Finite-time thermodynamics of fluctuations in microscopic heat engines

Gentaro Watanabe^{1,2} and Yuki Minami¹¹Department of Physics and Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, Zhejiang 310027, China²Zhejiang Province Key Laboratory of Quantum Technology and Device, Zhejiang University, Hangzhou, Zhejiang 310027, China

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Fluctuations of thermodynamic quantities become non-negligible and play an important role when the system size is small. We develop finite-time thermodynamics of fluctuations in microscopic heat engines whose environmental temperature and mechanical parameter are driven periodically in time. Within the slow-driving regime, this formalism universally holds in a coarse-grained timescale whose resolution is much longer than the correlation time of the fluctuations, and is shown to be consistent with the relation analogous to the fluctuation-dissipation relation. Employing a geometric argument, a scenario to simultaneously minimize both the average and fluctuation of the dissipation in the Carnot cycle is identified. For this simultaneous optimization, the existence of a zero eigenvalue of the singular metric for the scale invariant equilibrium state is found to be essential. Furthermore, we demonstrate that our optimized protocol can improve the dissipation and its fluctuation over the current experiment.

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Introduction. Technological development has enabled us to fabricate heat engines using small systems [1–9]. To understand the performance of such small engines, theory beyond the conventional thermodynamics for macroscopic systems, thermodynamics of small systems, is called for. Recent synergy between technology and the remarkable success of stochastic thermodynamics [10–16] in understanding thermodynamic properties of small systems [17] including micromachines using active matter [18–20] and biological systems [21] has led to a surge of activity on the study of microscopic heat engines [22–28], where effects of thermal fluctuation can play an important role [29–59].

For reversible heat engines under quasistatic operations, thermodynamics provides the celebrated Carnot bound on their efficiency, and some universal relations are also known even for fluctuations [38]. However, in the practical situations, heat engines are operated within a finite period of cycle. There, losses due to dissipation is inevitable, and how to reduce such dissipative loss is a crucial issue. Furthermore, since the amount of the dissipative loss in microscopic heat engines significantly varies for each trial (e.g., the fluctuation of the efficiency is of order unity in the experiment of Ref. [5]), how to suppress the fluctuation of the dissipation is an equally important agenda, which stands out as an open problem. Here, we formulate finite-time thermodynamics of fluctuations in microscopic heat engines which is applicable to the slow-driving regime and coarse-grained timescale. This formalism provides us a geometric description for fluctuation

of the dissipation under finite-time operations. Geometric descriptions [60–66] of the mean value of the dissipative loss are known for both the macroscopic [67–69] and microscopic systems [27, 70–73]. It is noted that such a geometric description is also possible for its fluctuation. We apply our framework to the current experiment and provide an optimum scheme to minimize both the mean and the fluctuation of the dissipation simultaneously unlike trade-off optimization (see also [74] for trade-off optimization between work fluctuation and efficiency and [75–78] for recent works on optimization of the mean value). Interestingly, it is found that the scale invariance of the equilibrium state is essential for such a simultaneous optimization, and this optimization is possible only for cycles solely consisting of isothermal and isentropic strokes as the Carnot cycle. Since the dissipation is closely related to the efficiency, which is also fluctuating in microscopic heat engines, the reduction of its mean and fluctuation leads to an enhancement and stabilization of the efficiency.

Setup. Consider a classical microscopic heat engine whose working substance is always in contact with an environment whose temperature T is controllable. This is a standard setting for the experiments of the classical microscopic heat engines [3, 5, 7, 9]. The Hamiltonian H_{λ_w} of the working substance has an external mechanical parameter λ_w (or V). Therefore, the system has two control parameters, λ_w and $\lambda_u \equiv T$, and the engine protocol is specified by a closed path $\mathcal{C}$ in the parameter space spanned by the vector $\lambda_\mu \equiv (\lambda_w, \lambda_u)$. The parameters are varied cyclically to drive the state (more precisely, the phase space distribution function p) of the working substance to be periodic in time with the period τ of the cycle.

The generalized forces (X_w, X_u) conjugate to the parameters (λ_w, λ_u) are $(X_w, X_u) \equiv (P, S) = (-\partial H_{\lambda_w}/\partial \lambda_w, -\ln p)$, where $P \equiv -\partial H_{\lambda_w}/\partial \lambda_w$ is the generalized pressure and $S \equiv -\ln p$ is the stochastic entropy. Note that X_w and X_u are random variables as functions of the phase space point Γ .

