Wakefields generated by electron beams passing through a waveguide loaded with a slightly dispersive active medium

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The wakefields of a relativistic electron beam passing through a waveguide loaded with an active medium with weak resonant dispersion are considered. For the calculations in this paper, the parameters of the medium are those of a solution of fullerene (C_{60}) in a nematic liquid crystal that exhibits activity in the X-band. It is shown that several of the TM accelerating modes can be amplified for the geometries under consideration; structures in which higher order modes are amplified exhibit essential advantages as PASERs. In particular, the amplification of the highest mode occurs in a structure loaded with a rather thick active medium layer that maximizes the energy stored by the active medium.

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I. INTRODUCTION

The possibility of using an active medium to amplify the wakefield generated by a charged particle beam and employing the amplified wakefield to accelerate a second beam has been recognized. This acceleration scheme is one of several related methods referred to as particle acceleration by stimulated emission of radiation (PASER), where an active medium is used to provide the energy for accelerating charged particles. Initial theoretical work in this area focused on acceleration in gaseous CO_2 and ammonia laser media [1,2]. The first proof-of-principle experiment on direct particle acceleration by stimulated emission of radiation has been published recently [3,4]. A fraction of a 45 MeV electron macrobunch was measured to have gained 200 keV from an excited CO_2 gas mixture. A new active material operating in the X-band has been proposed as well: a solution of fullerene (C_{60}) in a nematic liquid crystal has been found to exhibit a maser transition in the X-band frequency range [5,6]. The ability to employ a microwave frequency material simplifies the construction of test structures and facilitates beam experiments using dielectric loaded accelerator concepts [6].

We present here the results of analytical and numerical studies of the Cherenkov radiation from a Gaussian electron bunch moving through an accelerating structure loaded with an active medium. The analysis of the problem is based on the waveguide mode formalism applied to a resonant dispersive medium excited by external optical pumping. The theory of Cherenkov radiation in waveguides loaded with a dielectric medium was outlined previously in a variety of publications (see, for example, Refs. [7–10]). However, Cherenkov radiation in waveguides with dispersive media (passive or active) has not been extensively discussed in the literature. For example, in Ref. [11] the frequency dependence of the conductivity

is taken into account. Some papers are devoted to the investigation of radiation in a waveguide completely or partly filled with a passive dielectric exhibiting resonant dispersion [12,13].

The case of an active medium with strong resonant dispersion is considered in the paper [1]. Further expansion of the theory for the case of a strongly dispersive medium can be found in Ref. [2]. In contrast to the papers [1,2], the main goal of this work is the analysis of the case of a medium with relatively weak dispersion [14].

II. SOME BASIC RESULTS

The bunch is assumed to move along the axis z of a vacuum channel in a cylindrical waveguide with an active media layer. The charge density is determined by the expression $\rho(x, y, \zeta) = \frac{q}{\sqrt{2\pi}\sigma} \exp(-\frac{\zeta^2}{2\sigma^2})\delta(x)\delta(y)$, where $\zeta = z - Vt$, q is the charge of the bunch, and σ is a parameter characterizing the bunch length. The radius of the bunch is assumed to be negligible. The geometry of the problem is shown in Fig. 1, where a is the waveguide

FIG. 1. (Color) Geometry of the problem. The bunch moves along the axis of the waveguide with an active medium cylindrical layer of thickness d .

radius, b is the radius of the vacuum channel, and $d = a - b$ is the thickness of the active medium layer.

