$

$$\begin{aligned} T_{EE} &\equiv \frac{\Delta E/E_{\text{out}}}{\Delta E/E_{\text{in}}} \\ &= \frac{F_1 C_1 E_1}{E_{\text{out}}} - \frac{G_1 C_1^2 k_0 R_{56}^{(1)}}{E_{\text{out}}} - \frac{i F_1 C_1^2 e Z_1 I_0 k_0 R_{56}^{(1)}}{E_{\text{out}}}, \end{aligned} \quad (28)$$

where

$$\begin{aligned} F_1 &= \int \cos(k_0 C_1 R_{56}^{(1)} \delta / E_1) f(\delta) d\delta, \\ G_1 &= \int \delta \sin(k_0 C_1 R_{56}^{(1)} \delta / E_1) f(\delta) d\delta. \end{aligned} \quad (29)$$

The quantities F_1 and G_1 describe the suppression of gain at short wavelengths from the bunch's energy spread.

For a Gaussian energy distribution $f(\delta)$, the single-stage gain $(\Delta I/I)_{\text{out}}/(\Delta E/E)_{\text{in}}$ agrees with Eq. 4 of Ref. [4]. Equations (25)–(28) describe the modulation at any location between compression in the BC1 chicane and the entrance of the BC2 chicane, provided that E_{out} is the nominal energy at that location and Z_1 is the effective

longitudinal impedance experienced after compression in BC1, prior to reaching that location.

For a bunch entering BC1 at energy E_1 , 2-stage compression with $Z_{\text{zero}}(k_0) = 0$ obeys

$$T_{II} \equiv \frac{\Delta I/I_{\text{out}}}{\Delta I/I_{\text{in}}} = F_3 - \frac{iF_1 F_2 C_1^2 C_2 e Z_1 I_0 k_0 R_{56}^{(2)}}{E_2} \quad (30)$$

$$\begin{aligned} T_{EI} &\equiv \frac{\Delta E/E_{\text{out}}}{\Delta I/I_{\text{in}}} \\ &= -\frac{C_1 C_2 (F_1 F_2 Z_1 + F_3 Z_2) e I_0}{E_{\text{out}}} \\ &\quad + \frac{F_1 G_2 C_1^3 C_2^2 e Z_1 I_0 k_0 R_{56}^{(2)}}{E_2 E_{\text{out}}} \\ &\quad + \frac{iF_1 F_2 C_1^3 C_2^2 e^2 Z_1 Z_2 I_0^2 k_0 R_{56}^{(2)}}{E_2 E_{\text{out}}} + \frac{iG_3 C_1 C_2}{E_{\text{out}}} \end{aligned} \quad (31)$$

$$\begin{aligned} T_{IE} &\equiv \frac{\Delta I/I_{\text{out}}}{\Delta E/E_{\text{in}}} \\ &= \frac{F_1 F_2 C_1^3 C_2 e Z_1 I_0 k_0^2 R_{56}^{(1)} R_{56}^{(2)}}{E_2} + iF_3 C_1 k_0 R_{56}^{(1)} \\ &\quad + \frac{iF_3 C_1^2 C_2 k_0 R_{56}^{(2)} E_1}{E_2} \end{aligned} \quad (32)$$

$$\begin{aligned} T_{EE} &\equiv \frac{\Delta E/E_{\text{out}}}{\Delta E/E_{\text{in}}} \\ &= \frac{F_3 C_1 C_2 E_1}{E_{\text{out}}} - \frac{F_1 F_2 C_1^4 C_2^2 e^2 Z_1 Z_2 I_0^2 k_0^2 R_{56}^{(1)} R_{56}^{(2)}}{E_2 E_{\text{out}}} \\ &\quad - \frac{G_3 C_1^3 C_2 k_0 R_{56}^{(1)}}{E_{\text{out}}} - \frac{G_3 C_1^3 C_2^2 k_0 R_{56}^{(2)} E_1}{E_2 E_{\text{out}}} \\ &\quad - \frac{iF_1 F_2 C_1^2 C_2 e Z_1 I_0 k_0 R_{56}^{(1)}}{E_{\text{out}}} - \frac{iF_3 C_1^2 C_2 e Z_2 I_0 k_0 R_{56}^{(1)}}{E_{\text{out}}} \\ &\quad - \frac{iF_3 C_1^3 C_2^2 e Z_2 I_0 k_0 R_{56}^{(2)} E_1}{E_2 E_{\text{out}}} \\ &\quad + \frac{iF_1 G_2 C_1^4 C_2^2 e Z_1 I_0 k_0^2 R_{56}^{(1)} R_{56}^{(2)}}{E_2 E_{\text{out}}}, \end{aligned} \quad (33)$$

where

$$\begin{aligned} F_2 &= \int \cos(k_0 C_1^2 C_2 R_{56}^{(2)} \delta / E_2) f(\delta) d\delta, \\ G_2 &= \int \delta \sin(k_0 C_1^2 C_2 R_{56}^{(2)} \delta / E_2) f(\delta) d\delta, \end{aligned} \quad (34)$$

and

$$\begin{aligned} F_3 &= \int \cos[k_0 (C_1 R_{56}^{(1)} / E_1 + C_1^2 C_2 R_{56}^{(2)} / E_2) \delta] f(\delta) d\delta, \\ G_3 &= \int \delta \sin[k_0 (C_1 R_{56}^{(1)} / E_1 + C_1^2 C_2 R_{56}^{(2)} / E_2) \delta] f(\delta) d\delta. \end{aligned} \quad (35)$$

Equations (30)–(33) describe the modulation at any location after compression in the BC2 chicane, provided that E_{out} is the nominal energy at that location and Z_2 is the effective longitudinal impedance experienced by the bunch after compression in BC2, prior to reaching that location. Since the phase space entering BC2 is rotated by the previous compression in BC1, the microbunching growth in BC2 differs from that given by substituting the BC2 parameters into Eqs. (25)–(29).

Now consider effective impedance $Z_{\text{zero}}(k)$ upstream of BC1. In this case, a current modulation $\Delta I e^{-i\omega_0 t}$ entering BC1 is accompanied by an energy modulation $-e Z_{\text{zero}}(k_0) \Delta I e^{-i\omega_0 t}$ after traversing the impedance $Z_{\text{zero}}(k)$. The growth of each modulation is given by Eqs. (25)–(28) or Eqs. (30)–(33), so we easily obtain gain formulas which include the effect of $Z_{\text{zero}}(k)$:

$$\frac{\Delta I/I_{\text{out}}}{\Delta I/I_{\text{in}}} = T_{II} - \frac{e Z_{\text{zero}}(k_0) I_0}{E_1} T_{IE} \quad (36)$$

$$\frac{\Delta E/E_{\text{out}}}{\Delta I/I_{\text{in}}} = T_{EI} - \frac{e Z_{\text{zero}}(k_0) I_0}{E_1} T_{EE} \quad (37)$$

$$\frac{\Delta I/I_{\text{out}}}{\Delta E/E_{\text{in}}} = \frac{E_{\text{in}}}{E_1} T_{IE} \quad (38)$$

$$\frac{\Delta E/E_{\text{out}}}{\Delta E/E_{\text{in}}} = \frac{E_{\text{in}}}{E_1} T_{EE}. \quad (39)$$

For single-stage compression, T_{II} , T_{IE} , T_{EI} , and T_{EE} are given by Eqs. (25)–(28), while Eqs. (30)–(33) describe two-stage compression. Equations (36)–(39) may be used to calculate microbunching growth for a bunch with initial energy $E_{\text{in}} \leq E_1$, provided that the bunch is effectively frozen at energies between E_{in} and E_1 . In this case, $Z_{\text{zero}}(k_0)$ describes the total impedance between E_{in} and BC1, including the effective impedances of the first two magnets of BC1.

