General theory of quantum fingerprinting network

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The purpose of fingerprinting is to compare long messages with low communication complexity. Compared with its classical version, quantum fingerprinting can realize exponential reduction in communication complexity. Recently, multiparty quantum fingerprinting is studied to determine whether the messages from many parties are all the same. However, the complete relationships among them, including which ones are the same, are more general for future applications. Here, we propose the relationship state $\mathcal{G}$ to describe all possible relationships among N messages. Based on this, we provide a general quantum fingerprinting network and then take the four-party quantum fingerprinting protocol as an example for detailed analysis. We take the full parameter optimization for the quantum fingerprinting protocol through asymmetric channel. The method of encoding more than one bit on each coherent state is used to further improve the performance of our protocol. Compared with other protocols, our protocol provides more relationships among different parties with only linear increase in communication complexity.

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I. INTRODUCTION

The application of quantum mechanics in the field of communication brings benefits in many aspects, such as security and communication complexity. With respect to security, quantum key distribution (QKD) [1–7] is a representative example that can provide information-theoretic security for private communication, while the security of its classical counterpart is based on the assumptions of computational complexity. In terms of communication complexity [8,9], which is the minimum amount of communication required among participants to complete a certain task, compared with its classical version [10,11], quantum fingerprinting can exponentially reduce the amount of information required for the tasks. This is very useful in energy-saving communication [8,12,13].

The purpose of the fingerprinting protocol is to answer whether two n -bit messages x_1 and x_2 are the same based on their fingerprints. In the simultaneous message passing model [14], there are two senders sending the fingerprints of each one's original messages to the remote referee who needs to determine whether $x_1 = x_2$ with a small error probability P_e upper-bounded by ε . Quantum fingerprinting can

significantly reduce the communication complexity required to complete the comparison, i.e., the communication complexity of the classical version is $O(\sqrt{n})$ -bits [10,11], while that of the quantum version is only $O(\log_2 n)$ -qubits [12]. However, the original protocol requires high-dimensional entanglement which makes its implementation technically challenging [12]. Fortunately, a quantum fingerprinting protocol based on coherent states is proposed [15] and since then, there have been many advances in theory and experiment [16–18]. Recently, to reduce the communication time, wavelength division multiplexing (WDM) is used in quantum fingerprinting and the result shows that not only the communication time is reduced, but also the communication complexity is further reduced [19].

So far, most studies have focused on two-party quantum fingerprinting protocols [15–17,19] and there are only two relationships between the two inputs, $x_1 = x_2$ or $x_1 \neq x_2$. As the number of messages increases, things get more complicated and interesting. If we need to determine whether N messages are all the same, then a simple way is to use the two-party quantum fingerprinting to compare N inputs in pairs, which needs to be done $(N - 1)$ times. So is there a more efficient way? Recently, a multiparty quantum fingerprinting is proposed [20], which only needs to be run once to determine whether N inputs are the same. Obviously, in this case, the quantum fingerprinting network presents a huge advantage. However, the relationships among N inputs are in general more complicated than the case above and hence it is not enough just to know whether they are all the same. Usually, we also need to know more relationships or even complete

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relationships among these inputs. Therefore, it is of great interest to study the multiparty quantum fingerprinting network with more relationships.

In this work, based on the existing framework [15,20], we study the complete relationships by changing the combination of the interference beams. We propose the relationship state $\mathcal{G}$ to describe all possible relationships and provide a general quantum fingerprinting network by the ingenious design of our protocol. In particular, we take the four-party quantum fingerprinting protocol as an example for detailed analysis. The method of encoding more than one bit on each coherent state is used to further improve the performance of our protocol. Moreover, we take the full parameter optimization for the quantum fingerprinting protocol through asymmetric channel. Compared with other protocols, our protocol provides more relationships among different parties with similar communication complexity.

The rest of this paper is arranged as follows. In Sec. II, we present our N -party quantum fingerprinting protocol. In Sec. III, we show the details of our protocol through the special case of a four-party quantum fingerprinting network. In Sec. IV, we study the behavior of our protocol through asymmetric channels. After this, we show the advantages of our protocol and how to further improve its performance by the multibit encoding method in Sec. V. Finally, we present a brief summary in Sec. VI.

II. N -PARTY QUANTUM FINGERPRINTING PROTOCOL

A. Relationship state

In general, given N messages, they are not necessarily all the same. We can put the same messages into the same group (set). If there are $\mathcal{J}$ groups, then we name each group as $G_1, G_2, \dots, G_{\mathcal{J}}$ and, of course, $1 \leq \mathcal{J} \leq N$.

Relationship state. The set of all groups for those N messages. We shall denote it by $\mathcal{G} = \{G_1, G_2, \dots, G_{\mathcal{J}}\}$. Since $\mathcal{G}$ is a set, it is only determined by its elements rather than the positions of them. We shall always arrange the positions of groups by the rule $|G_a| \geq |G_b|$ if $a \leq b$.

For example, given $N = 4$, $x_1 = x_2 = x_3 \neq x_4$, $\mathcal{G} = \{G_1, G_2\}$, where $G_1 = \{x_1, x_2, x_3\}$, $G_2 = \{x_4\}$.

In Refs. [20,21], the following two special relationship states are studied.

