Speed limit for open systems coupled to general environments

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We investigate the bound on the speed of state transformation in the quantum systems that are coupled to general environments. We focus on the energy fluctuation of the target system and the coupling to the environment, and clarify the universal constraint on the state transformation deriving the general speed-limit inequality. This speed-limit inequality is applicable to any open systems including physical situations that have not been studied so far. We apply the inequality to the thermalization phenomena in isolated quantum systems, and the quantum cyclic engines. For the thermalization, we provide several inequalities to estimate the equilibration time. In the cyclic heat engines, we derive a rigorous constraint on the state change in the cycle where the Planck constant plays a fundamental role.

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I. INTRODUCTION

The rate at which a given initial state can be transformed into a desired final state has received significant research attention in the field of theoretical physics. Moreover, questions pertaining to the quantities limiting the speed of state change have been intensively investigated. One of the answers is obtained via the celebrated Mandelstam-Tamm (MT) quantum speed limit for isolated quantum systems [1], which is expressed by the following inequality:

$$\frac{\mathcal{L}(\rho(\tau), \rho(0))}{\langle \delta E \rangle_\tau / \hbar} \leq \tau. \quad (1)$$

Here, $\langle \delta E \rangle_\tau$ is the time-average of the energy fluctuation during the operation time τ , and $\mathcal{L}(\rho, \rho')$ is the Bures angle, which is a distance measure between the two states, i.e., two density matrices ρ and ρ' . This inequality clearly indicates that the amount of energy fluctuation restricts the speed of state change. Margolus and Levitin proposed another type of speed limit, where energy itself, and not energy fluctuation, bounds the state transformation [2]. These relations originate from the uncertainty relation between time and energy; hence, the bound on the operation-time inevitably accompanies quantities on energy [1–9].

In experiments the environmental effect is unavoidable. A typical example is solid-state quantum devices, which are frequently disturbed by thermal as well as nonthermal environments [10]. Therefore, it is fundamentally important to understand the effects of the environment on the speed of state transformation, and several studies have focused on this

aspect. When the dynamics of a system is well described by an effective equation, such as the Lindblad equation, mathematically compact expressions on the speed limit can be derived [11–15]. Nonunitary dynamics arising from the environment have also been discussed from the viewpoint of quantum information theory [16]. The speed of state transformation induced by the thermal environment can also be discussed in terms of the entropy production using the framework of stochastic thermodynamics [17–23].

It should be noted that in realistic situations, there are a wide variety of environments, coupling forms, and strengths between the system and environment. Not only the short-range interaction, but also the long-range interaction such as the dipole interaction are commonly observed in various materials. Recent experimental development has enabled us to control the interaction strength [24] and the range of interaction [25–28], and hence, various dynamics of the target small system surrounded by many particles can be experimentally scrutinized. In general, an environment does not necessarily act as a thermal reservoir, but as nonthermal quantum noise. The environmental effects often cannot be described with an effective equation in a simple form. In such cases, the physical limitation of speed cannot be understood within the framework using effective dynamics such as the Lindblad equation. From this background, we take the question one step further, i.e., is there any general and physically meaningful bound on the speed of state transformation that holds, irrespective of the type of environment, coupling form, and coupling strength? Given the prevalence of several relevant experimental situations, it is necessary to address this question for an in-depth understanding of the speed of state transformation observed in nature.

In this paper, we aim to clarify the limitation of state transformation by deriving the speed-limit expression for systems attached to the general environment through an arbitrary coupling. We establish that the energy fluctuation still plays an important role in limiting the speed of state transformation

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as in the MT bound. We show that the energy fluctuation of the coupling component as well as the system Hamiltonian is crucial. To derive the speed-limit inequality, we apply the decomposition on the time evolution operator, dividing it into two contributions: one from the system plus coupling, and the other from the environment. The speed-limit expression does not only show the fundamental mechanism of state transformation, but also has several potential applications. As simple applications, we discuss the equilibration time in thermalization phenomena of isolated quantum many-body systems, and the microscale cyclic engines.

II. MAIN RESULT: SPEED LIMIT FOR OPEN QUANTUM SYSTEMS

We consider a quantum system (S) interacting with the external environment (E) via coupling (SE) with an arbitrary interacting form. The total Hamiltonian is described as

$$H_{\text{tot}} = H_S + H_{SE} + H_E, \quad (2)$$

where H_S and H_E are the Hamiltonian of the system and that of the environment, respectively. The Hamiltonian H_{SE} is a coupling Hamiltonian between the system and the environment. We emphasize that H_{SE} is arbitrary, and its norm can be the same order as the system's Hamiltonian. Although we do not explicitly write the time dependence for the sake of notational simplicity, each Hamiltonian can be time dependent.

