Strong entanglement distribution of quantum networks

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Large-scale quantum networks have been employed to overcome practical constraints of transmission and storage for single entangled systems. Our goal in this paper is to explore the strong entanglement distribution of quantum networks. We firstly show that any connected network consisting of generalized Einstein-Podolsky-Rosen states and Greenberger-Horne-Zeilinger states satisfies the strong Coffman-Kundu-Wootters monogamy inequality in terms of the bipartite entanglement measure; in addition, the monogamy inequalities are also considered for generic entangled quantum networks. This reveals the interesting feature of high-dimensional entanglement with local tensor decomposition going beyond qubit entanglement. We then apply the entanglement distribution relation in entangled networks to get a quantum maximum-flow minimum-cut theorem in terms of von Neumann entropy and Rényi- α entropy. We finally classify entangled quantum networks by distinguishing network configurations under local unitary operations. These results provide insights into characterizing quantum networks in quantum information processing.

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I. INTRODUCTION

Entanglement as an intriguing phenomenon has been considered to be the heart of quantum mechanics. It provides a crucial resource for quantum information processing, including quantum teleportation [1], quantum dense coding [2], quantum secret sharing [3], and quantum cryptography [4]. The study of entanglement and its distribution reveals fundamental insights into the nature of quantum correlations [5,6], the features of many-body systems [7,8], and the potential limitations for quantum-enhanced technologies [9].

One interesting feature of entanglement is the impossibility of sharing entanglement freely in multiparty quantum systems. This, without any classical counterpart, is known as the monogamy of entanglement (MOE) [10,11]; that is, entanglements satisfy special constraints on how they can be distributed among multipartite systems. The monogamy inequality became synonymous due to Coffman, Kundu, and Wootters (CKW) [11] with

$$\mathcal{Q}_{A|BC}(\rho_{ABC}) \geq \mathcal{Q}_{A|B}(\rho_{AB}) + \mathcal{Q}_{A|C}(\rho_{AC}), \quad (1)$$

with specific entanglement measure $\mathcal{Q}$. Here, $\rho_{AB} = \text{Tr}_C(\rho_{ABC})$ denotes the reduced state of parties A and B, and analogously for ρ_{AC} . A|BC is for the bipartite split. The first example is related to the squared concurrence between bipartitions [10,11] for tripartite states and N -qubit states [12]. Some other monogamy inequalities are built for multiqubit systems

in terms of the α th power of entanglement of formation (EOF) ($\alpha \geq \sqrt{2}$) [13–15] or the α th power of concurrence ($\alpha \geq 2$) [16,17]. Similar results hold for qubit systems with the α ($\alpha \geq 2$)th powers of the Tsallis entropy [18], Rényi entropy [19], and unified entropy [20]. However, most of these well-known entanglement measures [12–15,18,21–24] fail to satisfy the monogamy relation (1) for qubit systems except for the squashed entanglement [25]. A natural problem is to explore which entangled states allow the CKW inequality. One intriguing example is from higher-dimensional quantum systems [18,26,27].

It is difficult to characterize general high-dimensional entangled states because of exponential parameters. One special scenario is from distributive constructions; that is, it allows local tensor structures further regarded as quantum networks [28]. These kinds of multipartite resources are applicable for large-distance quantum communication [29,30], quantum Internet [31,32], quantum secret sharing [26], distributed quantum computing [33], blind quantum computing [34], and quantum sensing [35,36]. This makes intriguing the interesting problem of universal monogamy relations for higher-dimensional systems from quantum networks.

Our motivation in this paper is to explore entangled quantum networks by using quantum entropy toward quantum communication. We firstly prove that the standard CKW monogamy (1) holds for generic entangled quantum networks in terms of EOF and Rényi entropy. This shows features of entanglement distribution for high-dimensional quantum states going beyond qubit entanglement [10,11,16] or single entanglement [14,22]. We then apply the proposed entanglement distribution to resolve the maximum-flow minimum-cut (max-flow min-cut) problem for entangled networks even if it fails with single multipartite entanglement [37]. The max-flow min-cut theorem presented here holds for any quantum networks consisting of Einstein-Podolsky-Rosen (EPR) states

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and Greenberger-Horne-Zeilinger (GHZ) states. Moreover, the entanglement entropy implies one way for classifying the configuration for entangled networks under local unitary operations. The present method can be adapted for cyclic networks going beyond recent results [38,39]. These results are interesting in entanglement theory, quantum communication, and quantum information processing.

II. ENTANGLEMENT MEASURE BASED ON QUANTUM ENTROPY

In this section, we recall necessary bipartite entanglement measures based on quantum entropies. As an entanglement measure characterizing bipartite entanglement through von Neumann entropy, the entanglement of formation (EOF) $\mathcal{E}$ for a pure state $|\phi\rangle_{AB}$ on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is defined as [40]

$$\mathcal{E}(|\phi\rangle_{AB}) = S(\rho_A) = -\text{Tr}(\rho_A \log_2 \rho_A), \quad (2)$$

where $\rho_A = \text{Tr}_B(|\phi\rangle_{AB}\langle\phi|)$ denotes the density operator of subsystem A obtained by tracing out subsystem B and $S(\rho_A)$ denotes the von Neumann entropy of system A. Note that the von Neumann entropy of two density operators ρ and σ satisfies the additivity [41]

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma). \quad (3)$$

For a bipartite mixed state ρ_{AB} on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, EOF is given by

$$\mathcal{E}(\rho_{AB}) = \inf_{\{p_i, |\phi_i\rangle\}} \sum_i p_i \mathcal{E}(|\phi_i\rangle_{AB}), \quad (4)$$

where the infimum takes over all possible pure-state decompositions of $\rho_{AB} = \sum_i p_i |\phi_i\rangle_{AB}\langle\phi_i|$ with $p_i \geq 0$, $\sum_i p_i = 1$, and $|\phi_i\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$.

As a generalization of EOF, the Rényi- α entanglement measure $\mathcal{R}_\alpha$ can be defined by using the Rényi- α entropy as [19]

$$\mathcal{R}_\alpha(|\phi\rangle_{AB}) = S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log_2 \text{Tr}(\rho_A^\alpha), \quad (5)$$

where $S_\alpha(\rho_A)$ denotes the Rényi- α entropy of system A. Similar to the von Neumann entropy, Rényi- α entropy has the additivity [42]

$$S_\alpha(\rho \otimes \sigma) = S_\alpha(\rho) + S_\alpha(\sigma) \quad (6)$$

for $\alpha > 0$ with $\alpha \neq 1$. For a bipartite mixed state ρ_{AB} , Rényi- α entanglement $\mathcal{R}_\alpha$ is defined via the convex-roof extension as [19]

$$\mathcal{R}_\alpha(\rho_{AB}) = \inf_{\{p_i, |\phi_i\rangle\}} \sum_i p_i \mathcal{R}_\alpha(|\phi_i\rangle_{AB}) \quad (7)$$

with the minimum taking over all possible decompositions of ρ_{AB} .

For a bipartite pure state $|\phi\rangle_{AB}$ on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, The Tsallis- q entanglement measure $\mathcal{T}^{(q)}$ derived from Tsallis- q entropy is defined as [43]

$$\mathcal{T}^{(q)}(|\phi\rangle_{AB}) = S_q(\rho_A) = \frac{1}{q-1} (1 - \text{Tr}(\rho_A^q)), \quad (8)$$

where $S_q(\rho_A)$ denotes the Tsallis- q entropy of the system A, $q > 0$ and $q \neq 1$. $S_q(\rho)$ converges to the von Neumann entropy when $q \rightarrow 1$; that is, $\lim_{q \rightarrow 1} S_q(\rho) = -\text{Tr} \rho \log_2(\rho) =$

$S(\rho)$ for any bipartite operator density ρ . For a bipartite mixed state ρ_{AB} , Tsallis- q entropy entanglement $\mathcal{T}^{(q)}$ is defined by

$$\mathcal{T}^{(q)}(\rho_{AB}) = \inf_{\{p_i, |\phi_i\rangle\}} \sum_i p_i \mathcal{T}^{(q)}(|\phi_i\rangle_{AB}), \quad (9)$$

where the infimum is taken over all possible pure-state decompositions of ρ_{AB} .

For $q > 1$, the quantum Tsallis entropy satisfies the subadditivity property [44]

$$S_q(\rho \otimes \sigma) \leq S_q(\rho) + S_q(\sigma), \quad (10)$$

while for $0 < q < 1$, it satisfies

$$S_q(\rho \otimes \sigma) \geq S_q(\rho) + S_q(\sigma). \quad (11)$$

Different from all the stated entropies, there exists a generalized entropy, i.e., unified (q, s) entropy [45], which involves two real parameters q and s :

$$S_{q,s}(\rho) = \frac{1}{(1-q)s} (\text{Tr}(\rho^q)^s - 1), \quad (12)$$

where $q, s \geq 0$ and $q \neq 1$, $s \neq 0$. This entropy includes Rényi- α entropy [46] and Tsallis- q entropy [47] as special cases of $s \rightarrow 0$ and $s \rightarrow 1$, respectively. Moreover, for $q \rightarrow 1$, it converges to the von Neumann entropy [41]. Using the unified (q, s) entropy, the unified (q, s) entanglement measure is defined for a bipartite pure state $|\phi\rangle_{AB}$ on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ by [20]

$$\mathcal{U}^{(q,s)}(|\phi\rangle_{AB}) = S_{q,s}(\rho_A) \quad (13)$$

for each $q, s \geq 0$. For a bipartite mixed state ρ_{AB} , its entanglement measure is given by

$$\mathcal{U}^{(q,s)}(\rho_{AB}) = \inf_{\{p_i, |\phi_i\rangle\}} \sum_i p_i \mathcal{U}^{(q,s)}(|\phi_i\rangle_{AB}), \quad (14)$$

where the infimum is taken over all possible pure-state decompositions of ρ_{AB} .

For $q \geq 1$ and $s \geq 0$, the quantum unified (q, s) entropy satisfies the subadditivity property [48]

$$S_{q,s}(\rho \otimes \sigma) \leq S_{q,s}(\rho) + S_{q,s}(\sigma), \quad (15)$$

while for $0 < q < 1$ and $s > 0$ it satisfies

$$S_{q,s}(\rho \otimes \sigma) \geq S_{q,s}(\rho) + S_{q,s}(\sigma). \quad (16)$$

From all the aforementioned bipartite entanglement measures, they are subadditive; that is,

$$\mathcal{Q}(\rho \otimes \sigma) \leq \mathcal{Q}(\rho) + \mathcal{Q}(\sigma), \quad (17)$$

where $\mathcal{Q}$ can be any one of $\mathcal{E}$, $\mathcal{R}_\alpha$, $\mathcal{T}^{(q)}$, and $\mathcal{U}^{(q,s)}$.

III. ENTANGLEMENT MONOGAMY RELATION OF QUANTUM NETWORKS

In comparison to single entanglement, a tripartite quantum network consists of various bipartite entangled (BE) states and GHZ states, as shown in Fig. 1. Here, the local unitary operations may entangle these independent states into an entanglement in the global version. This generates high-dimensional quantum entangled states that are important

FIG. 1. A monogamy relation of tripartite entanglement inspired by a tripartite quantum network consisting of one GHZ state and three arbitrary bipartite entangled (BE) states. U , V , and W are local unitary operations. $\mathcal{Q}$ is entanglement measure defined in the inequality (1).

resources in quantum information processing. Characterizing the general high-dimensional quantum systems is difficult because of the exponential number of parameters. Our motivation here is to feature quantum networks via entanglement distribution by using quantum entropy. The high-dimensional entanglement inspired by quantum networks allows local tensor decomposition that provides the possibility for evaluating new monogamy relations based on von Neumann entropy [41] and Rényi entropy [46], Tsallis entropy [47], and unified entropy [45].