All-equality relationship state. A special case that all messages are the same and we denote it as $\mathcal{G}^{\text{AE}}$,

$$\mathcal{G}^{\text{AE}} = \{G_1\}, \quad (1)$$

where $G_1 = \{x_1, x_2, \dots, x_N\}$.

Exists-equality relationship state. A special case that there is at least a pair of equal messages and we denote it as $\mathcal{G}^{\text{EE}}$,

$$\mathcal{G}^{\text{EE}} = \{G_1, G_2, \dots, G_{\mathcal{J}}\}, \quad (2)$$

where $|G_1| \geq 2$.

The referee's task in our protocol is to determine the relationship state $\mathcal{G}$ for N messages, given their fingerprints. The communication complexity of the protocol is the less the better, given ε .

FIG. 1. N -party quantum fingerprinting protocol. There are N senders $\{S_k\}$, $k = 1, 2, \dots, N$. They encode the original messages $\{x_k\}$, shown in gray, with the ECC and change them into $\{E(x_k)\}$, shown in yellow. Then they prepare fingerprint states $|s_k\rangle$ according to $E(x_k)$ and send them to a remote referee through the lossy channel of transmittance η_k . Finally, the referee can determine the relationship among N messages, that is, output the relationship state $\mathcal{G}$.

B. Protocol

Here, we introduce our N -party quantum fingerprinting protocol, as illustrated in Fig. 1.

Protocol: N -party quantum fingerprinting protocol

Input: N messages $\{x_k\}$, $k = 1, 2, \dots, N$.

Output: Relationship State $\mathcal{G}$.

1. Preparation.

N senders $\{S_k\}$ use the error correction code (ECC) $E(x_k) \in \{0, 1\}^m$ to encode messages x_k , where $m = \frac{n}{c}$ is the length of the $E(x_k)$, $c < 1$.

Remark 1: Since ECC is tolerable to errors up to a certain upper-bound, we shall regard the relationship of two $E(x_k)$ s to be different if the relative Hamming distance is larger than δ under which ECC runs successfully.

A table of the corresponding comparison results for all possible relationship states needs to be listed. (They are listed in Table I for the specific case $N = 4$.)

2. Encoding.

Based on $E(x_k)$, every sender encodes their own fingerprint state $|s_k\rangle$, i.e., a product state of m coherent states:

$$|s_k\rangle = \bigotimes_{j=1}^m |\beta\rangle_j, \quad |\beta\rangle_j = \left| (-1)^{E(x_k)_j} \frac{\alpha_k}{\sqrt{m}} \right\rangle. \quad (3)$$

Here $|\beta\rangle_j$ is a coherent state of amplitude $(-1)^{E(x_k)_j} \frac{\alpha_k}{\sqrt{m}}$, and $E(x_k)_j$ represents the j th component in the entire sequence $E(x_k)$. Then they send their fin-

gerprint state to the remote referee through the lossy channel of transmittance η_k , as illustrated in Fig. 1.

3. Measurement.

The referee uses several BSs and N detectors $\{D_k\}$ to perform the interference measurement of N fingerprint states and records the number of counts $\{C_k\}$ of each detector. Here, we use C_k to denote the number of counts of detector D_k .

Remark 2: The referee can adopt the extendable device or the optimal device [20], which requires a minimum number of BSs, and the key point of the device is to realize the multibeam interference. In this work, we mainly analyze the optimal design based on balanced BSs, as illustrated in Fig. 3.

4. Comparison.

The referee selects appropriate $(N - 1)$ threshold counts $\{C_k^{\text{th}}\}$, $k = 2, 3, \dots, N$. Then, for a certain detector D_k , if $C_k < C_k^{\text{th}}$, the referee records the result as $R_k = 0$, otherwise, records it as $R_k = 1$. If the referee observes $N - 1$ detectors, then he records it as $R_{2,3,\dots,N} = R_2 R_3 \cdots R_N$.

Remark 3: Ideally, if N messages $\{x_k\}$ are all the same, then only one detector D_1 has clicks and if they are not all the same, then any detector D_k is likely to have clicks. However, in practice, due to the device imperfection such as the dark count probability of detectors, even N messages are all the same, the detector D_1 must have clicks and other detectors except D_1 are also possible to click. Therefore, the referee needs to select the threshold $\{C_k^{\text{th}}\}$, $k = 2, 3, \dots, N$. If $C_k < C_k^{\text{th}}$ holds from $k = 2$ to N , then the referee concludes they are all the same; otherwise, he concludes they are not all the same.

5. Decision.

If the referee can get a complete relationship state $\mathcal{G}$, then he makes a decision of Yes and outputs $\mathcal{G}$.

Otherwise, the referee makes a decision of No. He changes the combination of the interference beams and returns to step 2 for next round.

The fingerprint state sent in round t is

$$|s_k^t\rangle = \bigotimes_{j=1}^m |\beta^t\rangle_j, \quad |\beta^t\rangle_j = \left| (-1)^{E(x_k)_j} \frac{\alpha_k^t}{\sqrt{m}} \right\rangle. \quad (4)$$

The result of the round t comparison can be recorded as $R_{2,3,\dots,N}^{(t)}$, where $t = 1, 2, \dots, T$ and T represents the actual number of rounds.