Let $\rho_{\text{tot}}(t)$ and $\rho_S(t)$ be the density matrix for the total system and the target system of interest, respectively. In the time evolution driven by H_{tot} , the total density matrix changes in time, and the system's reduced density matrix is obtained as $\rho_S(t) = \text{Tr}_E[\rho_{\text{tot}}(t)]$, where Tr_E implies the partial trace of the environment. Note that the system and the environment can be correlated at any time, including the initial time $t = 0$. Then, we consider a speed limit only for the system by analyzing the change in the system's density matrix. Let $\mathcal{L}(\rho, \rho')$ be the Bures angle, which quantifies the distance between two states ρ and ρ' , defined as [29]

$$\mathcal{L}(\rho, \rho') := \arccos \text{Tr}[\sqrt{\rho^{1/2} \rho' \rho^{1/2}}]. \quad (3)$$

In addition, we define the system plus the interaction Hamiltonian denoted by $\tilde{H}_S$ and the energy variance for this Hamiltonian at time t denoted by $\delta E_S(t)$:

$$\tilde{H}_S := H_S + H_{SE}, \quad (4)$$

$$\delta E_S(t) := \sqrt{\text{Tr}[\tilde{H}_S^2 \rho_{\text{tot}}(t)] - \text{Tr}[\tilde{H}_S \rho_{\text{tot}}(t)]^2}. \quad (5)$$

The Hamiltonian $\tilde{H}_S$ is a collection of all terms containing the system's operators. Then, we derive the following quantum speed limit for general open systems:

$$\frac{\mathcal{L}(\rho_S(\tau), \rho_S(0))}{\langle \delta E_S \rangle_\tau / \hbar} \leq \tau, \quad (6)$$

where $\langle \delta E_S \rangle_\tau$ is the time-average on the energy fluctuation, i.e., $\langle \delta E_S \rangle_\tau := (1/\tau) \int_0^\tau dt \delta E_S(t)$. The proof for this relation is provided at the end of this paper.

We explain the physical meaning behind this inequality. We note that the bound on the operation time in (6) is characterized not only by the system Hamiltonian but also

by the coupling component in the energy fluctuation term. This combination is essential for the correct estimation of the bound, which can be understood by comparing it with the direct application of the MT bound (1). Firstly, suppose that we employed the original MT bound (1), using the energy fluctuation only of the system Hamiltonian. Then, we immediately realize that such inequality does not hold. A simple counterexample is a case where the system Hamiltonian is constant, while the coupling Hamiltonian exists, in which the energy fluctuation of the system Hamiltonian is exactly zero and hence the bound diverges. Next, suppose that we employed the original MT bound (1) by setting the energy fluctuation for the total Hamiltonian and applying the monotonicity of the Bures angle $\mathcal{L}(\rho_S(\tau), \rho_S(0)) \leq \mathcal{L}(\rho_{\text{tot}}(\tau), \rho_{\text{tot}}(0))$. Then, the original MT bound can be written in terms of the state change only for the system: $\mathcal{L}(\rho_S(\tau), \rho_S(0)) / (\langle \delta E_{\text{tot}} \rangle_\tau / \hbar) \leq \tau$, where $\langle \delta E_{\text{tot}} \rangle_\tau$ is the time average of energy fluctuation for the total Hamiltonian. This bound is a correct inequality, while it is very different from the relation (6), especially when the size of the environment is very large. In the case of an infinite environment size, which is a typical setting for the thermal reservoir, the total energy fluctuation $\langle \delta E_{\text{tot}} \rangle_\tau$ diverges, leading to a trivial bound $0 \leq \tau$. On the other hand, the bound in (6) gives a finite value even in this case. In both cases described above, the original MT bound is not available, while the relation (6) works well.

We remark that if one requires further tightness, a similar inequality with replacing the energy fluctuation $\langle \delta E_S \rangle_\tau$ by the quantum Fisher information is also available [14] and it serves as a tighter bound. On the other hand, noting the relation $\|\tilde{H}_S\| \geq \langle \delta E_S \rangle_\tau$ where $\|\tilde{H}_S\|$ is the operator norm of the Hamiltonian $\tilde{H}_S$, one can derive less tighter but still important inequality, $\mathcal{L}(\rho_S(\tau), \rho_S(0)) / (\|\tilde{H}_S\| / \hbar) \leq \tau$. This relation is also useful since it does not contain the energy fluctuation and thus we need not to compute the time evolution.

III. APPLICATIONS

A. Application to thermalization in isolated quantum systems

The relation (6) has many implications to physical phenomena. As one of the applications, we consider the thermalization phenomena of isolated large quantum systems. We make a rough estimation of the *equilibration time* from a certain initial state [30–36]. To this end, we need to restrict our attention to macroscopic observables [30–33] or small subsystems [34–36]. We here consider the latter case, and regard the subsystem as a target system and the other component as the environment. (Later, we provide a simple example.) The reduced density matrix of the subsystem is expected to reach a steady state for a sufficiently large total system.

Let ρ_S' be a steady state of the subsystem inside the large total system. The Bures angle between the initial state and the steady state after relaxation is given by $\mathcal{L}_{\text{th}} := \arccos[\text{tr} \sqrt{(\rho_S(0)^{1/2} \rho_S' \rho_S(0)^{1/2})}]$. Regardless of the type of steady state (a Gibbs distribution or other distributions as generalized Gibbs distribution), the speed-limit relation (6) leads to the general lower bound of the subsystem equilibration time

τ_{eq} as

$$\frac{\mathcal{L}_{\text{th}}}{\|\tilde{H}_S\|/\hbar} \leq \frac{\mathcal{L}_{\text{th}}}{\langle \delta E_S \rangle_\tau / \hbar} \leq \tau_{\text{eq}}. \quad (7)$$

The second term suggests that the average energy fluctuation regarding the subsystem is crucial for the estimation of thermalization time [37]. The first term can also be meaningful if the operator norm is finite.