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The loss of the energy due to driving with nonzero speed is quantified by the *dissipated availability* A introduced by Salamon and Berry [67]: $A \equiv U - W$ with W being the work output by the engine and U being the effective energy input from the environment (not to be confused with the internal energy) defined respectively as

$$W \equiv \oint_C P dV = \int_0^\tau dt \left(-\frac{\partial H_{\lambda_w}}{\partial \lambda_w} \right) \dot{\lambda}_w = \int_0^\tau dt X_w \dot{\lambda}_w, \quad (1)$$

$$U \equiv \oint_C T dS = \int_0^\tau dt T \frac{d}{dt} (-\ln p) = \int_0^\tau dt \lambda_u \dot{X}_u, \quad (2)$$

where the dot denotes the time derivative. Here, A , W , and U are random variables. Note that T in U is the temperature of the environment instead of that of the working substance, and U is different from heat input in general, but agrees with the latter in the quasistatic limit.

In the following, we discuss the fluctuation of random variables. Fluctuation ΔY of a random variable Y is given by $\Delta Y \equiv Y - \langle Y \rangle$, where $\langle \dots \rangle$ means an ensemble average.

Since the working substance is always in contact with the thermal environment, correlations of the physical quantities decay exponentially in time provided the driving is slow enough. In the coarse-grained timescale whose resolution is much longer compared to the decay time, such an exponential decay function can be well approximated by a delta function multiplied by its decay time constant (or the correlation time). Therefore, in the slow-driving regime and coarse-grained timescale, the two-time correlation function of the fluctuations could universally be written in the form of

$$\langle \Delta X_\mu(t) \Delta X_\nu(t') \rangle = 2 \langle \Delta X_\mu(t) \Delta X_\nu(t) \rangle \tau_{\mu\nu}(t) \delta(t - t'), \quad (3)$$

where $\tau_{\mu\nu}(t) [= \tau_{\nu\mu}(t)]$ is the correlation time between ΔX_μ and ΔX_ν at time t . In the following, we will formulate finite-time thermodynamics of fluctuations of the dissipated availability based on Eq. (3). The ansatz (3) is the minimal prescription to introduce the time scale to thermodynamics through $\tau_{\mu\nu}$ which depends on microscopic details of the system.

First, we consider the variance $\langle \Delta A^2 \rangle$ of A for a cycle given by $\langle \Delta A^2 \rangle = \langle \Delta U^2 \rangle + \langle \Delta W^2 \rangle - 2 \langle \Delta U \Delta W \rangle$. Regarding the variance of W , using Eq. (3) and the condition of the closed cycle such that the phase space distribution function at $t = 0$ and τ are the same, $p(0) = p(\tau)$, we obtain

$$\begin{aligned} \langle \Delta W^2 \rangle &= \int_0^\tau dt \int_0^\tau dt' [\langle X_w(t) X_w(t') \rangle - \langle X_w(t) \rangle \langle X_w(t') \rangle] \\ &\quad \times \dot{\lambda}_w(t) \dot{\lambda}_w(t') \\ &= 2 \int_0^\tau dt \langle \Delta X_w^2(t) \rangle \tau_{ww}(t) \dot{\lambda}_w^2(t). \end{aligned} \quad (4)$$

By integrating by parts, U given by Eq. (2) can be rewritten as $U = -\int_0^\tau X_u(t) \dot{\lambda}_u(t) dt + [X_u(\tau) - X_u(0)] \lambda_u(0)$ since $\lambda_u(\tau) = \lambda_u(0)$ but $X_u(\tau) \neq X_u(0)$ in general. Thus, using Eq. (3) and the closed cycle condition as in the way to get Eq. (4), we obtain

$$\langle \Delta U^2 \rangle = 2 \int_0^\tau dt \langle \Delta X_u^2(t) \rangle \tau_{uu}(t) \dot{\lambda}_u^2(t) + 2 \langle \Delta X_u^2(0) \rangle \lambda_u^2(0), \quad (5)$$

$$\langle \Delta W \Delta U \rangle = -2 \int_0^\tau dt \langle \Delta X_w(t) \Delta X_u(t) \rangle \tau_{wu}(t) \dot{\lambda}_w(t) \dot{\lambda}_u(t). \quad (6)$$

