Universal-dephasing-noise injection via Schrödinger-wave autoregressive moving-average models

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We present and validate a method for noise injection of arbitrary spectra in quantum circuits that can be applied to any system capable of executing arbitrary single qubit rotations, including cloud-based quantum processors. As the consequences of temporally correlated noise on the performance of quantum algorithms are not well understood, the capability to engineer and inject such noise in quantum systems is paramount. To date, noise injection capabilities have been limited and highly platform specific, requiring low-level access to control hardware. We experimentally validate our method by comparing to a direct hardware-based noise-injection scheme. Using a combination of quantum noise spectroscopy and classical signal analysis, we show that the two approaches agree. These results showcase a highly versatile method for noise injection that can be utilized by theoretical and experimental researchers to verify, evaluate, and improve quantum characterization protocols and quantum algorithms for sensing and computing.

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I. INTRODUCTION

As qubit technology advances into the noisy intermediate-scale quantum era, spurred by quantum algorithms that promise advantages over classical alternatives, quantum hardware is progressively able to support circuits that are increasingly complex in circuit depth and system size. While numerous physical technologies and architectures are being developed, each system suffers from limitations and sensitivities due to environmental and systematic noise sources. Ultimately, these noise sources lead to computational errors that destroy the efficacy of a quantum computation. For this reason, a significant amount of research has focused on characterizing [1–8] and addressing (i.e., avoiding [9], suppressing [10–13], or correcting [14–18]) noise in quantum systems. These efforts have been complemented by evaluations of quantum algorithmic resilience to specific noise sources [19–23].

Engineered noise provides a necessary capability for investigating the effects of noise in a controlled environment. Beyond providing a means for validation and verification of system characterization protocols, engineered noise allows one to examine the susceptibility and robustness of noise characterization and mitigation protocols to particular noise types [24]. Furthermore, when sufficiently versatile, engineered noise can provide key insights into inherent noise resilience and vulnerabilities of quantum algorithms to targeted noise

sources. In these ways, engineered noise enables the development of robust system characterization techniques, noise mitigation protocols, and error-resilient quantum algorithms, all of which can be considered necessary for achieving fault-tolerant quantum computation. To date, engineered noise has been achieved in a few experimental platforms but a standard method for generating well-defined noise of arbitrary spectra has yet to be identified. Furthermore, methods for injecting engineered noise have been highly platform specific [4,5], with particular focus on laboratory-based experimental systems of superconducting transmon qubits [4,5] or trapped ions [25]. These noise-injection protocols require either a specific device design [4,5] or control of specific in-house hardware platforms [25].

In this paper, we present a platform-agnostic technique for injecting engineered dephasing noise of arbitrary spectra based on the Schrödinger wave autoregressive moving average (SchWARMA) model [26]. A generalization of the autoregressive moving average (ARMA) models used in classical signal processing, SchWARMA models have been previously used to numerically simulate spatiotemporally correlated noise processes in multiqubit systems. Exploiting the framework of SchWARMA-based numerical simulation, we show that the concept of inserting SchWARMA-designed error gates into an ideal quantum circuit can be leveraged for noise injection. This is a key advantage of SchWARMA over other noise-injection techniques, making it implementable in any situation where access to arbitrary single-qubit rotation gates is provided, including cloud devices such as the IBM Quantum Experience (IBMQE) [27]. Focusing, in particular, on phase noise, we show that SchWARMA can be utilized to introduce temporally correlated dephasing errors in *real* physical systems and that the SchWARMA approach has excellent predictive power as well. While this paper centers

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FIG. 1. Implementation of SchWARMA. (a) Phase noise on a qubit under ideal measurement conditions (upper Bloch sphere) is simulated by imparting controlled errors on the experimental clock (i.e., the reference frame) with no intentional additional noise on the qubit (lower Bloch sphere). (b) Gate injection. SchWARMA models an ideal quantum circuit in a noisy environment by interleaving error gates between each control pulse. We implement these error gates as instantaneous rotations of the qubit's reference frame. (c) SDR-based noise injection. In the SDR setup, the SDR signal is mixed with control pulses to impart errors in the control signal. The phase of the SDR signal is updated discretely, every 100 ns, but the signal itself runs continuously over all shots in an experiment. Unlike gate-injection experiments, each shot in an SDR-based noise injection experiment will have a unique phase trajectory of the control signal.

around stationary Gaussian phase noise, we emphasize that the SchWARMA approach is far-reaching and enables injection of control amplitude noise, as well as nonstationary, non-Gaussian, and spatiotemporally correlated noise.

To validate the SchWARMA model as both a statistical model for describing temporally correlated noise and as a generative model for mimicking phase noise on a qubit, we develop and implement two SchWARMA-motivated noise injection approaches: (1) a gate-based approach—faithful to the original SchWARMA concept—that inserts correlated phase errors into an ideal circuit and (2) a low-cost hardware implementation that uses a software defined radio (SDR) to impart errors directly on the master clock. Approach (2) is less universal and not platform agnostic, requiring direct access to hardware for implementation. However, unlike (1), method (2) allows for direct measurement of the injected noise spectrum. For this reason, we present method (1) as a proof-of-principle for arbitrary phase-noise spectra injection using SchWARMA and present method (2) as a validator for method (1). The IBMQE serves as a test bed for approach (1), where hardware access is purposefully restricted to preset gates to convey the universality of SchWARMA noise injection on systems with limited access. In addition, an in-house transmon-based superconducting qubit system, which we will refer to as the Applied Physics Laboratory (APL) system, is employed as an evaluation test bed that allows for both gate-based and direct access scenarios to be examined and compared.

Using the qubit as a probe, the injected noise spectra is reconstructed via quantum noise spectroscopy (QNS) [1–5,28]. We find strong agreement between the reconstructed and in-

jected noise spectra for both gate- and SDR-based approaches. The SDR approach is further validated by direct measurement of its output with a signal analyzer, displaying agreement with the desired spectrum. Additionally, we show that forward simulations of the corresponding SchWARMA models have strong correlation with the experimental data, closing the loop on the validation of the SchWARMA approach for modeling temporally correlated dephasing noise.

II. NOISE INJECTION PROTOCOL

A. Dephasing noise injection

The dephasing between a qubit and a master clock can be engineered through either dynamic control of a qubit's phase relative to the master clock or through the dynamic control of the phase of the master clock relative to the qubit. The former case is represented by the upper panel of Fig. 1(a) where the reference frame defined by the local oscillator (LO) is stationary and the qubit state wanders about the z axis. Conversely, in the latter case, represented by the lower panel of Fig. 1(a), the qubit state is stationary while the reference frame wanders [29]. We artificially induce correlated phase noise using this latter method through dynamical control over the master clock on a single qubit system. The result is an effective qubit response that mimics that of a qubit subject to a temporally correlated classical noise environment. The phase modulation of the master clock $\phi(t) = \phi_{\text{ctrl}}(t) + \phi_{\text{noise}}(t)$ is composed of the typical (potentially faulty) control phase $\phi_{\text{ctrl}}(t)$, with an additional $\phi_{\text{noise}}(t)$ representing the desired injected noise process. Next, we discuss how the SchWARMA formalism can be used to construct $\phi_{\text{noise}}(t)$.

