Efficient estimation of multipartite quantum coherence

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Quantification of coherence lies at the heart of quantum information processing and fundamental physics. Exact evaluation of coherence measures generally needs a full reconstruction of the density matrix, which becomes intractable for large-scale multipartite systems. Here, we propose a systematic theoretical approach to efficiently estimating lower and upper bounds of coherence in multipartite states. Under the stabilizer formalism, the lower bound is determined by the spectrum estimation method with a small number of measurements, and the upper bound is determined by a single measurement. We verify our theory with a four-qubit optical quantum system. We experimentally implement various multiqubit entangled states, including the Greenberger-Horne-Zeilinger state, the cluster state, and the W state, and show how their coherences are efficiently inferred from measuring a few observables.

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I. INTRODUCTION

Quantum coherence, being one of the defining features of quantum mechanics, underlies the fundamental phenomena of quantum interference and plays a significant role in physics and quantum information processing (QIP), such as quantum cryptography [1–3], quantum metrology [4,5], nanoscale thermodynamics [6–8], and energy transport in a biological system [9]. Based on the general framework of quantum resource theories [10–12], a systematic framework of coherence has been introduced [13,14], based on which various coherence measures have been defined [14–26]. Meanwhile, the framework of coherence has been extended from a single party to the multipartite scenario with several applications, such as quantum state merging [27], coherence localization [28], and incoherence teleportation [29]. Studies of the interconversion between coherence and other multipartite nonclassical correlations, such as entanglement [15,30,31], discord [32], and nonlocality [33], also highlight the fundamental role of quantum coherence. With the rapid development of quantum hardware in realizing large-scale multipartite systems, the ability to efficiently quantify the coherence would thus offer an operationally meaningful benchmarking tool and benefit our understanding of QIP tasks.

Several experiments have been reported regarding the efficient detection of robustness of coherence [34,35]. However, the experimental detection of a general coherence measure, such as the relative entropy of coherence [14], is still missing. Theoretical proposals to estimate general multipartite

coherence without costly state tomography have also been proposed [36–38]. While the initial proposals need either copies of the prepared multipartite state [36] or complicated postprocessing [37], the spectrum estimation method was recently proposed [38], which only requires local measurements and easy-to-compute postprocessing. Nevertheless, the performance of the spectrum estimation method highly depends on the choice of the measurements, and how it works for a general multipartite state still needs further study. Moreover, existing works generally focus on the lower bound of coherence. For a given quantum state, the maximal entanglement or discord it can generate is upper bounded by its coherence [15,32], which makes detecting the upper bound of coherence important as it can indicate whether the given quantum state can generate sufficient resources for a certain QIP task.

In this paper, we theoretically address these issues by proposing two methods that can respectively detect the lower and upper bound of coherence for all multiqubit stabilizer states. The lower bound detection is based on the spectrum estimation method [38] and the stabilizer theory [39,40], which only requires a few local observable measurements for stabilizer states. The upper bound detection is based on the monogamy of coherence with a single local measurement. Experimentally, we prepare five stabilizer states of up to four qubits and demonstrate how a small number of measurements could enable us to infer multipartite coherence.

II. THEORY

A. Lower bound estimation

Under the computational basis $\{|i\rangle : i \in \{0, 1\}^{\otimes n}\}$ of an n -qubit state, we consider the relative entropy of coherence [14]

$$C_{\text{RE}}(\rho) = S_{\text{VN}}(\rho_d) - S_{\text{VN}}(\rho), \quad (1)$$

with $S_{\text{VN}} = -\text{tr}[\rho \log_2 \rho]$ being the von Neumann entropy and $\rho_d = \sum_i \langle i | \rho | i \rangle |i\rangle \langle i|$ being the diagonal part of ρ . The

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relative entropy of coherence characterizes the asymptotic distillable coherence under different types of incoherent operations [41,42], quantifies the genuine randomness that can be extracted from measuring the quantum state in the computational basis [43–45], captures the deviation from thermodynamic equilibrium [46], etc. We thus focus on the estimation, in particular, the lower and upper bounds, of the relative entropy of coherence for general multipartite states.

The lower bound $l^c(\rho)$ of the coherence $C_{\text{RE}}(\rho)$ can be obtained by spectrum estimation and the majorization theory [47] as

$$C_{\text{RE}}(\rho) \geq l^c(\rho) = S_{\text{VN}}(\mathbf{d}) - S_{\text{VN}}[\mathbf{d} \vee (\wedge_{\mathbf{p} \in X} \mathbf{p})], \quad (2)$$

where $\mathbf{d} = (d_1, \dots, d_{2^n})$ are the diagonal elements of ρ , $\mathbf{p} = (p_1, \dots, p_{2^n})$ is the estimated probability distribution of the measurement on a certain entangled basis $\{|\psi_k\rangle\}_{k=1}^{2^n}$, $\vee$ is the majorization join, and $\wedge_{\mathbf{p} \in X} \mathbf{p}$ is the majorization meet of all probability distributions in X [38]. Here, the majorization join and meet are defined based on majorization. Specifically, given two probability distributions $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ with $a_1 \geq a_2 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq \dots \geq b_n$, $\mathbf{a}$ is majorized by $\mathbf{b}$ (written as $\mathbf{a} < \mathbf{b}$) if it satisfies $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for all $k = 1, 2, \dots, n$. A probability distribution $\mathbf{c}$ is called the majorization join (meet) of $\mathbf{a}$ and $\mathbf{b}$ if it satisfies (i) $\mathbf{c} > \mathbf{a}$, $\mathbf{b} < \mathbf{c}$ and (ii) $\mathbf{c} < \tilde{\mathbf{c}}$ ($\mathbf{c} > \tilde{\mathbf{c}}$) for any $\tilde{\mathbf{c}}$ that satisfies $\mathbf{a}, \mathbf{b} < \tilde{\mathbf{c}}$ ($\mathbf{a}, \mathbf{b} > \tilde{\mathbf{c}}$) [47]. Here, we consider that $\mathbf{p}$ is selected from the set X , which satisfies $X = \{\mathbf{p} | A\mathbf{p} \geq \boldsymbol{\alpha}, B\mathbf{p} = \boldsymbol{\beta}\}$. A and B are matrices and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectors. “ $\geq$ ” represents componentwise comparison.

To calculate $l^c(\rho)$ via Eq. (2), it is crucial to set constraints $A\mathbf{p} \geq \boldsymbol{\alpha}$ and $B\mathbf{p} = \boldsymbol{\beta}$ with experimentally collected data and then find the “largest” distribution $\mathbf{p}$ majorized by all probability distributions in X , i.e., $\wedge_{\mathbf{p} \in X} \mathbf{p}$. According to the Hermiticity of density matrix ρ , we can set A as a 2^n -dimensional identity matrix and $\boldsymbol{\alpha} = \mathbf{0}$ for $A\mathbf{p} \geq \boldsymbol{\alpha}$, by which $p_k \geq 0$ is guaranteed. In the following section, we introduce the procedure to construct constraint $B\mathbf{p} = \boldsymbol{\beta}$ via stabilizer formalism.

B. Constructing constraint via stabilizer formalism

An observable S_i stabilizes an n -qubit state $|\psi\rangle$ if $|\psi\rangle$ is an eigenstate with eigenvalue $+1$ of S_i , i.e., $S_i|\psi\rangle = |\psi\rangle$. The set $\mathcal{S}$ of operators S_i is the stabilizer of $|\psi\rangle$, and $|\psi\rangle$ is a so-called stabilizer state [40,48]. For an n -qubit state, there are n stabilizing operators $\{S_1, \dots, S_n\}$ that can uniquely determine $|\psi\rangle$. Here, $S_1, \dots, S_n$ are the generators of the set $\mathcal{S}$, and we denote $\mathcal{S} = \langle S_1, \dots, S_n \rangle$. Note that $|\psi\rangle$ is stabilized not only by $\{S_1, \dots, S_n\}$, but also by their products. Thus there could be in total 2^n stabilizer operators in $\mathcal{S}$.

Given an n -qubit stabilizer state $|\psi_1\rangle$ associated with stabilizer $\mathcal{S}$, there exists an orthonormal basis $\{|\psi_k\rangle\}_{k=1}^{2^n}$ including $|\psi_1\rangle = |\psi\rangle$, where $|\psi_k\rangle$ is uniquely specified by $\mathcal{S}$ but with different eigenvalues, i.e., $S_i|\psi_k\rangle = a_{ik}|\psi_k\rangle$ with eigenvalues $a_{ik} = \pm 1$. For example, the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ can be specified by $\mathcal{S} = \langle X^{(1)}X^{(2)}, Z^{(1)}Z^{(2)} \rangle$. Hereinafter, $X^{(j)}$, $Y^{(j)}$, and $Z^{(j)}$ denote the Pauli matrices σ_x , σ_y , and σ_z acting on the j th qubit. In the following, qubit index j may be omitted if there is no confusion. The four Bell states $|\Phi^\pm\rangle$ and $|\Psi^\pm\rangle$ are specified by $\mathcal{S} = \langle XX, ZZ \rangle$ as well, with

eigenvalues of $(+1, +1)$, $(-1, +1)$, $(+1, -1)$, and $(-1, -1)$, respectively. The basis $\{|\psi_k\rangle\}_{k=1}^{2^n}$ associated with the same stabilizer $\mathcal{S}$ is also called the graph-diagonal basis [49,50] as a stabilizer state is equivalent to a graph state under local Clifford operations [51]. In the graph-diagonal basis, the stabilizing operator can be written as $S_i = \sum_k a_{ik}|\psi_k\rangle\langle\psi_k|$, and its expected value on a given quantum state ρ is $\langle S_i \rangle = \text{tr}(S_i\rho) = \sum_k p_k a_{ik}$, where the parameters $p_k = \langle\psi_k|\rho|\psi_k\rangle$ form a probability distribution $\mathbf{p} = (p_1, \dots, p_{2^n})$ with $p_k \geq 0$ and $\sum_k p_k = 1$.