A. Quantum network with generalized GHZ states and EPR states

Consider an n -partite entangled quantum network $\mathcal{N}_q(\mathcal{A}, \xi)$, where $\mathcal{A}$ denotes parties $A_1, \dots, A_n$, and ξ denotes entangled states. Assume any two parties A_i and A_j share the states $\rho_{ij}(\theta_1), \dots, \rho_{ij}(\theta_s), \sigma_{ij}(\varphi_1), \dots, \sigma_{ij}(\varphi_t)$, and $\delta_{ij}(\vartheta_1), \dots, \delta_{ij}(\vartheta_k)$, where $\rho(\theta)$ denotes the density matrix of generalized EPR state [49] $|\phi(\theta)\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle$, $\sigma(\varphi)$ is the density matrix of generalized GHZ state [50] $|\phi(\varphi)\rangle = \cos \varphi |0\rangle^{\otimes m} + \sin \varphi |1\rangle^{\otimes m}$ with any integer $m \geq 3$, and $\delta(\vartheta) = \cos^2 \vartheta |00\rangle\langle 00| + \sin^2 \vartheta |11\rangle\langle 11|$ is the reduced density matrix of any two subsystems obtained by tracing out other subsystems in a generalized GHZ state. Here, each party can perform any local unitary operations to entangle the local systems. With this formation, the total system may be regarded as an n -partite entanglement in high-dimensional Hilbert spaces. Informally, we show that the entanglement distribution for quantum networks with local unitary operations satisfies the monogamy inequality (1).

Theorem 1. The entanglement distribution of quantum network $\mathcal{N}_q(\mathcal{A}, \xi)$ is given by

$$\mathcal{Q}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) \geq \sum_{j=1, j \neq i}^N \mathcal{Q}_{A_i|A_j}(\rho_{A_i A_j}), \quad (18)$$

FIG. 2. Schematic quantum network consisting of generalized EPR states or generalized GHZ states. (a) A chain quantum network consisting of $n-1$ generalized EPR states. (b) A cyclic quantum network consisting of n generalized EPR states. (c) A star quantum network consisting of generalized EPR states and generalized GHZ states. (d) A general quantum network consisting of generalized EPR states and generalized GHZ states.

where $\mathcal{Q}$ denotes the entanglement measure of EOF or Rényi- α entropy, $\mathcal{Q}_{X|Y}$ is an entanglement measure of a composite quantum system with respect to the bipartite cut between X and Y , and $\bar{A}_i$ denotes all parties except for A_i . The equality holds if and only if all GHZ states are being shared by two parties.

Proof. As shown in Fig. 2, we illustrate quantum networks consisting of generalized EPR states and GHZ states; these include a chain network [Fig. 2(a)], cyclic network [Fig. 2(b)], star network [Fig. 2(c)], and general network [Fig. 2(d)]. Now, we prove the monogamy for general network $\mathcal{N}_q(\mathcal{A}, \xi)$ as shown in Fig. 2(d). Suppose that any two parties A_i and A_j share the quantum state ξ_{ij} given by

$$\xi_{ij} = \bigotimes_{s=1}^{s_j} \rho_{ij}(\theta_s) \bigotimes_{t=1}^{t_j} \sigma_{ij}(\varphi_t) \bigotimes_{k=1}^{k_j} \delta_{ij}(\vartheta_k), \quad (19)$$

where $\rho(\theta_s)$ is the reduced density matrix of generalized EPR state [49] $|\phi(\theta_s)\rangle = \cos \theta_s |00\rangle + \sin \theta_s |11\rangle$, $\sigma(\varphi_t)$ is the reduced density matrix of generalized GHZ state [50] $|\phi(\varphi_t)\rangle = \cos \varphi_t |0\rangle^{\otimes m} + \sin \varphi_t |1\rangle^{\otimes m}$, and $\delta(\vartheta_k) = \cos^2 \vartheta_k |00\rangle\langle 00| + \sin^2 \vartheta_k |11\rangle\langle 11|$ is the reduced density matrix of any two subsystems by tracing out the remaining subsystems in a multipartite generalized GHZ state. Herein, s_j , t_j , and k_j denote the numbers of $\rho(\theta_s)$, $\sigma(\varphi_t)$, and $\delta(\vartheta_k)$ that are shared by any two parties A_i and A_j , respectively. Then, the total state $\rho_{A_1 \dots A_n}$ of an n -partite system is given by

$$\rho_{A_1 \dots A_n} = \bigotimes_{i,j=1, i \neq j}^n \xi_{ij}, \quad (20)$$

where ξ_{ij} has the form of Eq. (19).

Assume that any party A_i is entangled with N parties of A_j 's. From Eq. (2), we get

$$\begin{aligned}\mathcal{E}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) &= S(\rho_{A_i}) \\ &= S\left(\bigotimes_{j=1}^N \left(\bigotimes_{s=1}^{s_j} \rho_{i,j}(\theta_s) \bigotimes_{t=1}^{t_j} \rho_{i,j}(\varphi_t) \bigotimes_{k=1}^{k_j} \rho_{i,j}(\vartheta_k)\right)\right) \\ &= \sum_{j=1}^N \left(\sum_{s=1}^{s_j} S(\rho_{i,j}(\theta_s)) + \sum_{t=1}^{t_j} S(\rho_{i,j}(\varphi_t)) \right) \\ &\quad + \sum_{j=1}^N \sum_{k=1}^{k_j} S(\rho_{i,j}(\vartheta_k)),\end{aligned}\quad (21)$$

where Eq. (21) follows from the additivity of von Neumann entropy in Eq. (3). Here, $\rho_{i,j}(\theta_s)$ is the reduced density matrix of the subsystems owned by A_i , which is obtained by tracing out all the subsystems owned by A_j in a generalized EPR state $|\phi(\theta_s)\rangle_{i,j}$. $\rho_{i,j}(\varphi_t)$ is the reduced density operator of the subsystem owned by A_i with respect to the generalized GHZ state shared by two parties A_i and A_j . $\rho_{i,j}(\vartheta_k)$ is the reduced density matrix of the subsystems owned by A_i , which is obtained by tracing out the other subsystems in a generalized GHZ state shared by multipartite parties.

Note that the entanglement shared by the parties A_i and A_j is generated by generalized EPR states and GHZ states. It is sufficient to prove that

$$\begin{aligned}\mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) &= \mathcal{E}\left(\bigotimes_{s=1}^{s_j} \rho_{ij}(\theta_s) \bigotimes_{t=1}^{t_j} \sigma_{ij}(\varphi_t) \bigotimes_{k=1}^{k_j} \delta_{ij}(\vartheta_k)\right) \\ &\leq \mathcal{E}(\rho_{ij}(\theta_s)) + \mathcal{E}\left(\bigotimes_{t=1}^{t_j} \sigma_{ij}(\varphi_t)\right)\end{aligned}\quad (22)$$

$$= \sum_{s=1}^{s_j} S(\rho_{i,j}(\theta_s)) + \sum_{t=1}^{t_j} S(\rho_{i,j}(\varphi_t)). \quad (23)$$

From the additivity of the entanglement measure $\mathcal{E}$ in Eq. (17) and $\mathcal{E}(\delta_{ij}(\vartheta_k)) = 0$, the inequality (22) follows. Equation (23) is derived from the definition of the entanglement measure $\mathcal{E}$ in Eq. (2). Combining Eq. (21) with Eq. (23) yields

$$\mathcal{E}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) \geq \sum_{j=1, j \neq i}^N \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) \quad (24)$$

$$= \sum_{j=1, j \neq i}^n \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}). \quad (25)$$

Note that for $k_j = 0$, i.e., all m -qubit GHZ states are shared by two parties, we have

$$\mathcal{E}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) = \sum_{j=1}^N \left(\sum_{s=1}^{s_j} S(\rho_{i,j}(\theta_s)) + \sum_{t=1}^{t_j} S(\rho_{i,j}(\varphi_t)) \right). \quad (26)$$

Moreover, we obtain

$$\mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) = \sum_{j=1}^N \left(\sum_{s=1}^{s_j} S(\rho_{i,j}(\theta_s)) + \sum_{t=1}^{t_j} S(\rho_{i,j}(\varphi_t)) \right). \quad (27)$$

FIG. 3. Device-independent quantum key distribution with an attacker E. (a) The standard attack model. (b) The attack model based on the triangle network.

So, combining Eqs. (26) and (27), we get

$$\mathcal{E}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) = \sum_{j=1, j \neq i}^n \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}). \quad (28)$$

Similarly, the inequality (18) holds for the Rényi- α entropy entanglement in Eq. (5) from the additivity of Rényi- α entropy in Eq. (6), i.e.,

$$\mathcal{R}_\alpha^{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) \geq \sum_{j=1, j \neq i}^N \mathcal{R}_\alpha^{A_i|A_j}(\rho_{A_i A_j}). \quad (29)$$

This has completed the proof. $\blacksquare$

Theorem 1 indicates a mutually exclusive relation of quantum networks between any pair of parties A_i and A_j , which goes beyond the single entanglement of GHZ states being ruled out by the monogamy inequality (1). The more interesting part is that the present result provides a generic monogamous relation for high-dimensional entanglement beyond qubit states [10–12, 16, 18–20]. It is also stronger than the existence result of high-dimensional entanglement [26]. Theorem 1 may provide a better-than-single-entanglement performance in quantum tasks. One example is device-independent quantum key distribution [5] where the eavesdropper's attacks are basically limited by the monogamy of entanglement.

Example 1. Consider a device-independent quantum key distribution model with entanglement ρ_{AB} [6]. There is an eavesdropper Eve, represented by E, who may be correlated with ρ_{AB} . The total state is then denoted by ρ_{ABE} satisfying $\text{Tr}_E(\rho_{ABE}) = \rho_{AB}$, as shown in Fig. 3(a). Eve is assumed to control the source or measurement devices in a relaxed model. The entanglement between Eve and Alice satisfies the monogamy relation [16]

$$\mathcal{E}_{A|E}(\rho_{AE}) \leq (\mathcal{E}_{A|BC}^{\sqrt{2}}(\rho_{ABE}) - \mathcal{E}_{A|B}^{\sqrt{2}}(\rho_{AB}))^{\frac{1}{\sqrt{2}}}, \quad (30)$$

where $\mathcal{E}$ represents the EOF measure. The inequality (30) provides a monogamy relation for the information leakage in a device-independent model.

Instead, the eavesdropper may hold local systems correlated with ρ_{AB} . That is, Eve fakes entangled sources $|\varphi_\beta\rangle$ and $|\varphi_\gamma\rangle$ between her device and Alice's and Bob's respective devices, as shown in Fig. 3(b). This generates a triangle quantum

FIG. 4. Schematic difference between the upper bounds (30) and (31).

network which allows from Theorem 1 the relation

$$\mathcal{E}_{A|E}(\rho_{AE}) \leq \mathcal{E}_{A|BE}(\rho_{ABE}) - \mathcal{E}_{A|B}(\rho_{AB}). \quad (31)$$

The difference between the upper bounds (30) and (31) is denoted as

$$f(\mathcal{E}_{A|BE}, \mathcal{E}_{A|B}) = (\mathcal{E}_{A|BE}^{\sqrt{2}} - \mathcal{E}_{A|B}^{\sqrt{2}})^{\frac{1}{\sqrt{2}}} - (\mathcal{E}_{A|BE} - \mathcal{E}_{A|B}), \quad (32)$$

where $\mathcal{E}_{A|BE}$ and $\mathcal{E}_{A|B}$ satisfy the inequality $\mathcal{E}_{A|BE} > \mathcal{E}_{A|B}$. A schematic demonstration is given to state the difference between the upper bounds (30) and (31) as shown in Fig. 4. Obviously, $f(\mathcal{E}_{A|BE}, \mathcal{E}_{A|B})$ is always positive; this indicates that the inequality (31) provides a stronger constraint for the information leakage in a device-independent model, which makes it harder for the attacker to carry out the attack.