We remark that although the thermalization phenomena are widespread, estimating the equilibration time is considerably difficult. Calculations based on random Hamiltonians or typicality argument have been successfully performed, while they exhibit extremely fast equilibration time [38,39]. Analyses based on realistic Hamiltonians [40,41] are highly complicated, and the physical meaning of the estimation is not trivial. Compared to these, the inequality (7) provides a physically-reasonable equilibration time, in the sense that it is not extremely fast, and it consists of quantities whose physical meanings are clear. The first bound $\mathcal{L}_{\text{th}}/(\|\tilde{H}_S\|/\hbar)$ is easy to obtain, since it is given by the static quantities. The second bound $\mathcal{L}_{\text{th}}/(\langle \delta E_S \rangle_\tau / \hbar)$ is tighter than the first one, while the estimation of $\langle \delta E_S \rangle_\tau$ requires the information of the dynamics.

B. Application to systems connected to multiple reservoirs: Cyclic quantum engines

We can use the relation (6) to the systems attached to multiple reservoirs. We here consider cyclic quantum engines surrounded by large environments; examples include microscale quantum heat engines [42] and parametric electron pumping [43–45], both of which have been realized in recent experiments [46–49]. When we consider the heat engines driven by two different temperatures, we control the amplitude of the interactions between the system and two thermal reservoirs with different temperatures to switch the temperatures. By changing temperatures in this manner and controlling the system's parameters, we extract quantum work through a cycle process. In the parametric electron pumping, the small system, such as a quantum dot, is connected to two electrodes (particle reservoirs), and we control system's parameters in a cyclic way, to pump electrons from one electrode to another. We derive a fundamental limitation on the speed in the cyclic engines.

Let the period of the process be τ_c . The cyclic property requires that the initial and the final states of the system are identical, $\rho_S(t + \tau_c) = \rho_S(t)$ in the steady state. Then, the relation (6) leads to the following relation [50]

$$\max_{\rho_S(t), \rho_S(t')} \mathcal{L}(\rho_S(t), \rho_S(t')) \leq \frac{\tau_c \delta E_{S,\text{max}}}{2\hbar} \leq \frac{\tau_c \|\tilde{H}_S\|}{2\hbar}, \quad (8)$$

where $\delta E_{S,\text{max}}$ is the maximum value of $\delta E_S(t)$ during the cycle. The maximum in the left-hand side is taken over all time duration $0 \leq t, t' \leq \tau$. This relation is satisfied in any type of cyclic engines, including microscopic heat engines and parametric electron pumping, as long as the elementary time evolution of the total system can be described by the Schrödinger equation. This relation indicates a fundamental limitation of the speed of engines accompanying state change, which should be significant as the size of the target system

FIG. 1. Speed-limit bounds [left-hand side of (6)] versus actual time τ . The blue, green, and orange lines represent the plot with $(J, h) = (0.4, 1.0)$, $(1.2, 1.0)$, and $(2.0, 1.0)$. The dashed line is $y = \tau$.

becomes small, and as a result, the quantum effect becomes important.

IV. NUMERICAL DEMONSTRATION

We present numerical demonstration on the relation (6) and the bound for thermalization (7) with a simple model. We use the one-dimensional $S = 1/2$ Heisenberg chain with the periodic boundary condition with length N . We regard the first spin as a target system of interest and the other $N - 1$ spins as the environment. We consider the Hamiltonians as

$$H_S = hS_1^z, \quad H_{SE} = J(S_1 \cdot S_2 + S_1 \cdot S_N),$$

$$H_E = h \sum_{i=2}^N S_i^z + J \sum_{i=2}^{N-1} S_i \cdot S_{i+1}, \quad (9)$$

where S_i is a vector of spin-1/2 operator at the site i , i.e., $S_i = (S_i^x, S_i^y, S_i^z)$. Let $|\uparrow\rangle_i$ ($|\downarrow\rangle_i$) be the eigenstate of the z component of the spin at the site i , satisfying $S_i^z |\uparrow\rangle_i = (1/2) |\uparrow\rangle_i$ ($S_i^z |\downarrow\rangle_i = -(1/2) |\downarrow\rangle_i$). For simplicity, we consider the static Hamiltonian. We set the initial state $\rho(0) = |\psi(0)\rangle\langle\psi(0)|$ with $|\psi(0)\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \cdots |\downarrow\rangle_N$. We consider the system size $N = 200$, and demonstrate the relations (6) and (7) for various system parameters J and h .

We first show the left-hand side of (6) against the time τ in Fig. 1. The blue, green, and orange lines are results for the parameters $(J, h) = (0.4, 1.0)$, $(1.2, 1.0)$, and $(2.0, 1.0)$, respectively. The dashed line $y = \tau$ is drawn as a guide. As seen from the graph, the present speed-limit well bounds the actual time duration. As in the original MT inequality, the equality in the relation (6) does not hold in general. However, as in Fig. 1, one can clearly see that the bound becomes tighter as the time is larger. We checked this tendency for many parameter sets.