The expected values of stabilizer $\mathcal{S}$ on ρ lead to 2^n equations and can be represented in matrix form as

$$\underbrace{\begin{bmatrix} a_{11} & \dots & a_{12^n} \\ \vdots & \ddots & \vdots \\ a_{2^n 1} & \dots & a_{2^n 2^n} \end{bmatrix}}_B \cdot \underbrace{\begin{bmatrix} p_1 \\ \vdots \\ p_{2^n} \end{bmatrix}}_{\mathbf{p}} = \underbrace{\begin{bmatrix} \langle S_1 \rangle \\ \vdots \\ \langle S_{2^n} \rangle \end{bmatrix}}_{\boldsymbol{\beta}}, \quad (3)$$

from which we can construct the constraint $B\mathbf{p} = \boldsymbol{\beta}$. However, in practice, there do not always exist solutions of $\wedge_{\mathbf{p} \in X} \mathbf{p}$ in Eq. (2) with constraint equation (3) (as reflected by our experimental results). The experimentally generated state always has a distance to the target state due to the inevitable imperfections, which might lead to no solution of $\wedge_{\mathbf{p} \in X} \mathbf{p}$ with inputs of $\{\langle S_i \rangle\}$. On the other hand, experimentally obtained $\langle S_i \rangle$ is always associated with statistical errors. We address these issues by introducing the experimental standard deviation σ_i of $\langle S_i \rangle$ and relax the constraint equation (3) to an inequality form,

$$\underbrace{\begin{bmatrix} \langle S_1 \rangle - w\sigma_1 \\ \vdots \\ \langle S_{2^n} \rangle - w\sigma_{2^n} \end{bmatrix}}_{\boldsymbol{\beta}_-} \leq B \cdot \mathbf{p} \leq \underbrace{\begin{bmatrix} \langle S_1 \rangle + w\sigma_1 \\ \vdots \\ \langle S_{2^n} \rangle + w\sigma_{2^n} \end{bmatrix}}_{\boldsymbol{\beta}_+}, \quad (4)$$

where $w\sigma_i$ with $w \geq 0$ is the deviation from the mean value $\langle S_i \rangle$ represented in σ_i . To this end, an experimentally accessible constraint is formulated as $\boldsymbol{\beta}_- \leq B\mathbf{p} \leq \boldsymbol{\beta}_+$. In practice, instead of measuring all the stabilizers, which is impractical for a large quantum state, we can select a small subset of stabilizers so that the number of measurements does not scale exponentially with the number of qubits. Note that $\langle \mathbb{I}^n \rangle = 1$ must be set in Eq. (4) to ensure $\sum_k p_k = 1$. If we apply the scheme on the graph-diagonal states $\rho = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$, with $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{2^n})$ being the spectrum of ρ , we have $\mathbf{p} = \boldsymbol{\lambda}$. Thus $\mathbf{d} \vee (\wedge_{\mathbf{p} \in X} \mathbf{p}) = \boldsymbol{\lambda}$ implies $l^c(\rho) = C_{\text{RE}}(\rho)$, which indicates that the estimated lower bound of coherence is tight for graph-diagonal states.

We emphasize that relaxing the constraint to $\boldsymbol{\beta}_- \leq B\mathbf{p} \leq \boldsymbol{\beta}_+$ does not increase the risk of overestimation of $l^c(\rho)$. Suppose that X_1 and X_2 are two feasible sets of probability distributions and satisfy $X_1 \subseteq X_2$. X_1 and X_2 are restricted to $\mathbf{d} < \wedge_{\mathbf{p} \in X_1} \mathbf{p}$ and $\mathbf{d} < \wedge_{\mathbf{p} \in X_2} \mathbf{p}$; otherwise the result of Eq. (2) is 0. According to the definition of the majorization meet, $\wedge_{\mathbf{p} \in X} \mathbf{p}$ is the “largest” distribution majorized by all probability distributions in X . Therefore $\wedge_{\mathbf{p} \in X} \mathbf{p}$ becomes “smaller” when we enlarge the range of X , i.e., $\wedge_{\mathbf{p} \in X_1} \mathbf{p} > \wedge_{\mathbf{p} \in X_2} \mathbf{p}$ for $X_1 \subseteq X_2$. According to the strict Schur concavity of the Shannon entropy S , we obtain $S(\wedge_{\mathbf{p} \in X_1} \mathbf{p}) < S(\wedge_{\mathbf{p} \in X_2} \mathbf{p})$, which implies

$S(\mathbf{d}) - S(\wedge_{p \in X_1} \mathbf{p}) > S(\mathbf{d}) - S(\wedge_{p \in X_2} \mathbf{p})$. Thus we conclude that $l^c(\rho)$ decreases with enlarging range of X .

Similar constraints can also be formulated for multiqubit states that do not obviously fit the stabilizer formalism. The stabilizing operators of such kind of n -qubit states $|\psi\rangle$ could be determined by finding its unitary dynamics U^ψ acting on $|0\rangle^{\otimes n}$, i.e., $|\psi\rangle = U^\psi |0\rangle^{\otimes n}$ [48]. As $|0\rangle^{\otimes n}$ is stabilized by $S_i^{(0)\otimes n} = Z^{(i)}, \forall i \in \{1, 2, \dots, n\}$, the stabilizing operator of $|\psi\rangle$ is $S_i^\psi = U^\psi Z^{(i)} U^{\psi^\dagger}$.

C. Upper bound estimation

Theorem 1. Let $\mathcal{M}(\mathbf{d})$ be a set of states with the same diagonal part $\mathbf{d} = (d_1, \dots, d_{2^n})$; then $|\psi_d\rangle = \sum_{i=1}^{2^n} \sqrt{d_i} |i\rangle$ is the maximally coherent state in $\mathcal{M}(\mathbf{d})$.

Proof. It is equivalent to prove that $|\psi_d\rangle = \sum_{i=1}^{2^n} \sqrt{d_i} |i\rangle$ can be transformed into any $\rho \in \mathcal{M}(\mathbf{d})$ via incoherent operation.

We first consider the case in which there are no nonzero elements in $\mathbf{d}$, i.e., $d_i \neq 0$ for all i . Let $\{\lambda_\alpha, |\varphi_\alpha\rangle\}$ be the spectral decomposition of ρ , i.e.,

$$\rho = \sum_{\alpha} \lambda_{\alpha} |\varphi_{\alpha}\rangle \langle \varphi_{\alpha}| = \sum_{\alpha} \lambda_{\alpha} \sum_{i,j} c_{\alpha i} c_{\alpha j}^* |i\rangle \langle j|, \quad (5)$$

where $|\varphi_{\alpha}\rangle = \sum_{i=1}^{2^n} c_{\alpha i} |i\rangle$ with $\sum_{i=1}^{2^n} |c_{\alpha i}|^2 = 1$ for all α . Thus ρ form a set $\mathcal{M}(\mathbf{d})$ with $d_i = \sum_{\alpha} \lambda_{\alpha} |c_{\alpha i}|^2$. We define a completely positive and trace-preserving (CPTP) map $\Lambda_1(\cdot) = \sum_{\alpha} K_{\alpha} \cdot K_{\alpha}^{\dagger}$ associated with Kraus operators $K_{\alpha} = \sum_i \frac{\sqrt{\lambda_{\alpha} c_{\alpha i}}}{\sqrt{d_i}} |i\rangle \langle i|$, which transforms $|\psi_d\rangle \langle \psi_d|$ into ρ by

$$\begin{aligned} \Lambda_1(|\psi_d\rangle \langle \psi_d|) &= \sum_{\alpha} K_{\alpha} |\psi_d\rangle \langle \psi_d| K_{\alpha}^{\dagger} \\ &= \sum_{\alpha} \sum_{i,j,k,l} \frac{\lambda_{\alpha} c_{\alpha i} c_{\alpha j}^*}{\sqrt{d_i d_j}} \sqrt{d_k d_l} \delta_{ik} \delta_{jl} |i\rangle \langle j| \\ &= \sum_{\alpha} \lambda_{\alpha} \sum_{i,j} c_{\alpha i} c_{\alpha j}^* |i\rangle \langle j| \\ &= \rho. \end{aligned} \quad (6)$$

According to the definition of strictly incoherent operation [52,53], K_{α} is a strictly incoherent operator, and $\Lambda_1(\cdot)$ is a strictly incoherent operation.

For the case in which there are m ($m < 2^n$) nonzero elements in $\mathbf{d}$, we first transform $|\psi_d\rangle \langle \psi_d|$ into a block-diagonal matrix via a permutation matrix M ; that is, $M |\psi_d\rangle \langle \psi_d| M^{-1} = |\psi_m\rangle \langle \psi_m| \oplus \mathbf{0}_{2^n-m}$. $|\psi_m\rangle = \sum_{i'=1}^m \sqrt{d_{i'}} |i'\rangle$ is the vector in m -dimensional Hilbert space spanned by $\{|i'\rangle\}$, where i' is the rearranged index of m nonzero elements. $\mathbf{0}$ is a $(2^n - m) \times (2^n - m)$ all-0 matrix.