For an initial Gaussian energy distribution $f(\delta)$ with rms energy spread of σ_E [33], the growth suppression from the bunch's energy spread is characterized by

$$\begin{aligned}
F_1 &= \exp[-(k_0 C_1 R_{56}^{(1)} \sigma_E / E_1)^2 / 2], \\
F_2 &= \exp[-(k_0 C_1^2 C_2 R_{56}^{(2)} \sigma_E / E_2)^2 / 2], \\
F_3 &= \exp\{-[k_0(C_1 R_{56}^{(1)} / E_1 + C_1^2 C_2 R_{56}^{(2)} / E_2) \sigma_E]^2 / 2\}, \\
G_1 &= (k_0 C_1 R_{56}^{(1)} / E_1) \sigma_E^2 \exp[-(k_0 C_1 R_{56}^{(1)} \sigma_E / E_1)^2 / 2], \\
G_2 &= (k_0 C_1^2 C_2 R_{56}^{(2)} / E_2) \sigma_E^2 \exp[-(k_0 C_1^2 C_2 R_{56}^{(2)} \sigma_E / E_2)^2 / 2], \\
G_3 &= k_0(C_1 R_{56}^{(1)} / E_1 + C_1^2 C_2 R_{56}^{(2)} / E_2) \sigma_E^2 \\
&\quad \times \exp\{-[k_0(C_1 R_{56}^{(1)} / E_1 + C_1^2 C_2 R_{56}^{(2)} / E_2) \sigma_E]^2 / 2\}.
\end{aligned} \tag{40}$$

The curves in Fig. 9(a) show the microbunching gain of the WiFEL 2-stage compressor for a bunch with current of 50 A, normalized transverse emittances of 1 $\mu\text{m-rad}$, and an initial Gaussian energy distribution with $\sigma_E = 3$ keV, calculated for the impedances of Fig. 3. At an initial wavelength of 100 μm , the bunch's emittance reduces the calculated gain by 27% for an initial energy modulation and by 87% for an initial current modulation. For the wavelengths shown, the impedance Z_{zero} is suppressed by the bunch's emittance, so that similar gain curves are obtained if the impedance Z_{zero} is neglected.

When the initial energy spread results from a laser heater that is matched to the transverse beam size, the distribution $f(\delta)$ is approximately semicircular [8]:

$$f(\delta) = (2/\pi \delta_{\text{max}}) \sqrt{1 - (\delta/\delta_{\text{max}})^2}, \tag{41}$$

where [34] $\delta_{\text{max}} = 2\sigma_E$. In this case [35],

$$\begin{aligned}
F_1 &= (k_0 C_1 R_{56}^{(1)} \sigma_E / E_1)^{-1} J_1(2k_0 C_1 R_{56}^{(1)} \sigma_E / E_1), \\
F_2 &= (k_0 C_1^2 C_2 R_{56}^{(2)} \sigma_E / E_2)^{-1} J_1(2k_0 C_1^2 C_2 R_{56}^{(2)} \sigma_E / E_2), \\
F_3 &= [k_0(C_1 R_{56}^{(1)} / E_1 + C_1^2 C_2 R_{56}^{(2)} / E_2) \sigma_E]^{-1} \\
&\quad \times J_1[2k_0(C_1 R_{56}^{(1)} / E_1 + C_1^2 C_2 R_{56}^{(2)} / E_2) \sigma_E], \\
G_1 &= (k_0 C_1 R_{56}^{(1)} / 2E_1)^{-1} J_2(2k_0 C_1 R_{56}^{(1)} \sigma_E / E_1), \\
G_2 &= (k_0 C_1^2 C_2 R_{56}^{(2)} / 2E_2)^{-1} J_2(2k_0 C_1^2 C_2 R_{56}^{(2)} \sigma_E / E_2), \\
G_3 &= [k_0(C_1 R_{56}^{(1)} / E_1 + C_1^2 C_2 R_{56}^{(2)} / E_2) / 2]^{-1} \\
&\quad \times J_2[2k_0(C_1 R_{56}^{(1)} / E_1 + C_1^2 C_2 R_{56}^{(2)} / E_2) \sigma_E].
\end{aligned} \tag{42}$$

The curves in Fig. 9(b) show the microbunching gain for a semicircular energy distribution with $\sigma_E = 10$ keV, calculated for the impedances of Fig. 3. The zeroes of the Bessel functions in Eq. (42) result in localized dips in the gain curve.

To test the gain formulas, we performed ELEGANT tracking of an initial trapezoidal bunch with flattop of 2 mm, head and tail of length 1 mm, and peak current of 50 A. An initial current or energy modulation was applied to the flattop of the bunch, which was then tracked from the entrance of the BC1 chicane at 215 MeV to the exit of

FIG. 9. (Color) Curves show the analytical microbunching gain between the entrance of the BC1 chicane at 215 MeV and the exit of the L2 linac at 1.7 GeV, for a bunch with initial current of 50 A. The gain is shown as a function of the modulation's initial wavelength λ_0 . Plotted points show the gain in 4-million particle simulations of a 500-pC bunch with an initial trapezoidal longitudinal profile with flattop length of 2 mm and flattop current of 50 A. (a) Gaussian energy distribution with initial rms energy spread of 3 keV. (b) Semicircular energy distribution from a matched laser heater, with initial rms energy spread of 10 keV.

the L2 linac at 1.7 GeV. The tracking includes longitudinal coherent-radiation wakes (i.e., CSR and CER), the longitudinal LSC wakes of the L1 and L2 linacs, in addition to the longitudinal and transverse geometric wakes of the linacs. Noise in the simulations was reduced by tracking 4 million particles whose distributions are populated in each of the six dimensions of phase space by using Halton sequences [7,29]. High-frequency noise was further reduced by cutting off the CSR, CER, and LSC wakes at frequencies exceeding the third harmonic of the modulation. To avoid saturation effects, the current modulations in the simulations are $\leq 10\%$, while the initial energy modulations are smaller than the rms energy spread [4]. For simulations in which the output modulations exceeded numerical noise, the peak-to-peak current variations were measured from histograms of the bunch current and the peak-to-peak energy variations were measured from plots of the longitudinal phase space. The gain in simulations is shown as points in Fig. 9.

The analytic predictions agree with the simulations. For the case of a 10-keV laser heater, we suspect that the wavelengths with localized dips in the gain curve are affected by unequal compression of the bunch, which results in a 5% variation of the compressed bunch's flattop current.

We also tested the gain formulas for a trapezoidal bunch with normalized transverse emittances of 0.1 $\mu\text{m-rad}$ and

FIG. 10. (Color) Curves show the analytical microbunching gain between the entrance of the BC1 chicane at 215 MeV and the exit of the L2 linac at 1.7 GeV, for a bunch with initial current of 50 A. The gain is shown as a function of the modulation's initial wavelength λ_0 . Plotted points show the gain in 4-million particle simulations of a 200-pC parabolic bunch with initial rms bunch length of 400 μm and peak current of 50 A. (a) Gaussian energy distribution with initial rms energy spread of 3 keV. (b) Semicircular energy distribution from a matched laser heater, with initial rms energy spread of 10 keV.

rms Gaussian energy spread of 3 keV. The simulations were again in agreement with the formulas.

For a parabolic bunch whose wakes are compensated [as in Fig. 6(b)], we expect that the microbunching gain formulas will apply for wavelengths much shorter than the rms bunch length. For a 200-pC bunch with initial peak current of 50 A, rms length of 400 μm , and normalized transverse emittances of 1 μm -rad, Fig. 10 compares the analytic microbunching gain with that observed in ELEGANT simulations, showing approximate agreement

even for wavelengths as long as the bunch length. Since a parabolic bunch differs from the constant-current bunch modeled by the gain formulas, the agreement between the formulas and simulations is not as good as in Fig. 9.