Results for the analogous problem with a passive dispersive medium were included, for example, in Ref. [7]. It

can be shown that these results may be generalized for the case of an active medium. We obtain the following expression for the longitudinal component of the electric field behind the bunch at a distance $|\zeta| = |z - Vt| \gg \sigma$:

$$E_z = -\frac{4q(1 - \beta^2)}{c^2\beta^2} \sum_{m=1}^{\infty} \sum_{j=1}^2 \operatorname{Re} \left\{ \frac{\omega[sK_1(kb)\psi_1(s) + \varepsilon kK_0(kb)\psi_0(s)]}{\frac{d}{d\omega}[sI_1(kb)\psi_1(s) - \varepsilon kI_0(kb)\psi_0(s)]} I_0(kr) \exp\left(-\frac{\omega^2\sigma^2}{2V^2} + \frac{i\omega\zeta}{V}\right) \right\} \Bigg|_{\substack{\omega = \omega_{jm} \\ k = k_{jm} \\ s = s_m}} \quad (1)$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the waveguide axis,

$$\begin{aligned} \psi_0(s) &= J_1(sb)N_0(sa) - J_0(sa)N_1(sb), \\ \psi_1(s) &= J_0(sb)N_0(sa) - J_0(sa)N_0(sb), \\ k(\omega) &= \omega V^{-1} \sqrt{1 - \beta^2}, \\ k_{jm} &= k(\omega_{jm}), \quad \beta = V/c. \end{aligned}$$

Here $J_n(\xi)$ are the Bessel functions, $N_n(\xi)$ are the Neumann functions, $I_n(\xi)$ are the modified Bessel functions, and $K_n(\xi)$ are the modified Hankel functions. The mode frequencies ω_{jm} are connected with values s_m by the following system of equations:

$$s^2(\omega) = \omega^2(c\beta)^{-2}[\varepsilon(\omega)\mu(\omega)\beta^2 - 1], \quad (2)$$

$$s(\omega)I_1(k(\omega)b)\psi_1(s(\omega)) - \varepsilon(\omega)k(\omega)I_0(k(\omega)b)\psi_0(s(\omega)) = 0, \quad (3)$$

where $\varepsilon(\omega)$ is electric susceptibility, $\mu(\omega)$ is magnetic permeability. The value $s(\omega)$ is determined by the condition that $\operatorname{Im} s(\omega) > 0$. The quantities ω_{jm} are roots of Eq. (3). Note that the series (1) contains frequencies with positive real parts only: $\operatorname{Re} \omega_{jm} > 0$. The introduction of the summation over $j = 1, 2$ is convenient because in the case of the medium considered below each value of s_m corresponds to two frequencies $\omega_{1,2m}$.

III. THE CASE OF WEAK RESONANT DISPERSION

The results (1)–(3) are true both for passive and for active medium with arbitrary dispersion characteristics. We consider here a typical case when the dispersion has a resonant character and there is only one resonant frequency in the range of interest for us. In this case the refractive index squared has the form

$$n^2(\omega) = \varepsilon(\omega)\mu(\omega) = n_{c0}^2 + \frac{\omega_p^2}{\omega_r^2 - \omega^2 - 2i\omega_d\omega}, \quad (4)$$

where $\omega_r = 2\pi\nu_r$ is the resonant frequency, and $\omega_p = 2\pi\nu_p$ is a plasma frequency. It is noted that $\omega_p^2 > 0$ for a passive medium, but for an active medium $\omega_p^2 < 0$, i.e. ω_p is imaginary. In the passive medium case the dispersion is normal (i.e. $d \operatorname{Re} n^2/d\omega > 0$) almost everywhere except in the vicinity of the resonance but is anomalous (i.e. $d \operatorname{Re} n^2/d\omega < 0$) in the neighborhood of the resonance. In the second (active) case the dispersion is anomalous almost everywhere except in the resonance vicinity and is normal in the resonance vicinity where the imaginary part is negative.

We will consider the active medium case $\omega_p^2 < 0$. For the refractive index (4), Eq. (2) has the form

$$n_c^2\omega^4 + 2in_c^2\omega_d\omega^3 - (c^2s^2 + n_c^2\omega_r^2 + \omega_p^2)\omega^2 - 2i\omega_dc^2s^2\omega + \omega_r^2c^2s^2 = 0, \quad (5)$$

where $n_c = \sqrt{n_{c0}^2 - \beta^{-2}}$.