Similarly, we define the CPTP map $\Lambda_2(\cdot) = \sum_{\beta} K_{\beta} \cdot K_{\beta}^{\dagger}$ associated with Kraus operators $K_{\beta} = \sum_{i'} \frac{\sqrt{\lambda_{\beta} c_{\beta i'}}}{\sqrt{d_{i'}}} |i'\rangle \langle i'|$ acting on m -dimensional Hilbert space. Thus we can derive that

$$M^{-1} \Lambda_2(M |\psi_d\rangle \langle \psi_d| M^{-1}) M = \rho. \quad (7)$$

Note that any permutation matrix is strictly incoherent unitary [54]. For any two strictly incoherent operations Λ_{α} and Λ_{β} , the operation $\Lambda = \Lambda_{\alpha} \circ \Lambda_{\beta}$ is also a strictly incoherent operation [55]. Thus $|\psi_d\rangle = \sum_{i=1}^{2^n} \sqrt{d_i} |i\rangle$ can be transformed into any $\rho \in \mathcal{M}(\mathbf{d})$ via strictly incoherent operations $M^{-1} K_{\beta} M$. ■

To find the upper bound of the coherence of an n -qubit state ρ , we measure it in the computational basis $\{|i\rangle\}$, which yields a distribution $\mathbf{d} = (d_1, \dots, d_{2^n})$. Based on the monogamy of the relative entropy of coherence [14], the coherence of ρ is upper bounded by the coherence $u_c(\rho)$ of state $|\psi_d\rangle = \sum_{i=1}^{2^n} \sqrt{d_i} |i\rangle$, i.e.,

$$C_{\text{RE}}(\rho) \leq u_c(\rho) = C_{\text{RE}}(|\psi_d\rangle \langle \psi_d|). \quad (8)$$

Note that this upper bound is tight for pure states and can be used for various coherence measures.

III. EXPERIMENTAL DEMONSTRATION

Next, we demonstrate the capability of our scheme by estimating the coherence of several typical multiqubit states. We firstly generate photon pairs by a periodically poled potassium

FIG. 1. Schematic drawing of the experimental setup. (a) The setup to generate a polarization-entangled photon pair. (b)–(d) The setups to generate $|\text{GHZ}_4\rangle$, $|\text{C}_4\rangle$, and $|\text{W}_4\rangle$, respectively. (e) Symbols used in (a)–(d): PPKTP, periodically poled potassium titanyl phosphate; DM, dichroic mirror; HWP, half-wave plate; NBF, narrowband filter; QWP, quarter-wave plate; PBS, polarization beam splitter; BD, beam displacer; SPD, single-photon detector. More experimental details can be found in Appendix B.

titanyl phosphate (PPKTP) crystal in a Sagnac interferometer [56], which is bidirectionally pumped by an ultraviolet (UV) laser diode with central wavelength at 405 nm [as shown in Fig. 1(a)]. The two photons are entangled in the polarization degree of freedom (DOF), i.e., $|\Psi^+\rangle = (|HV\rangle + |VH\rangle)/\sqrt{2}$ with H being the horizontal polarization and V being the vertical polarization. We extend the photon to its path DOF using a beam displacer (BD), which transmits vertical polarization and deviates horizontal polarization, i.e., $|H\rangle \rightarrow |H\rangle|h\rangle$ and $|V\rangle \rightarrow |V\rangle|v\rangle$ with h and v being the path DOF [57–59]. The qubit is encoded in the polarization DOF as $|H(V)\rangle \rightarrow |0(1)\rangle$ and in the path DOF as $|h(v)\rangle \rightarrow |0(1)\rangle$. In our experiment, we denote the qubits encoded in the polarization DOF as 1 and 3, while the qubits encoded in the path DOF are denoted as 2 and 4. As shown in Figs. 1(b)–1(d), with different experimental setup configurations, we can generate various four-qubit states, including $|\text{GHZ}_4\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$ [a Greenberger-Horne-Zeilinger (GHZ) state], $|\text{C}_4\rangle = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)/2$, and $|\text{W}_4\rangle = (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)/2$. Moreover, $|\text{GHZ}_3\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ and $|\text{W}_3\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$ can be obtained by extending one photon to polarization and path DOFs while keeping the other in the polarization DOF (see Appendix B for more details).

For each experimentally generated state ρ_{expt}^ψ , we measure the expected values of its stabilizers $\langle S_i \rangle$ associated with the corresponding statistical errors σ_i . We refer the reader to Appendix A for details of stabilizing operators of $|\psi\rangle$ and the corresponding graph-diagonal basis. The measured expected values of stabilizing operators are presented in Appendix B. With the measured $\langle S_i \rangle$ and σ_i , we construct the constraint Eq. (4). Thus we can calculate $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ via solving Eq. (2), where ω represents the setting of deviations and m is the number of $\langle S_i \rangle$ with which we construct the constraint. We set ω as non-negative integers from 0 to 3, and m from 1 to $2^n - 1$ as $\langle \mathbb{I}^n \rangle = 1$ must be set in the constraint. In our calculation, we treat the case of no solution as $I_{\omega,m}^c(\rho_{\text{expt}}^\psi) = 0$ and the case of $I_{\omega,m}^c(\rho_{\text{expt}}^\psi) > 0$ as a valid solution. For a fixed m , there are $\binom{2^n-1}{m}$ subsets of $\{S_i^\psi\}$, and the maximum $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ is shown in Fig. 2. We observe that valid $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ cannot be obtained with $m \leq 3$ stabilizers for $\rho_{\text{expt}}^{\text{GHZ}_4}$, $m \leq 1$ stabilizers for $\rho_{\text{expt}}^{\text{W}_3}$, and $m \leq 2$ stabilizers for the other three states. With increase in m , the maximum $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ increases accordingly. When m gets close to $2^n - 1$, there exist situations in which we cannot obtain $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ for smaller ω [the values are 0 in the lower right corner of Figs. 2(b)–2(e)]. As previously mentioned, this is caused by experimental imperfections, such as slight misalignment of optical elements during data collection, which introduces small variation in prepared ρ_{expt}^ψ . The issue is improved by extending the range of $\langle S_i^\psi \rangle$, i.e., increasing ω . As shown in Figs. 2(b)–2(e), most $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ have valid solutions for large m by setting $\omega = 3$. Moreover, the accuracy of estimated $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ increases along with m as well. We investigate this by calculating the normalized distance between $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ and $C_{\text{RE}}(\rho_{\text{expt}}^\psi)$, i.e., $1 - I_{\omega,m}^c(\rho_{\text{expt}}^\psi)/C_{\text{RE}}(\rho_{\text{expt}}^\psi)$. $C_{\text{RE}}(\rho_{\text{expt}}^\psi)$ is calculated with

FIG. 2. (a)–(e) Experimental results of $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$. The color bar indicates the range of $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ from 0 to values of ideal $|\psi\rangle$, i.e., 1 for $|\text{GHZ}_3\rangle$ and $|\text{GHZ}_4\rangle$, 1.585 for $|\text{W}_3\rangle$, and 2 for $|\text{C}_4\rangle$ and $|\text{W}_4\rangle$.

Eq. (1) by reconstructing ρ_{expt}^ψ via quantum state tomographic technology (see Appendix B for the reconstructed ρ_{expt}^ψ). The distances between maximal $I_{\omega,m}^c(\rho_{\text{expt}}^\psi)$ and $C_{\text{RE}}(\rho_{\text{expt}}^\psi)$ are shown in Fig. 3(a), from which we observe that the distance drops down quickly with increasing m and tends to converge at $m = 5$.

As previously mentioned, the choice of selecting m stabilizers from $\{S_i^\psi\}$ is not unique except $m = 2^n - 1$. An important property is the successful probability of obtaining valid $I_{\omega,m}^c$ by randomly selecting m stabilizers. We show the percentage of valid $I_{\omega,5}^c(\rho_{\text{expt}}^\psi)$ [$I_{\omega,5}^c(\rho_{\text{expt}}^\psi) > 0$] for $m = 5$ in Fig. 3(b). By increasing ω , the probability of getting valid $I_{\omega,5}^c(\rho_{\text{expt}}^\psi)$ is enhanced, especially for $\rho_{\text{expt}}^{\text{W}_3}$. However, one cannot increase ω arbitrarily. A larger ω represents a smaller probability that we could obtain $\langle S_i \rangle$ in the range $[\langle S_i \rangle + (w-1)\sigma_i, \langle S_i \rangle + w\sigma_i]$ as well as $[\langle S_i \rangle - w\sigma_i, \langle S_i \rangle - (w-1)\sigma_i]$, which is less than 0.3% for $\omega = 3$. This is also reflected by the normalized distance of maximal $I_{\omega,5}^c(\rho_{\text{expt}}^\psi)$ shown in Fig. 3(c), which indicates that the inaccuracy of $I_{\omega,5}^c(\rho_{\text{expt}}^\psi)$ increases when we extend the range of $\langle S_i \rangle$. Also, the results in Fig. 3(c) agree with our claim that relaxing the constraint decreases the estimated value of $I^c(\rho)$. We conclude that setting $\omega = 3$ is reasonable in experiment, under which

FIG. 3. (a) The normalized distance between maximal $l_{3,m}^c(\rho_{\text{expt}}^{\psi})$ and $C_{\text{RE}}(\rho_{\text{expt}}^{\psi})$. (b) The percentage (Pct.) of valid $l_{\omega,5}^c$ obtained over all $\binom{2^n-1}{5}$ subsets. (c) The normalized distance between maximal $l_{\omega,5}^c(\rho_{\text{expt}}^{\psi})$ and $C_{\text{RE}}(\rho_{\text{expt}}^{\psi})$ of setting ω from 0 to 5. (d) The u^c (blue dash-dotted line) for $\rho_{\text{expt}}^{\text{GHZ}_3}$, $\rho_{\text{expt}}^{\text{W}_3}$, $\rho_{\text{expt}}^{\text{GHZ}_4}$, $\rho_{\text{expt}}^{\text{C}_4}$, and $\rho_{\text{expt}}^{\text{W}_4}$ are 1.117 ± 0.003 , 1.725 ± 0.004 , 1.198 ± 0.005 , 2.103 ± 0.004 , 1.725 ± 0.004 , and 2.259 ± 0.006 , respectively. The corresponding maximal $l_{\omega,m}^c$ (red dash-dotted line) are the results of $\omega = 3$ and $m = 5$. The solid circles represent $C_{\text{RE}}(\rho_{\text{expt}}^{\psi})$, and their error bars are too small compared with the marker size [see Appendix B for the values of $C_{\text{RE}}(\rho_{\text{expt}}^{\psi})$].