B. Noise injection via SchWARMA

ARMA models are widely used in the field of time-series analysis to model time correlations in data. SchWARMA provides a natural path for generalization of these classic techniques from signal processing to quantum information. While SchWARMA has been previously utilized as an approach for numerically simulating classical temporally correlated noise in quantum circuits, its utility can be readily extended to noise injection and model prediction.

The gate-based noise-injection protocol follows a similar recipe to the numerical approach demonstrated in Ref. [26], where the noise is injected via error gates interleaved in a quantum circuit, as demonstrated in Fig. 1(b). In the case of single qubit dephasing, a circuit with noise injection can be expressed as

$$U(\phi) = R_z(\phi_N)G_N R_z(\phi_{N-1})G_{N-1} \dots R_z(\phi_1)G_1, \quad (1)$$

where G_j denotes a single qubit operation in the noiseless circuit and the error gate $R_z(\phi_j)$ represents a z rotation with time-correlated ϕ_j . Upon specifying properties of the noise process, e.g., the mean and power spectral density, a SchWARMA model is used to generate a trajectory of temporally correlated phases $\phi = \{\phi_1, \dots, \phi_N\}$. Averaging an observable or fidelity metric over many realizations of ϕ results in the desired dephasing behavior.

In the SchWARMA approach, temporal correlations in the phases ϕ_j are accomplished by applying linear filtering of samples of white Gaussian noise or, equivalently, driving an ARMA model with white Gaussian noise. The ARMA model coefficients $\{a_i\}$ (the autoregressive coefficients) and $\{b_j\}$ (the moving average coefficients) define the output phase ϕ_k from the white Gaussian input w_k by

$$y_k = \underbrace{\sum_{i=1}^p a_i y_{k-i}}_{AR} + \underbrace{\sum_{j=0}^q b_j w_{k-j}}_{MA}. \quad (2)$$

This filtering approach yields a power spectrum $S_y(\omega)$ of

$$S_y(\omega) = \frac{|\sum_{k=0}^q b_k \exp(-ik\omega)|^2}{|1 + \sum_{k=1}^p a_k \exp(-ik\omega)|^2}, \quad (3)$$

and it is known that ARMA models can approximate any discrete-time power spectrum to arbitrary accuracy [30]. Here, these correlated phases are then converted to correlated error operators via $R_z(\phi_k) = \exp(-i\phi_k \sigma_z/2)$, although the SchWARMA approach supports a much wider class of correlated error models [26]. In general, there are many ways to define ARMA coefficients in a way that achieves a desired power spectrum and such a discussion is beyond the scope of this paper. Here, bandpass signals are MA models defined by length 128 Hamming windows of the desired bandwidth and cosine modulated (as needed) to the appropriate center frequency, and then scaled to achieve the overall desired power. The outputs of independently driven models are added to produce multiband spectra. ARMA coefficients for $1/f^\alpha$ noise were defined using the approach in Ref. [31].

This form of SchWARMA-based noise injection is equivalent to control master clock phase noise injection. In superconducting qubits, an $R_z(\phi)$ gate is implemented via a

virtual frame change that simply updates the control phase [Fig. 1(a)]. In this sense, ϕ represents a discretized implementation of $\phi_{\text{noise}}(t)$. When the sampling time of $\phi_{\text{noise}}(t)$ is the fundamental gate time, t_G , ϕ , and $\phi_{\text{noise}}(t)$ are identical. Note that all gates G_j do not need to be equivalent in duration but rather just integer multiples of t_G . For example, if gate G_j requires timing nt_G , the composition of n error gates are appended to the circuit following G_j . Note that this sampling time t_G limits the maximum frequency of noise that can be injected to $t_G^{-1}/2$.

Below, we also introduce an SDR-based SchWARMA noise injection technique that is distinct from the gate-based approach in that it occurs simultaneously and asynchronously with the control. Requiring lower level control access, the SDR approach involves injecting phase noise directly into the control lines. As a result, the sampling time of the noise is *not* necessarily restricted by the gate time t_G . However, even in the case where the sampling time is set by t_G , the injected noise and gates are inherently asynchronous; thus, phase updates may frequently occur during pulses. Because the engineered noise in the SDR-based noise injection technique is produced independently of the control pulses, this method allows a more direct comparison between the engineered and measured noise via a classical signal analyzer, a method that cannot be leveraged in the gate-based approach. In addition to these distinct features, we elaborate on other facets of the SDR approach in the discussion section.

C. Noise injection validation

The SchWARMA noise injection protocol is validated via QNS. We utilize a set of deterministically generated fixed total time pulse sequences (FTTPSs) designed to probe the noise spectrum across a frequency band of interest; see Appendix D for further details. Each sequence S_k is comprised of π pulses that produce a unique filter function g_k that is well-concentrated in frequency space [Fig. 2(a)]. The FTTPSs probe the noise spectrum by altering the number of pulses within a fixed total sequence time. This protocol is distinct from the Carr-Purcell-Meiboom-Gill-based (CPMG-based) QNS protocol [28] that probes the noise spectrum by varying the interpulse delay (and total sequence time) using a fixed number of π pulses and it is distinct from other fixed total time sequences found from a random search [4]. Further details on FTTPS are provided in Appendix D.

Dephasing noise [Fig. 2(b)] is injected into a single qubit system via the SDR or gate-based SchWARMA procedure, thereby simulating temporally correlated noise during the FTTPS evolution. Survival probabilities are then measured for the FTTPS [Fig. 2(c)], where the survival probability for each probe sequence exhibits a distinct (and sharp) dependence on noise of specific frequencies. The resulting survival probabilities are used to construct an estimated noise spectrum [Fig. 2(d)]. Note that we define the survival probability as the probability of the qubit being in a desired state at the end of a probe FTTPS.

Below, we demonstrate that QNS reconstructions agree with the various SchWARMA-generated and injected-noise spectra. In addition, we show that SchWARMA can be used to achieve a detailed prediction of the survival probabilities

FIG. 2. SchWARMA-based noise injection and reconstruction. (a) The filter functions of the fixed total-time pulse sequences have sharp and unique dependencies on noise of different frequencies. (b) Noise of a desired spectrum, in this example bandpass, is generated with either the gate-based or SDR-based injection method. Inset: A measurement of the frequency spectrum of the 2.4 GHz noisy carrier signal out of the SDR. This signal is mixed with an ideal control signal and upconverted to the qubit frequency. (c) The survival probabilities for each sequence can be analyzed to reconstruct the noise influencing the system (d).

resulting from the FTTPS. To separate injected noise from the native background, the native noise spectrum $S_{\text{nat}}(\omega)$ is measured using the procedure described above and subtracted from the injected noise spectrum reconstructions $S_{\text{inj}}(\omega)$ to obtain $\Delta S(\omega) = S_{\text{inj}}(\omega) - S_{\text{nat}}(\omega)$.