We also demonstrate the lower bound for the equilibration time (7) with the same setup. In Fig. 2, we plot the left-hand side $\mathcal{L}_{\text{th}}/(\|\tilde{H}_S\|/\hbar)$ (red triangles), the middle (i.e., $\mathcal{L}_{\text{th}}/(\langle \delta E_S \rangle_\tau / \hbar)$) (blue squares), and right-hand side τ_{eq} (black circles) of (7) for several J s with the fixed value $h = 1.0$. The

FIG. 2. Main: Plots of the left-hand side $\frac{\mathcal{L}_{\text{th}}}{\|\tilde{H}_S\|/\hbar}$ (red triangles), the middle $\frac{\mathcal{L}_{\text{th}}}{\langle \delta E_S \rangle \tau / \hbar}$ (blue squares), and right-hand side τ_{eq} (black circles) of (7) with respect to coupling strength J (horizontal axis). We set the strength of the magnetic field as $\hbar = 1.0$. Inset: Dynamics of $\langle S_1^z \rangle$ with respect to τ . The blue, green, and orange curves represent the dynamics of $(J, h) = (0.4, 1.0)$, $(1.2, 1.0)$, and $(2.0, 1.0)$ (same as Fig. 1).

behavior of equilibration is shown in the inset of Fig. 2, where we set the same parameters as those in Fig. 1 with the same color. The equilibration time is defined as the first-passage time that the magnetization of the target spin first reaches the microcanonical value, i.e., $\langle S_1^z(\tau_{\text{eq}}) \rangle = -(N-2)/(2N)$. Note that the reduced density matrix of the target spin computed from the microcanonical ensemble for the total system is described as $(1/N)|\uparrow\rangle_1\langle\uparrow|_1 + [(N-1)/N]|\downarrow\rangle_1\langle\downarrow|_1$. The figure clearly shows that the bound with energy fluctuation serves as a very tight bound for the equilibration time, and the bound with the operator norm also provides a useful bound. We emphasize that the red triangles $\mathcal{L}_{\text{th}}/(\|\tilde{H}_S\|/\hbar)$ can be obtained without calculating the time evolution of the state.

V. DERIVATION OF (6)

We present the proof of the speed limit for open quantum systems (6). We decompose a time interval τ into short-time intervals Δt as $\tau = N\Delta t$, and first consider the state change in Δt at time t . We then sum up contributions from N short-time intervals.

We define the following unitary operators with an infinitesimal time duration Δt :

$$U_1 := e^{-i(H_S + H_{SE})\Delta t/\hbar} = e^{-i\tilde{H}_S\Delta t/\hbar}, \quad (10)$$

$$U_2 := e^{-iH_E\Delta t/2\hbar}. \quad (11)$$

Here, note that H_S , H_{SE} , and H_E can be time dependent, although we omit to write the explicit time dependence for notational simplicity. We start with the following expression for the state of the total system at time $t + \Delta t$:

$$\rho_{\text{tot}}(t + \Delta t) = U_2 U_1 U_2 \rho_{\text{tot}}(t) U_2^\dagger U_1^\dagger U_2^\dagger + O(\Delta t^3). \quad (12)$$

We now introduce

$$\rho' := U_2 \rho_{\text{tot}}(t) U_2^\dagger, \quad (13)$$

$$\rho'' := U_1 U_2 \rho_{\text{tot}}(t) U_2^\dagger U_1^\dagger = e^{-i\tilde{H}_S\Delta t/\hbar} \rho' e^{i\tilde{H}_S\Delta t/\hbar}, \quad (14)$$

where ρ'' is given by the time evolution from ρ' with the Hamiltonian $\tilde{H}_S$. Below, we show that the quantum speed limit for the reduced density matrix ρ_S can be understood in terms of the density matrices ρ' and ρ'' . We first note the cyclic property of the partial trace [51],

$$\begin{aligned} \rho_S(t + \Delta t) &:= \text{Tr}_E[\rho_{\text{tot}}(t + \Delta t)] = \text{Tr}_E[U_2 \rho'' U_2^\dagger] + O(\Delta t^3) \\ &= \text{Tr}_E \rho'' + O(\Delta t^3), \end{aligned} \quad (15)$$

$$\rho_S(t) := \text{Tr}_E[\rho_{\text{tot}}(t)] = \text{Tr}_E[U_2^\dagger \rho_{\text{tot}}(t) U_2] = \text{Tr}_E \rho'. \quad (16)$$

We then note the monotonicity of the Bures angle $\mathcal{L}$ for the contraction within the first order of Δt :

$$\mathcal{L}(\rho_S(t + \Delta t), \rho_S(t)) \leq \mathcal{L}(\rho'', \rho'). \quad (17)$$

From the relation (14) between ρ' and ρ'' , the original MT speed limit (1) leads to

$$\mathcal{L}(\rho'', \rho') \leq \frac{1}{\hbar} \int_0^{\Delta t} \delta E'_S(t) dt = \frac{\Delta t}{\hbar} \delta E'_S(t), \quad (18)$$

up to the first order of Δt . Here, the energy fluctuation $\delta E'_S(t)$ is the energy fluctuation of $\tilde{H}_S$ at time t for ρ' defined as

$$\delta E'_S(t) := \sqrt{\text{Tr}[\tilde{H}_S^2 \rho'] - \text{Tr}[\tilde{H}_S \rho']^2}. \quad (19)$$

Furthermore, we can equally treat $\delta E'_S(t)$ and $\delta E_S(t)$ up to $O(\Delta t)$:

$$\frac{\Delta t}{\hbar} \delta E'_S(t) = \frac{\Delta t}{\hbar} \delta E_S(t) + O(\Delta t^{3/2}), \quad (20)$$

because the difference between $\rho_{\text{tot}}(t)$ and $\rho' = e^{-iH_E\Delta t/2\hbar} \rho_{\text{tot}}(t) e^{iH_E\Delta t/2\hbar}$ is $O(\Delta t)$. Finally, the triangle inequality of the Bures angle leads to

$$\begin{aligned} \mathcal{L}(\rho_S(\tau), \rho_S(0)) &\leq \sum_{n=0}^{N-1} \mathcal{L}(\rho_S((n+1)\Delta t), \rho_S(n\Delta t)) \\ &\leq \frac{\Delta t}{\hbar} \sum_{n=0}^{N-1} \delta E_S(n\Delta t), \end{aligned} \quad (21)$$

with $\tau = N\Delta t$. By considering the $\Delta t \rightarrow 0$ limit, we arrive at the desired result (6).