we observe that the probability of getting valid $l_{\omega,5}^c(\rho_{\text{expt}}^{\psi})$ is 100% for $\rho_{\text{expt}}^{\text{GHZ}_3}$, $\rho_{\text{expt}}^{\text{W}_3}$, $\rho_{\text{expt}}^{\text{C}_4}$, and $\rho_{\text{expt}}^{\text{W}_4}$ and 86% for $\rho_{\text{expt}}^{\text{GHZ}_4}$. Note that the estimated $l^c(\rho)$ of $\rho_{\text{expt}}^{\text{GHZ}_3}$, $\rho_{\text{expt}}^{\text{W}_3}$, $\rho_{\text{expt}}^{\text{C}_4}$, and $\rho_{\text{expt}}^{\text{GHZ}_4}$ is slightly more accurate than that of $\rho_{\text{expt}}^{\text{W}_4}$ as shown in Figs. 3(a) and 3(c). The main reason is that the fidelity of prepared $\rho_{\text{expt}}^{\text{W}_4}$ is slightly lower than that of the other four states as shown in Appendix B. The lower fidelity implies a larger distance between the prepared state and the target state, which indicates $\lambda \succ \mathbf{d} \vee (\wedge_{p \in X} p)$ so that $C_{\text{RE}}(\rho) = S(\mathbf{d}) - S(\lambda) > l^c(\rho) = S(\mathbf{d}) - S[\mathbf{d} \vee (\wedge_{p \in X} p)]$. However, it does not indicate that the lower fidelity always leads to the bigger gap between $C_{\text{RE}}(\rho)$ and $l^c(\rho)$. If the prepared state is still a graph-diagonal state after considering the experimental imperfections, the estimated lower bound on such state is tight as well, i.e., $l^c(\rho) = C_{\text{RE}}(\rho)$. In Appendix C, we analyze our experimental imperfections and show how they affect the prepared states.

Finally, we estimate the upper bound u^c of $\rho_{\text{expt}}^{\psi}$ by measuring the probability distribution $\mathbf{d}_{\text{expt}}^{\psi}$ on the basis of $Z^{\otimes n}$. The probability distributions $\mathbf{d}_{\text{expt}}^{\psi}$ are shown in Appendix B, by which we can calculate $u^c(\rho_{\text{expt}}^{\psi})$ according to Eq. (8). The results of $u^c(\rho_{\text{expt}}^{\psi})$ are shown with the blue dash-dotted line in Fig. 3(d). For all five states, we observe that $C_{\text{RE}}(\rho_{\text{expt}}^{\psi})$ lies within the range bounded by $l_{3,5}^c(\rho_{\text{expt}}^{\psi})$ and $u^c(\rho_{\text{expt}}^{\psi})$.

IV. CONCLUSION

To conclude, we introduce an efficient and experimentally friendly estimation method for detecting coherence of multipartite states. We demonstrate that the coherence with high accuracy as well as high successful probability can be efficiently estimated with a few measurements for various

multiqubit states. The procedure to obtain the lower bound is based on a few measurements of the stabilizing operators, similar to multipartite entanglement detection [60] and multipartite Bell inequalities [61]. This thus indicates that coherence and other resources can be inferred by the same set of measurements, which can further benefit our understanding of the connection between coherence and other resources [15,30,31,33].

There are several open follow-up problems. The scheme to detect the lower bound of multipartite coherence is efficient and tight for graph-diagonal states as well as special types of quantum states that admit efficient classical representation (such as the W state). Exploring whether our scheme will work for a more general multiqubit quantum state is an interesting future direction. Besides, one may be concerned as to whether our scheme can be generalized to high-dimensional cases. For the multiqubit stabilizer state, its stabilizing operators constructed by generalized Pauli group are generally non-Hermitian [62–64], which cannot be observed directly in experiment. Recent studies indicate that expected values of non-Hermitian operators could be measured via weak measurements [65,66]. However, a definite answer may require rather sophisticated analysis. Finally, it is worth noting that a fidelity-based method was recently proposed to detect the lower bound of multipartite coherence via the convex roof construction [67]. Thus another open follow-up question is whether our scheme can be further generalized to other coherence measures while maintaining its appealing features of efficiency and simplicity.

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APPENDIX A: STABILIZING OPERATORS AND GRAPH-DIAGONAL BASIS

1. GHZ state

For n -qubit GHZ states, the generators of $\mathcal{S}^{\text{GHZ}}$ are

$$S_1^{\text{GHZ}_n} = \prod_{i=1}^n X^{(i)},$$

$$S_i^{\text{GHZ}_n} = Z^{(i-1)} Z^{(i)} \quad \text{for } i = 2, 3, \dots, n, \quad (\text{A1})$$

and denoted as $\mathcal{S}^{\text{GHZ}_n} = \langle S_1^{\text{GHZ}_n}, S_2^{\text{GHZ}_n}, \dots, S_n^{\text{GHZ}_n} \rangle$. For $|\text{GHZ}_3\rangle$, the generators are $S_1^{\text{GHZ}_3} = XXX$, $S_2^{\text{GHZ}_3} = ZZ\mathbb{I}$, and $S_3^{\text{GHZ}_3} = \mathbb{I}ZZ$. Then, all stabilizing operators can be obtained by multiplying them with each other, i.e.,

$$\begin{aligned} S_4^{\text{GHZ}_3} &= S_1^{\text{GHZ}_3} S_2^{\text{GHZ}_3} = -YYX, \\ S_5^{\text{GHZ}_3} &= S_1^{\text{GHZ}_3} S_3^{\text{GHZ}_3} = -XY Y, \\ S_6^{\text{GHZ}_3} &= S_2^{\text{GHZ}_3} S_3^{\text{GHZ}_3} = Z\mathbb{I}Z, \\ S_7^{\text{GHZ}_3} &= S_1^{\text{GHZ}_3} S_2^{\text{GHZ}_3} S_3^{\text{GHZ}_3} = -YXY, \\ S_8^{\text{GHZ}_3} &= \mathbb{I}\mathbb{I}\mathbb{I}. \end{aligned} \quad (\text{A2})$$

The GHZ_3 -diagonal basis can be obtained from the computational basis $|k_1 k_2 k_3\rangle$ by performing a three-qubit unitary operation $U^{\text{GHZ}_3} = (\text{CNOT}_{12} \otimes \mathbb{I}_3) \cdot (\text{CNOT}_{13} \otimes \mathbb{I}_2) \cdot (H_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3)$, where H is the Hadamard operation and

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (\text{A3})$$

is the unitary matrix of the controlled-NOT (CNOT) operation. Eight GHZ_3 -diagonal bases are determined by $|\psi_i^{\text{GHZ}_3}\rangle = U^{\text{GHZ}_3} |k_1 k_2 k_3\rangle$ and shown in Table I.

The generators of $|\text{GHZ}_4\rangle$ are $S_1^{\text{GHZ}_4} = XXXX$, $S_2^{\text{GHZ}_4} = ZZ\mathbb{I}\mathbb{I}$, $S_3^{\text{GHZ}_4} = \mathbb{I}ZZ\mathbb{I}$, and $S_4^{\text{GHZ}_4} = \mathbb{I}\mathbb{I}ZZ$. Other stabilizing

TABLE I. The GHZ_3 -diagonal bases and the corresponding expected values of generators ($\langle S_1^{\text{GHZ}_3} \rangle$, $\langle S_2^{\text{GHZ}_3} \rangle$, $\langle S_3^{\text{GHZ}_3} \rangle$).

$k_1 k_2 k_3$	$ \psi_i^{\text{GHZ}_3}\rangle$	$(\langle S_1^{\text{GHZ}_3} \rangle, \langle S_2^{\text{GHZ}_3} \rangle, \langle S_3^{\text{GHZ}_3} \rangle)$
000	$(000\rangle + 111\rangle)/\sqrt{2}$	$(+1, +1, +1)$
001	$(001\rangle + 110\rangle)/\sqrt{2}$	$(+1, +1, -1)$
010	$(010\rangle + 101\rangle)/\sqrt{2}$	$(+1, -1, -1)$
011	$(011\rangle + 100\rangle)/\sqrt{2}$	$(+1, -1, +1)$
100	$(000\rangle - 111\rangle)/\sqrt{2}$	$(-1, +1, +1)$
101	$(001\rangle - 110\rangle)/\sqrt{2}$	$(-1, +1, -1)$
110	$(010\rangle - 101\rangle)/\sqrt{2}$	$(-1, -1, -1)$
111	$(011\rangle - 100\rangle)/\sqrt{2}$	$(-1, -1, +1)$

TABLE II. The GHZ_4 -diagonal bases and the corresponding expected values of generators ($\langle S_1^{\text{GHZ}_4} \rangle$, $\langle S_2^{\text{GHZ}_4} \rangle$, $\langle S_3^{\text{GHZ}_4} \rangle$, $\langle S_4^{\text{GHZ}_4} \rangle$).