From Eqs. (4)–(6), we finally obtain

$$\langle \Delta A^2 \rangle = \int_0^\tau dt g_{\mu\nu}^{(2)}(t) \dot{\lambda}_\mu(t) \dot{\lambda}_\nu(t) + 2 \langle \Delta X_u^2(0) \rangle \lambda_u^2(0) \quad (7)$$

with

$$g_{\mu\nu}^{(2)}(t) \equiv 2 \tau_{\mu\nu}(t) \langle \Delta X_\mu(t) \Delta X_\nu(t) \rangle \equiv 2 \tau_{\mu\nu}(t) \sigma_{\mu\nu}(t) \quad (8)$$

for each (μ, ν) element, where $\sigma_{\mu\nu} \equiv \langle \Delta X_\mu(t) \Delta X_\nu(t) \rangle$ is the covariance matrix element between X_μ and X_ν . Summation should be taken over (w, u) components for the repeated μ and ν in Eq. (7) and hereafter. For a continuous operation taking over consecutive N cycles, an average $\overline{\langle \Delta A^2 \rangle}$ of the fluctuation of A per cycle in the limit of $N \rightarrow \infty$ is given by

$$\begin{aligned} \overline{\langle \Delta A^2 \rangle} &= \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^{N\tau} dt g_{\mu\nu}^{(2)}(t) \dot{\lambda}_\mu(t) \dot{\lambda}_\nu(t) \\ &= \int_0^\tau dt g_{\mu\nu}^{(2)}(t) \dot{\lambda}_\mu(t) \dot{\lambda}_\nu(t), \end{aligned} \quad (9)$$

since the second term of $\langle \Delta A^2 \rangle$ in Eq. (7) comes solely from the end points and their contribution vanishes by taking the average over an infinite number of cycles.

Since the covariance matrix $\sigma_{\mu\nu}$ is positive definite, $g_{\mu\nu}^{(2)}$ is also positive definite when $\tau_{ww} \tau_{uu} = \tau_{wu}^2$. Therefore, as long as $\tau_{ww} \tau_{uu} = \tau_{wu}^2$ is satisfied, $g_{\mu\nu}^{(2)}$ can be regarded as a metric tensor. Note that the resulting expression of the variance $\overline{\langle \Delta A^2 \rangle}$ given by Eq. (9) is in the similar form as the average $\langle A \rangle = \int_0^\tau dt g_{\mu\nu}^{(1)}(t) \dot{\lambda}_\mu(t) \dot{\lambda}_\nu(t)$ derived in Ref. [27], but with a different metric tensor $g_{\mu\nu}^{(2)}$. From the Cauchy-Schwarz inequality, $[\int dt f(t) g(t)]^2 \leq [\int dt f^2(t)] [\int dt g^2(t)]$ with $f(t) = (g_{\mu\nu}^{(2)} \dot{\lambda}_\mu \dot{\lambda}_\nu)^{1/2}$ and $g(t) = 1$, we get

$$\overline{\langle \Delta A^2 \rangle} \geq \frac{(\mathcal{L}^{(2)})^2}{\tau} \quad (10)$$

with the thermodynamic length $\mathcal{L}^{(2)}$ for the variance of A :

$$\mathcal{L}^{(2)} \equiv \int_0^\tau dt \sqrt{g_{\mu\nu}^{(2)}(t) \dot{\lambda}_\mu(t) \dot{\lambda}_\nu(t)} = \oint \sqrt{g_{\mu\nu}^{(2)} d\lambda_\mu d\lambda_\nu}. \quad (11)$$

The equality in Eq. (10) holds if and only if $g_{\mu\nu}^{(2)}(t) \dot{\lambda}_\mu(t) \dot{\lambda}_\nu(t)$ is constant or identically equal to zero.

Relation between $g_{\mu\nu}^{(1)}$ and $g_{\mu\nu}^{(2)}$. Next, we discuss the relation between $g_{\mu\nu}^{(1)}$ and $g_{\mu\nu}^{(2)}$. Here, we assume that the phase space distribution p follows the Fokker-Planck equation (or more generally, the Kramers-Moyal equation):

$$\frac{\partial p(\Gamma, t)}{\partial t} = L(\Gamma, t) p(\Gamma, t) \quad (12)$$

with $L(\Gamma, t)$ being the derivative operator (so-called the Fokker-Planck operator or the Kramers-Moyal operator), so that the equilibrium distribution p^{eq} satisfies the detailed balance condition:

$$L(\Gamma, t) p^{\text{eq}}(\Gamma) \dots = p^{\text{eq}}(\Gamma) L^\dagger(\sigma \Gamma, t) \dots, \quad (13)$$

where $\dagger$ denotes the adjoint and $\sigma (= +1 \text{ or } -1)$ is the symmetry factor of the phase space variable under the time

reversal operation. For sufficiently slow driving such that its time scale $\lambda_\mu/\dot{\lambda}_\mu$ is much larger than the relaxation time of the working substance, we write