The basic thrust of this paper is the analysis of the active medium case, where the dispersion is relatively weak, i.e., the condition $|n^2 - n_{c0}^2| \ll n_{c0}^2$ is fulfilled at all frequencies (the opposite case of strong dispersion was analyzed in [1,2]). These conditions hold true if $|\omega_p^2| \ll \omega_r\omega_d$. It is assumed as well that $\omega_d \ll \omega_r$ and $\operatorname{Im} n_{c0}^2 \ll \operatorname{Re} n_{c0}^2$.

Using the method of successive approximations, one can show the following. The first series of mode frequencies [corresponding to $j = 1$ in (1)] are located in the vicinity of the resonant frequency and possess negative imaginary parts. The amplitudes of these harmonics decrease exponentially [as $\exp(-\omega_d t)$]. As a rule these harmonics excited by the beam exhibit relatively weak magnitudes, and therefore their impact is negligible. Usually the frequencies and amplitudes arising from the second series of harmonics [$j = 2$ in (1)] are close to the corresponding dispersion-free case, i.e. $\omega_{2m} \approx cs_m/n_{c0}$. However, these conditions occur if all frequencies of the second series are not close to the resonant frequency. If some mode frequency ω_{2M} is in the vicinity of the resonant frequency, its complex frequency is

$$\begin{aligned}\omega_{2M} &\approx \omega_r + 0.5i[\sqrt{(\omega_d - \chi\omega_r)^2 + |\omega_p^2|/\text{Re}(n_c^2)} - \omega_d - \chi\omega_r] \\ &= \omega_r + \frac{i}{2} \frac{|\omega_p^2| - 2\omega_d\omega_r\text{Im}(n_c^2)}{\text{Re}(n_c^2)\sqrt{(\omega_d - \chi\omega_r)^2 + |\omega_p^2|/\text{Re}(n_c^2)} + \omega_d + \chi\omega_r},\end{aligned}\quad (6)$$

where $\chi = \text{Im}(n_c^2)/(2\text{Re}(n_c^2))$. This mode can be decreased or increased depending on the parameters. The amplification condition is $\text{Im } \omega_{2M} > 0$, i.e.

$$|\omega_p^2| > 2\omega_d\omega_r\text{Im}(n_c^2). \quad (7)$$

As we see the amplification takes place necessarily if $\omega_d = 0$ or $\text{Im}(n_c^2) = 0$. In the case of finite values of ω_d and $\text{Im}(n_c^2)$, $|\omega_p^2|$ and hence the population inversion in the medium must be sufficiently large. An arbitrary mode M can be amplified if its frequency is close to the resonant frequency.

IV. NUMERICAL EXAMPLE: THE CASE OF A SOLUTION OF FULLERENE IN A LIQUID CRYSTAL

In the low GHz-frequency range, an example of an active medium with weak dispersion is a solution of fullerene (C_{60}) in a liquid crystal under optical pumping and an external magnetic field. This medium has been discussed as a possible “amplifier” for electromagnetic fields in Refs. [5,6]. In this medium, the amplification process is determined by the magnetic permeability. The dielectric permittivity is nearly constant and its imaginary part determines the dielectric losses of the structure. A model of the medium is given by the formulas

$$\mu = \mu_{c0} - \frac{|\tilde{\nu}_p^2|}{\nu_r^2 - \nu^2 - 2i\nu_d\nu}, \quad \varepsilon = \text{const}; \quad (8)$$

$$n^2 = n_{c0}^2 - \frac{|\nu_p^2|}{\nu_r^2 - \nu^2 - 2i\nu_d\nu}, \quad (9)$$

where $n_{c0}^2 = \varepsilon\mu_{c0}$, $|\nu_p^2| = \varepsilon|\tilde{\nu}_p^2|$. Based on the data obtained from electron paramagnetic resonance measurements, the parameters of this material are roughly estimated to be [6]: $\mu_{c0} = 1$, $\nu_r = 9 \times 10^9$ Hz, $|\tilde{\nu}_p| = 250 \times 10^6$ Hz, $\nu_d = 150 \times 10^6$ Hz, $\text{Re } \varepsilon = 2$. The applied magnetic field required is about 3 kG. The imaginary part of the permittivity (approximately 0.02 in this case) is proportional to the spin density in the population inversion that in turn is proportional to the optical pumping energy. Based on the present state of microwave active medium development, the required pump energy density for this parameter set is approximately 9 J/cm^3 . The real and imaginary parts of the refractive index in the vicinity of the resonance are shown in Fig. 2.