B. General quantum networks

In this section, we extend Theorem 1 to general quantum networks.

1. Quantum network with W states

Consider a genuine tripartite entangled W state [51]:

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle). \quad (33)$$

It is easy to obtain

$$\mathcal{E}(|W\rangle_{A|B_1B_2}) = H\left(\frac{1}{3}\right) \approx 0.918, \\ \mathcal{E}(\rho_{AB_i}) = H\left(\frac{1 + \sqrt{1 - C^2(\rho_{AB_i})}}{2}\right) \approx 0.55, \quad (34)$$

where $\mathcal{E}$ represents EOF in Eq. (2), $H(x)$ is the binary Shannon entropy given $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ for $0 \leq x \leq 1$ [41], and $C(\rho_{AB_i}) = \frac{2}{3}$ ($i = 1, 2$) is the concurrence [52] of the two-qubit mixed state ρ_{AB_i} . From Eq. (34), we have

$$\mathcal{E}(|W\rangle_{A|B_1B_2}) \leq \mathcal{E}(\rho_{AB_1}) + \mathcal{E}(\rho_{AB_2}); \quad (35)$$

that is, EOF $\mathcal{E}$ of the W state is not monogamous. However, we prove that the monogamy relation of $\mathcal{E}$ holds for quantum networks consisting of GHZ states and W states. The monogamy of the quantum network reaches a good saturation in the case that the quantum network consists of $|W\rangle^{\otimes 5} \otimes |GHZ\rangle$, i.e., five pairs of the W state and one GHZ state are shared by three parties, simultaneously. The main reason is as follows. For the GHZ state, we have

$$\mathcal{E}(|GHZ\rangle_{A|BC}) = 1, \\ \mathcal{E}(\rho_{AB}) = \mathcal{E}(\rho_{AC}) = 0. \quad (36)$$

From Eqs. (34) and (36), we get the inequality

$$\mathcal{E}(|GHZ\rangle_{A|BC}) + n\mathcal{E}(|W\rangle_{A|B_1B_2}) \geq n\mathcal{E}(\rho_{AB_1}) + n\mathcal{E}(\rho_{AB_2}) \quad (37)$$

for $n \leq 5$. The inequality (37) is optimal in the case of $n = 5$, i.e.,

$$1 + 5H\left(\frac{1}{3}\right) \geq 5\mathcal{E}(\rho_{AB_1}) + 5\mathcal{E}(\rho_{AB_2}). \quad (38)$$

From the inequality (38), we extend Theorem 1 to the network consisting of EPR states, GHZ states, and W states. Consider an n -partite entangled quantum network $\mathcal{N}_{q_1}(\mathcal{A}, \xi)$, where $\mathcal{A}$ denotes parties $A_1, \dots, A_n$, and ξ denotes entangled states. Assume any two parties A_i and A_j share the state ξ_{ij} given by

$$\xi_{ij} = \bigotimes_{s=1}^{s_j} \varrho_{ij}^s \bigotimes_{t=1}^{t_j} \sigma_{ij}^t \bigotimes_{k=1}^{k_j} \delta_{ij}^k \bigotimes_{l=1}^{l_j} \omega_{ij}^l, \quad (39)$$

where ϱ denotes the density matrix of EPR state [49] $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. σ is the density matrix of GHZ state [50] $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. $\delta = \frac{1}{\sqrt{2}}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ is the reduced density matrix of any two subsystems in a GHZ state, and $\omega = \frac{2}{3}|00\rangle\langle 00| + \frac{1}{3}|11\rangle\langle 11|$ is the reduced density matrix of any two subsystems in the W state in Eq. (33). s_j , t_j , k_j , and l_j denote the numbers of ϱ , σ , δ , and ω , respectively. All these entangled states are shared by any two parties A_i and A_j . Moreover, l_j and k_j satisfy $l_j \leq 5k_j$. Thus the total state $\rho_{A_1 \dots A_n}$ of an n -partite system is given by

$$\rho_{A_1 \dots A_n} = \bigotimes_{\substack{i,j=1, \\ i \neq j}}^n \left(\bigotimes_{s=1}^{s_j} \varrho_{ij}^s \bigotimes_{t=1}^{t_j} \sigma_{ij}^t \bigotimes_{k=1}^{k_j} \delta_{ij}^k \bigotimes_{l=1}^{l_j} \omega_{ij}^l \right). \quad (40)$$

Theorem 2. The entanglement distribution of quantum network $\mathcal{N}_{q_1}$ satisfies

$$\mathcal{E}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) \geq \sum_{j=1, j \neq i}^n \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}), \quad (41)$$

where $\mathcal{E}_{X|Y}$ represents EOF of a bipartite system X and Y , and $\bar{A}_i$ denotes all parties except for A_i .

Proof. Assume that any party A_i is entangled with N parties of A_j 's. The total numbers of EPR, GHZ, and W states are n_1 , n_2 , and n_3 , respectively. Any two parties A_i and A_j share s_j pairs of EPR states ϱ , t_j pairs of GHZ states σ , k_j pairs of δ , and l_j pairs of ω . As a result, it is easy to obtain that $\sum_{j=1}^N s_j = n_1$, $\sum_{j=1}^N t_j = n'_2$, $\sum_{k=1}^N k_j = 2(n_2 - n'_2)$, and $\sum_{l=1}^N l_j = 2n_3$, where n'_2 is the number of GHZ states shared by two parties. There are $n_2 - n'_2$ pairs of GHZ states shared by three parties. From the inequality (38), the entanglement

distribution holds for a quantum network consisting of GHZ states and W states from $l_j \leq 5k_j$. This yields the inequality $n_3 \leq 5(n_2 - n'_2)$. From Eq. (2), we get

$$\begin{aligned} \mathcal{E}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) &= S(\rho_{A_i}) \\ &= S\left(\left(\frac{\mathbb{1}}{2}\right)^{\otimes n_1} \otimes \left(\frac{\mathbb{1}}{2}\right)^{\otimes n_2} \otimes \omega^{\otimes n_3}\right) \\ &= n_1 S\left(\frac{\mathbb{1}}{2}\right) + n_2 S\left(\frac{\mathbb{1}}{2}\right) + n_3 S(\omega) \quad (42) \\ &= n_1 + n_2 + n_3 H\left(\frac{1}{3}\right), \quad (43) \end{aligned}$$

where Eq. (42) follows from the additivity of the von Neumann entropy in Eq. (3). In the equality (43), we have used $S(\frac{\mathbb{1}}{2}) = 1$ and $S(\omega) = H(\frac{1}{3})$.

Moreover, we have

$$\begin{aligned} \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) &= \mathcal{E}\left(\bigotimes_{s=1}^{s_j} \rho_{ij}^s \bigotimes_{t=1}^{t_j} \sigma_{ij}^t \bigotimes_{k=1}^{k_j} \delta_{ij}^k \bigotimes_{l=1}^{l_j} \omega_{ij}^l\right) \\ &\leq \mathcal{E}\left(\bigotimes_{s=1}^{s_j} \rho_{ij}^s\right) + \mathcal{E}\left(\bigotimes_{t=1}^{t_j} \sigma_{ij}^t\right) + \mathcal{E}\left(\bigotimes_{l=1}^{l_j} \omega_{ij}^l\right) \quad (44) \end{aligned}$$

$$\begin{aligned} &\leq \sum_{s=1}^{s_j} S\left(\frac{\mathbb{1}}{2}\right) + \sum_{t=1}^{t_j} S\left(\frac{\mathbb{1}}{2}\right) + \sum_{l=1}^{l_j} \mathcal{E}(\omega_{ij}^l) \\ &= \sum_{s=1}^{s_j} 1 + \sum_{t=1}^{t_j} 1 + \sum_{l=1}^{l_j} \mathcal{E}(\omega) \quad (45) \end{aligned}$$

$$= s_j + t_j + l_j \mathcal{E}(\omega), \quad (46)$$

where the inequality (44) follows from the additivity of EOF in Eq. (17). The inequality (45) is from the definition of EOF in Eq. (2) and the additivity of EOF in Eq. (17). We therefore have

$$\begin{aligned} \sum_{j=1}^N \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) &\leq \sum_{j=1}^N (s_j + t_j + l_j \mathcal{E}(\omega)) \\ &= n_1 + n'_2 + 2n_3 \mathcal{E}(\omega) \quad (47) \end{aligned}$$

$$\leq n_1 + n'_2 + \frac{n_3}{5} + n_3 H\left(\frac{1}{3}\right) \quad (48)$$

$$\leq n_1 + n_2 + n_3 H\left(\frac{1}{3}\right), \quad (49)$$

where the equality (47) is derived from the equalities $\sum_{j=1}^N s_j = n_1$, $\sum_{j=1}^N t_j = n'_2$, and $\sum_{j=1}^N l_j = 2n_3$. The inequality (48) follows from the inequality $5 \times 2\mathcal{E}(\omega) \leq 1 + 5H(\frac{1}{3})$, which is from the inequality (38). The inequality (49) holds for the inequality $n_3 \leq 5(n_2 - n'_2)$. Finally, we obtain

$$\begin{aligned} \mathcal{E}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) &\geq \sum_{j=1, j \neq i}^N \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) \\ &= \sum_{j=1, j \neq i}^n \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}). \quad (50) \end{aligned}$$

This completes the proof of Theorem 2. $\blacksquare$

2. Quantum network with arbitrary bipartite entangled pure states

Consider an n -partite entangled quantum network $\mathcal{N}_{q_2}(\mathcal{A}, \xi)$, where any two parties A_i and A_j share s_j pairs of arbitrary bipartite entangled pure states $|\varphi^1\rangle_{ij}, |\varphi^2\rangle_{ij}, \dots, |\varphi^{s_j}\rangle_{ij}$. The joint state shared by two parties A_i and A_j is denoted by $\bigotimes_{s=1}^{s_j} |\varphi^s\rangle_{ij}$. The total state $\rho_{A_1 \dots A_n}$ of $\mathcal{N}_{q_2}(\mathcal{A}, \xi)$ is given by

$$\rho_{A_1 \dots A_n} = \bigotimes_{i,j=1, i \neq j}^n \bigotimes_{s=1}^{s_j} |\varphi^s\rangle_{ij}. \quad (51)$$

We prove a strong monogamy relation for quantum network $\mathcal{N}_{q_2}$.

Theorem 3. The entanglement distribution of quantum network $\mathcal{N}_{q_2}$ satisfies

$$\mathcal{Q}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) = \sum_{j=1, j \neq i}^n \mathcal{Q}_{A_i|A_j}(\rho_{A_i A_j}), \quad (52)$$

where $\mathcal{Q}_{A_i|\bar{A}_i}$ and $\mathcal{Q}_{A_i|A_j}$ are defined in Theorem 1.