Remark 4: It is possible that after the first round of steps 2–5, the referee can only know N messages are not all the same, but he still cannot determine the complete relationship state $\mathcal{G}$. So, the combinations of the interference beams need to be changed and steps 2–5 need to be run for several rounds until the referee can determine a unique $\mathcal{G}$ based on the comparison results.

We can use a flow chart to describe the protocol, as illustrated in Fig. 2.

FIG. 2. The flow chart of N -party quantum fingerprinting protocol.

C. The design of N -party quantum fingerprinting network based on balanced BSs

We use the optimal device design that requires a minimum number of BSs [20]. When $N = 2^s$, $s \in \mathbb{Z}^+$, we show in Fig. 3 how to determine $\mathcal{G}$ based on quantum fingerprinting network composed of balanced BSs. Each bracket, which represents for a balanced BS, compares its two inputs. The first layer shows the comparison results of $E(x_k)$ and $E(x_{k+1})$, $k = 1, 3, \dots, N - 1$; the second layer shows the comparison results of $E(x_k) + E(x_{k+1})$ and $E(x_{k+2}) + E(x_{k+3})$, $k = 1, 5, \dots, N - 3$, and so on, and the s th layer shows the comparison results of $\sum_{k=1}^{2^{s-1}} E(x_k)$ and $\sum_{k=2^{s-1}+1}^N E(x_k)$.

FIG. 3. The design of N -party quantum fingerprinting network based on balanced BSs, where $N = 2^s$, $s \in \mathbb{Z}^+$. N senders, $\{S_k\}$, hold the messages to be compared. Each bracket represents a balanced BS and compares the input of the two parts. The 1st, 2nd, and s th layer are shown in green, purple, and black, respectively.

FIG. 4. Four-party quantum fingerprinting protocol based on balanced BSs. $\{S_k\}$: four message-senders. BS: balanced beam-splitter. If the amplitudes of two incident states are equal, then the output light is supposed to be on the upper side (lower side) of the BS; if the amplitudes of two incident states are opposite, then the output light is supposed to be on the lower side (upper side) of the BS. The referee can know whether the corresponding messages are equal or not by observing the clicks of the corresponding detector. $\{D_k\}$: detectors. The referee observes the clicking results of $(N - 1)$ detectors to determine the relationship state $\mathcal{G}$ of T rounds.

When $N = 4$, as shown in Fig. 4, the comparison results of the 1st layer are determined by the clicking results of detectors D_2 and D_4 , and the comparison result of the 2nd layer is determined by that of detector D_3 . In the 1st round, if $R_2 = 0$ ($R_4 = 0$), then the two parts compared through D_2 (D_4) are equal, i.e., $E(x_1) = E(x_2)$ ($E(x_3) = E(x_4)$); if $R_2 = 1$ ($R_4 = 1$), then the two parts compared through D_2 (D_4) are different, i.e., $E(x_1) \neq E(x_2)$ ($E(x_3) \neq E(x_4)$); if $R_3 = 0$ ($R_3 = 1$), then the two parts compared through D_3 are equal (different), i.e., $E(x_1) + E(x_2) = E(x_3) + E(x_4)$ ($E(x_1) + E(x_2) \neq E(x_3) + E(x_4)$). Based on this, the error probability of 1st round is

$$P_e^{(1)} = \max \{P_2^E(R_2 = 1), P_2^D(R_2 = 0), P_3^E(R_3 = 1), P_3^D(R_3 = 0), P_4^E(R_4 = 1), P_4^D(R_4 = 0)\}, \quad (5)$$

where $P_k^r(R_k)$ means the probability of the comparison result is R_k when the two parts compared through D_k are equal ($r = E$) and different ($r = D$). For example, $P_2^E(R_2 = 1)$ means the probability of the comparison result is $R_2 = 1$ when $E(x_1) = E(x_2)$.

The error probability of our protocol is

$$P_e = \max \{P_e^{(t)}\}, \quad (6)$$

where $t = 1, 2, \dots, T$, and $P_e^{(t)}$ represents the error probability of round t .

When $2^{s-1} < M < 2^s = N$, the design in Fig. 3 can also be used to determine $\mathcal{G}$ for M messages and we provide a detailed explanation in Appendix C. Therefore, our protocol can determine the relationship state $\mathcal{G}$ for any number of messages. In the following, we set $N = 2^s$, $s \in \mathbb{Z}^+$ if there is no special statement.

D. Communication complexity

If the referee only needs to judge whether N messages are all the same [20,21], then it is the special-case problem to judge whether $\mathcal{G} = \mathcal{G}^{\text{AE}}$ in our protocol, with the total communication complexity of the classical optimal fingerprinting

protocol C_0^{AE} and the classical limit C_1^{AE} :

$$C_0^{\text{AE}} = N \left[\frac{\log_2(P_e)}{\log_2 \left[1 - \frac{1}{9} (1 - e^{-\frac{1}{2}}) \right]} \right] \times \left[8 \sqrt{2 \left\lceil \frac{3n}{N} \right\rceil} + 4 \left\lceil \log_2 \left(3n / \left\lceil \frac{3n}{N} \right\rceil \right) \right\rceil \right], \quad (7)$$

$$C_1^{\text{AE}} = N \left[\frac{(1 - 2\sqrt{P_e})\sqrt{n}}{2\sqrt{N \ln 2}} - \frac{1}{N} \right], \quad (8)$$

where P_e is the error probability of the classical fingerprinting protocol.