We next describe some numerical results for the case of an ultrarelativistic beam ($\gamma = 22.37$, $\beta = 0.999$). First, we consider the case of a relatively thin waveguide with a radius of 2 cm to demonstrate the single mode amplification regime for the TM_{01} mode. Figure 3 shows the dependence of the imaginary and real parts of the frequency on the channel radius for different values of the imaginary part of the permittivity. Note that, for $\text{Im } \nu_{21}$, only the region of positive magnitudes is shown. We observe that the maximum amplification and the range of channel radii where the amplification takes place both decrease as the imaginary part of the permittivity increases. However, this decrease is not very significant when $\text{Im } \varepsilon < 0.02$.

The wakefield for the case of maximum amplification of the TM_{01} mode is shown in Fig. 4. From now on, we assume that the beam is relatively short ($\sigma = 1$ mm) and the total charge in the bunch is $q = -1$ nC. Note that the e-folding amplification distance for the first mode is $l_1 = 63.2$ cm for the case of $\text{Im } \varepsilon = 0.02$, and $l_1 = 40.7$ cm for $\text{Im } \varepsilon = 0$.

FIG. 2. (Color) Real and imaginary parts of refractive index. Solid, dotted, and dashed curves correspond to $\text{Im } \varepsilon = 0; 0.01; 0.02$ accordingly.

FIG. 3. Dependence of the imaginary and real parts of the mode frequency on the channel radius for $a = 2$ cm. The corresponding values of $\text{Im } \varepsilon$ are indicated.

FIG. 4. Longitudinal component of the electric field E_z as a function of the distance $\zeta = z - Vt$ behind the bunch. (left) The amplitudes of the harmonics $E_{mz}^{(0)}$ at $\zeta = 0$. (right) ($\sigma = 1$ mm, $q = -1$ nC, $\gamma = 22.37$, $\beta = 0.999$, $\text{Im } \varepsilon = 0.02$, $a = 2$ cm, $b = 1.405$ cm).

Next we consider the case of relatively thick waveguide with $a = 10$ cm, in which the amplification regime for several modes can be found. As we see from Fig. 5, one can obtain the amplification of the modes with numbers from 1 to 6 depending on the channel radius. The maximum values of $\text{Im } \omega_{2M}$ depend on M rather weakly, and one can use a set of modes for wakefield amplification. Meanwhile the amplification regime for the first mode

occurs only for the case of a very thin layer of active medium, and this regime is very sensitive to the geometric parameters of the structure (see curve 1 in Fig. 5 which is rather narrow). The amplification regime for the higher modes occurs in structures with thicker active medium layer. The thickest active medium layer corresponds to amplification of the 6th mode. This has the important advantage that the increased volume of the active medium

FIG. 5. (Color) Dependence of the imaginary parts of the mode frequencies on the channel radius for $a = 10$ cm. Solid and dotted curves correspond to the cases of $\text{Im } \varepsilon = 0$ and $\text{Im } \varepsilon = 0.02$ accordingly. The mode number m of the corresponding amplified TM_{0m} mode is indicated.

FIG. 6. Longitudinal component of the electric field E_z (MV/m) depending on the distance from the bunch $\zeta = z - Vt$ and amplitudes of the harmonics $E_{mz}^{(0)}$ (MV/m) at $\zeta = 0$. ($\sigma = 1 \text{ mm}$, $q = -1 \text{ nC}$, $\gamma = 22.37$, $\beta = 0.999$, $\text{Im } \varepsilon = 0.02$, $a = 10 \text{ cm}$)

stores more energy inside the structure. The amplification of the higher modes is also not very sensitive to the exact choice of channel radius (the curve for the 6th mode is the widest).