Proof. Assume that there exist N parties A_j who are entangled with the party A_i . Note that any two parties A_i and A_j share the state of $\bigotimes_{s=1}^{s_j} |\varphi^s\rangle_{ij}$. It is sufficient to prove that

$$\begin{aligned} \mathcal{E}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) &= \mathcal{E}\left(\bigotimes_{j=1}^N \bigotimes_{s=1}^{s_j} |\varphi^s\rangle_{ij}\right) \\ &= S(\rho_i^j) \\ &= S\left(\bigotimes_{j=1}^N \bigotimes_{s=1}^{s_j} \rho_{i,j}^s\right) \quad (53) \end{aligned}$$

$$= \sum_{j=1}^N \sum_{s=1}^{s_j} S(\rho_{i,j}^s), \quad (54)$$

where the equality (53) is due to the systems $\rho_{i,j}^s$ being uncorrelated, i.e., $\rho_i^j = \bigotimes_{j=1}^N \bigotimes_{s=1}^{s_j} \rho_{i,j}^s$, and $\rho_{i,j}^s$ is the reduced density matrix of the subsystems owned by the party A_i by tracing out the subsystems owned by A_j in arbitrary bipartite entangled pure state $|\varphi^s\rangle_{ij}$. The equality (54) is obtained by iteratively using the additivity of von Neumann entropy in Eq. (3).

Moreover, the entanglement shared by two parties A_i and A_j is from entangled sources of $\bigotimes_{s=1}^{s_j} |\varphi^s\rangle_{ij}$. According to Eq. (2), we obtain

$$\begin{aligned} \sum_{j=1}^N \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) &= \sum_{j=1}^N \mathcal{E}\left(\bigotimes_{s=1}^{s_j} |\varphi^s\rangle_{ij}\right) \\ &= \sum_{j=1}^N S(\rho_i^j) = \sum_{j=1}^N \sum_{s=1}^{s_j} S(\rho_{i,j}^s). \quad (55) \end{aligned}$$

Similar to Eqs. (53) and (54), we obtain Eq. (55). Thus, by combining Eqs. (54) and (55), we have

$$\begin{aligned} \mathcal{E}_{A_i|\bar{A}_i}(\rho_{A_1 \dots A_n}) &= \sum_{j=1}^N \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) \\ &= \sum_{j=1}^n \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}). \quad (56) \end{aligned}$$

Similar to Eq. (56), we get Eq. (52) for Rényi- α entropy entanglement from its definition in Eq. (5) and the additivity of Rényi- α entropy in Eq. (6). This has completed the proof. ■

Example 2. Consider a four-partite star quantum network consisting of three EPR states. It can be regarded as a four-partite pure state on $8 \otimes 2 \otimes 2 \otimes 2$ dimensional Hilbert space, i.e.,

$$|\phi\rangle_{ABCD} = \frac{1}{2\sqrt{2}}(|0000\rangle + |1001\rangle + |2010\rangle + |3011\rangle + |4100\rangle + |5101\rangle + |6110\rangle + |7111\rangle)_{ABCD}. \quad (57)$$

The reduced state of A is given by

$$\rho_A = \frac{1}{8} \sum_{i=0}^7 |i\rangle\langle i|. \quad (58)$$

The EOF of $|\phi\rangle_{ABCD}$ is given by $\mathcal{E}(|\phi\rangle_{ABCD}) = 3$. For mixed state ρ_{AB} , it should take an infimum for the EOF. Note that the reduced density matrix ρ_{AB} of $|\psi\rangle_{ABCD}$ in Eq. (57) has a spectral decomposition

$$\rho_{AB} = \frac{1}{4}(|x_1\rangle\langle x_1| + |x_2\rangle\langle x_2| + |x_3\rangle\langle x_3| + |x_4\rangle\langle x_4|),$$

where $|x_i\rangle$ are eigenstates given by

$$\begin{aligned} |x_1\rangle_{AB} &= \frac{1}{\sqrt{2}}(|00\rangle + |41\rangle), \\ |x_2\rangle_{AB} &= \frac{1}{\sqrt{2}}(|10\rangle + |51\rangle), \\ |x_3\rangle_{AB} &= \frac{1}{\sqrt{2}}(|20\rangle + |61\rangle), \\ |x_4\rangle_{AB} &= \frac{1}{\sqrt{2}}(|30\rangle + |71\rangle). \end{aligned} \quad (59)$$

By the Hughston-Jozsa-Wootters (HJW) theorem [53], any pure-state ensemble of ρ_{AB} can be realized as a superposition of $|x_i\rangle$ with $i = 1, \dots, 4$; that is, for arbitrary pure state $|\varphi\rangle_{AB} = \sum_i c_i |x_i\rangle$ with $\sum_i c_i^2 = 1$, its reduced density matrix $\rho_A = \text{Tr}_B(|\varphi\rangle_{AB}\langle\varphi|)$ has the same spectrum $\frac{1}{2}$ with double root [54] while all the others are 0. So, we have $\mathcal{E}(|\varphi\rangle_{AB}) = 1$.

Moreover, the state ρ_{AB} in Eq. (59) can be decomposed into

$$\rho_{AB} = \sum_i p_i |\varphi_i\rangle_{AB} \langle \varphi_i|. \quad (60)$$

Thus we have

$$\mathcal{E}(\rho_{AB}) = \sum_i p_i \mathcal{E}(|\varphi_i\rangle_{AB}) = 1 \quad (61)$$

from $\mathcal{E}(|\varphi_i\rangle_{AB}) = 1$ for each state $|\varphi_i\rangle_{AB}$.

From similar proofs, we show that the EOFs of ρ_{AC} and ρ_{AD} are

$$\mathcal{E}(\rho_{AC}) = \mathcal{E}(\rho_{AD}) = 1. \quad (62)$$

Consequently, we have

$$\mathcal{E}(|\psi\rangle_{ABCD}) = \mathcal{E}(\rho_{AB}) + \mathcal{E}(\rho_{AC}) + \mathcal{E}(\rho_{AD}). \quad (63)$$

Similarly, for the Rényi- α entanglement measure in Eq. (5), the entanglement distribution satisfies

$$\mathcal{R}_\alpha(|\psi\rangle_{ABCD}) = \mathcal{R}_\alpha(\rho_{AB}) + \mathcal{R}_\alpha(\rho_{AC}) + \mathcal{R}_\alpha(\rho_{AD}). \quad (64)$$

3. Quantum network with arbitrary multipartite entangled pure states

Generally, it is interesting to show the equality (52) for the multipartite entangled pure state shared by two parties. Consider quantum network $\mathcal{N}_{q_3}$, where any two parties A_i and A_j share the states $\tilde{\rho}_{ij}^1, \dots, \tilde{\rho}_{ij}^s, \sigma_{ij}(\varphi_1), \dots, \sigma_{ij}(\varphi_t)$, and $\delta_{ij}(\vartheta_1), \dots, \delta_{ij}(\vartheta_k)$. The joint state shared by any two parties A_i and A_j is given by

$$\xi_{ij} = \bigotimes_{s=1}^{s_j} \tilde{\rho}_{ij}^s \bigotimes_{t=1}^{t_j} \sigma_{ij}(\varphi_t) \bigotimes_{k=1}^{k_j} \delta_{ij}(\vartheta_k), \quad (65)$$

where $\tilde{\rho}_{ij}^s$ denotes the density matrix of the multipartite entangled pure state $|\psi\rangle_{ij}$ with bipartition, $\sigma(\varphi)$ is the density matrix of generalized GHZ state [50] $|\phi(\varphi)\rangle = \cos \varphi |0\rangle^{\otimes m} + \sin \varphi |1\rangle^{\otimes m}$ with any integer $m \geq 3$, and $\delta(\vartheta) = \cos^2 \vartheta |00\rangle\langle 00| + \sin^2 \vartheta |11\rangle\langle 11|$ is the reduced density matrix of any two subsystems obtained by tracing out other subsystems in a generalized GHZ state. Here, s_j , t_j , and k_j denote the numbers of $\tilde{\rho}$, $\sigma(\varphi)$, and $\delta(\vartheta)$ shared by two parties A_i and A_j .

Thus the total state of $\mathcal{N}_{q_3}$ is written as

$$\rho_{A_1 \dots A_n} = \bigotimes_{\substack{i,j=1 \\ i \neq j}}^n \bigotimes_{s=1}^{s_j} \tilde{\rho}_{ij}^s \bigotimes_{t=1}^{t_j} \sigma_{ij}(\varphi_t) \bigotimes_{k=1}^{k_j} \delta_{ij}(\vartheta_k). \quad (66)$$

We prove that the monogamy of entanglement measure $\mathcal{Q}$ is valid for $\rho_{A_1 \dots A_n}$ from $\mathcal{N}_{q_3}$.

Theorem 4. The entanglement distribution of quantum network $\mathcal{N}_{q_3}$ satisfies

$$\mathcal{Q}_{A_i|\overline{A_i}}(\rho_{A_1 \dots A_n}) \geq \sum_{j=1, j \neq i}^n \mathcal{Q}_{A_i|A_j}(\rho_{A_i A_j}), \quad (67)$$

where $\mathcal{Q}_{A_i|\overline{A_i}}$ and $\mathcal{Q}_{A_i|A_j}$ are defined in Theorem 3.

Proof. Suppose that there are N parties A_j who are entangled with the party A_i . According to Eq. (2), by iteratively using the additivity of von Neumann entropy in Eq. (6), we have

$$\begin{aligned} \mathcal{E}_{A_i|\overline{A_i}}(\rho_{A_1 \dots A_n}) &= S(\rho_{A_i}) \\ &= S\left(\bigotimes_{j=1}^N \left(\bigotimes_{s=1}^{s_j} \rho_{i,j}^s \bigotimes_{t=1}^{t_j} \rho_{i,j}(\varphi_t) \bigotimes_{k=1}^{k_j} \rho_{i,j}(\vartheta_k)\right)\right) \\ &= \sum_{j=1}^N S\left(\bigotimes_{s=1}^{s_j} \rho_{i,j}^s \bigotimes_{t=1}^{t_j} \rho_{i,j}(\varphi_t) \bigotimes_{k=1}^{k_j} \rho_{i,j}(\vartheta_k)\right) \\ &= \sum_{j=1}^N \left(\sum_{s=1}^{s_j} S(\rho_{i,j}^s) + \sum_{t=1}^{t_j} S(\rho_{i,j}(\varphi_t)) \right. \\ &\quad \left. + \sum_{k=1}^{k_j} S(\rho_{i,j}(\vartheta_k))\right). \end{aligned} \quad (68)$$

Moreover, it is easy to calculate that

$$\begin{aligned} \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) &= \mathcal{E}\left(\bigotimes_{s=1}^{s_j} \tilde{\rho}_{ij}^s \bigotimes_{t=1}^{t_j} \sigma_{ij}(\varphi_t) \bigotimes_{k=1}^{k_j} \delta_{ij}(\vartheta_k)\right) \\ &\leq \mathcal{E}\left(\bigotimes_{s=1}^{s_j} \tilde{\rho}_{ij}^s\right) + \mathcal{E}\left(\bigotimes_{t=1}^{t_j} \sigma_{ij}(\varphi_t)\right) \end{aligned} \quad (69)$$

$$= S\left(\bigotimes_{s=1}^{s_j} \rho_{i,j}^s\right) + S\left(\bigotimes_{t=1}^{t_j} \rho_{i,j}(\varphi_t)\right) \quad (70)$$

$$= \sum_{s=1}^{s_j} S(\rho_{i,j}^s) + \sum_{t=1}^{t_j} S(\rho_{i,j}(\varphi_t)). \quad (71)$$

From the additivity of EOF and $\mathcal{E}(\bigotimes_{k=1}^{k_j} \delta_{ij}(\vartheta_k)) = 0$, we get the inequality (69). The equality (70) follows from the definition of EOF in Eq. (2). The equality (71) is obtained by using the additivity of von Neumann entropy in Eq. (3) iteratively. Combining Eqs. (68) and (71), it yields

$$\begin{aligned} \mathcal{E}_{A_i|A_j}(\rho_{A_1 \dots A_n}) &\geq \sum_{j=1, j \neq i}^N \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) \\ &= \sum_{j=1, j \neq i}^n \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}). \end{aligned} \quad (72)$$

Similar to the inequality (72), from Eq. (5) and the additivity of Rényi- α entropy in Eq. (6), it is easy to get the result in Theorem 4. ■

It also includes entanglement distribution of quantum networks consisting of general entanglement in terms of Tsallis entropy and unified entropy (see Appendix A).