Completely determining the specific $\mathcal{G}$ is more difficult than just determining whether $\mathcal{G} = \mathcal{G}^{\text{AE}}$. Obviously,

$$C_0^R > C_0^{\text{AE}}, \quad C_1^R > C_1^{\text{AE}}, \quad (9)$$

where C_0^R and C_1^R represent the optimal communication complexity in classical version and classical limit when determining $\mathcal{G}$, respectively.

In prior-art N -party quantum fingerprinting protocol [20], the referee only needs to judge whether $\mathcal{G} = \mathcal{G}^{\text{AE}}$, the total communication complexity is

$$Q^{\text{AE}} = \sum_{k=1}^N \mu_k \log_2 n, \quad (10)$$

where $\mu_k = |\alpha_k|^2$ represents the total photon number of the fingerprint state in Eq. (3) sent by the k th sender.

In our protocol, if the referee needs to run steps 2–5 for T rounds to determine a unique $\mathcal{G}$, the total communication complexity Q^R is

$$Q^R(T) = \sum_{t=1}^T \sum_{k=1}^N \mu_k^t \log_2 n, \quad (11)$$

where $\mu_k^t = |\alpha_k^t|^2$ represents the total photon number of the fingerprint state sent by the k th sender of round t of steps 2–5 in the protocol. We denote the largest possible number of rounds for steps 2–5 as $\bar{T}(N)$ given N messages, obviously, the upper bound of the communication complexity Q^R for our

TABLE I. Determine $\mathcal{G}$ for four messages $\{x_k\}$ by three detectors D_2, D_3 and D_4 in Fig. 4. $R_{2,3,4}^{(i)}$ represent the comparison results of the three detectors after round i of the protocol. $[R_{2,3,4}^{(1)}, R_{2,3,4}^{(2)}, R_{2,3,4}^{(3)}] \in \{[101, 111, \emptyset], [111, 101, \emptyset], [111, 111, 101], [111, 111, 111]\}$, where $\emptyset$ means that there is no need for a third round.

Relationship State	$R_{2,3,4}^{(1)}$	$R_{2,3,4}^{(2)}$	$R_{2,3,4}^{(3)}$
1 $\{\{x_1, x_2, x_3, x_4\}\}$	000		
2 $\{\{x_1, x_2, x_3\}, \{x_4\}\}$	011	011	
3 $\{\{x_1, x_2, x_4\}, \{x_3\}\}$	011	110	
4 $\{\{x_1, x_3, x_4\}, \{x_2\}\}$	110	011	
5 $\{\{x_2, x_3, x_4\}, \{x_1\}\}$	110	110	
6 $\{\{x_1, x_2\}, \{x_3, x_4\}\}$	010		
7 $\{\{x_1, x_3\}, \{x_2, x_4\}\}$	101	010	
8 $\{\{x_1, x_4\}, \{x_2, x_3\}\}$	101	101	
9 $\{\{x_1, x_2\}, \{x_3\}, \{x_4\}\}$	011	111	
10 $\{\{x_1, x_3\}, \{x_2\}, \{x_4\}\}$	111	011	
11 $\{\{x_1, x_4\}, \{x_2\}, \{x_3\}\}$	111	111	011
12 $\{\{x_2, x_3\}, \{x_1\}, \{x_4\}\}$	111	111	110
13 $\{\{x_2, x_4\}, \{x_1\}, \{x_3\}\}$	111	110	
14 $\{\{x_3, x_4\}, \{x_1\}, \{x_2\}\}$	110	111	
15 $\{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\}$	$\widetilde{R_{2,3,4}^{(1)}}$	$\widetilde{R_{2,3,4}^{(2)}}$	$\widetilde{R_{2,3,4}^{(3)}}$

protocol is

$$\overline{Q^R} = \sum_{t=1}^T \sum_{k=1}^N \mu_k^t \log_2 n \geq Q^R(T), \quad (12)$$

where $Q^R(T)$ is the actual communication complexity with T -round steps 2–5.

To show the advantages over its classical version of our protocol in communication complexity, we compare $\overline{Q^R}$ with C_0^{AE} and C_1^{AE} . Note that $\overline{Q^R}$ is the upper bound of communication complexity of our protocol in Eq. (12), while C_0^{AE} and C_1^{AE} are lower bounds of communication complexity of the classical versions in Eq. (9).

III. RELATIONSHIP STATES OF FOUR-PARTY QUANTUM FINGERPRINTING PROTOCOL

Next, we take the four-party quantum fingerprinting protocol through symmetric channel as an example to introduce our protocol in detail, as illustrated in Fig. 4.

A. Relationship state $\mathcal{G}$ for four messages

There are 15 possible relationship states for four messages $\{x_k\}$ in Table I. Since $\{x_k\}$ and $\{E(x_k)\}$ have the same relationship state, the referee only needs to know the relationship state of $\{E(x_k)\}$. But he needs neither $E(x_k)$ nor x_k explicitly for the complete relationships.