$$p(\Gamma, t) = [1 + \xi(\Gamma, t)] p^{\text{eq}}(\Gamma; t), \quad (14)$$

where $p^{\text{eq}}(\Gamma; t) \equiv \exp[-\beta(t) H_{\lambda_w}(\Gamma; t)]/Z_t$ is the equilibrium state for the instantaneous parameter values $(\lambda_w(t), \lambda_u(t))$ at t , $\beta(t) \equiv 1/T(t)$ is the instantaneous inverse temperature, $Z_t \equiv \int d\Gamma \exp[-\beta(t) H_{\lambda_w}(\Gamma; t)]$ is the partition function, and $\xi \ll 1$ describes the deviation from p^{eq} . Within the linear approximation with respect to the small quantities $\dot{\lambda}_\mu$ and ξ , we obtain (see [79] for details)

$$\xi(\Gamma, t) \simeq -\beta(t) \int_0^\infty ds e^{L^\dagger(\sigma\Gamma, t)s} [X_\mu(t) - \langle X_\mu(t) \rangle_{t, \text{eq}}] \dot{\lambda}_\mu(t) \quad (15)$$

with $\langle \dots \rangle_{t, \text{eq}} \equiv \int d\Gamma p^{\text{eq}}(\Gamma; t) \dots$ being the average for the instantaneous equilibrium state at t . Within the linear response regime, the metric tensor $g_{\mu\nu}^{(1)}$ for $\langle A \rangle$ can be written as $g_{\mu\nu}^{(1)}(t) \equiv -2^{-1} [R_{\mu\nu}(t) + R_{\nu\mu}(t)]$ with the response coefficients $R_{\mu\nu}$ defined by $\langle X_\mu(t) \rangle = \int d\Gamma X_\mu(t) (1 + \xi) p^{\text{eq}} \equiv \langle X_\mu(t) \rangle_{t, \text{eq}} + R_{\mu\nu}(t) \dot{\lambda}_\nu(t)$ [27]. Using Eq. (15), it can be shown that the ansatz (3) leads to the relation between the two metric tensors $g_{\mu\nu}^{(1)}$ and $g_{\mu\nu}^{(2)}$ as

$$g_{\mu\nu}^{(2)}(t) = 2T(t) g_{\mu\nu}^{(1)}(t), \quad (16)$$

which is analogous to the fluctuation-dissipation relation.

Application to the Brownian Carnot cycle. An overdamped Brownian particle in a harmonic oscillator trapping potential is a typical setup commonly employed in experiments of microscopic heat engines [3,5,7,9,13]. Taking this setup as an example, we shall apply our framework to evaluate the fluctuation of the performance of the microscopic heat engine. For simplicity, we consider a one-dimensional case, and thus the phase space point Γ of the overdamped system can be specified solely by the position of the Brownian particle q : $\Gamma = \{q\}$. Suppose, with the mechanical control parameter λ_w , the external one-dimensional harmonic oscillator trap $V_{\lambda_w}(q)$ is given by

$$V_{\lambda_w}(q) = \frac{\lambda_w}{2} q^2. \quad (17)$$

This system can be described by the time-dependent Ornstein-Uhlenbeck process whose Fokker-Planck equation is [80,82]

$$\frac{\partial}{\partial t} p(q, t) = \frac{\lambda_w(t)}{\gamma} \frac{\partial}{\partial q} [q p(q, t)] + \frac{D(t)}{\gamma} \frac{\partial^2}{\partial q^2} p(q, t), \quad (18)$$

where D is the diffusion constant and γ is the friction coefficient.

Now, we consider a cycle whose driving speed $\dot{\lambda}_\mu$ is finite, but sufficiently slow (i.e., the time scale $\lambda_\mu/\dot{\lambda}_\mu$ is much larger than the relaxation time) so that the working substance is always close to the instantaneous equilibrium state and

$$D(t) = \beta^{-1}(t). \quad (19)$$

Thus, to obtain $\langle A \rangle$ and $\overline{\langle A^2 \rangle}$ up to the linear order of small quantities, $g_{\mu\nu}^{(i)}$ in their expressions can be evaluated for the instantaneous equilibrium state since it is already multiplied by a factor of the linear order of small quantities: $\int_0^\tau dt \dot{\lambda}_\mu \dot{\lambda}_\nu$.

FIG. 1. (a) Eigenvector field of $g_{\mu\nu}^{(2)}$ corresponding to the zero eigenvalue shown in the T - λ_w plane. The dotted straight lines are a guide for the eye showing the directions of the zero eigenvalue of $g_{\mu\nu}^{(2)}$ for different values of T/λ_w . (b) Brownian Carnot cycle, which consists of two isothermal strokes at T_h ($0 \rightarrow 1$) and T_c ($2 \rightarrow 3$), and two isentropic strokes ($1 \rightarrow 2$ and $3 \rightarrow 4$).