When a parabolic bunch with a 3-keV Gaussian energy distribution is heated by 10 keV in a matched laser heater upstream of BC1, the bunch's energy distribution differs from the semicircular distribution of Eq. (41), which applies for a laser heater acting upon a cold bunch. Figure 11 shows the energy distribution of a parabolic bunch before and after a 10-keV matched laser heater according to an ELEGANT simulation; the heated distribution appears to be a semicircular distribution with Gaussian tails. Figure 12 compares the microbunching gain when the laser heater is simulated by ELEGANT with that predicted analytically for a 10-keV semicircular laser-heater distribution. Approximate agreement is obtained. In this case, the parabolic bunch current distribution differs from the constant-current bunch of the analytic model, and the heated energy distribution differs from the semicircular distribution considered in the analytic model. Nonetheless, reasonable agreement between the formulas and simulations is obtained.

For WfEL, we require the current and average-energy modulations of the 1.7-GeV compressed bunch to be less than 10% and 500 keV at wavelengths shorter than the bunch length. Since our analytic gain formulas agree with simulations, we use them to calculate the allowable modulations on the bunch entering the compressor at 215 MeV. For a bunch with a 3-keV rms Gaussian energy distribution, the bunch entering the BC1 chicane must have current modulations less than 0.1 A and energy modulations less than 12 eV. For a bunch with a 10-keV rms energy distribution from a laser heater, the bunch entering BC1 must have current modulations less than 1 A and energy modulations less than 470 eV.

When a bunch is produced on a photocathode by a laser pulse whose duration equals that of the bunch, laser intensity variations may cause unacceptable modulations. To

FIG. 11. (a) Gaussian energy distribution with rms energy spread of 3 keV. (b) Energy distribution when a 3-keV Gaussian distribution is heated by 10 keV in a laser heater matched to the beam's transverse size.

FIG. 12. (Color) Curves show the analytical microbunching gain between the entrance of the BC1 chicane at 215 MeV and the exit of the L2 linac at 1.7 GeV, for a bunch with initial current of 50 A. The gain is calculated as a function of the modulation's initial wavelength λ_0 , for a semicircular energy distribution from a matched laser heater with initial rms energy spread of 10 keV. Plotted points show the gain in ELEGANT simulations of 1–3 million particles, for a 200-pC bunch with a parabolic longitudinal profile with initial rms bunch length of 400 μm and peak current of 50 A. In the simulations, a bunch with an initial 3-keV Gaussian energy distribution is heated by 10 keV in an ELEGANT simulation of a matched laser heater.

achieve low levels of modulation, we plan to use a laser pulse much shorter than the bunch duration to create a parabolic bunch profile with charge of 200 pC, peak current of 50 A, and rms bunch length of 400 μm . According to Figs. 6–8, the WiFEL preliminary bunch compressor design can compress such a bunch.

VI. SUMMARY

For a high-current bunch, wakefields affect the performance of a two-stage compression and acceleration system, where each stage compresses a chirped bunch in a magnetic chicane. The bunch may be affected by the wakes of coherent radiation from the chicane magnets, approximated as coherent steady-state synchrotron radiation (CSR) within the magnets and coherent edge radiation (CER) downstream of the magnets, in addition to the linac geometric and longitudinal space charge (LSC) wakes. An analytic model was obtained which represents the impedances within each chicane by effective impedances before and after the chicane. We applied our model to a preliminary bunch compressor design for the Wisconsin Free Electron Laser (WiFEL).

For macroscopic wavelengths, impedance calculations predict that the WiFEL wakes are dominated by the resistive impedance of CER. By approximating the effective impedances as resistances, we modeled the macroscopic effects. Tracking studies with resistive impedances confirm that this simple model approximates the macroscopic behavior of the WiFEL compressor.

With the resistive-impedance approximation, we calculated the minimum initial bunch length that can be compressed without producing an upright tail in phase space. We used resistive impedances to compensate wakes by

adjusting rf parameters. We also obtained approximate formulas for the jitter in the bunch arrival time caused by bunch-to-bunch variation in current, energy, and chirp.

Formulas for microbunching gain were derived for an initial small single-wavelength modulation, affected by frequency-dependent effective impedances before and after each chicane. For a given current or energy modulation at the entrance of the two-stage compressor, the formulas predict the output current and energy modulations, including the gain suppression at short wavelengths from the bunch's energy spread and emittance. The gain was calculated for a 3-keV rms Gaussian energy spread and for a 10-keV rms energy distribution produced by a matched laser heater. Tracking studies verify the approximate validity of the gain formulas.

For the preliminary WiFEL bunch compressor, we calculated the minimum compressible initial bunch length, the jitter from bunch-to-bunch variations in current, energy, and chirp, and the microbunching growth. We plan to compress a 200-pC bunch with peak current of 50 A and rms bunch length of 400 μm that is produced by illuminating a photocathode with a laser pulse much shorter than the temporal bunch length. With state-of-the-art control of the linac's superconducting cavities, we expect to obtain acceptable performance as we compress the bunch by a factor of 20, obtaining a peak current of 1 kA.

ACKNOWLEDGMENTS

The authors thank P. Emma, M. Borland, W. S. Graves, R. A. Legg, J. J. Bisognano, M. A. Green, and K. D. Jacobs for useful suggestions. This work of R. A. Bosch and K. J. Kleman was conducted at the Synchrotron Radiation Center, which is supported by National Science Foundation Award No. DMR-0537588 and the University of Wisconsin-Madison. The work of J. Wu was supported by the U.S. Department of Energy under Contract No. DE-AC02-76SF00515, and the work was performed in support of the Linac Coherent Light Source project at the Stanford Linear Accelerator Center.

APPENDIX A: RESISTIVE IMPEDANCE

Consider a cold, linearly chirped electron bunch that is compressed in the BC1 and BC2 chicanes. At the entrance of the first chicane BC1, let $z = ct$ parametrize the bunch (where t is the electron arrival time) with $z = 0$ for the reference particle and $z > 0$ at the tail, where c is the speed of light. Let $\varepsilon(z)$ be the energy profile, $I(z) > 0$ the bunch current magnitude, while the electron charge magnitude is $e > 0$.

In the absence of collective effects, BC1 compresses at energy E_1 , where an incoming bunch has linear energy chirp $d\varepsilon/dz = h_1 E_1$, with $h_1 \equiv (d\varepsilon/dz)/E_1$. After the bunch exits BC1, let z_1 parametrize the bunch. The bunch compression is characterized by the transfer matrix ele-

ment $R_{56}^{(1)} \equiv \partial z_1 / \partial \{[\varepsilon(z) - E_1] / E_1\} < 0$. The compression ratio C_1 obeys $C_1^{-1} = 1 + h_1 R_{56}^{(1)}$, where $C_1 > 0$ for undercompression and $C_1 < 0$ for overcompression (in which the head and tail of the bunch are exchanged). For compression in a chicane, $R_{56}^{(1)}$ is negative and h_1 is positive in our notation. The energy profile after BC1 is $\varepsilon_1(z_1)$.

The linac L1 between BC1 and BC2 imparts energy $E_2 - E_1$ and additional chirp $d\varepsilon/dz_1 = h_2 E_2$, and the energy profile becomes $\varepsilon_L(z_1)$.

The chicane BC2 compresses at energy E_2 . After the bunch exits BC2, let z_2 parametrize the bunch. The bunch compression is characterized by the transfer matrix element $R_{56}^{(2)} \equiv \partial z_2 / \partial \{[\varepsilon_L(z_1) - E_2] / E_2\} < 0$. The compression ratio C_2 obeys $C_2^{-1} = 1 + R_{56}^{(2)}(h_2 + C_1 h_1 E_1 / E_2)$.