As shown in Fig. 4 and Table I, on the first round, if the referee observes the any result in the 1st row or 6th row, he can determines the relationship state $\mathcal{G}$ completely.

Otherwise, he goes into the second round. If he observes any result in the range from 2nd row to 5th row, from 7th row to 10th row, from 13th row to 14th row, or a result of $\{[101, 111], [111, 101]\}$ in 15th row, then he can determine $\mathcal{G}$ completely.

TABLE II. Optimized parameters values of our four-party protocol. Here, $n = 10^{13}$, $\delta = 0.22$, $\eta = 0.1$, $P_d = 10^{-11}$, $c = 0.2$, $\varepsilon = 10^{-2}$.

μ	C_2^{th}	C_3^{th}	C_4^{th}	$\overline{Q^R}$
4961	602	553	602	2.57×10^6

Otherwise, he goes into the third round to finalize the complete relationship state $\mathcal{G}$.

B. Numerical results

If the channel is symmetric, then we set the four senders' channel transmittance as η and their average phonon number in every round as μ for our four-party protocol. We seek μ and $\{C_k^{\text{th}}\}$, $k = 2, 3, 4$, to minimize $\overline{Q^R}$ under the constraint $P_e \leq \varepsilon$, where P_e is the error probability in Eq. (6).

For the calculation of error probability, we denote C_k^r as the random variable corresponding to the number of clicks in D_k for the cases r , where $r = E$ or $r = D$ means the two parts compared through D_k is equal or different, respectively. Notation $\mathcal{P}_k^r$ represents the probability of observing a click in each mode for the case of r above through D_k . For example, in our four-party protocol, $\mathcal{P}_2^E$ represents the probability of observing a click in D_2 conditional that each mode of $E(x_1)$ and $E(x_2)$ is equal.

In our four-party protocol,

$$\begin{aligned} \mathcal{P}_2^E &= P_d, & \mathcal{P}_2^D &= \delta(1 - e^{-\frac{2\eta\mu}{m}}) + P_d, \\ \mathcal{P}_3^E &= P_d, & \mathcal{P}_3^D &= \delta(1 - e^{-\frac{\eta\mu}{m}}) + P_d, \\ \mathcal{P}_4^E &= P_d, & \mathcal{P}_4^D &= \delta(1 - e^{-\frac{2\eta\mu}{m}}) + P_d, \end{aligned} \quad (13)$$

where P_d represents the dark count probability of detectors and δ represents the smallest relative Hamming distance between any two distinct $E(x_k)$.

Once we know $\mathcal{P}_k^E$ and $\mathcal{P}_k^D$ of detector D_k , we can choose an appropriate threshold C_k^{th} to satisfy $P_e \leq \varepsilon$:

$$\begin{aligned} P_e &= \max \{P_2^E(C_2 > C_2^{\text{th}}), P_2^D(C_2 < C_2^{\text{th}}), \\ &P_3^E(C_3 > C_3^{\text{th}}), P_3^D(C_3 < C_3^{\text{th}}), \\ &P_4^E(C_4 > C_4^{\text{th}}), P_4^D(C_4 < C_4^{\text{th}})\}. \end{aligned} \quad (14)$$

The optimized values of μ , C_2^{th} , C_3^{th} , and C_4^{th} are listed in Table II. With these parameters, we can calculate the upper bound of the communication complexity $\overline{Q^R} = 2.57 \times 10^6$, while in the same condition, $C_0^{\text{AE}} = 1.29 \times 10^{10}$, $C_1^{\text{AE}} = 3.04 \times 10^6$. This shows, given the same four messages, our protocol has an advantage by four magnitude orders in communication complexity reduction compared with the optimal classical version and breaks the classical limit.

IV. OPTIMIZATION OVER ASYMMETRIC CHANNEL

Given an asymmetric channel, the optimization is more complicated, because there are more parameters to be optimized for minimization of $\overline{Q^R}$.

TABLE III. Optimized parameters values of the two-party quantum fingerprinting protocol through asymmetric channel. Here, $n = 3 \times 10^{12}$, $c = 0.2$, $P_d = 10^{-10}$, $\delta = 0.22$, $\sqrt{\eta_1} = 0.3$, $\sqrt{\eta_2} = 0.4$, $\varepsilon = 10^{-5}$, for ideal BS, $\nu = 1$, for realistic BS, $\nu = 0.99$.

	α_1	α_2	C_2^{th}	$\overline{Q^R}$
Our two-party protocol with ideal BS by one-bit method	85	78	1685	5.52×10^5
Our two-party protocol with ideal BS by two-bit method	69	70	898	3.91×10^5
Our two-party protocol with realistic BS by one-bit method	88	77	1695	5.67×10^5

A. Two-party quantum fingerprinting protocol through asymmetric channel

When $N = 2$, we optimize α_1 , α_2 , and C_2^{th} for the minimization of $\overline{Q^R}$. The analytical expression of $\mathcal{P}_k^r$ can be found in Appendix A 1.