Since the Hamiltonian of the overdamped system is given solely by the potential energy, i.e., $H_{\lambda_w} = V_{\lambda_w}$, the instantaneous equilibrium state is $p^{\text{eq}} = \exp[-\beta \lambda_w q^2/2]/Z_t$ with $Z_t = \sqrt{2\pi T/\lambda_w}$. For the equilibrium state, each element of the covariance matrix $\sigma_{\mu\nu} = \langle \Delta X_\mu \Delta X_\nu \rangle_{t, \text{eq}}$ reads $\sigma_{ww} = (T/\lambda_w)^2/2$, $\sigma_{wu} = \sigma_{uw} = -T/2\lambda_w$, and $\sigma_{uu} = 1/2$.

Next, we consider the correlation times $\tau_{\mu\nu}$ in the equilibrium state. The correlation functions $\langle \Delta X_\mu(t) \Delta X_\nu(0) \rangle_{\text{eq}}$ in the equilibrium state read $\langle \Delta X_w(t) \Delta X_w(0) \rangle_{\text{eq}} = 4^{-1} C_{q^2}(t)$, $\langle \Delta X_w(t) \Delta X_u(0) \rangle_{\text{eq}} = \langle \Delta X_u(t) \Delta X_w(0) \rangle_{\text{eq}} = -(\lambda_w/4T) C_{q^2}(t)$, and $\langle \Delta X_u(t) \Delta X_u(0) \rangle_{\text{eq}} = (\lambda_w/2T)^2 C_{q^2}(t)$ with $C_{q^2}(t) \equiv \langle q^2(t) q^2(0) \rangle_{\text{eq}} - \langle q^2 \rangle_{\text{eq}}^2$. Since the time-dependent part of these correlation functions $\langle \Delta X_\mu(t) \Delta X_\nu(0) \rangle_{\text{eq}}$ is given by $C_{q^2}(t)$, all the correlation times should be equal: $\tau_{ww} = \tau_{uu} = \tau_{wu} = \tau_{uw}$. Therefore, they satisfy $\tau_{ww} \tau_{uu} = \tau_{wu}^2$, so that $g_{\mu\nu}^{(2)}$ is positive definite and $g_{\mu\nu}^{(i)}$ ($i=1,2$) can indeed be regarded as metric tensors. Using the nonstationary solution $p(q, t|q', t')$ (with $t \geq t'$) of the Fokker-Planck equation (18) for constant λ_w and $D = \beta^{-1}$ at their instantaneous values [80] and the stationary solution $p_s = p^{\text{eq}}$ obtained for $t - t' \gg \tau_{\mu\nu}$, we get $\langle q^2(t) q^2(0) \rangle_{\text{eq}} = \iint dq_1 dq_2 q_1^2 q_2^2 p(q_1, t|q_2, 0) p_s(q_2) = (D/\lambda_w)^2 (1 + 2e^{-2\lambda_w t/\gamma})$ and $\langle q^2 \rangle_{\text{eq}} = \int dq q^2 p_s(q) = D/\lambda_w$, and thus $C_{q^2}(t) = 2(D/\lambda_w)^2 e^{-2\lambda_w t/\gamma}$. Therefore, we finally obtain

$$\tau_{ww} = \tau_{uu} = \tau_{wu} = \frac{\gamma}{2\lambda_w}. \quad (20)$$

From $\sigma_{\mu\nu}$ evaluated for the instantaneous equilibrium state and Eq. (20), the metric tensor $g_{\mu\nu}^{(2)}$ reads

$$g_{\mu\nu}^{(2)} = \frac{\gamma}{2\lambda_w} \begin{bmatrix} (T/\lambda_w)^2 & -T/\lambda_w \\ -T/\lambda_w & 1 \end{bmatrix}. \quad (21)$$

Remarkably, this metric is singular with a zero eigenvalue whose corresponding eigenvector (normalized to unity) is $[(\lambda_w/T)^2 + 1]^{-1/2} (\lambda_w/T, 1)^T$. Thus, the direction of the zero eigenvalue of $g_{\mu\nu}^{(2)}$ is $dT/d\lambda_w = T/\lambda_w$, which is along a straight line on the T - λ_w plane connecting each point at (λ_w, T) and the origin [see Fig. 1(a)]. Since $g_{\mu\nu}^{(1)} \propto g_{\mu\nu}^{(2)}$, $g_{\mu\nu}^{(1)}$ also has a zero eigenvalue and the above argument applies to $g_{\mu\nu}^{(1)}$ as well. The other eigenvalue of $g_{\mu\nu}^{(2)}$ is $\gamma(2\lambda_w)^{-1} [1 +$

$(T/\lambda_w)^2]$ and the corresponding eigenvector is orthogonal to the one for the zero eigenvalue, i.e., $\propto (-1, \lambda_w/T)^T$.