The linac L2 imparts energy $E_f - E_2$ and additional chirp $d\varepsilon/dz_2 = h_f E_f$, which may be chosen to obtain a dechirped compressed bunch at the final energy E_f .

To approximate the macroscopic effect of longitudinal wakes for a bunch with a nonzero current, we consider resistive impedance R_0 before the BC1 chicane, resistive impedance R_1 between the BC1 and BC2 chicanes, and resistive impedance R_2 after the BC2 chicane. We use the notation $T_1 \equiv R_{56}^{(1)} / E_1$, $T_2 \equiv R_{56}^{(2)} / E_2$, $S_1 \equiv h_1 E_1$, $S_2 \equiv h_2 E_2$, and $S_f \equiv h_f E_f$.

Consider a bunch current $I(z)$ at the entrance of the bunch compressor, where the bunch has energy profile

$$\varepsilon_{\text{in}}(z) = E_1 + S_1 z + \Delta E(z). \quad (\text{A1})$$

Here, $\Delta E(z)$ is the difference between the bunch energy profile and an ideal linear chirp. After passing through the effective resistance R_0 , the energy profile becomes

$$\varepsilon(z) = E_1 + S_1 z + \Delta E(z) - e R_0 I(z). \quad (\text{A2})$$

The compression in BC1 obeys

$$\begin{aligned} z_1(z) &= z + (R_{56}^{(1)} / E_1) [\varepsilon(z) - E_1] \\ &= z / C_1 + T_1 [\Delta E(z) - e R_0 I(z)], \end{aligned} \quad (\text{A3})$$

with derivative with respect to z of

$$z_1'(z) = C_1^{-1} \{1 + C_1 T_1 [\Delta E'(z) - e R_0 I'(z)]\}. \quad (\text{A4})$$

When the rhs of Eq. (A4) is zero, an upright bunch occurs after BC1. Thus, part of the bunch is upright after the first compressor when

$$\Delta E'(z) - e R_0 I'(z) = -1 / C_1 T_1 = -E_1 / C_1 R_{56}^{(1)}. \quad (\text{A5})$$

The function $z_1(z)$ is invertible if its derivative is nonzero for all values of z , with inverse $z(z_1)$. After compression in BC1, the current profile becomes

$$I_1(z_1) = \frac{I[z(z_1)]}{dz_1/dz} = \frac{C_1 I[z(z_1)]}{1 + C_1 T_1 \{\Delta E'[z(z_1)] - e R_0 I'[z(z_1)]\}}. \quad (\text{A6})$$

After passing through the resistive wake and linac downstream of BC1, the energy profile becomes

$$\begin{aligned} \varepsilon_L(z_1) &= \varepsilon[z(z_1)] - e R_1 I_1(z_1) + E_2 - E_1 + S_2 z_1 \\ &= E_2 + \left(S_1 + \frac{S_2}{C_1}\right) z(z_1) \\ &\quad + (1 + S_2 T_1) \{\Delta E[z(z_1)] - e R_0 I[z(z_1)]\} \\ &\quad - \frac{e R_1 C_1 I[z(z_1)]}{1 + C_1 T_1 \{\Delta E'[z(z_1)] - e R_0 I'[z(z_1)]\}}. \end{aligned} \quad (\text{A7})$$

Compression in BC2 obeys $z_2(z_1) = z_1 + (R_{56}^{(2)} / E_2) \times [\varepsilon_L(z_1) - E_2] = z_1 + T_2 [\varepsilon_L(z_1) - E_2]$, so that

$$\begin{aligned} z_2(z) &= \frac{1}{C_1 C_2} \left\{ z + (C_1 T_1 + C_1^2 C_2 T_2) [\Delta E(z) - e R_0 I(z)] \right. \\ &\quad \left. - \frac{C_1^2 C_2 T_2 e R_1 I(z)}{1 + C_1 T_1 [\Delta E'(z) - e R_0 I'(z)]} \right\} \end{aligned} \quad (\text{A8})$$

with derivative

$$\begin{aligned} \frac{dz_2}{dz} &= \frac{1}{C_1 C_2} \left\{ 1 + (C_1 T_1 + C_1^2 C_2 T_2) [\Delta E'(z) - e R_0 I'(z)] \right. \\ &\quad - \frac{C_1^2 C_2 T_2 e R_1 I'(z)}{1 + C_1 T_1 [\Delta E'(z) - e R_0 I'(z)]} \\ &\quad \left. + \frac{C_1^3 C_2 T_1 T_2 e R_1 I(z) [\Delta E''(z) - e R_0 I''(z)]}{\{1 + C_1 T_1 [\Delta E'(z) - e R_0 I'(z)]\}^2} \right\}. \end{aligned} \quad (\text{A9})$$

When the rhs of Eq. (A9) is zero, an upright bunch occurs after BC2. The function $z_2(z)$ is invertible if its derivative is nonzero for all values of z , in which case we let $z(z_2)$ denote the inverse. The current profile after BC2 is

$$I_2(z_2) = \frac{I[z(z_2)]}{dz_2/dz}. \quad (\text{A10})$$

After the resistive impedance R_2 and acceleration in the L2 linac, the energy profile becomes

$$\varepsilon_f(z_2) = \varepsilon_L[z_1(z_2)] - e R_2 I_2(z_2) + E_f - E_2 + S_f z_2. \quad (\text{A11})$$

APPENDIX B: MICROBUNCHING

In this Appendix, we consider microbunching from effective impedances $Z_1(k)$ and $Z_2(k)$ after each stage of compression, without considering the effective impedance $Z_{\text{zero}}(k)$ upstream of the first chicane. To model the gain suppression from the bunch's energy spread, we consider a distribution of electron energies. At the entrance of the first chicane, we assume that each slice of the bunch has the same symmetric normalized energy distribution $f(\delta)$, where δ is the energy deviation from an ideal linear chirp. This describes an uncompressed bunch whose energy spread results from the electron gun or a matched laser heater. After each chicane, the bunch distribution is rotated in phase space, so we do not make this assumption.

We use coordinates (z, δ) for the initial bunch, (z_1, δ) after the first stage of compression, and (z_2, δ) for the final bunch after two stages of compression, where δ is the initial energy deviation from an ideal linear chirp. For small modulations where the compressed bunch is not upright in phase space, these are valid coordinates with invertible transformations. The transformation from (z, δ) to (z_1, δ) has Jacobian $\partial z_1 / \partial z$ while the transformation from (z, δ) to (z_2, δ) has Jacobian $\partial z_2 / \partial z$. We will characterize the suppression of microbunching from energy spread with expressions F_1, F_2, F_3, G_1, G_2 , and G_3 , which depend upon the initial energy distribution $f(\delta)$.

For brevity, we use the notation $T_1 \equiv R_{56}^{(1)} / E_1$, $T_2 \equiv R_{56}^{(2)} / E_2$, $S_1 \equiv h_1 E_1$, $S_2 \equiv h_2 E_2$, and $S_f \equiv h_f E_f$. We let $\hat{Z}_1(k)$ represent the portion of the impedance $Z_1(k)$ that is experienced upstream of the L1 linac, where the bunch's nominal energy is E_1 . Similarly, we let $\hat{Z}_2(k)$ represent the portion of the impedance $Z_2(k)$ that is experienced upstream of the L2 linac, where the bunch's nominal energy is E_2 . For an initial modulation with wave number k_0 , let $R_1 \equiv Z_1(0)$, $Z_1 \equiv Z_1(C_1 k_0)$, $R_2 \equiv Z_2(0)$, and $Z_2 \equiv Z_2(C_1 C_2 k_0)$. Let $\hat{R}_1 \equiv \hat{Z}_1(0)$, $\hat{Z}_1 \equiv \hat{Z}_1(C_1 k_0)$, $\hat{R}_2 \equiv \hat{Z}_2(0)$, and $\hat{Z}_2 \equiv \hat{Z}_2(C_1 C_2 k_0)$.