VI. DISCUSSION

We established a Mandelstam-Tamm-type quantum speed-limit inequality for open quantum systems with arbitrary coupling to general environments. Similar to the original Mandelstam-Tamm speed limit, the energy fluctuation still plays a pivotal role in this speed limit. Notably, the energy of the system and the interaction energy are relevant to this speed limit, and the energy of the environment is irrelevant. In other words, the effect of the environment comes only through the interaction energy. As potential applications, we discussed microscale quantum engines and the equilibration time in

thermalization phenomena of isolated quantum systems. The technique shown in this paper is also useful in various problems: An extension to classical Hamiltonian dynamics [53] is presented in Appendix C.

It is worth comparing our result to previous quantum speed limits for open systems [11–16]. We here summarize two advantages of our inequality. First, same as the case of the celebrated Mandelstam-Tamm quantum speed limit for isolated quantum systems, our inequality consists of physically well-established quantity, energy fluctuation. In contrast, most of the previous results employ abstract quantities not easy to measure directly, such as quantities related to the time-evolution operator (e.g., the operator norm, variance, and singular values of the time-evolution operator) [11–13], information-geometric quantities [14], and quantities arising from the idea of quantum information theory [16]. Second, our inequality applies any environment with any types of

coupling, including strong coupling, although most of the previous results apply only specific situations, e.g., systems described with the Lindblad equation [11,15] or (effective) Markovian equation given in the form $\dot{\rho}_t = L_t(\rho_t)$ [12–14]. On the basis of these facts, we conclude that our speed-limit inequality is the most orthodox extension of the quantum speed limit of isolated quantum systems (Mandelstam-Tamm inequality) toward open quantum systems.

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APPENDIX A: DERIVATION OF MANDELSTAM-TAMM QUANTUM SPEED LIMIT

1. Case of pure state

We first consider a situation that a pure state $|\psi(0)\rangle$ evolves to $|\psi(\tau)\rangle$ with a unitary evolution with duration τ generated by a time-dependent Hamiltonian $H(t)$. The case of mixed states is considered later.

We expand $|\psi(t)\rangle$ as $|\psi(t)\rangle = \sum_n a_n |E_n\rangle$, where $|E_n\rangle$ is the energy eigenstate of $H(t)$. We here drop the time dependence of a_n for brevity. Then the inner product between the state at times t and $t + \Delta t$ is calculated as

$$\begin{aligned} |\langle\psi(t)|\psi(t + \Delta t)\rangle| &= \left| \sum_n |a_n|^2 e^{-iE_n \Delta t / \hbar} \right| \\ &= \left| \left(1 - \frac{1}{2\hbar^2} \sum_n |a_n|^2 E_n^2 \Delta t^2 \right) - \frac{i}{\hbar} \sum_n |a_n|^2 E_n \Delta t + O(\Delta t^3) \right| \\ &= 1 - \frac{1}{2\hbar^2} \sum_n |a_n|^2 (E_n - \bar{E})^2 \Delta t^2 + O(\Delta t^3), \end{aligned} \quad (\text{A1})$$

where we defined the energy fluctuation as

$$\bar{E} := \sum_n |a_n|^2 E_n. \quad (\text{A2})$$

We also denote the energy fluctuation by

$$\Delta E^2 := \sum_n |a_n|^2 (E_n - \bar{E})^2, \quad (\text{A3})$$

which is equivalent to the conventional definition $\Delta E^2 = \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2$. Using an elemental inequality $\arccos x \leq \sqrt{2(1-x)}$ and the triangle inequality for the Bures angle $\mathcal{L}(\rho, \rho') \leq \mathcal{L}(\rho, \rho'') + \mathcal{L}(\rho'', \rho')$, we find

$$\mathcal{L}(|\psi(0)\rangle, |\psi(\tau)\rangle) \leq \sum_{n=0}^{N-1} \mathcal{L}(|\psi(t_n)\rangle, |\psi(t_{n+1})\rangle) \leq \sum_{n=0}^{N-1} \sqrt{2(1 - |\langle\psi(t_n)|\psi(t_{n+1})\rangle|)}, \quad (\text{A4})$$

with $N\Delta t = \tau$ and $t_n := n\Delta t$. By combining Eqs. (A1) and (A3) and taking $\Delta t \rightarrow 0$ limit, we arrive at the desired Mandelstam-Tamm inequality:

$$\mathcal{L}(|\psi(0)\rangle, |\psi(\tau)\rangle) \leq \frac{1}{\hbar} \int_0^\tau \sqrt{\Delta E^2} dt. \quad (\text{A5})$$