III. EXPERIMENTAL IMPLEMENTATION

A. Gate-based noise injection

Gate-based noise injection provides a means for examining quantum algorithms in the presence of engineered noise when hardware access is limited to particular gate operations. Here, we demonstrate gate-based noise injection via SchWARMA. The IBMQE serves as a restricted access test bed, where we intentionally utilize the circuit-level interface to limit circuit operations to the standard IBM gate set [32], demonstrating the availability of the approach to the general user community. Phase noise is introduced through $U_1(\lambda)$ phase gates that are interleaved with the gates that comprise the sequences that define the probe (FTTPS). The time resolution of the noise is set by the X gate timing, given by the $U(\theta, \phi, \lambda)$ gate time. The U gate implements generic unitary operations using three configurable frame changes interleaved with $R_x(\pi/2)$ and $R_x(-\pi/2)$ rotations. The frame changes are used to set the phases θ, ϕ, λ .

Employing the five-qubit (T-shaped) Vigo processor for our experiment [33], where $t_G \approx 70$ ns, we prepare a single qubit in the equal superposition state, apply the FTTPS with injected phase noise, apply a gate that would return the state to $|0\rangle$ in the absence of noise, and measure. The experiment is performed over 50 independent realizations of SchWARMA trajectories for each probe sequence, where statistics for each experiment are collected over 1000 shots. The complete noise injection and noise spectrum estimation experiment is conducted over five calibration cycles and

bootstrapped to mitigate spectral anomalies solely due to variations in hardware characteristics across calibration cycles, thereby improving spectral estimates. Refer to Appendix E for further information regarding optimized experimental practices and procedures.

Through the SchWARMA gate-based injected noise protocol, we find good agreement between the desired and reconstructed noise spectra. In Fig. 3, we show the results for three distinct classes of noise spectra: (a) a single bandpass spectrum of varying width, (b) a multibandpass spectrum with varying center frequencies, and (c) $1/f^\alpha$ noise. We consider single bandpass bandwidths of $\Delta\omega = 0.18, 0.54, 0.89$ MHz, with a center frequency of $\omega_0 = 1.43$ MHz. We explore the versatility of SchWARMA by separating the injected noise into two bandpass regions centered at $\omega_0 = 1.07$ MHz and $\omega_1 = 1.79$ MHz and subsequently, $\omega_0 = 0.64$ MHz and $\omega_1 = 2.21$ MHz, while keeping the bandwidth of 0.18 MHz constant for each region. Lastly, we consider injected $1/f^\alpha$ -type noise with $\alpha = -2, -1, 0, 1, 2$. Such frequency dependence is commonly found in superconducting [34–38] and semiconductor [3,39,40] qubit systems.

The resulting spectral density estimates yield spectral features, such as peak height, bandwidth, and center frequency, that are found to be in good agreement with the desired noise profiles. Additional spectral features observed at high frequency have been determined to be artificial, resulting from increased gate error as higher frequency regimes are probed with sequences containing a greater number of pulses. The manifestation of gate errors in dephasing noise spectra has been previously noted and mitigated using spin-locking (SL) techniques [41]. IBM’s quantum control toolbox [42] provides a means for exploring SL-based methods, as well as pulse-error compensating sequences for QNS. We focus on the latter and create a set of robust FTTPSs (RFTTPSs) composed of calibrated, true $R_x(\pi)$, and $R_x(-\pi)$ pulses. It has long been known in the NMR and dynamical decoupling communities that pulse-phase adjustments can aid in the suppression of pulse imperfection errors (e.g., over-/under-rotation errors) for pulse-based error mitigation schemes [43–46]. By constructing RFTTPSs composed of pulses of alternating phases [i.e., $R_x(\pi)$ and $R_x(-\pi)$], we effectively reduce the effect of pulse imperfections on the estimated spectra. RFTTPSs are employed on the APL system in the subsequent section, and an additional comparison between FTTPSs and RFTTPSs using the IBMQE is included in Appendix G. The use of RFTTPSs along with additional simulations (see Supplemental Material) support the conclusion that the observed high-frequency noise is primarily an artifact of gate error.

B. Verification with SDR

Our SDR-based approach to noise injection relies on direct hardware access. This approach is not platform agnostic and is presented as a means to validate SchWARMA gate-based noise injection by directly comparing results against a control line signal with the synthesized noise spectrum. To meet the requirements of lower-level access, we turn to the APL system to perform this comparison.

FIG. 3. SchWARMA gate-based noise injection on the IBMQE. Noise reconstructions are shown for (a) bandpass, (b) double bandpass, and (c) $1/f^\alpha$ noise for different bandwidths, center frequencies, and spectral decays. Ideal spectra [dotted lines in (a) and (b) and solid lines in (c)] are found to be in good agreement with estimated spectra [solid lines in (a) and (b) and symbols in (c)]. The data shown here are compiled over five calibration cycles, with solid lines denoting bootstrapped medians and shaded regions specifying 95% confidence intervals in (a) and (b). Note that the confidence intervals are small for a majority of the estimated spectra. Data has been offset for clarity.

An SDR is configured to implement a SchWARMA model and produce $\phi_{\text{noise}}(t)$. The control signal sums the desired control phase with the phase-noise spectrum output of the SDR, acting as a faulty control signal with a drifting master clock. The SDR outputs a continuous 2.4 GHz carrier signal with pseudorandom phase changes that are updated every 100 ns. While the phase changes may be random, one advantage of SchWARMA is that temporal correlations are permitted, meaning that the pseudorandom phase changes are biased by previously applied phase updates that mimic the dephasing error gates in the gate-based approach. In software, these fluctuations are shaped from white noise into an engineered, SchWARMA-generated noise spectrum, mimicking the gate-based approach described above. Again, the primary difference between the approaches is that the SDR-based approach interacts with the qubit asynchronously, and given that we have the same SDR-sampling rate (limited by our specific SDR-hardware and computation power) and t_G , we can use the same ARMA coefficients between them. We measure the SDR output on a signal analyzer with a 90 kHz bandwidth as a ground-truth comparison for spectral reconstructions [Fig. 2(b) inset].

After confirmation via signal analyzer, we perform a series of RFTTPS qubit experiments on the APL system with the SDR as the master clock and the clock phase controlled by various SchWARMA models. The APL system includes a fixed-frequency transmon qubit held at 20 mK at the mixing chamber stage of a dilution refrigerator. The qubit (≈ 5.4 GHz) is coupled to a readout resonator (≈ 7 GHz) and the state of the qubit is determined by dispersive measurement techniques.

See Appendices B and C for further details about SDR injection and the APL experimental setup. Experiments using the SDR approach are performed over 10,000 unique shots, and the $R_x(\pm\pi)$ pulses are calibrated to have $t_G = 100$ ns for both SDR injection and gate-based injection in the APL experimental setup. The fixed total time of noise injection experiments in the APL setup is $12.8 \mu\text{s}$, well within the measured coherence times of the qubit ($T_1 \approx 47 \mu\text{s}$ and $T_2 \approx 58 \mu\text{s}$). At the start of an experiment, we apply a $R_x(\pi/2)$ to initialize the qubit in $|+\rangle$. Subsequently, an RFTTPS sequence is implemented to probe a desired frequency range. Before measurement, we apply a $R_x(\pi/2)$ gate which would excite an ideally behaved qubit in the absence of noise to the $|1\rangle$ state. Using the same QNS analysis applied in the gate-based approach, we reconstruct the injected noise spectra. As is the case with the gate-based approach, the SDR injection approach can be used to produce a variety of phase-noise spectra. In Fig. 4, we show the results for (a) a single bandpass spectrum of varying width and noise power, (b) a multibandpass spectrum, and (c) $1/f^\alpha$ noise. We consider single bandpass bandwidths of $\Delta\omega = 0.2, 1$ MHz with a center frequency $\omega_0 = 1$ MHz. The double bandpass noise spectrum is engineered to have center frequencies $\omega_0 = 0.5$ MHz and $\omega_1 = 1$ MHz, with noise bandwidths $\Delta\omega = 0.5$ MHz. Lastly, we consider $1/f^\alpha$ -type noise with $\alpha = -2, -1, 0, 1, 2$, as we did with gate-based experiments.