IV. COMMUNICATION CAPACITIES OF QUANTUM NETWORKS

Quantum networks provide new capabilities for generating and applying quantum entanglement. One fundamental problem is to determine the maximal rate achievable in end-to-end transmission of states. This has so far been addressed for single entanglement [1]. Our goal here is to investigate the capacities of general quantum networks $\mathcal{N}_q$ consisting of EPR states and GHZ states. One simple method is firstly to transform all GHZ states into EPR states. The analysis that follows is based on the recent method of Ref. [37]. In this case, all the transformed networks should be explored to get the network capacity. This may cost exponential complexity for a large number of GHZ states. Instead, the present method implies a one-shot scheme for characterizing the network capacity of the hybrid network. The present method has always polynomial complexity.

For any bipartite entanglement $|\varphi\rangle_{AB}$, it can be regarded as a quantum channel with the capacity [1] $\mathcal{C}_{AB} = S(\rho_A) = \mathcal{E}_{A|B}(|\varphi\rangle_{AB})$, where $S(\rho_A)$ is the von Neumann entropy of party A. As a result, the total capacity of quantum channel $\{\xi_{ij}\}$ connecting the parties A_i and A_j in quantum network $\mathcal{N}_q$ is given by

$$\begin{aligned} S(\rho_i^j) &= \mathcal{E}_{A_i|A_j}(\rho_{A_i A_j}) \\ &= n_1 + n_2, \end{aligned} \quad (73)$$

where n_1 and n_2 denote the respective number of EPR states and GHZ states shared by parties A_i and A_j and $S(\rho_i^j)$ represents the von Neumann entropy of the reduced density matrix ρ_i^j by tracing out the system of j , i.e., $\rho_i^j = \text{Tr}_j(|\xi_{ij}\rangle\langle\xi_{ij}|)$. This equality allows us to evaluate the capacity of quantum networks via von Neumann entropy. It is noteworthy that Rényi entropy is also available for evaluating channel capacity.

Similar to classical network theory [55], consider an n -partite entangled quantum network $\mathcal{N}_q(\mathcal{A}, \Omega)$. Let $\mathbf{s}$ and $\mathbf{t}$ be the source and sink, respectively. Define the flow $f_{i \rightarrow j}$ in a quantum network $\mathcal{N}_q$ as the number of qubits which are reliably transmitted from A_i to A_j along the channel ξ_{ij} ; that is,

$$f_{i \rightarrow j} \geq 0. \quad (74)$$

For the quantum channel consisting of EPR states [49] and GHZ states [50], the capacity between A_i and A_j is defined in Eq. (73), which means the maximal number of qubits transmitted. Inspired by the method in Ref. [37], the flow is bounded by the two-way quantum capacity of the associated channel ξ_{ij} , i.e.,

$$f_{i \rightarrow j} \leq \mathcal{C}(\xi_{ij}) = S(\rho_i^j), \quad (75)$$

where $S(\rho_i^j)$ has the formation of Eq. (73). In addition, at any intermediate party, the number of qubits simultaneously received must be equal to the number of qubits simultaneously transmitted through all the party-to-party communications with neighbor parties. This means that for any $i \in \mathcal{A} \setminus \{\mathbf{s}, \mathbf{t}\}$, we have

$$\sum_{j \in \mathcal{A}} f_{i \rightarrow j} = 0. \quad (76)$$

This does not hold for the source $\mathbf{s}$ and the sink $\mathbf{t}$, for which we impose

$$\sum_{j \in \mathcal{A}} f_{\mathbf{s} \rightarrow j} = - \sum_{j \in \mathcal{A}} f_{\mathbf{t} \rightarrow j} = f, \quad (77)$$

where f is known as the value of the flow. This is an achievable end-to-end rate since it represents the total number of qubits which are transmitted by the source and correspondingly received by the sink via all of the end-to-end communication. Thus, analogous to a classical network, we define a flow in a quantum network as follows.

Definition 1. A flow of a quantum network $\mathcal{N}_q(\mathcal{A}, \Omega)$ from $\mathbf{s}$ to $\mathbf{t}$ is given by a function $f : \Omega \rightarrow \mathbb{R}^+$, such that f satisfies the following.

(i) In the capacity constraint, the flow is bounded by the capacity of the channel; that is, for any $\xi_{ij} \in \Omega$, $f_{i \rightarrow j}$ satisfies

$$f_{i \rightarrow j} \leq \mathcal{C}(\xi_{ij}). \quad (78)$$

(ii) In the conservation of flow, the flow leaving the party j is equal to the flow entering the party j ; that is,

$$\sum_{\{i: \xi_{ij} \in \Omega\}} f_{i \rightarrow j} = \sum_{\{k: \xi_{jk} \in \Omega\}} f_{j \rightarrow k}, \quad (79)$$

for any party $j \in \mathcal{A}$, other than the sender (source) $\mathbf{s}$ and receiver (sink) $\mathbf{t}$.

Herein, $f_{i \rightarrow j}$ denotes the number of qubits transmitted from party i to party j by the quantum channel ξ_{ij} . If $f_{i \rightarrow j} = \mathcal{C}(\xi_{ij})$, then the quantum channel ξ_{ij} is saturated. Otherwise, ξ_{ij} is unsaturated.

Definition 2. A maximum flow $\text{MF}(\mathcal{N})$ of quantum network $\mathcal{N}_q(\mathcal{A}, \Omega)$ is the maximum amount of flow f from the source $\mathbf{s}$ to the sink $\mathbf{t}$; that is,

$$f_{\max} = \max\{f\}. \quad (80)$$

To address the maximal flow of given network $\mathcal{N}_q(\mathcal{A}, \Omega)$, the main idea is to find the cut of associated undirected graph $\mathcal{G} = (\mathcal{V}, \Omega)$, where $\mathcal{V}$ consists of all vertices and Ω consists of all edges. Here, one vertex schematically denotes one party. One edge (i, j) connecting two vertices schematically denotes an EPR state shared by the corresponding parties $\mathbf{A}_i$ and $\mathbf{A}_j$. The GHZ state shared by m parties is associated with the hyperedge connecting m vertices. This correspondence is reasonable because of the equality $S(\rho_{\mathbf{A}_i}) = S(\rho_{\mathbf{A}_j})$ for any GHZ state $\rho_{\mathbf{A}_1 \dots \mathbf{A}_m}$, $1 \leq i < j \leq m$. Thus the quantum network $\mathcal{N}_q$ is associated with an undirected finite hypergraph $\mathcal{G}$. With this correspondence, the weight of each edge (i, j) or hyperedge is the unit, i.e., the entanglement entropy of the EPR state or GHZ state. We then adopt the standard definition of a cut set for a graph [55]. A cut T_{cut} of $\mathcal{G}$ is a bipartition $(\mathbf{S}_1, \mathbf{S}_2)$ of $\mathcal{V}$ such that $\mathbf{s} \in \mathbf{S}_1$ and $\mathbf{t} \in \mathbf{S}_2$. The cut set $\tilde{T}_{\text{cut}}$ of T_{cut} is the set of edges so that the removal of these edges disconnects the source $\mathbf{s}$ and the sink $\mathbf{t}$.

Definition 3. An edge cut set $\tilde{T}_{\text{cut}}$ consists of edges such that there exists a bipartition $(\mathbf{S}_1, \mathbf{S}_2)$ of all the vertices in $\mathcal{V}$ satisfying $\mathbf{s} \in \mathbf{S}_1$ and $\mathbf{t} \in \mathbf{S}_2$; that is, $\tilde{T}_{\text{cut}} = \{(i, j) : i \in \mathbf{S}_1, j \in \mathbf{S}_2\}$. Then the capacity of an edge cut set T_{cut} is given by

$$\mathcal{C}(\mathbf{S}_1, \mathbf{S}_2) = \sum_{(i,j) \in \tilde{T}_{\text{cut}}} \mathcal{C}(i, j), \quad (81)$$

where $\mathcal{C}(i, j)$ denotes the weight of edge (i, j) .

Definition 4. If $\mathcal{C}(T_{\text{cut}}) \geq \mathcal{C}(T'_{\text{cut}})$ holds for any cut T_{cut} in graph $\mathcal{G}$, then T'_{cut} is a minimal cut of $\mathcal{G}$; that is, a minimum cut is a cut with the minimum capacity, with the capacity given by

$$\mathcal{C}(T'_{\text{cut}}) = \min_{T_{\text{cut}}} \mathcal{C}(\mathbf{S}_1, \mathbf{S}_2). \quad (82)$$

Significantly, the flow in a classical network obeys the celebrated max-flow min-cut theorem [55]. We prove a similar result for quantum settings.

Theorem 5. For a given quantum network $\mathcal{N}_q$ associated with undirected graph $\mathcal{G}$, the maximal flow $f_{\max}$ is equal to the minimal cut of $\mathcal{G}$; that is,

$$f_{\max} = \min_{T_{\text{cut}}} \mathcal{C}(\mathbf{S}_1, \mathbf{S}_2). \quad (83)$$

In what follows, we prove Theorem 5. Firstly, we resort to following Lemma 1 in a classical network [56,57].

Lemma 1. Let f be a flow on a network $\mathcal{N}$ and T_{cut} be a cut $T_{\text{cut}} = (\mathbf{S}_1, \mathbf{S}_2)$ with the capacity $\mathcal{C}(T_{\text{cut}})$. Then the flow satisfies

$$f \leq \mathcal{C}(T_{\text{cut}}), \quad (84)$$

where the equality holds if and only if each edge (i, j) in the cut $T_{\text{cut}} = (\mathbf{S}_1, \mathbf{S}_2)$ is saturated, i.e., $f_{i \rightarrow j} = \mathcal{C}(i, j)$, and each edge (j, i) in $T_{\text{cut}} = (\mathbf{S}_2, \mathbf{S}_1)$ has zero flow with $f_{j \rightarrow i} = 0$.

The proof of Lemma 1 is presented in Appendix B.

Proof of Theorem 5. Now, consider a quantum network $\mathcal{N}_q(\mathcal{A}, \Omega)$ consisting of quantum channels defined as EPR states and GHZ states. There exists a cut T_{cut} in $\tilde{T}_{\text{cut}}$ of the associated graph $\mathcal{G}$ separating $\mathbf{S}_1$ and $\mathbf{S}_2$. Similar to Lemma 1, we get

$$f(\mathbf{S}_1, \mathbf{S}_2) \leq \mathcal{C}(T_{\text{cut}}), \quad (85)$$

where the equality holds if and only if each channel ξ_{ij} in the cut $T_{\text{cut}} = (\mathbf{S}_1, \mathbf{S}_2)$ is saturated, and $f_{i \rightarrow j} = \mathcal{C}(\xi_{ij}) = \mathcal{C}(i, j)$. As is shown in Refs. [1,58], one can perfectly transmit one qubit by using the quantum channel of the EPR state [49] or GHZ state [50]. Thereby, it is feasible to achieve $f_{i \rightarrow j} = \mathcal{C}(i, j)$ from the point of view of quantum communication. Note that the transmission of information in quantum communication is one way, i.e., $f(\mathbf{S}_2, \mathbf{S}_1) = 0$. Furthermore, we get the following result.

Lemma 2. If a flow f and cut $T_{\text{cut}} = (\mathbf{S}_1, \mathbf{S}_2)$ on a quantum network $\mathcal{N}_q(\mathcal{A}, \Omega)$ satisfy $f = \mathcal{C}(T_{\text{cut}})$, then f is a maximum flow and the cut T_{cut} is a minimum cut.