1. Current modulation

Consider an initial current modulation $\Delta I e^{-i\omega_0 t} = \Delta I e^{-ik_0 z}$, where t is the arrival time and the coordinate z increases toward the tail of the bunch. The bunch current is

$$I(z) = I_0 + \Delta I e^{-ik_0 z}. \quad (\text{B1})$$

We assume that the current density in phase space at the entrance of BC1 is $I(z, \delta) = I(z)f(\delta)$, where $I(z)$ is the bunch current. At the entrance of BC1, let $\varepsilon(z, \delta)$ equal the energy at coordinate (z, δ) , while $\varepsilon(z)$ is the average energy at longitudinal position z . We have

$$\varepsilon(z, \delta) = E_1 + S_1 z + \delta, \quad (\text{B2})$$

$$\varepsilon(z) = \int \varepsilon(z, \delta) f(\delta) d\delta = E_1 + S_1 z. \quad (\text{B3})$$

The compression in BC1 gives the transformation

$$z_1(z, \delta) = z + T_1[\varepsilon(z, \delta) - E_1] = z/C_1 + T_1 \delta, \quad (\text{B4})$$

with inverse

$$z(z_1, \delta) = C_1 z_1 - C_1 T_1 \delta. \quad (\text{B5})$$

After BC1, the current density in coordinates (z_1, δ) is

$$\begin{aligned} I_1(z_1, \delta) &= I[z(z_1, \delta), \delta] / (\partial z_1 / \partial z) \\ &= C_1 (I_0 + \Delta I e^{-ik_0(C_1 z_1 - C_1 T_1 \delta)}) f(\delta). \end{aligned} \quad (\text{B6})$$

The current after BC1 equals

$$I_1(z_1) = \int I_1(z_1, \delta) d\delta = C_1 (I_0 + F_1 \Delta I e^{-ik_0 C_1 z_1}), \quad (\text{B7})$$

where

$$F_1 = \int \cos(k_0 C_1 T_1 \delta) f(\delta) d\delta. \quad (\text{B8})$$

The bunch passes through the CSR and CER impedance upstream of the L1 linac, $\hat{Z}_1(k)$, where $\hat{Z}_1(0) \equiv \hat{R}_1$ and $\hat{Z}_1(C_1 k_0) \equiv \hat{Z}_1$. Using the relation $C_1 = 1 - S_1 C_1 T_1$ gives the energy at the entrance of the L1 linac in coordinates (z_1, δ)

$$\begin{aligned} \varepsilon_1(z_1, \delta) &= \varepsilon[z(z_1, \delta), \delta] - (e/2\pi) \int dk \hat{Z}_1(k) I_1(k) e^{-ik z_1} \\ &= E_1 + S_1 C_1 z_1 + C_1 \delta \\ &\quad - e C_1 (\hat{R}_1 I_0 + F_1 \hat{Z}_1 \Delta I e^{-ik_0 C_1 z_1}). \end{aligned} \quad (\text{B9})$$

The average energy of electrons at coordinate z_1 , $\varepsilon_1(z_1)$, obeys $\varepsilon_1(z_1) I_1(z_1) = \int d\delta \varepsilon_1(z_1, \delta) I_1(z_1, \delta)$. To first order in ΔI , this gives

$$\begin{aligned} \varepsilon_1(z_1) &= E_1 + S_1 C_1 z_1 - e C_1 \hat{R}_1 I_0 - e F_1 C_1 \hat{Z}_1 \Delta I e^{-ik_0 C_1 z_1} \\ &\quad + i G_1 C_1 (\Delta I / I_0) e^{-ik_0 C_1 z_1}, \end{aligned} \quad (\text{B10})$$

where

$$G_1 = \int \delta \sin(k_0 C_1 T_1 \delta) f(\delta) d\delta. \quad (\text{B11})$$

After passing through the L1 linac, the energy is

$$\begin{aligned} \varepsilon_L(z_1, \delta) &= E_2 + (S_1 C_1 + S_2) z_1 + C_1 \delta \\ &\quad - e C_1 (R_1 I_0 + F_1 Z_1 \Delta I e^{-ik_0 C_1 z_1}). \end{aligned} \quad (\text{B12})$$

To first order in ΔI , the average energy of electrons at coordinate z_1 , $\varepsilon_L(z_1)$, is

$$\begin{aligned} \varepsilon_L(z_1) &= E_2 + (S_1 C_1 + S_2) z_1 - e C_1 R_1 I_0 \\ &\quad - e F_1 C_1 Z_1 \Delta I e^{-ik_0 C_1 z_1} \\ &\quad + i G_1 C_1 (\Delta I / I_0) e^{-ik_0 C_1 z_1}. \end{aligned} \quad (\text{B13})$$

From Eqs. (B7), (B10), and (B13), the current and average-energy modulations for single-stage compression obey

$$T_{II} \equiv \frac{\Delta I / I_{\text{out}}}{\Delta I / I_{\text{in}}} = F_1, \quad (\text{B14})$$

$$T_{EI} \equiv \frac{\Delta E / E_{\text{out}}}{\Delta I / I_{\text{in}}} = \frac{-F_1 C_1 e Z_1 I_0}{E_{\text{out}}} + \frac{i G_1 C_1}{E_{\text{out}}}. \quad (\text{B15})$$

Equations (B14) and (B15) describe the modulation at any location between compression in BC1 and the entrance of BC2, provided that E_{out} is the nominal bunch energy at that location and Z_1 is the longitudinal impedance experienced after compression in BC1, prior to reaching that location.

The compression in BC2 gives the transformation

$$\begin{aligned} z_2(z_1, \delta) &= z_1 + T_2[\varepsilon_L(z_1, \delta) - E_2] \\ &= z_1/C_2 + C_1 T_2 \delta - C_1 T_2 e(R_1 I_0 \\ &\quad + F_1 Z_1 \Delta I e^{-ik_0 C_1 z_1}). \end{aligned} \quad (B16)$$

We can relate z_2 and z by using Eq. (B4),

$$\begin{aligned} z_2(z, \delta) &= z/C_1 C_2 + (T_1/C_2 + T_2 C_1) \delta \\ &\quad - C_1 T_2 e(R_1 I_0 + F_1 Z_1 \Delta I e^{-ik_0(z+C_1 T_1 \delta)}), \end{aligned} \quad (B17)$$

with derivative

$$\begin{aligned} \partial z_2 / \partial z(z, \delta) &= (C_1 C_2)^{-1} \\ &\quad \times (1 + i F_1 C_1^2 C_2 T_2 e Z_1 k_0 \Delta I e^{-ik_0(z+C_1 T_1 \delta)}). \end{aligned}$$

When the real part of this derivative equals zero, upright bunch regions occurs after BC2. The energy spread modifies the threshold value of ΔI by the factor F_1^{-1} .