$k_1 k_2 k_3 k_4$	$ \psi_i^{\text{GHZ}_4}\rangle$	$(\langle S_1^{\text{GHZ}_4} \rangle, \langle S_2^{\text{GHZ}_4} \rangle, \langle S_3^{\text{GHZ}_4} \rangle, \langle S_4^{\text{GHZ}_4} \rangle)$
0000	$(0000\rangle + 1111\rangle)/\sqrt{2}$	$(+1, +1, +1, +1)$
0001	$(0001\rangle + 1110\rangle)/\sqrt{2}$	$(+1, +1, +1, -1)$
0010	$(0010\rangle + 1101\rangle)/\sqrt{2}$	$(+1, +1, -1, -1)$
0011	$(0011\rangle + 1100\rangle)/\sqrt{2}$	$(+1, +1, -1, +1)$
0100	$(0100\rangle + 1011\rangle)/\sqrt{2}$	$(+1, -1, -1, +1)$
0101	$(0101\rangle + 1010\rangle)/\sqrt{2}$	$(+1, -1, -1, -1)$
0110	$(0110\rangle + 1001\rangle)/\sqrt{2}$	$(+1, -1, +1, -1)$
0111	$(0111\rangle + 1000\rangle)/\sqrt{2}$	$(+1, -1, +1, +1)$
1000	$(0000\rangle - 1111\rangle)/\sqrt{2}$	$(-1, +1, +1, +1)$
1001	$(0001\rangle - 1110\rangle)/\sqrt{2}$	$(-1, +1, +1, -1)$
1010	$(0010\rangle - 1101\rangle)/\sqrt{2}$	$(-1, +1, -1, -1)$
1011	$(0011\rangle - 1100\rangle)/\sqrt{2}$	$(-1, +1, -1, +1)$
1100	$(0100\rangle - 1011\rangle)/\sqrt{2}$	$(-1, -1, -1, +1)$
1101	$(0101\rangle - 1010\rangle)/\sqrt{2}$	$(-1, -1, -1, -1)$
1110	$(0110\rangle - 1001\rangle)/\sqrt{2}$	$(-1, -1, +1, -1)$
1111	$(0111\rangle - 1000\rangle)/\sqrt{2}$	$(-1, -1, +1, +1)$

operators are

$$\begin{aligned} S_5^{\text{GHZ}_4} &= -YYXX, & S_6^{\text{GHZ}_4} &= -XXYY, \\ S_7^{\text{GHZ}_4} &= -XXYY, & S_8^{\text{GHZ}_4} &= Z\mathbb{I}Z\mathbb{I}, \\ S_9^{\text{GHZ}_4} &= ZZZZ, & S_{10}^{\text{GHZ}_4} &= \mathbb{I}Z\mathbb{I}Z, \\ S_{11}^{\text{GHZ}_4} &= -YXXY, & S_{12}^{\text{GHZ}_4} &= YYY Y, \\ S_{13}^{\text{GHZ}_4} &= -YXXY, & S_{14}^{\text{GHZ}_4} &= Z\mathbb{I}\mathbb{I}Z, \\ S_{15}^{\text{GHZ}_4} &= -YXXY, & S_{16}^{\text{GHZ}_4} &= \mathbb{I}\mathbb{I}\mathbb{I}\mathbb{I}. \end{aligned} \quad (\text{A4})$$

The GHZ_4 -diagonal basis can be obtained from the computational basis $|k_1 k_2 k_3 k_4\rangle$ by performing a four-qubit unitary operation $U^{\text{GHZ}_4} = (\mathbb{I}_1 \otimes H_2 \otimes H_3 \otimes H_4) \cdot (\text{CZ}_{14}) \cdot (\text{CZ}_{13}) \cdot (\text{CZ}_{12}) \cdot (H_1 \otimes H_2 \otimes H_3 \otimes H_4)$, where

$$\text{CZ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{A5})$$

is the unitary matrix of the controlled-Z (CZ) operation. Sixteen GHZ_4 -diagonal bases are determined by $|\psi_i^{\text{GHZ}_4}\rangle = U^{\text{GHZ}_4} |k_1 k_2 k_3 k_4\rangle$ and shown in Table II.

2. Cluster state

The generators of a n -qubit cluster state $|\tilde{\mathcal{C}}_n\rangle$ are

$$\begin{aligned} S_1^{\tilde{\mathcal{C}}_n} &= X^{(1)} Z^{(2)}, \\ S_i^{\tilde{\mathcal{C}}_n} &= Z^{(i-1)} X^{(i)} Z^{(i+1)} \quad \text{for } i = 2, 3, \dots, n-1, \\ S_n^{\tilde{\mathcal{C}}_n} &= Z^{(n-1)} X^{(n)}. \end{aligned} \quad (\text{A6})$$

The generators of a four-qubit linear graph are $S_1^{\tilde{\mathcal{C}}_4} = XZ\mathbb{I}\mathbb{I}$, $S_2^{\tilde{\mathcal{C}}_4} = ZXZ\mathbb{I}$, $S_3^{\tilde{\mathcal{C}}_4} = \mathbb{I}ZXZ$, and $S_4^{\tilde{\mathcal{C}}_4} = \mathbb{I}\mathbb{I}ZX$, and the corresponding state is $|\tilde{\mathcal{C}}_4\rangle = (|+00+\rangle + |+01-\rangle + |-10+\rangle - |-11-\rangle)/2$. Note that $|\tilde{\mathcal{C}}_4\rangle$ can be transformed to the

TABLE III. The cluster-diagonal bases and the corresponding expected values of generators ($\langle S_1^{C_4} \rangle$, $\langle S_2^{C_4} \rangle$, $\langle S_3^{C_4} \rangle$, $\langle S_4^{C_4} \rangle$).

$k_1 k_2 k_3 k_4$	$ \psi_i^{C_4}\rangle$	$(\langle S_1^{C_4} \rangle, \langle S_2^{C_4} \rangle, \langle S_3^{C_4} \rangle, \langle S_4^{C_4} \rangle)$
0000	$\frac{ 0000\rangle + 0011\rangle + 1100\rangle - 1111\rangle}{2}$	(+1, +1, +1, +1)
0001	$\frac{ 0001\rangle + 0010\rangle + 1101\rangle - 1110\rangle}{2}$	(+1, +1, +1, -1)
0010	$\frac{ 0000\rangle - 0011\rangle + 1100\rangle + 1111\rangle}{2}$	(+1, +1, -1, +1)
0011	$\frac{ 0001\rangle - 0010\rangle + 1101\rangle + 1110\rangle}{2}$	(+1, +1, -1, -1)
0100	$\frac{ 0100\rangle - 0111\rangle + 1000\rangle + 1011\rangle}{2}$	(-1, +1, +1, +1)
0101	$\frac{ 0101\rangle - 0110\rangle + 1001\rangle + 1010\rangle}{2}$	(-1, +1, +1, -1)
0110	$\frac{ 0100\rangle + 0111\rangle + 1000\rangle - 1011\rangle}{2}$	(-1, +1, -1, +1)
0111	$\frac{ 0101\rangle + 0110\rangle + 1001\rangle - 1010\rangle}{2}$	(-1, +1, -1, -1)
1000	$\frac{ 0000\rangle + 0011\rangle - 1100\rangle + 1111\rangle}{2}$	(+1, -1, +1, +1)
1001	$\frac{ 0001\rangle + 0010\rangle - 1101\rangle + 1110\rangle}{2}$	(+1, -1, +1, -1)
1010	$\frac{ 0000\rangle - 0011\rangle - 1100\rangle - 1111\rangle}{2}$	(+1, -1, -1, +1)
1011	$\frac{ 0001\rangle - 0010\rangle - 1101\rangle - 1110\rangle}{2}$	(+1, -1, -1, -1)
1100	$\frac{ 0100\rangle - 0111\rangle - 1000\rangle - 1011\rangle}{2}$	(-1, -1, +1, +1)
1101	$\frac{ 0101\rangle - 0110\rangle - 1001\rangle - 1010\rangle}{2}$	(-1, -1, +1, -1)
1110	$\frac{ 0100\rangle + 0111\rangle - 1000\rangle + 1011\rangle}{2}$	(-1, -1, -1, +1)
1111	$\frac{ 0101\rangle + 0110\rangle - 1001\rangle + 1010\rangle}{2}$	(-1, -1, -1, -1)

common representation $|C_4\rangle = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)/2$ by the local unitary $H\mathbb{I}\mathbb{I}H$, i.e., $|C_4\rangle = H\mathbb{I}\mathbb{I}H|C_4\rangle$. Accordingly, the generators of $|C_4\rangle$ are transformed to $ZZ\mathbb{I}\mathbb{I}$, $XXZ\mathbb{I}$, $\mathbb{I}ZXX$, and $\mathbb{I}\mathbb{I}ZZ$. Other stabilizing operators are

$$\begin{aligned}
 S_5^{C_4} &= -YYZ\mathbb{I}, & S_6^{C_4} &= Z\mathbb{I}XX, \\
 S_7^{C_4} &= ZZZZ, & S_8^{C_4} &= XY YX, \\
 S_9^{C_4} &= XX\mathbb{I}Z, & S_{10}^{C_4} &= -\mathbb{I}ZYY, \\
 S_{11}^{C_4} &= YXYX, & S_{12}^{C_4} &= -YY\mathbb{I}Z, \\
 S_{13}^{C_4} &= -Z\mathbb{I}YY, & S_{14}^{C_4} &= XYXY, \\
 S_{15}^{C_4} &= YXXY, & S_{16}^{C_4} &= \mathbb{I}\mathbb{I}\mathbb{I}\mathbb{I}.
 \end{aligned} \quad (A7)$$

The cluster-diagonal basis can be obtained from the computational basis $|k_1 k_2 k_3 k_4\rangle$ by performing a four-qubit unitary operation $U^{C_4} = (\mathbb{I}_1 \otimes CZ_{23} \otimes \mathbb{I}_4) \cdot (\text{CNOT}_{12} \otimes \text{CNOT}_{34}) \cdot (H_1 \otimes \mathbb{I}_2 \otimes H_3 \otimes \mathbb{I}_4)$. Then, the cluster-diagonal bases are determined by $|\psi_i^{C_4}\rangle = U^{C_4}|k_1 k_2 k_3 k_4\rangle$ and shown in Table III.