Note that any flow f' of the network $\mathcal{N}_q$ satisfies $f' \leq \mathcal{C}(T_{\text{cut}}) = f$. On the other hand, for any cut T'_{cut} of the associated graph $\mathcal{G}$, it follows that $\mathcal{C}'(T_{\text{cut}}) \geq f = \mathcal{C}(T_{\text{cut}})$. Thus f is the maximum flow, and the cut T_{cut} is the minimum cut. As a result, we obtain the claim in Theorem 5 for the quantum network $\mathcal{N}_q(\mathcal{A}, \Omega)$. This means that the maximal flow of $\mathcal{N}_q(\mathcal{A}, \Omega)$ is equal to the minimum-capacity cut of the associated graph $\mathcal{G}$. This completes the proof of Theorem 5. ■

Theorem 5 states that the maximum flow from $\mathbf{s}$ to $\mathbf{t}$ in quantum network $\mathcal{N}_q$ is equal to the minimum cut that separates $\mathbf{s}$ and $\mathbf{t}$ in the corresponding graph $\mathcal{G}$. This provides a general method similar to the point-to-point quantum protocol [1] for general quantum networks beyond the recent result [37]. In applications, for a given quantum network $\mathcal{N}_q$ consisting of EPR states and GHZ states under local unitary operations, the network capacity can be evaluated by extending the Ford-Fulkerson algorithm [59] as shown in Algorithm 1.

Algorithm 1.

Input: An n -partite quantum network $\mathcal{N}_q$.

Output: A maximal flow.

- (i) Find the associated undirected graph $\mathcal{G} = (\mathcal{V}, \Omega)$.
 - (ii) Initialize $f(i, j) = 0$ for all edges and hyperedges.
 - (iii) Find a path $p_{\mathbf{s}, \mathbf{t}}$ from $\mathbf{s}$ to $\mathbf{t}$ with $f(i, j) \leq \mathcal{C}(i, j)$ (augmenting path) on every edge and hyperedge.
 - (iv) Augment flow f on edges or hyperedge along the path $p_{\mathbf{s}, \mathbf{t}}$.
 - (v) Repeat the procedure until there is no augmenting path from $\mathbf{s}$ to $\mathbf{t}$.
-

Example 3. Consider the maximization of the flow from $\mathbf{s}$ to $\mathbf{t}$ as shown in Fig. 5(a) by applying the extended Ford-Fulkerson algorithm. It is easy to check that the maximum flow from $\mathbf{s}$ to $\mathbf{t}$ is equal to 7, which is consistent with the minimum capacity of all source-sink cuts in a corresponding

FIG. 5. (a) Maximum flow of six-partite quantum network $\mathcal{N}_q$. s is the source, while t is the sink. Three parties, A_2 , A_4 , and t , are connected by two GHZ states, and the remaining parties are connected by multiple EPR states. The red integer on a quantum channel denotes its capacity, i.e., the number of EPR states or GHZ states. The blue and green arrows are two maximal flows. (b) Minimum cut of the associated graph of $\mathcal{N}_q$. The EPR state is denoted as one edge, while the GHZ state is denoted as one hyperedge (red lines). A source-sink cut achieves the minimum cut of 7.

undirected graph as shown in Fig. 5(b). The iterations of Example 3 are given in Appendix C.

V. NETWORK TOPOLOGY CLASSIFICATION

Actual quantum networks have a complex topology which significantly affects the communication efficiency. It is natural to consider how to identify the network configuration for a given network. One method is to explore the measurement statistics by using the covariance matrix [38,39] or the generalized Finner inequality [60] in a device-independent manner. The other is from coherence theory [61]. These methods can be applied to characterize quantum networks assisted by specific measurement.

A. Classifying networks based on the von Neumann entropy

In this section, our goal is to consider a beyond problem of classifying configurations of a given set of networks with lower complexity. It is well known that two local unitary equivalent multipartite entangled states have the same entanglement entropy with respect to any bipartition. We show that the converse holds for quantum networks consisting of EPR states and GHZ states. Especially, for a given n -partite quantum network $\mathcal{N}_q$, denote the characteristic vector of quantum network $\mathcal{N}_q$ as

$$S_{\mathcal{N}_q} = (S(\rho_{A_1}), S(\rho_{A_2}), \dots, S(\rho_{A_n})), \quad (86)$$

where $S(\rho_{A_i})$ denotes the von Neumann entropy of the reduced state of party A_i of $\mathcal{N}_q$. We present a configuration classification under the local unitary equivalence by virtue of the characteristic vector of quantum networks.

Theorem 6. Assume $\mathcal{N}_1$ and $\mathcal{N}_2$ are two quantum networks consisting of EPR states and multipartite GHZ states, where any two parties A_i and A_j in each network share no more than one entanglement. Then, $\mathcal{N}_1$ and $\mathcal{N}_2$ have unitary equivalence if and only if their characteristic vectors are equal to each other, i.e., $S_{\mathcal{N}_1} = S_{\mathcal{N}_2}$.

Proof. Consider the following two cases.

(i) *Case 1: $n = 3$.* In this case, there are three types of quantum networks, $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$, as shown in Fig. 6. The

FIG. 6. Three types of tripartite quantum network, $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$, consisting of EPR states or GHZ states, where any two parties share no more than an entanglement. (a) A chain quantum network consisting of two EPR states. (b) A triangle quantum network consisting of three EPR states. (c) A triangle quantum network consisting of one GHZ state.

vectors of von Neumann entropy of quantum networks are given by

$$\begin{aligned} S_{\mathcal{N}_1} &= (1, 2, 1), \\ S_{\mathcal{N}_2} &= (2, 2, 2), \\ S_{\mathcal{N}_3} &= (1, 1, 1). \end{aligned} \quad (87)$$

It follows that the characteristic vectors of von Neumann entropy for tripartite quantum networks consisting of EPR states or GHZ states are different from each other; that is, they are unique for each kind of tripartite quantum network. Note that the von Neumann entropy is invariant under the local unitary operations. This has proved the result.

(ii) *Case 2: $n > 3$.* Assume that the result holds for $n \leq k - 1$; that is, for any pair of quantum networks $\mathcal{N}_1$ and $\mathcal{N}_2$ there exist local unitary operations U_j such that

$$\otimes_{j=1}^n U_j \rho_{\mathcal{N}_1} U_j^\dagger = \rho_{\mathcal{N}_2} \quad (88)$$

if $S_{\mathcal{N}_1} = S_{\mathcal{N}_2}$, where $S_{\mathcal{N}_j}$ denotes the characteristic vector defined by von Neumann entropies for quantum network $\mathcal{N}_j$ consisting of EPR states or GHZ states. Here, each party in the networks may perform local unitary operations. This means that the quantum network consisting of EPR states and GHZ states is unique under local unitary operations.

In what follows, we will prove the result for quantum networks $\mathcal{N}_3$ with $n = k$. In fact, this will be proved by two subcases. The main idea is that the new network $\mathcal{N}_3$ can be regarded as an extended network of $\mathcal{N}_1$.

Assume that $\mathcal{N}_3$ consists of k parties $A_1, A_2, \dots, A_k$ who share EPR states and GHZ states, as shown in Fig. 7(a). The quantum network $\mathcal{N}_3$ will be decomposed into two subnetworks $\mathcal{N}_{3,1}$ as shown in Fig. 7(b) and $\mathcal{N}_{3,2}$ as shown in Fig. 7(c), where $\mathcal{N}_{3,1}$ consists of $k - 1$ parties ($A_2, \dots, A_k$, for example) of $\mathcal{N}_3$, and $\mathcal{N}_{3,2}$, which has at most $k - 1$ parties, consists of at least one party A_1 and some parties $A_{s_1}, \dots, A_{s_m}$ entangled with A_1 of $\mathcal{N}_3$.

Consider the subnetwork $\mathcal{N}_{3,1}$ with $k - 1$ parties. The characteristic vector of $\mathcal{N}_{3,1}$ is given by $S_{\mathcal{N}_{3,1}}$. For the subnetwork $\mathcal{N}_{3,2}$, the characteristic vector is given by $S_{\mathcal{N}_{3,2}}$. These networks consist of EPR states and GHZ states. Moreover, the local von Neumann entropy of each party satisfies the additivity, namely,

$$S_j = m_{\text{EPR}}^{(j)} + m_{\text{GHZ}}^{(j)}, \quad (89)$$

where m_{EPR} and m_{GHZ} denote the respective number of EPR states and GHZ states shared by the party A_j . Note that the

FIG. 7. Schematic quantum network $\mathcal{N}_3$ decomposed into two subnetworks, $\mathcal{N}_{3;1}$ and $\mathcal{N}_{3;2}$. (a) $\mathcal{N}_3$ consisting of $A_1, A_2, \dots, A_{k-1}, A_k$. (b) Subnetwork $\mathcal{N}_{3;1}$ consisting of $k-1$ parties ($A_2, \dots, A_k$, for example) of $\mathcal{N}_1$ and the party A_k . (c) $\mathcal{N}_{3;2}$ consisting of at least one party A_1 and some parties $A_{s1}, \dots, A_{sm}$ entangled with A_1 of $\mathcal{N}_3$.

von Neumann entropy is invariant under the local unitary operations. It follows that the characteristic vector of quantum networks consisting of EPR and GHZ states satisfies the additivity; that is,

$$S_{\mathcal{N}_3} = S_{\mathcal{N}_{3;1}} + S_{\mathcal{N}_{3;2}}. \quad (90)$$

Moreover, $\mathcal{N}_{3;1}$ and $\mathcal{N}_{3;2}$ have at most $k-1$ parties. Hence we get that the new network generated by combinations of the subnetworks $\mathcal{N}_{3;1}$ and $\mathcal{N}_{3;2}$ is equivalent to the network $\mathcal{N}_3$. From the assumption, it follows that the characteristic vectors of $\mathcal{N}_{3;1}$ and $\mathcal{N}_{3;2}$ are unique under the local unitary operations. This means that for each of the other networks $\mathcal{N}'_{3;1}$ and $\mathcal{N}'_{3;2}$ consisting of EPR and GHZ states we have

$$\otimes_j U_j \rho_{\mathcal{N}_{3;s}} U_j^\dagger = \rho_{\mathcal{N}'_{3;s}}, \quad s = 1, 2, \quad (91)$$

if $S_{\mathcal{N}_{3;s}} = S_{\mathcal{N}'_{3;s}}$. This means that the characteristic vectors of $\mathcal{N}_{3;s}$ and $\mathcal{N}'_{3;s}$ are equal to each other, where U_j is the unitary operation. This implies that

$$S_{\mathcal{N}_3} = S_{\mathcal{N}'_3}, \quad (92)$$

where

$$S_{\mathcal{N}'_3} = S_{\mathcal{N}'_{3;1}} + S_{\mathcal{N}'_{3;2}}. \quad (93)$$

Note Eq. (91) means that there exist local unitary operations satisfying

$$\otimes_j U_j \rho_{\mathcal{N}_3} U_j^\dagger = \rho_{\mathcal{N}'_3}. \quad (94)$$

So, $\mathcal{N}_3$ is local unitary (LU) equivalent to the network $\mathcal{N}'_3$; that is, the characteristic vector of quantum network $\mathcal{N}_3$ is unique. This completes the proofs of Theorem 6. ■

According to Theorem 6, we present a quantum network classification method by virtue of the characteristic vector of

FIG. 8. A multipartite complete graphlike network consisting of EPR states.

von Neumann entropy. We present some examples to explain the main idea.