The optimized parameter values in the two-party protocol through asymmetric channel are listed in 2nd row of Table III. With these parameter values, we can calculate the upper bound of the communication complexity $\overline{Q^R} = 5.52 \times 10^5$. Compared with the communication complexity of the classical optimal version $C_0^{\text{AE}} = 1.24 \times 10^{10}$ and the classical limit $C_1^{\text{AE}} = 1.46 \times 10^6$, our two-party protocol keeps its advantages of five-order of magnitude and breaks the classical limit.

B. Fingerprinting network through asymmetric channel

Here we study the optimization of four-party fingerprinting network through asymmetric channel. Detailed analytical expression of each $\mathcal{P}_k^r$ can be found in the Appendix A 2. In the asymmetric case, the fingerprint states sent by each sender in each round of steps 2–5 are always different from each other. To obtain the optimized result, the value of $\mathcal{P}_k^r$ is supposed to be different in different rounds, we use $\mathcal{P}_k^{r,t}$ to represent $\mathcal{P}_k^r$ in round t .

The optimized values of $\{\alpha_k^t\}$ and $\{C_k^{\text{th}}\}$ for different senders and in different rounds are listed in Table IV. We can calculate the upper bound of the communication complexity $\overline{Q^R} = 4.43 \times 10^6$, while in optimal classical protocol $C_0^{\text{AE}} = 1.01 \times 10^{11}$ and the classical limit is $C_1^{\text{AE}} = 1.19 \times 10^7$. Obviously, our four-party protocol through asymmetric channel shows an advantage by five orders of magnitude reduction in communication complexity compared with the optimal classical version. It also breaks the classical limit.

V. DISCUSSION

Given the same N messages, the comparison of three multiparty fingerprinting protocols can be found in Table V. The

TABLE IV. Optimized parameter values of our four-party protocol through asymmetric channel. Here, $n = 10^{14}$, $c = 0.2$, $P_d = 10^{-11}$, $\delta = 0.22$, $\sqrt{\eta_1} = 0.3$, $\sqrt{\eta_2} = 0.4$, $\sqrt{\eta_3} = 0.5$, $\sqrt{\eta_4} = 0.6$, $\varepsilon = 10^{-5}$.

Round	α_1	α_2	α_3	α_4	C_2^{th}	C_3^{th}	C_4^{th}
1st	109	109	69	69	5367	5700	5332
2nd	97	77	99	78	5519	5600	5439
3rd	90	84	85	91	5699	5600	5347

prior-art multiparty quantum fingerprinting protocol [20] can be regarded as the special case of our multiparty protocol, performing steps 2–5 only once and obtaining the partial relationship of whether $\mathcal{G} = \mathcal{G}^{\text{AE}}$. In our protocol, by changing the combination of interference beams, the referee can determine all possible $\mathcal{G}$ for N messages in $O(N)$ rounds. Compared with the prior-art multiparty quantum protocol [20], our protocol can provide complete relationships among all messages with only linear increase in communication complexity.

A complete result for N parties in general is very hard, although we can have results for some special cases. When the number of messages M satisfies $M = N = 2^s$, $s \in \mathbb{Z}^+$, we can use the device based on balanced BSs, as shown in Fig. 3, and the largest possible number of rounds $\overline{T} = N - 1$. When $2^{s-1} < M < 2^s = N$, the device in Fig. 3 can also be used and we analyze this in detail in Appendix C.

To further improve the performance of the protocol, the senders can encode multiple bits of $E(x_k)$ in one $|\beta'\rangle_j$. For example, we consider the following two-bit encoding method

$$|s_k\rangle = \bigotimes_{j=1, \text{odd}}^m |\beta'\rangle_j, \quad (15)$$

$$|\beta'\rangle_j = \left| i^{E(x_k)_j \oplus E(x_k)_{j+1}} (-1)^{E(x_k)_j} \frac{\alpha_k}{\sqrt{m/2}} \right\rangle_j,$$

where $i^2 = -1$ and $E(x_k)_j \oplus E(x_k)_{j+1}$ means $E(x_k)_j + E(x_k)_{j+1} \bmod 2$.

For our one-bit encoding two-party protocol, the upper bound of the communication complexity $\overline{Q^R} = 5.52 \times 10^5$, while it is 3.91×10^5 in two-bit encoding two-party protocol as shown in Table III. Not only $\overline{Q^R}$ in two-bit encoding protocol is smaller, but also the number of fingerprinting states sent by the senders is half that of the one-bit encoding method, saving the communication time. If the senders encode more bits in each coherent state, then it saves more communication time, but the technical difficulty in preparing the fingerprint state $|s_k\rangle$ may also increase. So in this work we only analyze one-bit and two-bit encoding method.

TABLE V. The comparison of three multiparty fingerprinting protocols. The symbol $\checkmark$ indicates that the corresponding relationship state can be determined. Q stands for communication complexity per sender per run.