TABLE IV. The W_3 -diagonal bases and the corresponding expected values of generators ($\langle S_1^{W_3} \rangle$, $\langle S_2^{W_3} \rangle$, $\langle S_3^{W_3} \rangle$).

$k_1 k_2 k_3$	$ \psi_i^{W_3}\rangle$	$(\langle S_1^{W_3} \rangle, \langle S_2^{W_3} \rangle, \langle S_3^{W_3} \rangle)$
000	$(001\rangle + 010\rangle + 100\rangle)/\sqrt{3}$	(+1, +1, +1)
001	$(000\rangle - 011\rangle + 101\rangle)/\sqrt{3}$	(+1, +1, -1)
010	$(000\rangle + 011\rangle - 110\rangle)/\sqrt{3}$	(+1, -1, +1)
011	$(- 001\rangle + 010\rangle - 111\rangle)/\sqrt{3}$	(+1, -1, -1)
100	$(000\rangle - 101\rangle + 110\rangle)/\sqrt{3}$	(-1, +1, +1)
101	$(001\rangle - 100\rangle - 111\rangle)/\sqrt{3}$	(-1, +1, -1)
110	$(- 010\rangle + 100\rangle - 111\rangle)/\sqrt{3}$	(-1, -1, +1)
111	$(- 011\rangle - 101\rangle - 110\rangle)/\sqrt{3}$	(-1, -1, -1)

TABLE V. The W_4 -diagonal bases and the corresponding expected values of generators ($\langle S_1^{W_4} \rangle$, $\langle S_2^{W_4} \rangle$, $\langle S_3^{W_4} \rangle$, $\langle S_4^{W_4} \rangle$).

$k_1 k_2 k_3 k_4$	$ \psi_i^{W_4}\rangle$	$(\langle S_1^{W_4} \rangle, \langle S_2^{W_4} \rangle, \langle S_3^{W_4} \rangle, \langle S_4^{W_4} \rangle)$
0000	$\frac{ 0001\rangle + 0010\rangle + 0100\rangle + 1000\rangle}{2}$	(+1, +1, +1, +1)
0001	$\frac{ 0000\rangle + 0011\rangle - 0101\rangle + 1001\rangle}{2}$	(-1, +1, +1, +1)
0010	$\frac{ 0000\rangle - 0011\rangle - 0110\rangle + 1010\rangle}{2}$	(+1, -1, +1, +1)
0011	$\frac{ 0001\rangle - 0010\rangle - 0111\rangle + 1011\rangle}{2}$	(-1, -1, +1, +1)
0100	$\frac{ 0000\rangle - 0101\rangle - 0110\rangle + 1100\rangle}{2}$	(+1, +1, -1, +1)
0101	$\frac{ 0001\rangle - 0100\rangle - 0111\rangle + 1101\rangle}{2}$	(-1, +1, -1, +1)
0110	$\frac{ 0010\rangle - 0100\rangle - 0111\rangle + 1110\rangle}{2}$	(1, -1, -1, +1)
0111	$\frac{ 0011\rangle - 0101\rangle - 0110\rangle + 1111\rangle}{2}$	(-1, -1, -1, +1)
1000	$\frac{ 0000\rangle - 1001\rangle - 1010\rangle - 1100\rangle}{2}$	(+1, +1, +1, -1)
1001	$\frac{ 0001\rangle - 1000\rangle - 1011\rangle - 1101\rangle}{2}$	(-1, +1, +1, -1)
1010	$\frac{ 0010\rangle - 1000\rangle - 1011\rangle - 1110\rangle}{2}$	(+1, -1, +1, -1)
1011	$\frac{ 0011\rangle - 1001\rangle - 1010\rangle - 1111\rangle}{2}$	(-1, -1, +1, -1)
1100	$\frac{ 0100\rangle - 1000\rangle - 1101\rangle + 1110\rangle}{2}$	(+1, +1, -1, -1)
1101	$\frac{ 0101\rangle - 1001\rangle - 1100\rangle + 1111\rangle}{2}$	(-1, +1, -1, -1)
1110	$\frac{ 0110\rangle - 1010\rangle - 1100\rangle - 1111\rangle}{2}$	(+1, -1, -1, -1)
1111	$\frac{ 0111\rangle - 1011\rangle - 1101\rangle - 1110\rangle}{2}$	(-1, -1, -1, -1)

3. W state

$|W_3\rangle$ can be transformed from $|000\rangle$ by the unitary $U^{W_3} = (XZ\mathbb{I} + \mathbb{I}XZ + Z\mathbb{I}X)/\sqrt{3}$, i.e., $|W_3\rangle = U^{W_3}|000\rangle$ [60]. Thus the generators of $|W_3\rangle$ are derived by

$$\begin{aligned}
 S_1^{W_3} &= U^{W_3} Z\mathbb{I}\mathbb{I} U^{W_3\dagger} = \frac{1}{3}(Z\mathbb{I}\mathbb{I} + 2YYZ + 2XZX), \\
 S_2^{W_3} &= U^{W_3} \mathbb{I}Z\mathbb{I} U^{W_3\dagger} = \frac{1}{3}(\mathbb{I}Z\mathbb{I} + 2ZYY + 2XXZ), \\
 S_3^{W_3} &= U^{W_3} \mathbb{I}\mathbb{I}Z U^{W_3\dagger} = \frac{1}{3}(\mathbb{I}\mathbb{I}Z + 2YZY + 2ZXX). \quad (A8)
 \end{aligned}$$

FIG. 4. Illustration of experimental setup to generate a polarization-entangled photon pair. LD, laser diode.

Other stabilizing operators are

$$\begin{aligned} S_4^{W_3} &= \frac{1}{3}(2X\mathbb{I}X + 2\mathbb{I}YY - ZZ\mathbb{I}), \\ S_5^{W_3} &= \frac{1}{3}(2\mathbb{I}XX + 2YY\mathbb{I} - Z\mathbb{I}Z), \\ S_6^{W_3} &= \frac{1}{3}(2XX\mathbb{I} + 2Y\mathbb{I}Y - \mathbb{I}ZZ), \\ S_7^{W_3} &= -ZZZ, \quad S_8^{W_3} = \mathbb{I}\mathbb{I}\mathbb{I}. \end{aligned} \quad (\text{A9})$$

The W_3 -diagonal bases are shown in Table IV.

Similarly, we can find a possible unitary operator $U^{W_4} = (ZZZX + ZZX\mathbb{I} + ZX\mathbb{I}\mathbb{I} + X\mathbb{I}\mathbb{I}\mathbb{I})/2$ to generate $|W_4\rangle$ from $|0000\rangle$. Thus the generators of $|W_4\rangle$ can be obtained, i.e.,

$$\begin{aligned} S_1^{W_4} &= \frac{1}{2}(YZZY + \mathbb{I}YZY + \mathbb{I}\mathbb{I}YY + \mathbb{I}\mathbb{I}\mathbb{I}Z), \\ S_2^{W_4} &= \frac{1}{2}(YZY\mathbb{I} + \mathbb{I}YY\mathbb{I} + \mathbb{I}\mathbb{I}Z\mathbb{I} + \mathbb{I}\mathbb{I}XX), \\ S_3^{W_4} &= \frac{1}{2}(YY\mathbb{I}\mathbb{I} + \mathbb{I}Z\mathbb{I}\mathbb{I} + \mathbb{I}XZX + \mathbb{I}XX\mathbb{I}), \\ S_4^{W_4} &= \frac{1}{2}(Z\mathbb{I}\mathbb{I}\mathbb{I} + XZZX + XZX\mathbb{I} + XX\mathbb{I}\mathbb{I}), \end{aligned} \quad (\text{A10})$$

and the other stabilizing operators are

$$\begin{aligned} S_5^{W_4} &= \frac{1}{2}(YZYZ + YZ\mathbb{I}Y + \mathbb{I}YYZ + \mathbb{I}Y\mathbb{I}Y), \\ S_6^{W_4} &= \frac{1}{2}(YY\mathbb{I}Z + Y\mathbb{I}ZY + \mathbb{I}ZYY + \mathbb{I}XXZ), \\ S_7^{W_4} &= \frac{1}{2}(ZYZY + Z\mathbb{I}YY + XZXZ + XX\mathbb{I}Z), \\ S_8^{W_4} &= \frac{1}{2}(YYZ\mathbb{I} + Y\mathbb{I}Y\mathbb{I} + \mathbb{I}ZXX + \mathbb{I}X\mathbb{I}X), \\ S_9^{W_4} &= \frac{1}{2}(ZYY\mathbb{I} + Z\mathbb{I}XX + XZ\mathbb{I}X + XXZ\mathbb{I}), \end{aligned}$$

$$S_{10}^{W_4} = \frac{1}{2}(ZXZX + ZXX\mathbb{I} + X\mathbb{I}ZX + X\mathbb{I}X\mathbb{I}),$$

$$S_{11}^{W_4} = \frac{1}{2}(YYZZ + Y\mathbb{I}YZ + Y\mathbb{I}\mathbb{I}Y - \mathbb{I}ZZZ),$$

$$S_{12}^{W_4} = \frac{1}{2}(ZYYZ + ZY\mathbb{I}Y - Z\mathbb{I}ZZ + XXZZ),$$

$$S_{13}^{W_4} = \frac{1}{2}(ZZYY - ZZ\mathbb{I}Z + ZXXZ + X\mathbb{I}XZ),$$

$$S_{14}^{W_4} = \frac{1}{2}(-ZZZ\mathbb{I} + ZZXX + ZX\mathbb{I}Z + X\mathbb{I}\mathbb{I}X),$$

$$S_{15}^{W_4} = -ZZZZ, \quad S_{16}^{W_4} = \mathbb{I}\mathbb{I}\mathbb{I}\mathbb{I}. \quad (\text{A11})$$

The W_4 -diagonal bases are shown in Table V.