Reconstructed spectra are compared against the measured frequency-noise spectrum of the SDR output signal (measured on a signal analyzer) for validation. To achieve fits, the measured frequency spectra of the SDR output signal are

FIG. 4. Verification of SDR-based SchWARMA noise injection. Noise reconstructions are shown for (a) bandpass, (b) double bandpass, and (c) $1/f^\alpha$ noise for different bandwidths, center frequencies, and spectral decays. The data shown are reconstructed from the average survival probability of 10,000 unique shots per sequence, each with a unique phase trajectory. Points represent noise reconstructions, and solid lines are measurements performed on the generated noise by a signal analyzer. In (a) and (c), plots are offset for clarity.

each multiplied by a scaling factor, constant over experimental setup, to account for attenuation of the signal between room temperature and the qubit. Overall, the reconstructed noise spectra match the injected spectra well. The measured bandwidths, peak heights, and center frequencies of the injected bandpass and double bandpass spectra are in good agreement with the injected noise spectra. The behavior of $1/f^\alpha$ spectra match the expected power dependencies best within approximately $0.1 \text{ MHz} \leq \omega_0 \leq 2 \text{ MHz}$ [Fig. 4(c)]. At frequencies above 2 MHz, the innate roll off in the SDR output distorts the injected noise, leading to inaccuracies in the reconstructed noise spectra. Despite these distortions, the overall shape of the reconstructed spectra remains in good agreement with the expected noise spectra measured at the output of the SDR (solid lines).

The combination of the signal analyzer output and the reconstructed spectra provides confidence that the SDR-based approach injects phase noise in the desired manner. We can transfer this confidence to the gate-based approach by comparing reconstructions of gate-based noise injection of identical spectra on the same device. Figure 5 shows reconstructions of gate-based SchWARMA injected noise plotted on top of nominally identical noise injected using the SDR experimental approach. Gate-based injection experiments performed on the APL experimental setup are measured over 200 unique SchWARMA trajectories, with 1000 shots per trajectory. We reiterate that both measurements in Fig. 5 are performed on the same APL experimental setup. These results show excellent agreement between the two methods, with the primary discrepancy between the approaches due to finite sampling

effects on the measurements and number of SchWARMA trajectories used in the gate-based approach. Further comparisons between the gate-based and SDR approaches are included in Appendix F.

IV. SCHWARMA MODEL PREDICTION

Much like classical ARMA, a SchWARMA model's associated power spectrum can be used to model real physical systems and predict in closed form their average response to given stimuli (here pulse sequences). This effectively closes the loop between the generation and injection processes. Here, using the known injected SchWARMA model, we fit a few ancillary parameters to experimentally generated FTTPS data. Figure 6 shows the results of this fitting process for sample experiments performed via SDR and gate-based noise injection (details of this process are found in the following paragraphs). The top panel shows SDR and gate-based data for the 1 MHz bandpass experiments whose reconstructions are shown in Figs. 4 and 5, and the bottom panel displays gate-based data from the double bandpass experiment with $\omega_0=1.07 \text{ MHz}$ shown in Fig. 3(a). The accuracy of these fits show that SchWARMA models can be used to effectively predict the expected survival probabilities. This capability conveys that SchWARMA can be a powerful tool for parametric modeling of temporally correlated noise environments and, therefore, improved quantum system dynamics simulation informed by experimental data.

Given a FTTPS probe sequence k [with $n_k R_x(\pi)$ pulses] and filter function g_k , we predict the survival probability p_k

FIG. 5. Comparison between gate-based (solid lines) and SDR-based (dashed lines) SchWARMA noise injection for bandpass dephasing noise with three different bandwidths. Good agreement between the approaches is found for all spectra considered; thus, validating SchWARMA gate-based noise injection. Data for $\Delta\omega = 0.5, 1.0$ MHz are vertically offset by 0.2 for survival probabilities (left) and 1 MHz for spectrum reconstructions (right).

using the model

$$p_k = \frac{1}{2} + \frac{1}{2} \exp[-g_k(S_{\text{nat}}(\alpha) + S_{\text{inj}}) - c_1 n_k - c_2 n_k^2], \quad (4)$$

where $S_{\text{nat}}(\alpha)$ is a parametric model of the native dephasing noise power spectrum (see below), S_{inj} is the (known) injected spectrum and c_1 and c_2 capture the effects of stochastic (i.e., X dephasing) and coherent errors on the $R_x(\pi)$ pulses, respectively. The pulse error terms are meant to capture the artificial high frequency features observed in the spectral estimates shown above, most notably in the case of the IBMQE results where FTFPSs without pulse-error compensation are used. We use Eq. (4) to set up a nonlinear least-squares problem to estimate the parameters α and c_i using a set of FTFPSs. We further expand upon the motivation for the model and present additional results in Appendix G.

The native noise S_{nat} is, in general, qubit and system dependent. For the APL system (Fig. 6 top), we found that a combination of a Lorentzian spectrum $A/(1 + \omega^2/\omega_c^2)$, with

unknown amplitude A and cutoff frequency ω_c , and a white noise floor with power σ^2 captured the observed native dephasing noise (i.e., $\alpha = \{A, \omega_c^2, \sigma^2\}$), which exhibited low-frequency energy that was not well modeled by the white noise floor nor the control noise terms c_i . For the IBM system (Fig. 6, bottom), we found that there were isolated resonances at the zeroth and eighth sequences, and beyond that the noise was well approximated by a white noise floor. Thus, instead of introducing additional model terms in S_{nat} to account for these two sequences (i.e., the native noise in those frequency bands), we ignored these sequences in the estimation (as the introduction of two more ancillary terms would perfectly match these points without changing the estimates for the unknown white noise floor power σ^2 and c_i). Again, we emphasize that Fig. 6 shows strong agreement of the least-squares fits to the measured experimental data and that we are only fitting over the ancillary background and control noise terms, not on any terms associated with the injected spectrum.