The following equations are valid to first order in ΔI . The inverse of Eq. (B16) is

$$z_1(z_2, \delta) = C_2 z_2 - C_1 C_2 T_2 \delta + C_1 C_2 T_2 e R_1 I_0 + F_1 C_1 C_2 T_2 e Z_1 \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)}. \quad (B19)$$

The inverse of Eq. (B17) is

$$z(z_2, \delta) = C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - (C_1 T_1 + C_1^2 C_2 T_2) \delta + F_1 C_1^2 C_2 T_2 e Z_1 \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)}. \quad (B20)$$

The current density after BC2 in coordinates (z_2, δ) is

$$\begin{aligned} I_2(z_2, \delta) &= \frac{I[z(z_2, \delta), \delta]}{\partial z_2 / \partial z} = \frac{C_1 C_2 [I_0 + \Delta I e^{-ik_0 z(z_2, \delta)}] f(\delta)}{1 + i F_1 C_1^2 C_2 T_2 e Z_1 k_0 \Delta I e^{-ik_0[z(z_2, \delta) + C_1 T_1 \delta]}} \\ &= C_1 C_2 f(\delta) I_0 [1 + (\Delta I / I_0) e^{-ik_0[C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - (C_1 T_1 + C_1^2 C_2 T_2) \delta]} \\ &\quad - i F_1 C_1^2 C_2 T_2 e Z_1 k_0 \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)}]. \end{aligned} \quad (B21)$$

The current is

$$I_2(z_2) = \int I_2(z_2, \delta) d\delta = C_1 C_2 I_0 [1 + (F_3 / I_0 - i F_1 F_2 C_1^2 C_2 T_2 e Z_1 k_0) \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}], \quad (B22)$$

where

$$F_2 = \int \cos(k_0 C_1^2 C_2 T_2 \delta) f(\delta) d\delta, \quad F_3 = \int \cos[k_0(C_1 T_1 + C_1^2 C_2 T_2) \delta] f(\delta) d\delta. \quad (B23)$$

After passing through the CSR and CER impedance $\hat{Z}_2(k)$, where $\hat{Z}_2(0) \equiv \hat{R}_2$ and $\hat{Z}_2(C_1 C_2 k_0) \equiv \hat{Z}_2$, using the relation $C_2 = 1 - (S_1 C_1 + S_2) T_2 C_2$ gives the energy in coordinates (z_2, δ) :

$$\begin{aligned} \varepsilon_2(z_2, \delta) &= \varepsilon_L[z_1(z_2, \delta), \delta] - (e/2\pi) \int dk \hat{Z}_2(k) I_2(k) e^{-ik z_2} \\ &= E_2 - C_1 C_2 e R_1 I_0 - C_1 C_2 e \hat{R}_2 I_0 + (S_1 C_1 + S_2) C_2 z_2 + C_1 C_2 \delta \\ &\quad - F_1 C_1 C_2 e Z_1 \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)} - F_3 C_1 C_2 e \hat{Z}_2 \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)} \\ &\quad + i F_1 F_2 C_1^3 C_2^2 T_2 e^2 Z_1 \hat{Z}_2 k_0 I_0 \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}. \end{aligned} \quad (B24)$$

The average energy of electrons at coordinate z_2 , $\varepsilon_2(z_2)$, obeys $\varepsilon_2(z_2) I_2(z_2) = \int d\delta \varepsilon_2(z_2, \delta) I_2(z_2, \delta)$, so that

$$\begin{aligned} \varepsilon_2(z_2) &= E_2 - C_1 C_2 e R_1 I_0 - C_1 C_2 e \hat{R}_2 I_0 + (S_1 C_1 + S_2) C_2 z_2 + (-F_1 F_2 C_1 C_2 e Z_1 - F_3 C_1 C_2 e \hat{Z}_2 \\ &\quad + i F_1 F_2 C_1^3 C_2^2 T_2 e^2 Z_1 \hat{Z}_2 k_0 I_0 + i G_3 C_1 C_2 / I_0 + F_1 G_2 C_1^3 C_2^2 T_2 e Z_1 k_0) \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}, \end{aligned} \quad (B25)$$

where

$$G_2 = \int \delta \sin(k_0 C_1^2 C_2 T_2 \delta) f(\delta) d\delta, \quad G_3 = \int \delta \sin[k_0(C_1 T_1 + C_1^2 C_2 T_2) \delta] f(\delta) d\delta. \quad (B26)$$

After passing through the L2 linac, the final energy is

$$\begin{aligned}\varepsilon_f(z_2, \delta) = & E_f - C_1 C_2 e R_1 I_0 - C_1 C_2 e R_2 I_0 + (S_1 C_1 C_2 + S_2 C_2 + S_f) z_2 + C_1 C_2 \delta \\ & - F_1 C_1 C_2 e Z_1 \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)} - F_3 C_1 C_2 e Z_2 \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)} \\ & + i F_1 F_2 C_1^3 C_2^2 T_2 e^2 Z_1 Z_2 k_0 I_0 \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}.\end{aligned}\quad (\text{B27})$$

The average energy at coordinate z_2 , $\varepsilon_f(z_2)$, obeys

$$\begin{aligned}\varepsilon_f(z_2) = & E_f - C_1 C_2 e R_1 I_0 - C_1 C_2 e R_2 I_0 + (S_1 C_1 C_2 + S_2 C_2 + S_f) z_2 + (-F_1 F_2 C_1 C_2 e Z_1 - F_3 C_1 C_2 e Z_2 \\ & + i F_1 F_2 C_1^3 C_2^2 T_2 e^2 Z_1 Z_2 k_0 I_0 + i G_3 C_1 C_2 / I_0 + F_1 G_2 C_1^3 C_2^2 T_2 e Z_1 k_0) \Delta I e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}.\end{aligned}\quad (\text{B28})$$

For two-stage compression, the current and average-energy modulations given by Eqs. (B22), (B25), and (B28) obey

$$T_{II} \equiv \frac{\Delta I / I_{\text{out}}}{\Delta I / I_{\text{in}}} = F_3 - \frac{i F_1 F_2 C_1^2 C_2 e Z_1 I_0 k_0 R_{56}^{(2)}}{E_2} \quad (\text{B29})$$

$$T_{EI} \equiv \frac{\Delta E / E_{\text{out}}}{\Delta I / I_{\text{in}}} = -\frac{C_1 C_2 (F_1 F_2 Z_1 + F_3 Z_2) e I_0}{E_{\text{out}}} + \frac{F_1 G_2 C_1^3 C_2^2 e Z_1 I_0 k_0 R_{56}^{(2)}}{E_2 E_{\text{out}}} + \frac{i F_1 F_2 C_1^3 C_2^2 e^2 Z_1 Z_2 I_0^2 k_0 R_{56}^{(2)}}{E_2 E_{\text{out}}} + \frac{i G_3 C_1 C_2}{E_{\text{out}}}. \quad (\text{B30})$$

Equations (B29) and (B30) describe the modulation at any location after compression in BC2, provided that E_{out} is the nominal energy at that location and Z_2 is the longitudinal impedance experienced after compression in BC2, prior to reaching that location.

2. Energy modulation

Consider an initial symmetric distribution of energies $f(\delta)$, where δ is the initial energy deviation from that of a cold, chirped beam with energy modulation. For constant current,

$$I(z) = I_0. \quad (\text{B31})$$

We again assume that the current distribution at the entrance of BC1 is $I(z, \delta) = I(z)f(\delta)$, where $I(z)$ is the bunch current. At the entrance of BC1, let $\varepsilon(z, \delta)$ equal the energy at coordinates (z, δ) , while the average energy at longitudinal position z is denoted $\varepsilon(z)$. For an energy modulation of $\Delta E e^{-i\omega_0 t} = \Delta E e^{-ik_0 z}$,

$$\varepsilon(z, \delta) = E_1 + S_1 z + \Delta E e^{-ik_0 z} + \delta, \quad (\text{B32})$$

$$\varepsilon(z) = \int \varepsilon(z, \delta) f(\delta) d\delta = E_1 + S_1 z + \Delta E e^{-ik_0 z}. \quad (\text{B33})$$

The compression in BC1 gives the transformation

$$\begin{aligned}z_1(z, \delta) &= z + T_1[\varepsilon(z, \delta) - E_1] \\ &= z/C_1 + T_1 \delta + T_1 \Delta E e^{-ik_0 z},\end{aligned}\quad (\text{B34})$$

with derivative

$$\partial z_1 / \partial z = 1/C_1 - ik_0 T_1 \Delta E e^{-ik_0 z}. \quad (\text{B35})$$

When the real part of this derivative is zero, upright bunch regions and current spikes occur after BC1. The following equations are valid to first order in ΔE . The inverse of