APPENDIX B: DETAILS OF EXPERIMENTAL REALIZATIONS AND RESULTS

In this Appendix, we provide further details about our experimental setup. It would be useful to bear in mind the following:

(i) A half-wave plate (HWP) at θ performs the unitary transformation $U_{\text{HWP}} = \cos 2\theta(|H\rangle\langle H| - |V\rangle\langle V|) + \sin 2\theta(|H\rangle\langle V| + |V\rangle\langle H|)$ on a polarization state, where θ is the angle between the fast axis of the HWP and vertical polarization.

(ii) A beam displacer (BD) transmits a vertically polarized photon but deviates a horizontally polarized one.

(iii) A polarized beam splitter (PBS) transmits a horizontally polarized photon but reflects a vertically polarized one.

(iv) A quarter-wave plate (QWP) at θ performs the unitary transformation $U_{\text{QWP}} = \frac{1}{\sqrt{2}}[\mathbb{I}_2 + i \cos 2\theta(|H\rangle\langle H| - |V\rangle\langle V|) +$

FIG. 5. Illustration of the experimental setups to generate a multiqubit state: (a) the four-qubit GHZ state $|GHZ_4\rangle$, (b) the four-qubit cluster state $|C_4\rangle$, (c) the four-qubit W state $|W_4\rangle$, (d) the three-qubit GHZ state $|GHZ_3\rangle$, and (e) the three-qubit W state $|W_3\rangle$.

FIG. 6. Experimental setups of measurement on a photon encoded in the polarization DOF and path DOF.

$i \sin 2\theta(|H\rangle\langle V| + |V\rangle\langle H|)$, where $\mathbb{I}_2 = |H\rangle\langle H| + |V\rangle\langle V|$ and θ is the angle between the fast axis of the QWP and vertical polarization, on a polarization state.

1. Polarization-entangled photon source

As shown in Fig. 4, the power of the pump light can be adjusted by a HWP and a PBS. After the PBS, horizontal

polarization $|H_p\rangle$ is rotated to $|+_p\rangle = \frac{1}{\sqrt{2}}(|H_p\rangle + |V_p\rangle)$ by a HWP set at 22.5° . The pump beam is focused into a PPKTP crystal with a beam waist of $74 \mu\text{m}$ by two lenses L_1 , whose focal lengths are 75 and 125 mm, respectively. The PPKTP crystal, with dimensions of $10 \times 2 \times 1 \text{ mm}$ (length $\times$ width $\times$ thickness) and poling period of $\Lambda = 10.025 \mu\text{m}$, is held in a home-built copper oven, and the temperature is controlled by a homemade temperature controller, which is set at 29°C to realize the optimum type-II phase matching at 810 nm. Then, the pump beam is split by a dual-wavelength PBS and coherently pumps PPKTP in the clockwise and counterclockwise direction, respectively. The clockwise and counterclockwise photons are recombined at the dual-wavelength PBS to generate polarization-entangled photons with an ideal form of $|\Psi_{13}^+\rangle = \frac{1}{\sqrt{2}}(|H_1V_3\rangle + |V_1H_3\rangle)$. Two photons are filtered by narrowband filters (NBFs) with full width at half maximum (FWHM) of 3 nm and coupled into single-mode fibers (SMFs) by lenses of focal length 200 mm (L_2 and L_3) and objective lenses (not shown in Fig. 4).

2. Experimental setups to generate a multiqubit state

We extend a photon to its path degree of freedom (DOF) using a BD with length of 28.3 mm and clear aperture of $10 \times 10 \text{ mm}$. The BD transmits vertical polarization and deviates horizontal polarization by 3 mm. Specifically, the experimental setups to generate the five multiqubit states $|\text{GHZ}_4\rangle$, $|\text{C}_4\rangle$, $|\text{W}_4\rangle$, $|\text{GHZ}_3\rangle$, and $|\text{W}_3\rangle$ are shown in Figs. 5(a)–5(e), respectively. The step-by-step calculations are shown in Eqs. (B1)–(B3).

$$|\Psi_{ab}^+\rangle = \frac{1}{\sqrt{2}}(|H_1V_3\rangle + |V_1H_3\rangle) \xrightarrow[\text{on path of photon } a]{\text{HWP at } 45^\circ} \frac{1}{\sqrt{2}}(|V_1V_3\rangle + |H_1H_3\rangle) \xrightarrow{\text{BD1}} \frac{1}{\sqrt{2}}(|V_1\rangle|v_2\rangle|V_3\rangle + |H_1\rangle|h_2\rangle|H_3\rangle) = |\text{GHZ}_3\rangle$$

$$\xrightarrow{\text{BD2}} \frac{1}{\sqrt{2}}(|V_1\rangle|v_2\rangle|V_3\rangle|v_4\rangle + |H_1\rangle|h_2\rangle|H_3\rangle|h_4\rangle) = |\text{GHZ}_4\rangle, \quad (\text{B1})$$

$$|\Psi_{ab}^+\rangle = \frac{1}{\sqrt{2}}(|H_1V_3\rangle + |V_1H_3\rangle) \xrightarrow[\text{on path of photon } a]{\text{HWP at } 45^\circ} \frac{1}{\sqrt{2}}(|V_1V_3\rangle + |H_1H_3\rangle) \xrightarrow[\text{on path of photon } b]{\text{HWP at } 22.5^\circ} \frac{1}{2}(|V_1H_3\rangle - |V_1V_3\rangle + |H_1H_3\rangle + |H_1V_3\rangle)$$

$$\xrightarrow{\text{BD1, BD2}} \frac{1}{2}(|V_1\rangle|v_2\rangle|H_3\rangle|h_4\rangle - |V_1\rangle|v_2\rangle|V_3\rangle|v_4\rangle + |H_1\rangle|h_2\rangle|H_3\rangle|h_4\rangle + |H_1\rangle|h_2\rangle|V_3\rangle|v_4\rangle) = |\text{C}_4\rangle, \quad (\text{B2})$$

$$|\Psi_{ab}^+\rangle = \frac{1}{\sqrt{2}}(|H_1V_3\rangle + |V_1H_3\rangle) \xrightarrow{\text{BD1}} \frac{1}{\sqrt{2}}(|H_1\rangle|h_2\rangle|V_3\rangle + |V_1\rangle|v_2\rangle|H_3\rangle)$$

$$\xrightarrow[\text{on path } h \text{ and } v \text{ of photon } a]{\text{HWP at } 67.5^\circ} \frac{1}{2}[(-|H_1\rangle + |V_1\rangle)|h_2\rangle|V_3\rangle + (|H_1\rangle + |V_1\rangle)|v_2\rangle|H_3\rangle]$$

$$\xrightarrow{\text{BD2}} \frac{1}{\sqrt{3}}(|V_1\rangle|h_2\rangle|V_3\rangle + |H_1\rangle|h_2\rangle|H_3\rangle + |V_1\rangle|v_2\rangle|H_3\rangle)$$

$$\xrightarrow[\text{on path } h \text{ and } v \text{ of photon } a]{\text{HWP at } 45^\circ} \frac{1}{\sqrt{3}}(|H_1\rangle|h_2\rangle|V_3\rangle + |V_1\rangle|h_2\rangle|H_3\rangle + |H_1\rangle|v_2\rangle|H_3\rangle) = |\text{W}_3\rangle$$

$$\xrightarrow{\text{BD3}} \frac{1}{\sqrt{3}}(|H_1\rangle|h_2\rangle|V_3\rangle|v_4\rangle + |V_1\rangle|h_2\rangle|H_3\rangle|h_4\rangle + |H_1\rangle|v_2\rangle|H_3\rangle|h_4\rangle)$$

$$\xrightarrow[\text{on path } h \text{ and } v \text{ of photon } b]{\text{HWP at } 67.5^\circ} \frac{1}{\sqrt{6}}[|H_1\rangle|h_2\rangle(|H_3\rangle + |V_3\rangle)|v_4\rangle + |V_1\rangle|h_2\rangle(-|H_3\rangle + |V_3\rangle)|h_4\rangle + |H_1\rangle|v_2\rangle(-|H_3\rangle + |V_3\rangle)|h_4\rangle]$$

$$\xrightarrow{\text{BD4}} \frac{1}{2}(|H_1\rangle|h_2\rangle|H_3\rangle|h_4\rangle + |H_1\rangle|h_2\rangle|V_3\rangle|v_4\rangle + |V_1\rangle|h_2\rangle|V_3\rangle|h_4\rangle + |H_1\rangle|v_2\rangle|V_3\rangle|h_4\rangle)$$

$$\xrightarrow[\text{on path } h \text{ and } v \text{ of photon } b]{\text{HWP at } 45^\circ} \frac{1}{2}(|H_1\rangle|h_2\rangle|V_3\rangle|h_4\rangle + |H_1\rangle|h_2\rangle|H_3\rangle|v_4\rangle + |V_1\rangle|h_2\rangle|H_3\rangle|h_4\rangle + |H_1\rangle|v_2\rangle|H_3\rangle|h_4\rangle) = |\text{W}_4\rangle. \quad (\text{B3})$$

FIG. 7. Experimental results of (a) expected values of $\langle S_i^\psi \rangle$ and (b) probabilities of d^{ψ}_{expt} for states $\rho_{\text{expt}}^{\text{GHZ}_3}$, $\rho_{\text{expt}}^{\text{GHZ}_4}$, $\rho_{\text{expt}}^{C_4}$, $\rho_{\text{expt}}^{W_3}$, and $\rho_{\text{expt}}^{W_4}$, respectively. The black grids represent the values for ideal states.