V. DISCUSSION

A. Beyond stationary, Gaussian dephasing noise injection

The focus of the paper presented here was on stationary, Gaussian dephasing noise with a variety of parametric spectra, but there are a number of straightforward generalizations of the gate-based approach that can be applied using the same tool set. For example, non-Gaussian noise can be introduced by applying an arbitrary nonlinearity on the output of the underlying ARMA model or by considering non-Gaussian Levy-stable distributions as inputs. Given access to arbitrary unitary gates in a system, one could look beyond Z -gate dephasing errors and have dephasing noise on the X or Y axes (or on the angle defining the axis of the control), or multiaxis dephasing errors (potentially each with its own spectrum). Nonstationary noise can be generated using heteroskedastic generalizations of ARMA (itself a potential source of $1/f$ noise) to vary the noise power over time or by explicitly varying the terms of the SchWARMA model to produce, e.g., bandpass signals that move in frequency or drifting α terms in $1/f^\alpha$ noise. Additionally, the gate-based approach trivially allows for the introduction of gate-dependent errors, for example, by injecting error gates only after nonidentity gates

FIG. 6. Comparison of SchWARMA predicted to actual experimentally measured survival probabilities using FTFPS. Top: SDR- and gate-based noise injection of a wide bandpass spectrum on the APL experimental platform. Bottom: Gate-based injection of a double bandpass spectrum on the IBMQE.

or mimicking multiplicative control noise by using errors that are proportional to the desired rotation angles of ideal control gates. We note that errors that move beyond the free Z-gate phase updates should be compiled into the ideal gates to keep the overall circuit length the same. Finally, it is clear that this approach is readily extendable to any number of qubits and, furthermore, the injected errors could be themselves spatially correlated between qubits.

B. Complexity and scalability considerations for noise injection approaches

While the versatility of the gate-based SchWARMA approach allows for noise injection on any cloud-based platform, there are some advantages to implementing the SDR approach when physical access to the hardware is available. The primary limitation of the gate-based approach is the time cost associated with uploading the circuit (for cloud systems) or arbitrary waveform generator output (for in-house systems). Ideally, each measurement shot of the system should be generated using a different SchWARMA-generated noise trajectory, and the SDR-based approach does not require changing the underlying circuit to impart a new noise sequence. As a compromise, we use a limited number of SchWARMA noise trajectories (typically 50) and use repeated measurements of these trajectories to build up statistics. Obviously, this trade-off in time has costs in the error variance of the spectrum reconstructions which is not present in the SDR-based approach, and despite this approximation, the SDR-based experiments are still roughly $10\times$ faster to complete than the gate-based experiments (in-house). A potential avenue to circumvent these statistical issues would be to build SchWARMA into specialized hardware in a similar fashion to randomized compilation [47,48], where instead of randomly inserting Pauli frame changes for each circuit shot, SchWARMA-driven error gates are compiled into the ideal circuit in real-time for each circuit shot. Another advantage of the SDR-based approach is the potential (with more capable/additional hardware) to inject noise at frequencies much higher than the gate-based limit and explore alternative, non-gate-based spectroscopic techniques.

Beyond the requirement of physical access, there are some limitations to consider with the SDR-based approach that limits its universality. Modifications of the underlying ARMA models can be performed to produce similar extensions as in the gate-based approach, but the ability to move beyond Z-dephasing errors to, e.g., control amplitude errors would require additional experimental apparatuses. Furthermore, while we expect to achieve simultaneous noise injection using the SDR-based approach on a few qubits in the near future, the hardware burden limits the scalability of this approach, whereas the gate-based approach is essentially infinitely scalable to multiqubit experimentation. Again, the incorporation of SchWARMA into a randomized compilation-like approach could overcome these limitations.

VI. CONCLUSIONS

Although quantum systems are commonly subject to temporally correlated noise, the impact of such noise processes

on the performance of quantum algorithms is not well understood. Here, we present a method for engineering and injecting temporally correlated dephasing noise that can be used to elucidate key features of noise resilience and vulnerability in quantum algorithms for noise characterization, sensing, and computing. The approach enables noise injection of arbitrary noise spectra via SchWARMA, a statistical tool for modeling, estimating, and generating semiclassical noise in quantum systems. SchWARMA is used to synthesize engineered phase noise spectra that are introduced by adjusting the control master clock to emulate classical temporally correlated dephasing noise processes. We show that the experimental system dynamics generated by engineered noise can be predicted by SchWARMA-based models; therefore, closing the loop between generation and injection processes. Our results indicate that SchWARMA is a highly flexible tool for injecting engineered noise and developing experimental data-informed parametric noise models for simulating quantum system dynamics. The techniques presented here can be readily extended to multiaxis noise, non-Gaussian and non-stationary noise, and multiqubit scenarios. This points toward the generalizability and applicability of SchWARMA as a tool for understanding, evaluating, and improving quantum algorithms subjected to a wide range of temporally correlated noise processes relevant to current and future hardware platforms.

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APPENDIX A: CHOICE OF SPECTRA

To emulate dephasing with a particular noise spectrum, we use a SchWARMA model to generate Z gate errors that are interleaved into a quantum circuit. These errors map to noise spectra that are observable with QNS techniques. A survey of QNS experiments shows numerous noise characteristics that often follow a $1/f^\alpha$ curve. However, α has been observed to

FIG. 8. Example GNU Radio flow graph for bandpass noise. The parameters of the noise shaping filter block determine the power spectrum of the injected noise.

of each gate, t_{gate} , is held constant. This gives a fixed total time of $(t_{\text{gate}} 2^{m+1})$ for each sequence. For experiments on the APL system, we typically used $t_{\text{gate}} = 100$ ns and $m = 6$ for a total control duration of $12.8 \mu\text{s}$. Each gate in these sequences is either an identity gate or a $\pm\pi$ pulse. The sign changes of the function $\text{sgn}\{\cos(\pi nk/2^m)\}$ where sgn is the signum function and k is the sequence number (ranging here from 0–63) determine which pulses n (here ranging from 0–127) are π pulses. The remaining pulses are identity gates. In Fig. 9, we illustrate the gate sequences applied for $n = 0, 3, 30$, and 60 . The motivation behind these sequences is that single-axis π pulses should create a so-called modulation function whose values are ± 1 , and this particular set of sequences should have good frequency concentration and little overlap.

Noise reconstruction is performed by implementing non-negative least-squares regression to solve for the noise S

which gives a minimum to $\|\Phi S - \chi\|_2$, where Φ is a matrix whose rows sample the controls sequences' filter functions at discrete points. For the FTTPS used here, we have an excellent condition number for Φ of ≈ 1.63 , indicating that this regression problem is very well-posed. The filter function is related to the squared Fourier transform of the pulse sequence (assuming instantaneous pulses), and χ is related to survival probability p by $p = 1/2 + 1/2e^{-\chi}$.

Fixed total time gate sequences have a constant T_1 dependence, avoid long-time steady state values, and produce better-conditioned matrices for inversion than traditional CMPG sequences [28]. The primary disadvantage of this method is that the highest frequency for reconstruction analysis is limited to $2/t_{\text{gate}}$, which for the IBM system used is 7.14 MHz and for the in-house system is 5 MHz. However, higher frequencies can be addressed with other techniques such as SL measurements [41].

FIG. 9. Robust fixed total time pulse sequences 0, 3, 30, and 60, with $m = 6$. Each sequence consists of $2^{m+1} = 128$ pulses, bookended by $\pi/2$ pulses. The sequences are composed of equally timed (100 ns for in-house experiments) identity, or $\pm\pi$ gates. For the zeroth sequence, all pulses between the two bookend $\pi/2$ pulses are identity gates. Subsequent sequences have increasing number of $\pm\pi$ pulses. The $+\pi$ gates are represented by positive square pulses, $-\pi$ gates are represented by negative square pulses. Zeros represent identity gates. Note that zero for each sequence is offset for clarity. Sequence 3 is annotated for clarity. The function $\cos(3\pi n/64)$ determines the placement of $\pm\pi$ gates in the sequence. Each gate following a zero of this function is a π gate.