It is noted that, along the path of the zero eigenvalue of $g_{\mu\nu}^{(i)}$ (i.e., a straight line on the T - λ_w plane for a given value of T/λ_w), the mean value $\langle S \rangle_{\text{eq}}$ of the stochastic entropy is constant:

$$\langle S \rangle_{\text{eq}} = -\langle \ln p^{\text{eq}} \rangle_{\text{eq}} = 2^{-1} [1 + \ln 2\pi - \ln(\lambda_w/T)]. \quad (22)$$

Therefore, the isentropic process defined as the one that conserves $\langle S \rangle_{\text{eq}}$ [5,30] is along the path of the zero eigenvalue of $g_{\mu\nu}^{(i)}$.

Now we consider what is called the Brownian Carnot cycle (hereafter, we refer to it as the Carnot cycle for simplicity) realized in an experiment by Martínez *et al.* [5], which consists of two isothermal strokes at hot (T_h) and cold (T_c) temperatures, and two isentropic strokes [83] as shown in Fig. 1(b). The duration of the hot ($0 \rightarrow 1$) and cold ($2 \rightarrow 3$) isothermal strokes is denoted by τ_h and τ_c , and that of the isentropic strokes $1 \rightarrow 2$ and $3 \rightarrow 4$ is denoted by $\tau_{1 \rightarrow 2}$ and $\tau_{3 \rightarrow 4}$, respectively. Thus the period τ of the cycle is given by $\tau = \tau_{\text{isoth}} + \tau_{\text{isen}}$ with $\tau_{\text{isoth}} \equiv \tau_h + \tau_c$ and $\tau_{\text{isen}} \equiv \tau_{1 \rightarrow 2} + \tau_{3 \rightarrow 4}$ being the total duration of the isothermal strokes and isentropic strokes in a cycle, respectively.

Suppose the duration of each stroke in the cycle is given and it is large enough so that the linear approximation is valid. Now, we shall optimize the protocol of each stroke under its given duration to minimize $\langle A \rangle$ and $\langle \Delta A^2 \rangle$. First of all, it is noted that, since the isentropic strokes are along the path of the zero eigenvalue of $g_{\mu\nu}^{(1)}$ and $g_{\mu\nu}^{(2)}$, these strokes have no contribution to $\langle A \rangle$, $\langle \Delta A^2 \rangle$, and the thermodynamic lengths $\mathcal{L}^{(i)} \equiv \oint \sqrt{g_{\mu\nu}^{(i)}} d\lambda_\mu d\lambda_\nu$ of the cycle [84]. Therefore, the isentropic strokes are completely irrelevant to this optimization, and what we have to do is to optimize only the isothermal strokes to fulfill the lower bounds of the inequalities:

$$\langle A \rangle = \langle A_h \rangle + \langle A_c \rangle \geq \frac{(\mathcal{L}_h^{(1)})^2}{\tau_h} + \frac{(\mathcal{L}_c^{(1)})^2}{\tau_c} \quad (23)$$

[27,85] and

$$\langle \Delta A^2 \rangle = \langle \Delta A_h^2 \rangle + \langle \Delta A_c^2 \rangle \geq \frac{(\mathcal{L}_h^{(2)})^2}{\tau_h} + \frac{(\mathcal{L}_c^{(2)})^2}{\tau_c}, \quad (24)$$

where A_h and A_c are the dissipated availability in the hot and cold isothermal strokes, and $\mathcal{L}_h^{(i)} \equiv \int_{0 \rightarrow 1} \sqrt{g_{\mu\nu}^{(i)}} d\lambda_\mu d\lambda_\nu$ and $\mathcal{L}_c^{(i)} \equiv \int_{2 \rightarrow 3} \sqrt{g_{\mu\nu}^{(i)}} d\lambda_\mu d\lambda_\nu$ are the thermodynamic length of the path of the hot and cold isothermal strokes, respectively. Writing $\lambda_{w,i}$ for λ_w at node i , from Eqs. (21) and (16) we obtain $\mathcal{L}_h^{(i)} = \sqrt{2^{i-1} \gamma T_h^i (\lambda_{w,1}^{-1/2} - \lambda_{w,0}^{-1/2})}$ ($i = 1, 2$). Since $T_h/\lambda_{w,1} = T_c/\lambda_{w,2}$ and $T_h/\lambda_{w,0} = T_c/\lambda_{w,3}$, we further get $\mathcal{L}_c^{(i)} = (T_c/T_h)^{(i-1)/2} \mathcal{L}_h^{(i)}$ ($i = 1, 2$). As has been discussed for Eq. (10), the equalities of Eqs. (23) and (24) hold when $g_{\mu\nu}^{(i)}(t) \dot{\lambda}_\mu(t) \dot{\lambda}_\nu(t)$ is constant in time, or equivalently, when we sweep the parameter λ_μ in a way such that the time interval to traverse the path element $d\lambda_\mu$ is proportional to the thermodynamic length $d\mathcal{L}^{(i)}$ of this path element. Since $g_{\mu\nu}^{(i)} \dot{\lambda}_\mu \dot{\lambda}_\nu = g_{\mu\nu}^{(i)} \dot{\lambda}_w^2$ in the isothermal strokes, the above condition is satisfied by sweeping as