Eq. (B34) is

$$z(z_1, \delta) = C_1 z_1 - C_1 T_1 \delta - C_1 T_1 \Delta E e^{-ik_0(C_1 z_1 - C_1 T_1 \delta)}. \quad (\text{B36})$$

After BC1, the current density in coordinates (z_1, δ) is

$$\begin{aligned}I_1(z_1, \delta) &= I[z(z_1, \delta), \delta] / (\partial z_1 / \partial z) \\ &= C_1 I_0 (1 + i C_1 T_1 k_0 \Delta E e^{-ik_0(C_1 z_1 - C_1 T_1 \delta)}) f(\delta).\end{aligned}\quad (\text{B37})$$

Therefore, the current is

$$\begin{aligned}I_1(z_1) &= \int I_1(z_1, \delta) d\delta \\ &= C_1 I_0 (1 + i F_1 C_1 T_1 k_0 \Delta E e^{-ik_0 C_1 z_1}).\end{aligned}\quad (\text{B38})$$

The bunch passes through the CSR and CER impedance upstream of the L1 linac, $\hat{Z}_1(k)$, where $\hat{Z}_1(0) \equiv \hat{R}_1$ and $\hat{Z}_1(C_1 k_0) \equiv \hat{Z}_1$. The energy at the entrance of the L1 linac is

$$\begin{aligned}\varepsilon_1(z_1, \delta) &= \varepsilon[z(z_1, \delta), \delta] - (e/2\pi) \int dk \hat{Z}_1(k) I_1(k) e^{-ik z_1} \\ &= E_1 + S_1 C_1 z_1 + C_1 \delta + C_1 \Delta E e^{-ik_0(C_1 z_1 - C_1 T_1 \delta)} \\ &\quad - e C_1 I_0 (\hat{R}_1 + i F_1 C_1 T_1 \hat{Z}_1 k_0 \Delta E e^{-ik_0 C_1 z_1}).\end{aligned}\quad (\text{B39})$$

The average electron energy at coordinate z_1 , $\varepsilon_1(z_1)$, obeys $\varepsilon_1(z_1) I_1(z_1) = \int d\delta \varepsilon_1(z_1, \delta) I_1(z_1, \delta)$, so that

$$\begin{aligned}\varepsilon_1(z_1) &= E_1 + S_1 C_1 z_1 + C_1 (F_1 - G_1 C_1 T_1 k_0) \Delta E e^{-ik_0 C_1 z_1} \\ &\quad - e C_1 I_0 (\hat{R}_1 + i F_1 C_1 T_1 \hat{Z}_1 k_0 \Delta E e^{-ik_0 C_1 z_1}).\end{aligned}\quad (\text{B40})$$

After passing through the L1 linac between BC1 and BC2, the energy becomes

$$\begin{aligned}\varepsilon_L(z_1, \delta) = & E_2 + (S_1 C_1 + S_2) z_1 + C_1 \delta \\ & + C_1 \Delta E e^{-ik_0(C_1 z_1 - C_1 T_1 \delta)} \\ & - e C_1 I_0 (R_1 + i F_1 C_1 T_1 Z_1 k_0 \Delta E e^{-ik_0 C_1 z_1}).\end{aligned}\quad (\text{B41})$$

The average electron energy at coordinate z_1 , $\varepsilon_L(z_1)$, obeys

$$\begin{aligned}\varepsilon_L(z_1) = & E_2 + (S_1 C_1 + S_2) z_1 \\ & + C_1 (F_1 - G_1 C_1 T_1 k_0) \Delta E e^{-ik_0 C_1 z_1} \\ & - e C_1 I_0 (R_1 + i F_1 C_1 T_1 Z_1 k_0 \Delta E e^{-ik_0 C_1 z_1}).\end{aligned}\quad (\text{B42})$$

For single-stage compression, the current and average-energy modulations obey

$$T_{IE} \equiv \frac{\Delta I / I_{\text{out}}}{\Delta E / E_{\text{in}}} = i F_1 C_1 k_0 R_{56}^{(1)}, \quad (\text{B43})$$

$$\begin{aligned}T_{EE} & \equiv \frac{\Delta E / E_{\text{out}}}{\Delta E / E_{\text{in}}} \\ & = \frac{F_1 C_1 E_1}{E_{\text{out}}} - \frac{G_1 C_1^2 k_0 R_{56}^{(1)}}{E_{\text{out}}} - \frac{i F_1 C_1^2 e Z_1 I_0 k_0 R_{56}^{(1)}}{E_{\text{out}}}.\end{aligned}\quad (\text{B44})$$

Equations (B43) and (B44) describe the modulation at any location between compression in BC1 and the entrance of BC2, provided that E_{out} is the nominal energy at that location and Z_1 is the longitudinal impedance experienced after compression in BC1, prior to reaching that location.

The compression in BC2 gives the transformation

$$\begin{aligned}z_2(z_1, \delta) & = z_1 + T_2 [\varepsilon_L(z_1, \delta) - E_2] \\ & = z_1 / C_2 - C_1 T_2 e R_1 I_0 + C_1 T_2 \delta \\ & \quad + C_1 T_2 \Delta E e^{-ik_0(C_1 z_1 - C_1 T_1 \delta)} \\ & \quad - i F_1 C_1^2 T_1 T_2 e Z_1 I_0 k_0 \Delta E e^{-ik_0 C_1 z_1},\end{aligned}\quad (\text{B45})$$

with inverse

$$\begin{aligned}z_1(z_2, \delta) = & C_2 z_2 + C_1 C_2 T_2 e R_1 I_0 - C_1 C_2 T_2 \delta - C_1 C_2 T_2 \Delta E e^{-ik_0[C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - (C_1 T_1 + C_1^2 C_2 T_2) \delta]} \\ & + i F_1 C_1^2 C_2 T_1 T_2 e Z_1 I_0 k_0 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)}.\end{aligned}\quad (\text{B46})$$

Using Eqs. (B34) and (B45) to relate z_2 and z gives

$$z_2(z, \delta) = z / C_1 C_2 - C_1 T_2 e R_1 I_0 + (T_1 / C_2 + T_2 C_1) (\delta + \Delta E e^{-ik_0 z}) - i F_1 C_1^2 T_1 T_2 e Z_1 I_0 k_0 \Delta E e^{-ik_0(z + C_1 T_1 \delta)}. \quad (\text{B47})$$

The inverse of Eq. (B47) is

$$\begin{aligned}z(z_2, \delta) = & C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - (C_1 T_1 + C_1^2 C_2 T_2) \delta - (C_1 T_1 + C_1^2 C_2 T_2) \Delta E e^{-ik_0[C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - (C_1 T_1 + C_1^2 C_2 T_2) \delta]} \\ & + i F_1 C_1^3 C_2 T_1 T_2 e Z_1 I_0 k_0 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)},\end{aligned}\quad (\text{B48})$$

while its derivative is

$$\partial z_2 / \partial z(z, \delta) = (C_1 C_2)^{-1} [1 - i(C_1 T_1 + C_1^2 C_2 T_2) k_0 \Delta E e^{-ik_0 z} - F_1 C_1^3 C_2 T_1 T_2 e Z_1 I_0 k_0^2 \Delta E e^{-ik_0(z + C_1 T_1 \delta)}]. \quad (\text{B49})$$

Therefore, the current distribution is

$$\begin{aligned}I_2(z_2, \delta) & = I[z(z_2, \delta), \delta] / (\partial z_2 / \partial z) \\ & = C_1 C_2 I_0 f(\delta) [1 + i(C_1 T_1 + C_1^2 C_2 T_2) k_0 \Delta E e^{-ik_0 z(z_2, \delta)} + F_1 C_1^3 C_2 T_1 T_2 e Z_1 I_0 k_0^2 \Delta E e^{-ik_0[z(z_2, \delta) + C_1 T_1 \delta]}] \\ & = C_1 C_2 I_0 f(\delta) [1 + i(C_1 T_1 + C_1^2 C_2 T_2) k_0 \Delta E e^{-ik_0[C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - (C_1 T_1 + C_1^2 C_2 T_2) \delta]} \\ & \quad + F_1 C_1^3 C_2 T_1 T_2 e Z_1 I_0 k_0^2 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)}].\end{aligned}\quad (\text{B50})$$