FIG. 10. Survival probabilities for injected noise experiments on the IBM quantum experience. (a) Bandpass noise injection survival probabilities shown for each bandwidth considered. (b) Double bandpass injection survival probabilities for different center frequencies ω_0 and ω_1 , where ω_0 denotes the peak closest to probe sequence zero. (c) $1/f^\alpha$ noise injection survival probabilities for different α values. Note that in all subfigures, the native noise survival probability is denoted by the dashed line.

APPENDIX E: EXPERIMENTAL PRACTICES FOR IBM EXPERIMENTS

There are a number of experimental practices that we enforce to establish accurate comparisons and evaluations of native and injected noise spectra. Perhaps the most notable feature of the IBM quantum experience is the daily calibration, where qubit transition frequencies are measured and single and multiqubit pulses are calibrated. We observed that qubit performance can vary significantly between, and even within, calibration cycles. Calibration errors can appear as large shifts in qubit survival probability when a suite of experiments are performed across a calibration. Furthermore, subtle changes in survival probabilities are observed, depending upon when one performs the experiments, e.g., experiments performed soon after a calibration can be quite distinct from those performed close to the next calibration due; this is likely due to unwanted drifts in qubit frequency and the control master clock.

To mitigate the effects of calibration for the large suite of experiments required to obtain the data shown in the main text, we enforce the following set of experimental practices. Let C_j be a particular comparison which contains experiments for a set of FTTSPs used to probe the native and injected noise spectra; let S_j denote the individual probe sequence experiment. We require each C_j to be performed within a calibration cycle and then collect each C_j for five distinct calibration cycles to capture typical hardware behavior. For a comparison C_j , the ordering of S_j is randomized to average out shifts in survival probability due to qubit drift within a calibration cycle. We further attempt to mitigate the effects of qubit drift by performing S_j for both the native and injected noise experiments back-to-back. This last procedure allows for a direct comparison between native and injected noise spectra each FTTSP and, thus, each frequency bin. Together, these practices provide confidence in the capability of SchWARMA noise injection as a universal noise injection technique.

APPENDIX F: NOISE INJECTION SURVIVAL PROBABILITIES

Noise spectrum reconstructions shown in the main text are calculated based on survival probabilities obtained from

FTTSP experiments performed with and without injected noise. In Fig. 10, the survival probabilities for each of the gate-based noise injection experiments performed on the IBM quantum experience are shown as a function of probe sequence number. Note that this particular point of view offers a similar, yet inverted, representation of the noise spectral density due to the relationship between probe sequence number and probing frequency. More concretely, the increasing probe sequence index directly corresponds to an increasing number of π pulses, which in turn corresponds to shifting the location of the fundamental harmonic of the probing sequence filter function to higher frequency. Thus, one can observe native or injected noise spectral features from the survival probabilities without performing an inversion or least-squares optimization. Furthermore, an attractive feature of examining the survival probability versus the spectral estimate is that the former does not amplify the effect of spurious spectral features. In many scenarios, the injected noise spectra are found to be in closer agreement with the desired spectra from the perspective of the survival probability.

In Fig. 11, we present the survival probabilities associated with noise reconstructions measured on the in-house experimental setup that are presented in the main text. The native noise background (red dashed lines) was measured once, while measurements of injected spectra were taken over many days. We do not account for changes over time in the native noise background apparent in the high probe sequence numbers of Fig. 11(d). Figures 11(a)–11(c) correspond to survival probabilities using the SDR injection method, while Fig. 11(d) corresponds to gate-based injection experiments.

APPENDIX G: EFFECT OF GATE ERROR ON NOISE SPECTRUM ESTIMATION

Pulse-error compensating probe sequences dramatically reduce artificial spectral features at high frequency. Here, we provide support for this statement by examining the effect of pulse error on FTTSP noise reconstruction from both a theoretical and experimental perspective. The theoretical analysis relies on previous results from Ref. [51] to examine the effect of pulse error on survival probability. Subsequent experiments focus on the comparison between FTTSP and its robust counterpart on the IBMQE.

FIG. 11. Survival probabilities for injected noise experiments on the in-house setup. (a) Bandpass noise injection survival probabilities using the SDR injection method. The two curves with $\Delta\omega = 0.2$ MHz correspond to differing noise powers, as shown in the main text. (b) Double bandpass injection survival probabilities using the SDR injection method. (c) $1/f^\alpha$ noise injection survival probabilities for different α values using the SDR injection method. (d) Bandpass noise injection survival probabilities using the gate-based injection method.

1. Theoretical analysis

We model the system as a single qubit subject to semi-classical dephasing noise and multiplicative control noise. Assuming only amplitude control, the control Hamiltonian is given by

$$H_C(t) = \frac{\Omega(t)}{2} \sigma^X, \quad (\text{G1})$$

while the error Hamiltonian

$$H_E(t) = \frac{\Omega(t)}{2} \epsilon(t) \sigma^X + \frac{\beta(t)}{2} \sigma^Z \quad (\text{G2})$$

explicitly defines the noise contributions. In $H_E(t)$, the control amplitude $\Omega(t)$ is augmented by an error function

$\epsilon(t)$ that dictates the characteristics of the control noise. Both $\epsilon(t)$ and $\beta(t)$ are assumed to be wide-sense stationary, Gaussian random variables, with means $\overline{\epsilon(t)}$ and $\overline{\beta(t)}$ and two-point correlation functions $C_\epsilon(\tau) = \overline{\epsilon(\tau)\epsilon(0)}$ and $C_\beta(\tau) = \overline{\beta(\tau)\beta(0)}$, respectively; $\overline{\cdots}$ denotes classical ensemble averaging.

We consider the weak noise limit and examine the dynamics of $H(t) = H_C(t) + H_E(t)$ via time-dependent perturbation theory. This is accomplished by moving into the rotating frame with respect to the control Hamiltonian such that

$$\begin{aligned} \tilde{H}_E(t) &= U_C^\dagger(t) H_E(t) U_C(t) \\ &= \frac{\Omega(t)}{2} \epsilon(t) \sigma^X + \frac{\beta(t)}{2} f(t) \sigma^Z. \end{aligned} \quad (\text{G3})$$

FIG. 12. Approximate analytical survival probabilities derived via the filter function formalism assuming a single qubit system with multiplicative control noise and dephasing. Results for coherent control errors are shown in the first column for (a) FTTPS and (c) RFTTPS. The second column displays results for uncorrelated stochastic control noise for FTTPS and RFTTPS in (b) and (d), respectively. While coherent errors appear to induce larger artificial spectral features at higher frequency than stochastic errors, they are easily corrected by the alternating phases used in RFTTPS. In contrast, RFTTPS is not effective against stochastic errors, appearing to enhance the contribution of the control noise as opposed to reducing it.