$$|\dot{\lambda}_w| \propto \lambda_w^{3/2}, \quad (25)$$

and we get $\langle A \rangle = (\tau_h^{-1} + \tau_c^{-1})(\mathcal{L}_h^{(1)})^2$ and $\langle \Delta A^2 \rangle = [\tau_h^{-1} + (T_c/T_h)\tau_c^{-1}](\mathcal{L}_h^{(2)})^2$. It is noted that, since $g_{\mu\nu}^{(1)}$ and $g_{\mu\nu}^{(2)}$ are related through Eq. (16) and T is constant in each isothermal stroke, the optimal protocol (25) applies to both $\langle A \rangle$ and $\langle \Delta A^2 \rangle$. Namely, once the duration of each stroke is given, both $\langle A \rangle$ and $\langle \Delta A^2 \rangle$ for the Carnot cycle are optimized simultaneously, which is in contrast to the trade-off optimization between, e.g., work fluctuation and efficiency [74].

As can be seen from the above discussion, such simultaneous minimization of $\langle A \rangle$ and $\langle \Delta A^2 \rangle$ by the same protocol within finite time is possible only for cycles solely consisting of isothermal strokes and isentropic strokes. First of all, isentropic strokes have zero contribution to $\langle A \rangle$ and $\langle \Delta A^2 \rangle$ because of the zero eigenvalue of the metric in our system. In addition, only isothermal strokes are nontrivial strokes with nonzero $\langle A \rangle$ and $\langle \Delta A^2 \rangle$ that allow us to minimize both $\langle A \rangle$ and $\langle \Delta A^2 \rangle$ simultaneously by the same protocol $\lambda_\mu(t)$. Among cycles whose number of strokes N_s is less than or equal to 5, the Carnot cycle is the only possibility to form a meaningful cycle with nonzero work output solely with isothermal and isentropic strokes [86]. Therefore, when the duration of each stroke is given, the Carnot cycle is the only case with $N_s \leq 5$ which allows us to simultaneously minimize $\langle A \rangle$ and $\langle \Delta A^2 \rangle$.

It is noted that the existence of the zero eigenvalue is essential for the above simultaneous minimization, and such a singular metric is due to the scale invariance of the equilibrium state. In the overdamped case, once the trapping potential is scale invariant, i.e., V_{λ_w} satisfies $V_{\lambda_w}(aq) = |a|^\alpha V_{\lambda_w}(q)$ for some α with a being the scaling factor, so is the system itself, and as a consequence, the metric becomes singular. A power-law function is a representative example of the scale invariant function, and the above simultaneous minimization of $\langle A \rangle$ and $\langle \Delta A^2 \rangle$ is possible also for a general power-law trapping potential $V_{\lambda_w}(q) \propto \lambda_w |q|^k$ with positive k [79].

If only the total duration of the isothermal strokes τ_{isoth} is given instead of τ_h and τ_c separately, we can further optimize to reduce either $\langle A \rangle$ or $\langle \Delta A^2 \rangle$, respectively. This optimization holds irrespective of the duration τ_{isen} and the protocol of the isentropic strokes, provided the driving in these strokes is slow enough. From the condition that $g_{\mu\nu}^{(i)}(t) \dot{\lambda}_\mu(t) \dot{\lambda}_\nu(t) = g_{\mu\nu}^{(i)} \dot{\lambda}_w^2$ is constant at the same value for both the two isothermal strokes, the optimal protocol is $|\dot{\lambda}_w| \propto \lambda_w^{3/2} T^{-i+1}$ ($i = 1, 2$), which yields $\tau_h = \tau_c = \tau_{\text{isoth}}/2$ to optimize $\langle A \rangle$ and $\tau_h/\sqrt{T_h} = \tau_c/\sqrt{T_c}$ [i.e., $\tau_h = \tau_{\text{isoth}}/(1 + \sqrt{T_c/T_h})$ and $\tau_c = \tau_{\text{isoth}} - \tau_h$] to optimize $\langle \Delta A^2 \rangle$ with their minimum values $\langle A \rangle = (\mathcal{L}^{(1)})^2/\tau_{\text{isoth}}$ and $\langle \Delta A^2 \rangle = (\mathcal{L}^{(2)})^2/\tau_{\text{isoth}}$, respectively.