Figure 6 shows the wakefields in the cases of maximum amplification of the $m = 1, 4$, and 6 modes. The channel radii and e-folding amplification distances of the corresponding modes are indicated. The right-hand plot in Fig. 6 gives the corresponding spectra at the distance $\zeta = 0$. The single mode wakefield is excited in the case of a very thin medium layer. In the other cases, the wakefields are multi-mode. It is interesting that for a distance of $|\zeta| = 200 \text{ cm}$ the amplification of the 6th mode is the most effective provided this mode can also be excited effectively.

V. CONCLUSION

In this paper a general analysis of the amplification of wakefields in cylindrical structures by weakly dispersive

active media has been presented. The wakefields are generated by a relativistic electron beam passing through an accelerating structure loaded with an active medium. The parameters of a proposed active solution of fullerene (C_{60}) in a nematic liquid crystal operating in the X-band [5] have been assumed for these calculations. A proof-of-principle experiment has recently been performed using an infrared-active CO_2 medium [3,4].

It was shown that in the case of a medium with weak dispersion the amplification effect is determined by the imaginary part of the refractive index. (In contrast to this, for the case of medium with strong dispersion the amplification effect is determined by the anomalous character of the dispersion and the contribution of the imaginary part of refractive index was less significant [1,2].) The amplification of several modes has been demonstrated, and the dependence of the maximum increment on the mode number was found to be relatively small. The waveguide ge-

ometry determines the amplification conditions for a given mode. The amplification of higher order modes exhibits noticeable advantages: (1) amplification of the higher modes occurs when an electron beam traverses a structure loaded with a thick active medium layer and hence more energy is available from the medium; (2) the dependence of the amplification magnitude on the geometrical parameters of the structure for the higher modes is not very strong; (3) as a rule, the excitation of the highest amplified mode is rather effective.

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[1] L. Schächter, Phys. Rev. E **62**, 1252 (2000).

[2] N. V. Ivanov and A. V. Tyukhtin, Tech. Phys. Lett. **32**, 449 (2006).

- [3] S. Banna, V. Berezovsky, and L. Schächter, Phys. Rev. Lett. **97**, 134801 (2006).
- [4] S. Banna, V. Berezovsky, and L. Schächter, Phys. Rev. E **74**, 046501 (2006).
- [5] A. Blank, R. Kastner, and H. Levanon, IEEE Trans. MTT **46**, 2137 (1998).
- [6] P. Schoessow, A. Kanareykin, L. Schächter, Yu. Bogachev, A. Tyukhtin, E. Bagryanskaya, and N. Yevlampieva, American Institute of Physics, CP877, Advanced Accelerator Concepts: 12th Workshop, 2006, pp. 452–460.
- [7] B. M. Bolotovskiy, Sov. Phys. Usp. **4**, 781 (1962).
- [8] K.-Y. Ng, Phys. Rev. D **42**, 1819 (1990).
- [9] A. Burov and A. Novokhatski, Proceedings of XVth International Linac Conference on High Energy Accelerators, 1992, p. 537.
- [10] A. M. Altmark, A. D. Kanareykin, and I. L. Sheinman, Tech. Phys. **50**, 87 (2005).
- [11] A. D. Kanareykin and A. V. Tyukhtin, Nucl. Instrum. Methods Phys. Res., Sect. A **558**, 62 (2006).
- [12] A. V. Tyukhtin, Tech. Phys. Lett. **30**, 605 (2004).
- [13] A. V. Tyukhtin, Tech. Phys. Lett. **31**, 150 (2005).
- [14] A. Tyukhtin, A. Kanareykin, and P. Schoessow, American Institute of Physics, CP877, Advanced Accelerator Concepts: 12th Workshop, 2006, pp. 468–474.