Example 4. Consider a simple network with n parties $A_1, A_2, \dots, A_n$. Here, adjacent parties share no more than one EPR pair as shown in Fig. 2(a) for a chain quantum network, Fig. 2(b) for a cyclic network, and Fig. 2(c) for a star quantum network. The von Neumann entropy vectors of these networks are given by $(1, 2, \dots, 2, 1)$, $(2, \dots, 2)$, and $(n-1, 1, \dots, 1)$, respectively, which are different. This shows that these networks are inequivalent to each other under local unitary operations.

Example 5. Consider an n -partite complete graphlike network $\mathcal{N}_A$, as shown in Fig. 8. Any pair of two parties shares an EPR state. Each party shares $n-1$ EPR states with others. There are $n(n-1)/2$ EPR states. The characteristic vector of $\mathcal{N}_A$ is given by $(n-1, n-1, \dots, n-1)$, which is different from that for the networks in Fig. 2.

Theorem 6 provides a simple method to classify the given set of networks consisting of EPR states and GHZ states. In application, it only requires one to evaluate n number of von Neumann entropies $S(\rho_{A_i})$, which has linear complexity beyond the exponential complexity for evaluating all entanglement entropies of $S(\rho_{A_{i_1} \dots A_{i_t}})$, $1 \leq i_1 < \dots < i_t \leq n$. The drawback is from the assumption of no more than one entanglement shared by a given group of parties. Otherwise, two counterexamples are shown in Fig. 9. From Fig. 9(a), the triangle network consisting of three EPR states is inequivalent to its consisting of two GHZ states even for the same characteristic vector $(2, 2, 2)$. The other counterexample is cyclic networks consisting of four EPR states or two GHZ states and one EPR state in Fig. 9(b) with characteristic vector $(2, 2, 2, 2)$. These can be extended for any cyclic quantum network. Thus a new method beyond the characteristic vector of von Neumann entropy should be explored to verify these cyclic quantum networks. One possibility is from Shannon mutual information [62] derived from local projection measurements.

B. Classifying networks based on the mutual information

In this section, we classify quantum networks based on the mutual information. We firstly recall the definition of mutual information.

FIG. 9. Different networks with the same characteristic vector. (a) Triangle network consisting of three EPR states (black lines) or two GHZ states (blue lines). (b) Four-partite network consisting of four EPR states or two GHZ states and one EPR state.

Shannon mutual information [62] describes how much information can be determined about a random variable X , by knowing the value of a correlated random variable Y . The Shannon mutual information for a tripartite system can be written as

$$I(X : Y : Z) = H(XYZ) + H(X) + H(Y) + H(Z) - H(XY) - H(YZ) - H(XZ), \quad (95)$$

where $H(X) = -\sum_{x \in X} p(x) \log_2 p(x)$ is the entropy of X and $H(XY) = -\sum_{x \in X, y \in Y} p(x, y) \log_2 p(x, y)$ is the joint entropy of X and Y ; $p(x, y)$ is the joint probability distribution of X and Y .

Lemma 3. The two types of cyclic networks in Figs. 9(a) and 9(b) can be verified by Shannon mutual information derived from local projection measurements.

The proof of Lemma 3 is given in Appendix D. From Lemma 3, we obtain the following theorem.

Theorem 7. For any quantum network $\mathcal{N}_A$ consisting of EPR states contains some cyclic subnetworks, there is another quantum network $\mathcal{N}_B$ with the same characteristic vector of von Neumann entropy as $\mathcal{N}_A$ such that they have different network topologies under local operations.

Proof. As is illustrated in Fig. 10(a), quantum network $\mathcal{N}_A$ contains four subnetworks, $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$, and $\mathcal{N}_4$. Here, three-partite subnetwork $\mathcal{N}_1$ has parties A_1, A_3 , and A_5 . $\mathcal{N}_2$ contains parties A_2, A_4 , and A_6 . $\mathcal{N}_3$ contains parties A_{n-2}, A_{n-1} , and A_n . All of these subnetworks consist of three EPR states. A four-partite cyclic subnetwork $\mathcal{N}_4$ consists of parties A_1, A_2, A_3 , and A_4 , who share an EPR state with adjacent parties. By replacing EPR states with GHZ states, we can construct another quantum network $\mathcal{N}_B$ as shown in Fig. 10(b), where the subnetwork $\mathcal{N}_i$ is transformed into the corresponding subnetwork $\mathcal{N}'_i, i = 1, \dots, 4$.

From Lemma 3, it is easy to show that the subnetworks $\mathcal{N}_i$ are inequivalent to the subnetworks $\mathcal{N}'_i$. This follows that two networks $\mathcal{N}_A$ and $\mathcal{N}_B$ are inequivalent even if they have the same characteristic vector of von Neumann entropies as $S(\rho_{A_i}) = S(\rho_{B_i}) = (4, 4, 5, 5, 2, 2, \dots, 3, 3, 2)$. Thus, for any quantum network containing a triangle network or four-partite cyclic subnetwork consisting of EPR states, there is another

FIG. 10. Inequivalent networks with the same characteristic vector defined by von Neumann entropies. (a) Quantum network $\mathcal{N}_A$ consisting of EPR states. (b) Quantum network $\mathcal{N}_B$ consisting of EPR states and GHZ states.

quantum network with the same characteristic vector of von Neumann entropies such that they are inequivalent under local unitary operations. This completes the proof. ■

Theorem 7 shows that the characteristic vector of von Neumann entropies cannot be used for classifying all quantum networks. Fortunately, this problem can be resolved by using Lemma 3.

VI. DISCUSSION AND CONCLUSION

Quantum entropy provides an easy way to characterize the entanglement distribution of specific quantum networks. This strong monogamy relation implies interesting applications in

quantum communications and quantum network configurations. The present method is useful for generic settings of a quantum network consisting of EPR states and GHZ states. These kinds of networks are universal for measurement-based quantum computation [63]. A natural problem is for general entangled networks. Specially, it should be useful for exploring quantum networks consisting of mixed states, which may bring to light new features going beyond pure systems. Another is for the maximum-flow problem on noisy networks including dephasing and erasure channels.

To sum up, we have proved the strong monogamy relation for quantum networks consisting of any bipartite entangled pure states and generalized GHZ states. These relations provide the feature of high-dimensional entanglement beyond qubit states. This implies the celebrated max-flow min-cut theorem on quantum networks based on von Neumann entropy. Finally, we presented a network classification using Shannon or von Neumann entropies of the measurement statistics. This provides a device-independent manner to verify the quantum network configuration under local unitary operations. The present results are interesting in quantum entanglement, quantum networks, and quantum information processing.

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APPENDIX A: EXTENSION OF THEOREM 1 WITH OTHER ENTROPIES

In this Appendix, we investigate the entanglement distribution of quantum network $\mathcal{N}_{q_2}(\mathcal{A}, \xi)$ in terms of Tsallis- q entanglement and unified (q, s) -entropy entanglement.

1. Tsallis- q entropy

Theorem 8. Assume that a quantum network $\mathcal{N}_{q_2}(\mathcal{A}, \xi)$ is connected by arbitrary multipartite entangled pure states.

(i) For $0 < q < 1$, the entanglement distribution of $\mathcal{N}_{q_2}$ satisfies

$$\mathcal{T}_{\mathbf{A}_i|\overline{\mathbf{A}_i}}^{(q)}(\rho_{\mathbf{A}_1 \dots \mathbf{A}_n}) \geq \sum_{j=1, j \neq i}^n \mathcal{T}_{\mathbf{A}_i|\mathbf{A}_j}^{(q)}(\rho_{\mathbf{A}_i \mathbf{A}_j}). \quad (\text{A1})$$

(ii) For $q > 1$, the entanglement distribution of $\mathcal{N}_{q_2}$ satisfies

$$\mathcal{T}_{\mathbf{A}_i|\overline{\mathbf{A}_i}}^{(q)}(\rho_{\mathbf{A}_1 \dots \mathbf{A}_n}) \geq \sum_{j=1, j \neq i}^n \mathcal{T}_{\mathbf{A}_i|\mathbf{A}_j}^{(q)}(\rho_{\mathbf{A}_i \mathbf{A}_j}), \quad (\text{A2})$$

where $\mathcal{T}_{\mathbf{X}|\mathbf{Y}}^{(q)}$ is the Tsallis- q entanglement in Eq. (8) with respect to the bipartition $\mathbf{X}$ and $\mathbf{Y}$, and $\overline{\mathbf{A}_i}$ denotes all parties except for $\mathbf{A}_i$.

Proof. Suppose that there exist N parties $\mathbf{A}_j$ who are entangled with the party $\mathbf{A}_i$. From Eq. (8), for the case $0 < q < 1$,

we have

$$\begin{aligned} \mathcal{T}_{\mathbf{A}_i|\overline{\mathbf{A}_i}}^{(q)}(\rho_{\mathbf{A}_1 \dots \mathbf{A}_n}) &= S_q(\rho_{\mathbf{A}_i}) \\ &= S_q\left(\bigotimes_{j=1}^N \left(\bigotimes_{s=1}^{s_j} \rho_{i,j}^s\right)\right) \\ &\leq \sum_{j=1}^N S_q\left(\bigotimes_{s=1}^{s_j} \rho_{i,j}^s\right), \end{aligned} \quad (\text{A3})$$

where the inequality (A3) is derived from using the additivity of the Tsallis- q entropy in Eq. (10) iteratively, $\rho_{i,j}^s$ is the reduced density matrix of the subsystems owned by party $\mathbf{A}_i$ by tracing out the subsystems owned by party $\mathbf{A}_j$ in a multipartite entangled pure state $|\psi^s\rangle$, i.e., $\rho_{i,j}^s = \text{Tr}_j(|\psi^s\rangle_{ij}\langle\psi^s|)$.

In addition, we have

$$\begin{aligned} \sum_{j=1}^N \mathcal{T}_{\mathbf{A}_i|\overline{\mathbf{A}_i}}^{(q)}(\rho_{\mathbf{A}_i \mathbf{A}_j}) &= \sum_{j=1}^N \mathcal{T}^{(q)}\left(\bigotimes_{s=1}^{s_j} |\psi^s\rangle_{ij}\right) \\ &= \sum_{j=1}^N S_q(\rho_i^j) \end{aligned} \quad (\text{A4})$$

$$= \sum_{j=1}^N S_q\left(\bigotimes_{s=1}^{s_j} \rho_{i,j}^s\right). \quad (\text{A5})$$

Here, the equality (A4) follows from the definition of Tsallis- q entanglement in Eq. (8), and the equality (A5) is due to $\rho_i^j = \bigotimes_{s=1}^{s_j} \rho_{i,j}^s$, i.e., all the states of $\rho_{i,j}^s$'s are not entangled. The inequality (A3) and equality (A5) yield to the inequality (A1).

Similarly, we obtain the inequality (A2) for $q > 1$. This completes the proof. ■

2. Unified (q, s) entropy

Similar to Theorem 8, we obtain the following theorem.

Theorem 9. Assume that a quantum network $\mathcal{N}_{q_2}(\mathcal{A}, \xi)$ consists of arbitrary multipartite entangled pure states.