3. Measurement and quantum state tomography

If the photon is encoded in either the polarization DOF or the path DOF, we first perform the measurement on the polarization DOF and then on the path DOF. As illustrated in Fig. 6, the measurement basis of a qubit on the polarization DOF $\alpha|H\rangle + \beta|V\rangle$ is determined using the HWP at θ_1 and the QWP at θ_2 . The measurement basis of a qubit on the path DOF $\gamma|h\rangle + \delta|v\rangle$ is determined using the HWP at θ_3 and the QWP at θ_4 . Finally, a PBS is applied before the photon arrives at the detector. With this setting, the measurement on the basis $(\alpha|H\rangle + \beta|V\rangle) \otimes (\gamma|h\rangle + \delta|v\rangle)$ is achieved. The specific calculations are shown in eq. (B4). If the photon is only encoded in the polarization DOF, the measurement on the basis $\alpha|H\rangle + \beta|V\rangle$ is implemented by a QWP, HWP, and PBS.

$$\begin{aligned}
 & (\alpha|H\rangle + \beta|V\rangle) \otimes (\gamma|h\rangle + \delta|v\rangle) \\
 & \xrightarrow[\text{QWP at } \theta_2]{\text{HWP at } \theta_1} |H\rangle \otimes (\gamma|h\rangle + \delta|v\rangle) \\
 & \xrightarrow[\text{on path } h]{\text{HWP at } 45^\circ} \gamma|V\rangle|h\rangle + \delta|H\rangle|v\rangle \\
 & \xrightarrow{\text{BD}} \gamma|V\rangle|h\rangle + \delta|H\rangle|h\rangle
 \end{aligned}$$

$$\begin{aligned}
 & \xrightarrow{\text{HWP at } 45^\circ} (\gamma|H\rangle + \delta|V\rangle)|h\rangle \\
 & \xrightarrow[\text{QWP at } \theta_4]{\text{HWP at } \theta_3} |H\rangle|h\rangle \\
 & \xrightarrow{\text{PBS}} |H\rangle|h\rangle.
 \end{aligned} \tag{B4}$$

With this experimental setting, we can perform the measurement on an arbitrary basis. The experimental results of $\langle S_i^\psi \rangle$ and d^{ψ}_{expt} are shown in Figs. 7(a) and 7(b), respectively. Moreover, we reconstruct experimentally generated states $\rho_{\text{expt}}^{\text{GHZ}_4}$, $\rho_{\text{expt}}^{C_4}$, $\rho_{\text{expt}}^{W_4}$, $\rho_{\text{expt}}^{\text{GHZ}_3}$, and $\rho_{\text{expt}}^{W_3}$ by quantum state tomography [48]. The results are shown in Fig. 8, from which we calculate the fidelity $F^\psi = \text{tr}(\rho_{\text{expt}}^\psi |\psi\rangle\langle\psi|)$. We observe that $F^{\text{GHZ}_3} = 0.9643 \pm 0.0003$, $F^{W_3} = 0.9589 \pm 0.0005$, $F^{\text{GHZ}_4} = 0.9571 \pm 0.0003$, $F^{C_4} = 0.9497 \pm 0.0002$, and $F^{W_4} = 0.915 \pm 0.001$. Also, the relative entropy of coherence $C_{\text{RE}}(\rho_{\text{expt}}^\psi)$ of ρ_{expt}^ψ can be calculated by $C_{\text{RE}}(\rho) = S_{\text{VN}}(\rho_d) - S_{\text{VN}}(\rho)$, according to which we obtain $C_{\text{RE}}(\rho_{\text{expt}}^{\text{GHZ}_3}) = 0.875 \pm 0.002$, $C_{\text{RE}}(\rho_{\text{expt}}^{W_3}) = 1.479 \pm 0.004$, $C_{\text{RE}}(\rho_{\text{expt}}^{\text{GHZ}_4}) = 0.906 \pm 0.002$, $C_{\text{RE}}(\rho_{\text{expt}}^{C_4}) = 1.806 \pm 0.002$, and $C_{\text{RE}}(\rho_{\text{expt}}^{W_4}) = 1.964 \pm 0.003$.

APPENDIX C: EXPERIMENTAL IMPERFECTIONS

In our experiment, the main imperfection of prepared multiqubit states comes from the imperfection of the polarization-entangled photon pair, which is caused by the mode mismatch of overlapping lights on the PBS in Fig. 4. The noisy state can be described by a dephasing channel acting on ideal $|\Phi^+\rangle = (|HH\rangle + |VV\rangle)/\sqrt{2}$ by

$$\rho^{\Psi^+} = \mathcal{E}(|\Phi^+\rangle\langle\Phi^+|) = (1 - \mu)|\Phi^+\rangle\langle\Phi^+| + \frac{\mu}{2}(|HH\rangle\langle HH| + |VV\rangle\langle VV|). \tag{C1}$$

With such a noisy state and following the procedure of generating a multiqubit state in Appendix B 2, we calculate the noisy multiqubit states in the computational basis (CB)

$$\rho_{\text{CB}}^{\text{GHZ}_3} = (1 - \mu)|\text{GHZ}_3\rangle\langle\text{GHZ}_3| + \frac{\mu}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|), \tag{C2}$$

FIG. 8. Reconstructed density matrices of generated multiqubit states.

$$\rho_{CB}^{GHZ_4} = (1 - \mu)|GHZ_4\rangle\langle GHZ_4| + \frac{\mu}{2}(|0000\rangle\langle 0000| + |1111\rangle\langle 1111|), \quad (C3)$$

$$\rho_{CB}^{C_4} = (1 - \mu)|C_4\rangle\langle C_4| + \frac{\mu}{4}(|0000\rangle\langle 0000| + |0011\rangle\langle 0011| + |1100\rangle\langle 1100| + |1111\rangle\langle 1111|), \quad (C4)$$

$$\rho_{CB}^{W_3} = (1 - \mu)|W_3\rangle\langle W_3| + \frac{\mu}{3}(|100\rangle\langle 100| + |010\rangle\langle 010| + |001\rangle\langle 001|), \quad (C5)$$

$$\rho_{CB}^{W_4} = (1 - \mu)|W_4\rangle\langle W_4| + \frac{\mu}{4}(|1000\rangle\langle 1000| + |0100\rangle\langle 0100| + |0010\rangle\langle 0010| + |0001\rangle\langle 0001|). \quad (C6)$$

In the graph-diagonal basis (GDB),

$$\rho_{GDB}^{GHZ_3} = U^{GHZ_3} \rho_{CB}^{GHZ_3} U^{GHZ_3 \dagger} = \sum_{k=1}^8 \lambda_k |\psi_k^{GHZ_3}\rangle\langle \psi_k^{GHZ_3}| = \left(1 - \frac{\mu}{2}\right) |\psi_1^{GHZ_3}\rangle\langle \psi_1^{GHZ_3}| + \frac{\mu}{2} |\psi_8^{GHZ_3}\rangle\langle \psi_8^{GHZ_3}|, \quad (C7)$$

$$\rho_{GDB}^{GHZ_4} = U^{GHZ_4} \rho_{CB}^{GHZ_4} U^{GHZ_4 \dagger} = \sum_{k=1}^{16} \lambda_k |\psi_k^{GHZ_4}\rangle\langle \psi_k^{GHZ_4}| = \left(1 - \frac{\mu}{2}\right) |\psi_1^{GHZ_4}\rangle\langle \psi_1^{GHZ_4}| + \frac{\mu}{2} |\psi_{16}^{GHZ_4}\rangle\langle \psi_{16}^{GHZ_4}|, \quad (C8)$$

$$\rho_{GDB}^{C_4} = U^{C_4} \rho_{CB}^{C_4} U^{C_4 \dagger} = \sum_{k=1}^{16} \lambda_k |\psi_k^{C_4}\rangle\langle \psi_k^{C_4}| = \left(1 - \frac{3\mu}{4}\right) |\psi_1^{C_4}\rangle\langle \psi_1^{C_4}| + \frac{\mu}{4} (|\psi_4^{C_4}\rangle\langle \psi_4^{C_4}| + |\psi_{13}^{C_4}\rangle\langle \psi_{13}^{C_4}| + |\psi_{16}^{C_4}\rangle\langle \psi_{16}^{C_4}|) \quad (C9)$$

are still graph-diagonal states under the dephasing channel, which means that the estimated lower bounds on such states are tight, and it is easy to check that $\rho_{GDB}^{W_3} = U^{W_3} \rho_{CB}^{W_3} U^{W_3 \dagger} \neq \sum_k \lambda_k |\psi_k^{W_3}\rangle\langle \psi_k^{W_3}|$ and $\rho_{GDB}^{W_4} = U^{W_4} \rho_{CB}^{W_4} U^{W_4 \dagger} \neq \sum_k \lambda_k |\psi_k^{W_4}\rangle\langle \psi_k^{W_4}|$.

In experiment, we observe that the normalized distance of estimated $l^c(\rho^{W_4})$ is larger than that of other states as the prepared $\rho_{expt}^{W_4}$ has lower fidelity compared with other states. More importantly, it is not a graph-diagonal state anymore under the dephasing channel, which causes there to be only a 5% difference in state fidelity (shown in Appendix B) but a 15% difference in normalized distance [shown in Fig. 3(c)].

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