Finally, we discuss how much the optimized protocol given by Eq. (25) improves the dissipated availability and its fluctuation of the Carnot cycle compared to the one in the current experiment [5]. In this experiment, $\lambda_w(t)$ in Eq. (17) is controlled by the following protocol: $\lambda_w(t) = f(t)$ for $0 \leq t \leq \tau/2$ and $\lambda_w(t) = f(\tau - t)$ for $\tau/2 \leq t \leq \tau$ with $f(t) = 4(\lambda_{w,2} - \lambda_{w,0})(t/\tau)^2 + \lambda_{w,0}$. The experimental parameter values are as follows: $T_c = 300$ K, $T_h = 525$ K, $\lambda_{w,0} = 20.0$ pN μm^{-1} , $\lambda_{w,2} = 2.0$ pN μm^{-1} , the cycle period $\tau = 200$ ms, and the duration of each stroke is $\tau_{0 \rightarrow 1} = 0.26\tau$, $\tau_{1 \rightarrow 2} = 0.24\tau$, $\tau_{2 \rightarrow 3} = 0.25\tau$, and $\tau_{3 \rightarrow 4} = 0.25\tau$. In addition, the friction coefficient is estimated as $\gamma = 8.4$ pN μm^{-1} ms

[79]. From these parameter values, the ratio of the dissipated availabilities for the experimental protocol and the optimized protocol given by Eq. (25) and that for their fluctuations are calculated to be [79]

$$\frac{\langle A_{\text{opt}} \rangle}{\langle A_{\text{expt}} \rangle} = 0.65, \quad \frac{\overline{\langle \Delta A_{\text{opt}}^2 \rangle}}{\overline{\langle \Delta A_{\text{expt}}^2 \rangle}} = 0.70. \quad (26)$$

Here, the subscripts “opt” and “expt” represent quantities obtained by the optimized and the experimental protocols, respectively. The optimized protocol improves the dissipated availability by 35% and its fluctuation by 30%. This means that the above optimization improves not only the efficiency ϵ , which is given by $\epsilon \equiv \langle W \rangle / \langle U \rangle \simeq 1 - \langle A \rangle / \mathcal{W}$ with $\mathcal{W} (= \langle W \rangle)$ being the work output by the quasistatic cycle [27], but also the stability of the fluctuating efficiency $\mathcal{E} \equiv W/U \simeq 1 - A/\mathcal{W}$ characterized by $\overline{\langle \Delta \mathcal{E}^2 \rangle} \simeq \overline{\langle \Delta A^2 \rangle} / \mathcal{W}^2$. Since the fluctuation of the stochastic efficiency $\mathcal{E}$ is non-negligible in the Brownian Carnot engine (in the experiment of [5], the variance of $\mathcal{E}$ is of order unity, and the distribution function of $\mathcal{E}$ spreads even to the negative side), it is crucially important to reduce $\overline{\langle \Delta A^2 \rangle}$ as well as $\langle A \rangle$.

Concluding remarks. In summary, we have formulated finite-time thermodynamics of fluctuations in microscopic heat engines, which universally holds for the slow-driving regime and the coarse-grained timescale. This formalism provides a geometric description of the fluctuation of the dissipation whose metric is found to be consistent with the relation

analogous to the fluctuation-dissipation relation. Applying our framework to the Carnot cycle, it has been found that both the average and the fluctuation of the dissipation can be minimized simultaneously when the duration of each stroke is given. Interestingly, we have seen that the scale invariance of the equilibrium state is essential for this simultaneous minimization. The benefit of this optimized protocol has been demonstrated for the current experiment [5].

Our framework should find broad applications to design not only energy efficient but also stable microscopic thermal machines. While we have considered classical systems, developing a corresponding formalism for quantum systems constitutes an important and challenging future problem. Further analysis based on Riemannian geometry for systems with a nonsingular metric, such as a classical or quantum two-level system, etc., would lead to a deeper understanding on thermodynamics of fluctuations. For example, geometric interpretation of thermodynamics of fluctuations provided in our work may lead to another type of universal relations and bounds on fluctuations of the performance of microscopic heat engines distinct from thermodynamic uncertainty relations [87–91].

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