Protocol	$\mathcal{G}^{\text{AE}}$	$\mathcal{G}^{\text{EE}}$	$\mathcal{G}$	Q
Multiparty classical protocol [20,21]	$\checkmark$			$O(\sqrt{n})$
Original multiparty quantum protocol [20]	$\checkmark$			$O(\mu \log_2 n)$
Our multiparty protocol	$\checkmark$	$\checkmark$	$\checkmark$	$O(\mu \log_2 n)$

We analyze our protocol based on the ideal BSs with interference visibility $\nu = 1$. The interference visibility of the realistic BS is not equal to 1, for example, $\nu = 0.99$ in Ref. [16]. We can extend our method with ideal BSs to that with realistic BSs. The imperfection of BSs only affects $\mathcal{P}_k^r$ as analyzed in the Appendix B. As shown in the last line of Table III, when taking this imperfection into consideration, $\overline{Q^R} = 5.67 \times 10^5$ increases a little, but still satisfies $\overline{Q^R} < C_1^{\text{AE}} < C_0^{\text{AE}}$, keeping the advantages of our protocol in communication complexity.

VI. CONCLUSION

We propose the relationship state $\mathcal{G}$ to describe all possible relationships among N messages. Based on this, we provide a general protocol of quantum fingerprinting network. We take

the four-party quantum fingerprinting protocol as an example to analyze in detail how to determine $\mathcal{G}$ by changing the combination of interference beams. Compared with the optimal classical version, our protocol reduces the communication complexity exponentially. Different from prior-art multiparty quantum protocol, our protocol can provide complete relationships among all messages with only linear increase in communication complexity.

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APPENDIX A: OPTIMIZATION OF QUANTUM FINGERPRINTING PROTOCOL OVER ASYMMETRIC CHANNEL

1. Two-party quantum fingerprinting protocol through asymmetric channel

When $N = 2$, the referee has two detectors D_1 and D_2 . Ideally, when $x_1 = x_2$, only D_1 has clicks; when $x_1 \neq x_2$, both D_1 and D_2 have clicks. The probabilities of observing a click for the case of each mode of $E(x_1)$ and $E(x_2)$ are equal and different are

$$\mathcal{P}_2^E = [1 - e^{-\frac{1}{2m}(\sqrt{\eta_1}\alpha_1 - \sqrt{\eta_2}\alpha_2)^2}] + P_d, \quad \mathcal{P}_2^D = \delta[1 - e^{-\frac{1}{2m}(\sqrt{\eta_1}\alpha_1 + \sqrt{\eta_2}\alpha_2)^2}] + (1 - \delta)[1 - e^{-\frac{1}{2m}(\sqrt{\eta_1}\alpha_1 - \sqrt{\eta_2}\alpha_2)^2}] + P_d. \quad (\text{A1})$$

To further improve the performance of the protocol, we consider the two-bit encoding method. The following is the corresponding probability

$$\mathcal{P}_2^E = [1 - e^{-\frac{1}{m}(\sqrt{\eta_1}\alpha_1 - \sqrt{\eta_2}\alpha_2)^2}] + P_d, \\ \mathcal{P}_2^D = (1 - \delta)^2[1 - e^{-\frac{1}{m}(\sqrt{\eta_1}\alpha_1 - \sqrt{\eta_2}\alpha_2)^2}] + 2\delta(1 - \delta)[1 - e^{-\frac{1}{m}((\sqrt{\eta_1}\alpha_1)^2 + (\sqrt{\eta_2}\alpha_2)^2)}] + \delta^2[1 - e^{-\frac{1}{m}(\sqrt{\eta_1}\alpha_1 + \sqrt{\eta_2}\alpha_2)^2}] + P_d. \quad (\text{A2})$$

2. Four-party quantum fingerprinting protocol through asymmetric channel

Similar to Table I, the referee can determine $\mathcal{G}$ for four messages based on three detectors $\{D_k\}$, $k = 2, 3, 4$. The corresponding probability in round t is

$$\mathcal{P}_2^{E,t} = [1 - e^{-\frac{1}{2m}(-\sqrt{\eta_p}\alpha_p^t + \sqrt{\eta_q}\alpha_q^t)^2}] + P_d, \quad \mathcal{P}_2^{D,t} = \delta[1 - e^{-\frac{1}{2m}(\sqrt{\eta_p}\alpha_p^t + \sqrt{\eta_q}\alpha_q^t)^2}] + (1 - \delta)[1 - e^{-\frac{1}{2m}(-\sqrt{\eta_p}\alpha_p^t + \sqrt{\eta_q}\alpha_q^t)^2}] + P_d, \\ \mathcal{P}_3^{E,t} = [1 - e^{-\frac{1}{4m}(-\sqrt{\eta_p}\alpha_p^t - \sqrt{\eta_q}\alpha_q^t + \sqrt{\eta_r}\alpha_r^t + \sqrt{\eta_s}\alpha_s^t)^2}] + P_d, \\ \mathcal{P}_3^{D,t} = \delta[1 - e^{-\frac{1}{4m}(x^t)^2}] + (1 - \delta)[1 - e^{-\frac{1}{4m}(-\sqrt{\eta_p}\alpha_p^t - \sqrt{\eta_q}\alpha_q^t + \sqrt{\eta_r}\alpha_r^t + \sqrt{\eta_s}\alpha_s^t)^2}] + P_d, \\ \mathcal{P}_4^{E,t} = [1 - e^{-\frac{1}{2m}(-\sqrt{\eta_r}\alpha_r^t + \sqrt{\eta_s}\alpha_s^t)^2}] + P_d, \quad \mathcal{P}_4^{D,t} = \delta[1 - e^{-\frac{1}{2m}(\sqrt{\eta_r}\alpha_r^t + \sqrt{\eta_s}\alpha_s^t)^2}] + (1 - \delta)[1 - e^{-\frac{1}{2m}(-\sqrt{\eta_r}\alpha_r^t + \sqrt{\eta_s}\alpha_s^t)^2}] + P_d, \quad (\text{A3})$$