(i) For $q \geq 1$ and $s \geq 0$, the entanglement distribution of $\mathcal{N}_{q_2}$ satisfies

$$\mathcal{U}_{\mathbf{A}_i|\overline{\mathbf{A}_i}}^{(q,s)}(\rho_{\mathbf{A}_1 \dots \mathbf{A}_n}) \geq \sum_{j=1, j \neq i}^n \mathcal{U}_{\mathbf{A}_i|\mathbf{A}_j}^{(q,s)}(\rho_{\mathbf{A}_i \mathbf{A}_j}). \quad (\text{A6})$$

(ii) For $0 < q < 1$ and $s > 0$, the entanglement distribution of $\mathcal{N}_{q_2}$ satisfies

$$\mathcal{U}_{\mathbf{A}_i|\overline{\mathbf{A}_i}}^{(q,s)}(\rho_{\mathbf{A}_1 \dots \mathbf{A}_n}) \geq \sum_{j=1, j \neq i}^n \mathcal{U}_{\mathbf{A}_i|\mathbf{A}_j}^{(q,s)}(\rho_{\mathbf{A}_i \mathbf{A}_j}), \quad (\text{A7})$$

where $\mathcal{U}_{\mathbf{X}|\mathbf{Y}}^{(q,s)}$ is the unified (q, s) -entropy entanglement in Eq. (13) with respect to the bipartition $\mathbf{X}$ and $\mathbf{Y}$, and $\overline{\mathbf{A}_i}$ denotes all parties except for $\mathbf{A}_i$.

Proof. The proof is similar to that for Theorem 8. Similar to the inequality (A3), from Eq. (13) and the subadditivity of the unified (q, s) -entropy, for $q \geq 1$ and $s \geq 0$, we get

$$\mathcal{U}_{A_i|A_i}^{(q,s)}(\rho_{A_1 \dots A_n}) = S_{q,s}(\rho_{A_i}) \quad (\text{A8})$$

$$\begin{aligned} &= S_{q,s} \left(\bigotimes_{j=1}^N \left(\bigotimes_{s=1}^{s_j} \rho_{i,j}^s \right) \right) \\ &\leq \sum_{j=1}^N S_{q,s} \left(\bigotimes_{s=1}^{s_j} \rho_{i,j}^s \right). \end{aligned} \quad (\text{A9})$$

Similar to Eq. (A5), from Eq. (13) we have

$$\mathcal{U}_{A_i|A_j}^{(q,s)}(\rho_{A_i A_j}) = \sum_{j=1}^N S_{q,s} \left(\bigotimes_{s=1}^{s_j} \rho_{i,j}^s \right). \quad (\text{A10})$$

Thus Eqs. (A9) and (A10) imply the inequality (A6) for $q \geq 1$ and $s \geq 0$.

Similarly, we can obtain the inequality (A7) for $0 < q < 1$ and $s > 0$. This completes the proof. ■

APPENDIX B: PROOF OF LEMMA 1

Proof. Denote $f = f(\mathbf{S}_1, \mathbf{S}_2) - f(\mathbf{S}_2, \mathbf{S}_1)$. On the one hand, the flow conservation law implies that $f(\mathbf{S}_1, \mathbf{S}_2) \leq C(T_{\text{cut}})$ and $f(\mathbf{S}_2, \mathbf{S}_1) \geq 0$. Note that $f = f(\mathbf{S}_1, \mathbf{S}_2) - f(\mathbf{S}_2, \mathbf{S}_1) \leq C(T_{\text{cut}})$, i.e.,

$$f \leq C(T_{\text{cut}}). \quad (\text{B1})$$

For $f(\mathbf{S}_1, \mathbf{S}_2) \leq C(T_{\text{cut}})$, the equality holds if and only if each edge (i, j) in cut $T_{\text{cut}} = (\mathbf{S}_1, \mathbf{S}_2)$ is saturated. Besides, $f(\mathbf{S}_2, \mathbf{S}_1) \geq 0$, where the equality holds if and only if each edge (j, i) in $(\mathbf{S}_2, \mathbf{S}_1)$ has zero flow. So, $f = C(T_{\text{cut}})$ holds if and only if each edge (i, j) in the cut $T_{\text{cut}} = (\mathbf{S}_1, \mathbf{S}_2)$ is saturated and each edge (j, i) in $(\mathbf{S}_2, \mathbf{S}_1)$ has zero flow. ■

APPENDIX C: THE ITERATION OF EXAMPLE 3

In this Appendix, embarking on presenting the iteration of Example 3, we recall the following lemma.

Lemma 4 (Ref. [57]). Let f be a flow on a classical network $\mathcal{N}$. Then f is the maximum flow only if f has no augmenting paths.

This lemma is also available for a quantum network $\mathcal{N}_q$ consisting of EPR states and GHZ states. Hence the main idea for the maximal flow of quantum network $\mathcal{N}_q$ is to find augmenting paths. We apply the Ford-Fulkerson algorithm [57] that sends the maximal possible amount of flow at each iteration; that is, the flow equals the capacity of the path under consideration, where f_i denotes the amount of flow sent in iteration i .

Iteration 1. We find a path $s \rightarrow 1 \rightarrow 2 \rightarrow t$ that can carry a positive flow; then the maximum flow we can send along this path is $f_1 = \min\{4, 3, 2\} = 2$, as shown in Fig. 11(a). In Fig. 11(b), we obtain a residual network with updated link capacities resulting from pushing the flow along the path.

Iteration 2. Similarly, we find an augmenting path $s \rightarrow 1 \rightarrow \{2, 4\} \rightarrow t$ to send a positive flow. The maximum flow

FIG. 11. The iterations for finding the maximum flow in Example 2. (a) The augmenting path $s \rightarrow 1 \rightarrow 2 \rightarrow t$. (b) The residual network in the first iteration by adding the maximum flow f_1 . (c) The augmenting path $s \rightarrow 1 \rightarrow \{2, 4\} \rightarrow t$. (d) The residual network in the second iteration by adding a maximum flow f_2 . (e) The augmenting path $s \rightarrow 1 \rightarrow 3 \rightarrow t$. (f) The residual network in the third iteration by adding a maximum flow f_3 . (g) The augmenting path $s \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow t$. (h) The residual network in the fourth iteration by adding a maximum flow f_4 . (i) The augmenting path $s \rightarrow \{2, 4\} \rightarrow t$. (j) The residual network in the fifth iteration by adding a maximum flow f_5 .

along this path is then given by $f_2 = \min\{4 - 2, 3 - 2, 2\} = 1$, as shown in Fig. 11(c). We get a residual network with updated capacities shown in Fig. 11(d).

Iteration 3. We find a path $s \rightarrow 1 \rightarrow 3 \rightarrow t$ which sends a maximum flow along this path given by $f_3 = \min\{4 - 3, 3, 5\} = 1$, as shown in Fig. 11(e). A residual network is updated as shown in Fig. 11(f).

Iteration 4. For the path $s \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow t$, a maximum flow along this path is given by $f_4 = \min\{6, 3, 3 - 1, 5 - 1\} = 2$ as shown in Fig. 11(g). A residual network with updated capacities resulting from pushing the flow along the path is given by Fig. 11(h).

Iteration 5. We find a path $s \rightarrow \{2, 4\} \rightarrow t$ for sending a positive flow. The maximum flow we can send along this path is given by $f_5 = \min\{6 - 2, 2 - 1\} = 1$, as shown in Fig. 11(i). We obtain a residual network with updated link capacities resulting from pushing the flow along the path, as shown in Fig. 11(j).

APPENDIX D: PROOF OF LEMMA 3

Proof. The quantum networks $\mathcal{N}_1$ and $\mathcal{N}_2$ shown in Fig. 9(a) are inequivalent under local unitary operations. The total state of the system for network $\mathcal{N}_1$ is given by

$$|\mu^1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{A_1B_1} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{A_1C_1} \\ \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{B_1C_1}, \quad (D1)$$

which may be written as

$$|\mu^1\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |201\rangle + |012\rangle + |213\rangle \\ + |120\rangle + |321\rangle + |132\rangle + |333\rangle)_{A_1B_1C_1}, \quad (D2)$$

where we regard the local two-qubit systems as a four-dimensional system with the computation basis $\{|0\rangle, \dots, |4\rangle\}$. After projection measurements are performed on quantum state $|\mu^1\rangle$ in Eq. (D2), we get a joint probability distribution

$$p(x_1x_2x_3) = \frac{1}{8}, \quad (D3)$$

where $x_1x_2x_3 \in \{000, 201, 012, 213, 120, 321, 132, 333\}$.

For the network $\mathcal{N}_2$ as shown in Fig. 9(b) the total state is given by

$$|\mu^2\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{A_1B_1C_1}^{\otimes 2}; \quad (D4)$$

that is,

$$|\mu^2\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle + |222\rangle + |333\rangle)_{A_1B_1C_1}. \quad (D5)$$

For quantum state $|\mu^2\rangle$ in Eq. (D5), after projection measurements are performed, a joint probability distribution is obtained as

$$p(y_1y_2y_3) = \frac{1}{4}, \quad (D6)$$

where $y_1y_2y_3 \in \{000, 111, 222, 333\}$.

From Eq. (95), it is easy to check that the Shannon mutual information of subsystems in $\mathcal{N}_1$ and $\mathcal{N}_2$ is given by

$$I(A_1 : B_1 : C_1) = 3, \quad (D7)$$

$$I(A_2 : B_2 : C_2) = 2. \quad (D8)$$

Similarly, the Shannon mutual information of subsystems in $\mathcal{N}_3$ and $\mathcal{N}_4$ in Fig. 9(b) is given by

$$I(A_3 : B_3 : C_3 : D_3) = 4, \quad (D9)$$

$$I(A_4 : B_4 : C_4 : D_4) = 3, \quad (D10)$$

which is different from that in Eqs. (D7) and (D8). This means that the Shannon mutual information as a character can be used to distinguish two networks.

Note that the von Neumann entropy is unitary invariant. Another interesting method for distinguishing network topologies is to use the von Neumann entropy of the extended system by copying the original quantum state into two copies. Specially, by introducing an auxiliary qubit in the state $|0\rangle$ locally, each party performs a controlled-NOT operation on local systems, where the original qubit is the controlling qubit while the auxiliary qubit is the target. The total system $|\mu^1\rangle$ in Eq. (D1) is then changed into

$$|\nu^1\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes 4} + |1\rangle^{\otimes 4})_{A_1a_1B_1b_1} \\ \otimes \frac{1}{\sqrt{2}}(|0\rangle^{\otimes 4} + |1\rangle^{\otimes 4})_{A_1a_1C_1c_1} \\ \otimes \frac{1}{\sqrt{2}}(|0\rangle^{\otimes 4} + |1\rangle^{\otimes 4})_{B_1b_1C_1c_1}. \quad (D11)$$

Similarly, for the network $\mathcal{N}_2$ in Fig. 9(b), the state $|\mu^2\rangle$ in Eq. (D4) is changed into

$$|\nu^2\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes 6} + |1\rangle^{\otimes 6})_{A_2a_2B_2b_2C_2c_2} \\ \otimes \frac{1}{\sqrt{2}}(|0\rangle^{\otimes 6} + |1\rangle^{\otimes 6})_{A_2a_2B_2b_2C_2c_2}. \quad (D12)$$

For quantum states in Eqs. (D11) and (D12), we obtain

$$S(\rho_{A_1B_1C_1}) = 3, \quad (D13)$$

$$S(\rho_{A_2B_2C_2}) = 2, \quad (D14)$$

which are different from each other, where $S(X)$ is the von Neumann entropy of the subsystem X . This means that we can identify two different quantum networks by using the von Neumann entropy for the system $A_iB_iC_i$ with $i = 1, 2$.

Similarly, the von Neumann entropies of the subsystems A_i, B_i, C_i , and D_i ($i = 3, 4$) for $\mathcal{N}_3$ and $\mathcal{N}_4$ are given by

$$S(\rho_{A_3B_3C_3D_3}) = 4, \quad (D15)$$

$$S(\rho_{A_4B_4C_4D_4}) = 3, \quad (D16)$$

respectively, which elucidates two inequivalent networks $\mathcal{N}_3$ and $\mathcal{N}_4$ under local unitary operations. This provides another method for classifying quantum networks. ■

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