2. Case of mixed state

We now derive the Mandelstam-Tamm inequality for the case of mixed states by using the obtained relation for pure states (A5). The key ingredient of this extension is the purification of mixed states to pure states by introducing an auxiliary system A. For a given initial mixed state $\rho(0)$, there exists a proper auxiliary system and a pure state $|\Psi(0)\rangle$ on a composite system of the system S and the auxiliary system A satisfying $\text{Tr}_A[|\Psi(0)\rangle\langle\Psi(0)|] = \rho(0)$. An important fact is that if the state $\rho(0)$ evolves to

$\rho(\tau)$ with the Hamiltonian $H(t)$, then $|\Psi(0)\rangle$ evolves to $|\Psi(\tau)\rangle$ with the Hamiltonian $H(t) \otimes 1$ satisfying $\text{Tr}_A[|\Psi(\tau)\rangle \langle \Psi(\tau)|] = \rho(\tau)$. We note that both the expectation energy and the energy fluctuation take the same value between the original state and the purified pure state:

$$\text{Tr}_S[H(t)\rho(t)] = \text{Tr}_{SA}[(H(t) \otimes 1)|\Psi(t)\rangle \langle \Psi(t)|], \quad (\text{A6})$$

$$\text{Tr}_S[H(t)^2\rho(t)] = \text{Tr}_{SA}[(H(t) \otimes 1)^2|\Psi(t)\rangle \langle \Psi(t)|]. \quad (\text{A7})$$

Then, the combination of the Mandelstam-Tamm relation for pure states (A5) and the monotonicity of the Bures angle directly implies the desired Mandelstam-Tamm relation for mixed states:

$$\begin{aligned} \mathcal{L}(\rho(0), \rho(\tau)) &\leq \mathcal{L}(|\Psi(0)\rangle, |\Psi(\tau)\rangle) \leq \frac{1}{\hbar} \int_0^\tau \sqrt{\text{Tr}_{SA}[(H(t) \otimes 1)^2|\Psi(t)\rangle \langle \Psi(t)|] - \text{Tr}_{SA}[(H(t) \otimes 1)|\Psi(t)\rangle \langle \Psi(t)|]^2} dt \\ &= \frac{1}{\hbar} \int_0^\tau \sqrt{\Delta E^2} dt, \end{aligned} \quad (\text{A8})$$

where ΔE^2 is the energy fluctuation of the original system defined as $\Delta E^2 := \text{Tr}_S[H(t)^2\rho(t)] - \text{Tr}_S[H(t)\rho(t)]^2$.

APPENDIX B: ANOTHER SOLVABLE TOY EXAMPLE: INFINITE-RANGE INTERACTION

We here demonstrate the speed limit in the open quantum system (6) with another solvable model in addition to the Heisenberg chain presented in the main text. We use a globally coupled spin-1/2 system (also known as the mean-field model) that demonstrates the validity of (6), even for long-range coupling with arbitrary strength. We regard a single spin as a principal system and other N spins attached to this single spin as the environment. We focus on the state transformation of the single spin. We set the Hamiltonians as

$$\begin{aligned} H_S &= hS_1^z, \quad H_{SE} = J \sum_{i=2}^{N+1} \mathbf{S}_1 \cdot \mathbf{S}_i, \\ H_E &= h \sum_{i=2}^{N+1} S_i^z + J \sum_{i=2}^{N+1} \sum_{j>i}^{N+1} \mathbf{S}_i \cdot \mathbf{S}_j, \end{aligned} \quad (\text{B1})$$

where $\mathbf{S}_i$ is a vector of spin-1/2 operator at the site i , i.e., $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$. Let $|\uparrow\rangle_i$ ($|\downarrow\rangle_i$) be the eigenstate of the z component of the spin at the site i , satisfying $S_i^z|\uparrow\rangle_i = \frac{1}{2}|\uparrow\rangle_i$ ($S_i^z|\downarrow\rangle_i = -\frac{1}{2}|\downarrow\rangle_i$). For simplicity, we set the Hamiltonian as a static one. We set the initial state $\rho(0) = |\psi(0)\rangle \langle \psi(0)|$ with $|\psi(0)\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \cdots |\downarrow\rangle_{N+1}$. Using the conservation law on the total magnetization, it is straightforward to calculate the time-evolution of the state. The Bures angle for the one spin system during the time τ and the energy fluctuation for the Hamiltonian $\tilde{H}_S$ at time t ($0 \leq t \leq \tau$) is computed as

$$\begin{aligned} \mathcal{L}(\rho_S(\tau), \rho_S(0)) &= \arccos \left[\frac{\sqrt{(N-1)^2 + 4N \cos^2\left(\frac{JN\tau}{4\hbar}\right)}}{N+1} \right], \\ [\delta E_S(t)]^2 &= N \left\{ \left[\frac{J}{2} - \frac{2h(N-1)}{(N+1)^2} \sin^2\left(\frac{JNt}{4\hbar}\right) \right]^2 \right. \\ &\quad \left. + \left[\frac{h}{N+1} \sin\left(\frac{JNt}{4\hbar}\right) \right]^2 \right\}. \end{aligned}$$

For $N = 1$, these expressions are reduced to $\mathcal{L} = J\tau/4\hbar$ and $\delta E_S(t) = [(J/2)^2 + (h/2) \sin(Jt/4\hbar)]^{1/2}$. These clearly satisfy the relation (6). For a sufficiently small τ , the left-hand

side in (6) is approximately equal to $N\tau/(N+1)$ for any N , which is tight for a large N . In Fig. 3, we plot the left-hand side of (6) versus τ for several sizes of N with the fixed parameter $h/J = 1$, which clearly indicates the validity of (6) at all times.