The control noise term commutes with the control while the dephasing term does not. As a result, the dephasing term undergoes a rotation that is dictated by the specifications of $\Omega(t)$. We consider the case where $\Omega(t)$ defines a sequence of instantaneous pulses and, thus, the dephasing term in the rotating frame reduces to its original form with an additional multiplicative function $f(t)$. Commonly referred to as the switching function, $f(t) = \pm 1$ under the conditions we have defined. Note that the FTTPSs are composed of either identity or $R_x(\pi)$ gates, hence, our focus on pulse sequences. The additional assumption of instantaneous pulses focuses the analysis on the effect of imperfections in the control amplitude rather than the effects of finite pulse width. While both contribute to the dynamics of the system in practice, we argue that the amplitude error is the dominant source of pulse error that contributes to the spectral features observed at high frequency in the FTTPS reconstructions.

Spectrum estimates are calculated via survival probabilities generated by initializing the qubit in $|+\rangle_y$, implementing a sequence of pulses to probe the environment spectrum, and measuring $|+\rangle_y$. More concretely, our experiments measured

$$P(+y, t) = \overline{|\langle +_y | U(t) | +_y \rangle|^2}, \quad (\text{G4})$$

where $U(t) = \mathcal{T}_+ \exp[-i \int_0^t H(s) ds]$ is the time propagator for the system, with $\mathcal{T}_+$ designating the time-ordering operator. In the rotating frame, $U(t) = U_C(t) \tilde{U}_E(t)$ is factorized into a propagator corresponding to the ideal control Hamiltonian, $U_C(t) = \mathcal{T}_+ \exp[-i \int_0^t H_C(s) ds]$, and an error propagator, $\tilde{U}_E(t) = \mathcal{T}_+ \exp[-i \int_0^t \tilde{H}_E(s) ds]$. Following Ref. [51], we parametrize the error dynamics in terms of a real, time-dependent error vector $\vec{a}(t) \equiv (a_X(t), a_Y(t), a_Z(t))$, such that

$$\tilde{U}_E(t) = e^{-i\vec{a}(t) \cdot \vec{\sigma}}, \quad (\text{G5})$$

with $\vec{\sigma} = (\sigma^X, \sigma^Y, \sigma^Z)$. Typically, the error vector is expressed in terms of a perturbative magnus expansion, where $\vec{a}(t) = \sum_{k=0}^{\infty} \vec{a}^{(k)}(t)$. For sufficiently weak noise and short timescales, the expansion can be truncated to first order such that $\vec{a}(t) \approx \vec{a}^{(1)}(t)$, where

$$a_{\mu}^{(1)}(t) = \frac{1}{2} \text{Tr} \left[\sigma^{\mu} \int_0^t \tilde{H}_E(s) ds \right]. \quad (\text{G6})$$

The approximate dynamics of $\tilde{U}_E(t)$ combined with $U_C(t)$ lead to two distinct survival probabilities for the FTTPS sequences. FTTPSs of total time T generate $U_C(T) = I$ for the first $N/2$ sequences or $U_C(T) = -i\sigma^X$ for the second half. Combining the first-order approximation of the error dynamics with the two possible ideal control evolutions, we obtain two approximate expressions for the survival probabilities of the FTTPS sequences:

$$P(+y|T) \approx 1 - \overline{[a_X^{(1)}(t)]^2} - \overline{[a_Z^{(1)}(t)]^2} \quad (\text{G7})$$

and $P(-y|T) = 1 - P(+y|T)$. These expressions exhibit dependence only on the x and z components of the error vector. Note that in our model, the x component corresponds to the control noise, while the z component captures the dephasing contribution.

Moving to the frequency domain, we obtain a relationship between filter functions and the noise spectral densities. In particular, it can be shown that

$$\overline{[a_X^{(1)}(T)]^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_X(\omega, T) S_{\epsilon}(\omega) d\omega, \quad (\text{G8})$$

$$\overline{[a_Z^{(1)}(T)]^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_Z(\omega, T) S_{\beta}(\omega) d\omega, \quad (\text{G9})$$

where the filter functions are

$$F_X(\omega, T) = \frac{1}{4} \left| \int_0^T \Omega(t) e^{i\omega t} dt \right|^2, \quad (\text{G10})$$

$$F_Z(\omega, T) = \frac{1}{4} \left| \int_0^T f(t) e^{i\omega t} dt \right|^2, \quad (\text{G11})$$

and the $S_{\mu}(\omega) = \int_{-\infty}^{\infty} C_{\mu}(\tau) e^{-i\omega\tau} d\tau$, $\mu = \epsilon, \beta$ denotes the power spectral density for the control and dephasing noise, respectively.

Here, we consider two forms of control noise: coherent errors, where $\bar{\epsilon}(t) = \epsilon$ and $S_{\epsilon}(\tau) = \epsilon^2 \delta(\tau)$, and zero-mean stochastic errors characterized by white noise with cutoff frequency ω_c , i.e., $\bar{\epsilon}(t) = 0$ and $S_{\epsilon}(\tau) = \sigma^2 [\Theta(\omega + \omega_c) - \Theta(\omega - \omega_c)]$. In the case of instantaneous pulses, we find

$$\overline{[a_X^{(1)}(t)]^2} = \begin{cases} \frac{\epsilon^2 \pi}{8} M^2, & \text{coherent} \\ \frac{\sigma^2 \omega_c \pi}{4} \sum_{i,j} \text{sinc}[\omega_c(s_i - s_j)], & \text{stochastic,} \end{cases} \quad (\text{G12})$$

where M corresponds to the total number of $R_x(\pi)$ pulses and s_j is the location of the j th $R_x(\pi)$ pulse in the sequence. The above expressions can be augmented to account for phase flips in the pulses, e.g., in such cases as RFTTPSs, by considering $\Omega(t) \mapsto \Omega(t)\alpha(t)$. The phase flipping function $\alpha(t) = \pm 1$, where the function's value is determined by whether the pulse sequence implements an $R_x(\pi)$ or $R_x(-\pi)$, respectively, at time t . When phase flips are considered, the resulting error vector component for coherent errors is

$$\overline{[a_X^{(1)}(t)]^2} = \frac{\epsilon^2 \pi}{8} \left(\sum_j \alpha(s_j) \right)^2 \quad (\text{G13})$$

and

$$\overline{[a_X^{(1)}(t)]^2} = \frac{\sigma^2 \omega_c \pi}{4} \sum_{i,j} \alpha(s_i) \alpha(s_j) \text{sinc}[\omega_c(s_i - s_j)] \quad (\text{G14})$$

for stochastic errors. As expected, in the coherent error case, phase-flip modulation results in a pulse error that is determined in part by the accumulated sum of the phases. While a similar quadratic dependence on the accumulated phase is observed in the stochastic error case, the expressions are more complex and reliant on interactions between phases at distinct points in time.