This gives the current profile as

$$\begin{aligned}
I_2(z_2) &= \int I_2(z_2, \delta) d\delta \\
&= C_1 C_2 I_0 [1 + i(C_1 T_1 + C_1^2 C_2 T_2) F_3 k_0 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)} \\
&\quad + F_1 F_2 C_1^3 C_2 T_1 T_2 e Z_1 I_0 k_0^2 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}].
\end{aligned} \tag{B51}$$

After passing through the CSR and CER impedance upstream of the L2 linac, $\hat{Z}_2(k)$, where $\hat{Z}_2(0) \equiv \hat{R}_2$ and $\hat{Z}_2(C_1 C_2 k_0) \equiv \hat{Z}_2$, the energy is

$$\begin{aligned}
\varepsilon_2(z_2, \delta) &= \varepsilon_L[z_1(z_2, \delta), \delta] - (e/2\pi) \int dk \hat{Z}_2(k) I_2(k) e^{-ikz_2} \\
&= E_2 - C_1 C_2 e R_1 I_0 - C_1 C_2 e \hat{R}_2 I_0 + (S_1 C_1 + S_2) C_2 z_2 + C_1 C_2 \delta \\
&\quad + C_1 C_2 \Delta E e^{-ik_0[C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - (C_1 T_1 + C_1^2 C_2 T_2) \delta]} - i F_1 C_1^2 C_2 T_1 e Z_1 I_0 k_0 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)} \\
&\quad - i F_3 C_1 C_2 (C_1 T_1 + C_1^2 C_2 T_2) e \hat{Z}_2 I_0 k_0 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)} \\
&\quad - F_1 F_2 C_1^4 C_2^2 T_1 T_2 e^2 Z_1 \hat{Z}_2 I_0^2 k_0^2 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}.
\end{aligned} \tag{B52}$$

The average energy of electrons at coordinate z_2 , $\varepsilon_2(z_2)$, obeys $\varepsilon_2(z_2) I_2(z_2) = \int d\delta \varepsilon_2(z_2, \delta) I_2(z_2, \delta)$. This gives

$$\begin{aligned}
\varepsilon_2(z_2) &= E_2 - C_1 C_2 e R_1 I_0 - C_1 C_2 e \hat{R}_2 I_0 + (S_1 C_1 + S_2) C_2 z_2 + [F_3 C_1 C_2 - i F_1 F_2 C_1^2 C_2 T_1 e Z_1 I_0 k_0 \\
&\quad - i F_3 C_1 C_2 (C_1 T_1 + C_1^2 C_2 T_2) e \hat{Z}_2 I_0 k_0 - F_1 F_2 C_1^4 C_2^2 T_1 T_2 e^2 Z_1 \hat{Z}_2 I_0^2 k_0^2 - G_3 C_1 C_2 (C_1 T_1 + C_1^2 C_2 T_2) k_0 \\
&\quad + i F_1 G_2 C_1^4 C_2^2 T_1 T_2 e Z_1 I_0 k_0^2] \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}.
\end{aligned} \tag{B53}$$

After passing through the L2 linac, the final energy is

$$\begin{aligned}
\varepsilon_f(z_2, \delta) &= E_f - C_1 C_2 e R_1 I_0 - C_1 C_2 e R_2 I_0 + (S_1 C_1 C_2 + S_2 C_2 + S_f) z_2 + C_1 C_2 \delta \\
&\quad + C_1 C_2 \Delta E e^{-ik_0[C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - (C_1 T_1 + C_1^2 C_2 T_2) \delta]} - i F_1 C_1^2 C_2 T_1 e Z_1 I_0 k_0 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0 - C_1^2 C_2 T_2 \delta)} \\
&\quad - i F_3 C_1 C_2 (C_1 T_1 + C_1^2 C_2 T_2) e Z_2 I_0 k_0 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)} \\
&\quad - F_1 F_2 C_1^4 C_2^2 T_1 T_2 e^2 Z_1 Z_2 I_0^2 k_0^2 \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}.
\end{aligned} \tag{B54}$$

The average energy at coordinate z_2 , $\varepsilon_f(z_2)$, obeys

$$\begin{aligned}
\varepsilon_f(z_2) &= E_f - C_1 C_2 e R_1 I_0 - C_1 C_2 e R_2 I_0 + (S_1 C_1 C_2 + S_2 C_2 + S_f) z_2 + [F_3 C_1 C_2 - i F_1 F_2 C_1^2 C_2 T_1 e Z_1 I_0 k_0 \\
&\quad - i F_3 C_1 C_2 (C_1 T_1 + C_1^2 C_2 T_2) e Z_2 I_0 k_0 - F_1 F_2 C_1^4 C_2^2 T_1 T_2 e^2 Z_1 Z_2 I_0^2 k_0^2 - G_3 C_1 C_2 (C_1 T_1 + C_1^2 C_2 T_2) k_0 \\
&\quad + i F_1 G_2 C_1^4 C_2^2 T_1 T_2 e Z_1 I_0 k_0^2] \Delta E e^{-ik_0(C_1 C_2 z_2 + C_1^2 C_2 T_2 e R_1 I_0)}.
\end{aligned} \tag{B55}$$

For two-stage compression, the current and average-energy modulations given by Eqs. (B51), (B53), and (B55) obey

$$T_{IE} \equiv \frac{\Delta I / I_{\text{out}}}{\Delta E / E_{\text{in}}} = \frac{F_1 F_2 C_1^3 C_2 e Z_1 I_0 k_0^2 R_{56}^{(1)} R_{56}^{(2)}}{E_2} + i F_3 C_1 k_0 R_{56}^{(1)} + \frac{i F_3 C_1^2 C_2 k_0 R_{56}^{(2)} E_1}{E_2} \tag{B56}$$

$$\begin{aligned}
T_{EE} \equiv \frac{\Delta E / E_{\text{out}}}{\Delta E / E_{\text{in}}} &= \frac{F_3 C_1 C_2 E_1}{E_{\text{out}}} - \frac{F_1 F_2 C_1^4 C_2^2 e^2 Z_1 Z_2 I_0^2 k_0^2 R_{56}^{(1)} R_{56}^{(2)}}{E_2 E_{\text{out}}} - \frac{G_3 C_1^2 C_2 k_0 R_{56}^{(1)}}{E_{\text{out}}} - \frac{G_3 C_1^3 C_2^2 k_0 R_{56}^{(2)} E_1}{E_2 E_{\text{out}}} \\
&\quad - \frac{i F_1 F_2 C_1^2 C_2 e Z_1 I_0 k_0 R_{56}^{(1)}}{E_{\text{out}}} - \frac{i F_3 C_1^2 C_2 e Z_2 I_0 k_0 R_{56}^{(1)}}{E_{\text{out}}} - \frac{i F_3 C_1^3 C_2^2 e Z_2 I_0 k_0 R_{56}^{(2)} E_1}{E_2 E_{\text{out}}} + \frac{i F_1 G_2 C_1^4 C_2^2 e Z_1 I_0 k_0^2 R_{56}^{(1)} R_{56}^{(2)}}{E_2 E_{\text{out}}}.
\end{aligned} \tag{B57}$$

Equations (B56) and (B57) describe the modulation at any location after compression in BC2, provided that E_{out} is the nominal energy at that location and Z_2 is the longitudinal impedance experienced after compression in BC2, prior to reaching that location.

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