APPENDIX C: THE SPEED LIMIT FOR CLOSED CLASSICAL HAMILTONIAN SYSTEMS

We here extend the speed limit (6) to classical Hamiltonian systems. Although several speed limits for classical Hamiltonian systems have been proposed, we employ the technique given in Ref. [53]. Consider a classical version of the total Hamiltonian corresponding to (2) in the main text,

$$H_{\text{tot}} = H_S + H_{SE} + H_E. \quad (\text{C1})$$

The distribution function in the phase space at time t is denoted by $\rho_{\text{tot}}(t)$. The dynamics of $\rho_{\text{tot}}(t)$ is given by the Liouville equation

$$(\partial/\partial t)\rho_{\text{tot}}(t) = \{H_{\text{tot}}, \rho_{\text{tot}}(t)\}, \quad (\text{C2})$$

FIG. 3. Speed-limit bounds [left-hand side of (6)] versus actual time τ for different sizes of N with the parameter $h/J = 1$. The unit of time is $\hbar/J$. The thin gray line represents $y = \tau$. All plots are below the gray line, as shown in (6).

where $\{A, B\} := \sum_{\alpha=x,y,z} \sum_i (\partial A / \partial q_{i,\alpha}) (\partial B / \partial p_{i,\alpha}) - (\partial A / \partial p_{i,\alpha}) (\partial B / \partial q_{i,\alpha})$ is the Poisson bracket, and $q_{i,\alpha}$ and $p_{i,\alpha}$ are, respectively, the position and momentum variables of the i th particle in the α th direction. We employ the classical Bures angle (Bhattacharyya angle) as the distance between two classical distribution:

$$\mathcal{L}_B(\rho, \rho') := \arccos \mathcal{B},$$

$$\mathcal{B} := \int d\Gamma \sqrt{\rho} \sqrt{\rho'}, \quad (\text{C3})$$

where $d\Gamma = \prod_{\alpha} \prod_i dp_{i,\alpha} dq_{i,\alpha}$, and $\mathcal{B}$ is the Bhattacharyya coefficient, which serves as a classical counterpart of the fidelity and affinity [52].

The distribution function for the system is defined as $\rho_S := \int d\Gamma_E \rho_{\text{tot}}$, where $\int d\Gamma_E$ is integration with respect to the degree of freedom in the environment. We then derive the speed limit for classical open Hamiltonian systems:

$$\frac{(2/\pi) \mathcal{L}_B(\rho_S(\tau), \rho_S(0))}{\langle \sqrt{\int d\Gamma \{ \tilde{H}_S, \sqrt{\rho_{\text{tot}}} \}^2} \rangle_{\tau}} \leq \tau, \quad (\text{C4})$$

where $\langle \dots \rangle_{\tau}$ implies the average over time. Similar to the quantum version (6), the bound contains the Hamiltonian $\tilde{H}_S$.

Before proving (C4), we first derive the speed-limit expression for the isolated classical system, which is first proved by Shanahan *et al.* [53]. Let $\rho(t)$ be a distribution function over

the whole phase space at time t . The dynamics is given by the Liouville equation:

$$\frac{d}{dt} \rho(t) = \{H(t), \rho(t)\} =: -i \mathbb{L}_t \rho(t). \quad (\text{C5})$$

Note that this type of equation is also satisfied for $\sqrt{\rho(t)}$, i.e., $(d/dt) \sqrt{\rho_t} = \{H(t), \sqrt{\rho(t)}\}$.

We define the Bures angle via the Bhattacharyya coefficient

$$\mathcal{L}_B(\rho(t), \rho(0)) := \arccos \mathcal{B}(t), \quad (\text{C6})$$

$$\mathcal{B}(t) := \int d\Gamma \sqrt{\rho(t)} \sqrt{\rho(0)}, \quad (\text{C7})$$

where $\int d\Gamma$ stands for the integration over the whole phase space. From these definitions, it is straightforward to derive the following relations:

$$\dot{\mathcal{B}} = -\sin \mathcal{L}_B \frac{d\mathcal{L}_B}{dt} \quad (\text{C8})$$

$$\dot{\mathcal{B}} = \int d\Gamma \sqrt{\rho(0)} \{H(t), \sqrt{\rho(t)}\}$$

$$= \int d\Gamma (\sqrt{\rho(0)} - \sqrt{\rho(t)}) \{H(t), \sqrt{\rho(t)}\}, \quad (\text{C9})$$

where we used an identity $\int d\Gamma \sqrt{\rho(t)} \{H(t), \sqrt{\rho(t)}\} = \int d\Gamma \frac{1}{2} \frac{d}{dt} (\sqrt{\rho(t)})^2 = 0$. Using the Schwartz inequality, one can bound $\dot{\mathcal{B}}$ from above:

$$\dot{\mathcal{B}} \leq \left[\int d\Gamma (\sqrt{\rho(0)} - \sqrt{\rho(t)})^2 \right]^{1/2} \left[\int d\Gamma (\{H(t), \sqrt{\rho(t)}\})^2 \right]^{1/2}$$

$$= [2(1 - \mathcal{B}(t))]^{1/2} \left[\int d\Gamma (\{H(t), \sqrt{\rho(t)}\})^2 \right]^{1/2}$$

$$= 2 \sin(\mathcal{L}_B/2) \left[\int d\Gamma (\{H(t), \sqrt{\rho(t)}\})^2 \right]^{1/2}, \quad (\text{C10})$$