where $t = 1, 2, 3$ and α_k^t represents the total amplitude of the fingerprint states sent by the k th sender in round t of steps 2–5; $(p, q)(r, s) = (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)$ when $t = 1, 2, 3$, respectively, and $x^t = \min\{|\sqrt{\eta_p}\alpha_p^t - \sqrt{\eta_q}\alpha_q^t + \sqrt{\eta_r}\alpha_r^t + \sqrt{\eta_s}\alpha_s^t|, |-\sqrt{\eta_p}\alpha_p^t + \sqrt{\eta_q}\alpha_q^t + \sqrt{\eta_r}\alpha_r^t + \sqrt{\eta_s}\alpha_s^t|, |-\sqrt{\eta_p}\alpha_p^t - \sqrt{\eta_q}\alpha_q^t - \sqrt{\eta_r}\alpha_r^t + \sqrt{\eta_s}\alpha_s^t|, |-\sqrt{\eta_p}\alpha_p^t - \sqrt{\eta_q}\alpha_q^t - \sqrt{\eta_r}\alpha_r^t - \sqrt{\eta_s}\alpha_s^t|\}$. This means that α_k^t and C_k^{th} in each run can be different with the purpose of minimizing $\overline{Q^R}$ under the condition that $P_e \leq \varepsilon$.

APPENDIX B: THE IMPACT OF INTERFERENCE VISIBILITY

We can make some modifications to the probability of observing a click on detectors when considering the realistic BSs. Take the two-party quantum fingerprinting as an example. The corresponding probability is

$$\mathcal{P}_2^E = \nu[1 - e^{-\frac{1}{2m}(\sqrt{\eta_1}\alpha_1 - \sqrt{\eta_2}\alpha_2)^2}] + (1 - \nu)[1 - e^{-\frac{1}{2m}(\sqrt{\eta_1}\alpha_1 + \sqrt{\eta_2}\alpha_2)^2}] + P_d, \\ \mathcal{P}_2^D = \delta[\nu(1 - e^{-\frac{1}{2m}(\sqrt{\eta_1}\alpha_1 + \sqrt{\eta_2}\alpha_2)^2}) + (1 - \nu)(1 - e^{-\frac{1}{2m}(\sqrt{\eta_1}\alpha_1 - \sqrt{\eta_2}\alpha_2)^2})] + (1 - \delta)[\nu(1 - e^{-\frac{1}{2m}(\sqrt{\eta_1}\alpha_1 - \sqrt{\eta_2}\alpha_2)^2}) \\ + (1 - \nu)(1 - e^{-\frac{1}{2m}(\sqrt{\eta_1}\alpha_1 + \sqrt{\eta_2}\alpha_2)^2})] + P_d. \quad (\text{B1})$$

Obviously, when $\nu = 1$, the expressions above becomes the probabilities in the ideal case, as shown in Eq. (A1).

As shown in Table III, when taking this imperfection into consideration, $\overline{Q^R}$ increases a little, but still satisfies $\overline{Q^R} < C_1^{\text{AE}} < C_0^{\text{AE}}$, keeping the advantages in communication complexity of our protocol.

The analysis in the multiparty quantum fingerprinting protocol is similar.

APPENDIX C: DETERMINE $\mathcal{G}$ FOR M MESSAGES WITH THE DEVICE OF N -PARTY QUANTUM FINGERPRINTING PROTOCOL

The device composed of balanced BSs, which can be used for N -party quantum fingerprinting, $N = 2^s$, $s \in \mathbb{Z}^+$, can also be used to determine $\mathcal{G}$ for M messages ($M < N$). We show how to determine $\mathcal{G}$ for three messages using the four-party device in Fig. 4.

For example, when there are three messages for comparison, the sender S_1 prepares two fingerprint states $|s_1\rangle$, then sends them to the remote referee. The referee combines $|s_1\rangle$ with $|s_2\rangle$ and $|s_1\rangle$ with $|s_3\rangle$ for the interference measurement. He can determine $\mathcal{G}$ of three messages according to $R_{2,3,4}^{(1)}$, as shown in Table VI.

The referee needs to change combination of interference beams to determine $\mathcal{G}$ in three-party quantum fingerprinting

TABLE VI. Determine $\mathcal{G}$ for three messages $\{x_1, x_2, x_3\}$ by detectors D_2, D_3 , and D_4 in Fig. 4.

Relationship state	$R_{2,3,4}^{(1)}$
$\{\{x_1, x_2, x_3\}\}$	000
$\{\{x_1, x_2\}, \{x_3\}\}$	011
$\{\{x_1, x_3\}, \{x_2\}\}$	110
$\{\{x_2, x_3\}, \{x_1\}\}$	101
$\{\{x_1\}, \{x_2\}, \{x_3\}\}$	111

based on extendable design [20], which can be completed for three rounds at most. Our method above can determine $\mathcal{G}$ in one round and does not require unbalanced BSs, which is very beneficial for experiments.

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