In Fig. 12, we show comparisons of survival probabilities for FTTPSs and RFTTPSs based on the analytical expressions derived above. We consider zero-mean, uncorrelated dephasing noise such that $\bar{\beta}(t) = 0$ and $C_{\beta}(\tau) = \Gamma^2 \delta(\tau)$. In the first column, results are shown for coherent errors for different magnitudes of the error ϵ . In the second column, we show results for uncorrelated, white cutoff control noise with

FIG. 13. Effects of errors on $R_x(\theta)$ gates in simulated FTTSPs in the presence of a white-noise Z-dephasing error. Markers indicate averages of 1000 Monte Carlo runs of SchWARMA-driven noisy circuits, with lines showing regression fits using the gate-error penalty terms. (a) Ideal $R_x(\theta)$ gates are followed by uncorrelated $R_x(\vartheta)$ where ϑ are zero mean normally distributed with standard deviation σ , resulting in X-dephasing errors on the R_x gates, when averaged. The solid lines show least-squares fits of the survival probabilities using the parameter c_1 to the model $\frac{1}{2} + B_0 \exp(-c_1 n_k)$, where B_0 is determined by the fixed Z-dephasing noise power and n_k the number of $R_x(\pi)$ pulses in the k th FTTSP. (b) Ideal $R_x(\pi)$ gates are replaced by over-rotated gates $R_x(\pi + \vartheta)$ and $R_x(\pi/2)$ gates with $R_x((\pi + \vartheta)/2)$. The dashed line shows least-squares fits using the parameter c_2 to the model $\frac{1}{2} + B_0 \exp(-c_2 n_k^2)$. The solid lines show fits using an additional parameter β for a model $\frac{1}{2} + B_0 \exp(-c_2 n_k^\beta)$ which improves the fits for the larger errors.

different standard deviations σ . As expected from the analytical expressions, the FTTSP conveys reductions in survival probability that increase with probe sequence number, i.e., number of pulses. In addition, comparing the FTTSP (top row) and RFTTSP (bottom row), we find that the robust sequences suppress coherent errors extremely well, while stochastic errors typically do not benefit from the phase alterations used in RFTTSPs.

We corroborate our analytical results with a series of SchWARMA-driven numerical experiments using FTTSP under different $R_x(\pi)$ -gate error models. In the first set of experiments, we consider fixed power white noise-driven Z-dephasing errors and followed each $R_x(\theta)$ gate with a random $R_x(\vartheta)$ gate, with ϑ drawn independently from a zero-mean normal distribution with standard deviation σ (we note that this is an example of a gate-dependent, multi-axis SchWARMA model). Figure 13(a) shows the resulting survival probabilities for various σ , along with fits to an exponential decay model proportional to n_k , the number of $R_x(\pi)$ gates in a given sequence. These exponential decays fit the resulting simulated survival probabilities quite well, and we note that for modest σ , the decay appears quite linear.

The next error model we considered was a fixed, coherent over-rotation error on the $R_x(\theta)$ gates, subject to the same Z-dephasing error as above; this models a long-running calibration error. The over-rotation was held constant for the full set of FTTSPs and all Monte Carlo trajectories. Specifically, we replaced ideal $R_x(\pi)$ gates with $R_x(\pi + \vartheta)$ and $R_x(\pi/2)$ gates with $R_x((\pi + \vartheta)/2)$ for several values of ϑ , see Fig. 13(b). Again, we performed least-squares fits to the survival probabilities, but instead of an exponential decay pro-

portional to n_k , we intuitively expected a decay proportional to n_k^2 . We found that this model fits quite well for modest ϑ , but was not a great fit for $\vartheta = 0.01$ rad. To see if we could produce a better fit, we added an additional regression parameter β and performed fits of exponential decay proportional to n_k^β . We saw that this improved the mean squared error of the fit to $\vartheta = 0.01$ rad by nearly an order of magnitude, produced a modest improvement in the $\vartheta = 0.0075$ rad case, and negligible differences in the remaining.

In summary, these simulations further support the hypothesis that the high-frequency noise observed in the FTTSP-based spectrum reconstructions are likely due to presence of gate errors. Thus, when performing SchWARMA-driven predictions and fits, it is necessary to incorporate terms that account for effects which the results above indicate can be quite well modeled using only a few terms (i.e., c_1 , c_2 , and β). Furthermore, since the decays we observed in the experimental data are not at the extremes of what we explored in simulation, the inclusion of the β term is not needed. We also note that we found that the inclusion of the c_1 term can improve prediction accuracy even when the coherent errors are canceled out using alternating directions in the $R_x(\pi)$ pulses.

2. Experimental implementation

We perform a comparison between FTTSPs and RFTTSPs on the IBM Armonk processor under the condition of native and injected noise. FTTSPs are constructed from U gates of duration $t_G \approx 248$ ns, while RFTTSP are composed of true, precalibrated π pulses of duration $t_G \approx 142$ ns. The Armonk processor maintains a library of precalibrated pulses that

FIG. 14. Comparison of QNS protocols in the presence of native and injected $1/f$ noise. Left: Survival probability as a function of probe sequence index for FTTPSs and RFTTPSs for both native and injected noise. Right: Spectrum estimates for FTTPS and RFFPS for injected noise. Experiments were performed on the IBMQE.

includes an $R_x(\pi)$ DRAG pulse that is consistently updated each calibration cycle. By altering the phase of the precalibrated DRAG pulse, we obtain the additional $R_x(-\pi)$ used in the RFTTPSs. In Fig. 14, the top-left panel displays the survival probabilities for both sets of probe sequences in the presence of native and injected triple bandpass noise bootstrapped over five calibration cycles. The first two bandpass signals possess the same bandwidth of $\Delta\omega = 0.18$ MHz, while the third, high-frequency bandpass signal has a bandwidth of $\Delta\omega = 0.54$ MHz. The center frequencies for the three bandpass signals are $\omega_0 = 0.53, 1.23, 2.64$ MHz. The top-right panel of Fig. 14 shows a similar comparison for injected $1/f$ noise. Native noise survival probabilities display a clear distinction between FTTPSs and RFTTPSs, most notably at large sequence index (i.e., high frequency) where pulse errors are more evident for FTTPSs than RFTTPSs.

By injecting noise and comparing estimated spectra to the expected injected noise spectra, we determine which protocol supplies the more accurate spectral estimates. The bottom panels of Fig. 14 show the resulting estimated injected spectra for triple bandpass (bottom left) and $1/f$ (bottom right) noise. FTTPS spectral estimations exhibit deviations from the ideal spectrum at high frequency despite correction for the native spectrum. Specifically, these deviations can be ob-

served in the bottom panels of Fig. 14 at frequencies greater than 1.5 MHz. Divergences from the ideal spectrum are less substantial for RFTTPSs, indicating that this particular set of probe sequences yields more accurate spectral estimates. Together, these comparisons convey the potential benefit of utilizing true π pulses and phase adjustments to improve QNS estimation.

The comparison performed above and in the main text indicate that survival probabilities are correlated with the number of gates in a given FTTPS. Noting that this was inconsistent with the established literature on the dephasing spectra of superconducting qubits, we conjectured that this was primarily due to errors on the $R_x(\pi)$ gates. The comparison between nonrobust and robust variations of FTTPSs supports the claim that pulse errors indeed contribute to spurious features in the dephasing noise spectrum. However, while the RFTTPSs provide insight into ways of suppressing such features, they do not elucidate details about the noise characteristics of the pulse error. Based on our analytical and numerical analyses, we conjecture that the pulse errors are likely characterized by a correlation time that is more consistent with slow drift rather than rapidly fluctuating, uncorrelated errors. Estimating this correlation time is left for future studies of multiaxis noise spectroscopy protocols.

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