where we note that $0 \leq \mathcal{L}_B \leq \pi$. Combining (C8) and this inequality, we have

$$2 \frac{d}{dt} \sin(\mathcal{L}_B/2) \leq \left[\int d\Gamma (\{H(t), \sqrt{\rho(t)}\})^2 \right]^{1/2}, \quad (\text{C11})$$

which directly leads to

$$2 \sin[\mathcal{L}_B(\rho(\tau), \rho(0))/2] = 2 \int_0^{\tau} dt \frac{d}{dt} \sin(\mathcal{L}_B(\rho(t), \rho(0))/2)$$

$$\leq \int_0^{\tau} dt \left[\int d\Gamma (\{H(t), \sqrt{\rho(t)}\})^2 \right]^{1/2}. \quad (\text{C12})$$

With noting the inequality $(1/\pi) \mathcal{L}_B \leq \sin(\mathcal{L}_B/2)$, we arrive at the following relation:

$$\frac{\mathcal{L}_B(\rho(\tau), \rho(0))}{\frac{\pi}{2\tau} \int_0^{\tau} \left[\int d\Gamma (\{H(t), \sqrt{\rho(t)}\})^2 \right]^{1/2}} \leq \tau. \quad (\text{C13})$$

We now provide the proof of (C4). We follow a similar procedure to the quantum case. We divide the Liouville operator into three contributions, i.e.,

$$\mathbb{L}_{\text{tot}}(t) = \mathbb{L}_S(t) + \mathbb{L}_{SE}(t) + \mathbb{L}_E(t), \quad (\text{C14})$$

where the subscripts S, SE, and E mean that the contribution from the system, coupling part between the system and environment, and the environment. Note the following identity

$$\int d\Gamma_E e^{-i\mathbb{L}_E(t)dt} A(\Gamma) = \int d\Gamma_E A(\Gamma), \quad (\text{C15})$$

where Γ_E is the phase space for the environment, and the function A is arbitrary function dependent on distribution function.

We note the relations

$$d\mathcal{L}_B(\rho', \rho'') \leq dt \frac{\pi}{2} \left[\int d\Gamma (\{\tilde{H}_S(t), \sqrt{\rho_{\text{tot}}(t)}\})^2 \right]^{1/2}, \quad (\text{C16})$$

$$\rho' = e^{-idt(\mathbb{L}_S(t) + \mathbb{L}_{SE}(t))} e^{-idt\mathbb{L}_E(t)/2} \rho_{\text{tot}}(t), \quad (\text{C17})$$

$$\rho'' = e^{-idt\mathbb{L}_E(t)/2} \rho_{\text{tot}}(t). \quad (\text{C18})$$

Subsequently, we note that the expression of the reduced distribution function

$$\begin{aligned} \rho_S(t) &= \int \Gamma_E \rho_{\text{tot}}(t) \\ &= \int \Gamma_E e^{-idt\mathbb{L}_E(t)/2} \rho_{\text{tot}}(t) = \int \Gamma_E \rho'', \end{aligned} \quad (\text{C19})$$

and

$$\begin{aligned} \rho_S(t + dt) &= \int \Gamma_E \rho_{\text{tot}}(t + dt) \\ &= \int \Gamma_E e^{-idt\mathbb{L}_E(t)/2} e^{-i(\mathbb{L}_S(t) + \mathbb{L}_{SE}(t))dt} e^{-idt\mathbb{L}_E(t)/2} \rho_{\text{tot}}(t) + O(dt^2) \\ &= \int \Gamma_E e^{-i(\mathbb{L}_S(t) + \mathbb{L}_{SE}(t))dt} e^{-idt\mathbb{L}_E(t)/2} \rho_{\text{tot}}(t) + O(dt^2) \\ &= \int \Gamma_E \rho' + O(dt^2). \end{aligned} \quad (\text{C20})$$

Through the contraction, the following relation is satisfied up to $O(dt)$:

$$\mathcal{L}_B(\rho_S(t + dt), \rho_S(t)) \leq \mathcal{L}_B(\rho', \rho''). \quad (\text{C21})$$

Then we have

$$\mathcal{L}_B(\rho_S(t + dt), \rho_S(t)) \leq dt \frac{\pi}{2} \left[\int d\Gamma (\{H_S(t) + H_{SE}(t), \sqrt{\rho_{\text{tot}}(t)}\})^2 \right]^{1/2}. \quad (\text{C22})$$

By integrating this from 0 to τ , the desired result (C4) is obtained.

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- [50] Suppose that in the cycle, two times t_1 and $t_2 (> t_1)$ give a maximum value of Bures angle. Then, the relation (6) leads to $t_2 - t_1 \geq \mathcal{L}(\rho(t_2), \rho(t_1)) / (\langle \delta E_S \rangle_\tau / \hbar) \geq \mathcal{L}(\rho(t_2), \rho(t_1)) / (\delta E_{S, \max} / \hbar)$. Considering the rest of time in the cycle whose time duration is $\tau_c - (t_2 - t_1)$, we get the desired

result. We assume that the state of the system can be the same only after the period τ_c .

- [51] If an operator A on SE acts nontrivially only on E (i.e., A can be expressed as $A = I_S \otimes A_E$), the cyclic property $\text{Tr}_E[AB] = \text{Tr}_E[BA]$ is satisfied.
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