Fractonic superfluids. II. Condensing subdimensional particles

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As a series of work about “fractonic superfluids,” in this paper, we develop an exotic fractonic superfluid phase in d -dimensional space where subdimensional particles—their mobility is *partially* restricted—are condensed. The off-diagonal long range order (ODLRO) is investigated. To demonstrate, we consider “lineons”—a subdimensional particle whose mobility is free only in certain one-dimensional directions. We start with a d -component microscopic Hamiltonian model. The model respects a higher-rank symmetry such that both particle numbers of each component and angular charge moments are conserved quantities. By performing the Hartree-Fock-Bogoliubov approximation, we derive a set of Gross-Pitaevskii equations and a Bogoliubov–de Gennes (BdG) Hamiltonian, which leads to a description of both condensed components and unification of gapless phonons and gapped rotons. With the coherent-path-integral representation, we also derive the long-wavelength effective field theory of gapless Goldstone modes and analyze quantum fluctuations around classical ground states. The Euler-Lagrange equations and Noether charges/currents are also studied. In two spatial dimensions and higher, such an ODLRO stays stable against quantum fluctuations. Finally, we study vortex configurations. The higher-rank symmetry enforces a hierarchy of thermal vortex excitations whose structures are dominated by two guiding statements. Specially, we construct two types of vortex excitations, the conventional and dipole vortices. The latter carries a charge with dimension as a momentum. The two statements can be more generally applicable. Further perspectives are discussed.

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I. INTRODUCTION

As exotic states of matter, fracton topological order can be characterized by noise-immune ground state degeneracy that unconventionally depends on the system size on a nontrivial compact manifold [1–4]. Recently, fracton topological order or *fracton physics* in a more general sense has been intensively investigated; see, e.g., Refs. [1–62]. A recent review can be found in Refs. [63,64]. Topological excitations of fracton topological order include fractons, subdimensional particles [1–4,26], and more complicated spatially extended excitations [27]. One of the remarkable features of these excitations is topological restriction on mobility: Their geometrical locations cannot be freely changed by any local operators. More concretely, mobility of fractons is completely frozen while subdimensional particles are still allowed to move but within a certain cluster of lower-dimensional subspace. As two examples of subdimensional particles in the X -cube lattice model [1–4], lineons and planeons can move along certain one-dimensional directions and two-dimensional parallel

planes, respectively. Instead of the interpretation as “topological excitations,” one can also regard all these strange particles, i.e., fractons, lineons, and planeons as original bosons, which leads to unconventional many-body physics. In this context, the restriction on mobility is ascribed to the implementation of so-called “higher-rank symmetry.” The latter guarantees a set of higher moments are conserved [25,33,65].

In a previous work [47] of many-body physics of fractons, the authors of the present work proposed a *fractonic superfluid phase* formed by nonrelativistic bosonic fractons in d spatial dimensions (dD). The phase can be regarded as a result of spontaneous breakdown of higher-rank symmetry. Due to a higher-rank symmetry, both total dipole moments and the total particle number (charge) are conserved. The microscopic Hamiltonian that respects such a symmetry must be non-Gaussian, which naturally forbids a single fracton from freely propagating. Starting with the first-order time derivative just like a conventional superfluid phase, we add a usual Mexican-hat potential for fractons. When the chemical potential is turned from a negative to positive value, the system undergoes a quantum phase transition from the normal state to the superfluid phase. The latter is manifested by occupation of a macroscopic number of fractons on the same quantum state, which leads to the formation of an off-diagonal long range order (ODLRO) [66]. As a direct consequence of non-Gaussianity, the corresponding Euler-Lagrange equation is highly nonlinear, from which one can extract the

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TABLE I. Comparison between a conventional superfluid phase (denoted as $d\text{SF}^d$), fractonic superfluid phase (denoted as $d\text{SF}^0$; see Ref. [47]) via condensing fractons, and a fractonic superfluid phase (denoted as $d\text{SF}^1$) via condensing lineons. In these three types of superfluids, the condensed particles are, respectively, usual bosons of full mobility, fractons without any mobility, and lineons with partial mobility. Vortex excitations in 2D form a hierarchy where ℓ , ℓ_1 , ℓ_2 denote winding numbers and p , p_1 , p_2 are quantized as momenta with $\varphi(\mathbf{x})$ being the relative angle of site $\mathbf{x}$ to the vortex core.

	$d\text{SF}^d$	$d\text{SF}^0$	$d\text{SF}^1$
Conserved quantities	Total charge	Total charge, total dipole moment	Total charges, angular charge moments
Order parameter	$\sqrt{\rho_0}e^{i\theta_0}$	$\sqrt{\rho_0}e^{i(\theta_0+\sum_a\beta_ax^a)}$	$\sqrt{\rho_0}e^{i(\theta_0+\sum_b\beta_{ab}x^a)}(\beta_{ab}=-\beta_{ba})$
Plane-wave dispersion	Dispersive	Dispersionless	Partially dispersive
Ground state	$e^{\int d^d x \sqrt{\rho_0} e^{i\theta_0} \hat{\Phi}^\dagger(\mathbf{x})} 0\rangle$	$e^{\int d^d x \sqrt{\rho_0} e^{i(\theta_0+\sum_a\beta_ax^a)} \hat{\Phi}^\dagger(\mathbf{x})} 0\rangle$	$\prod_a e^{\int d^d x \sqrt{\rho_0} e^{i(\theta_0+\sum_b\beta_{ab}x^a)} \hat{\Phi}_a^\dagger(\mathbf{x})} 0\rangle$
Specific capacity heat	$c_v \propto T^d$	$c_v \propto T^{\frac{d}{2}}$	$c_v \propto T^d$
Number of Goldstone modes	1	1	d
Dispersion of Goldstone modes	$\omega \propto \mathbf{k} $	$\omega \propto \mathbf{k} ^2$	$\omega \propto \mathbf{k} $
Stable dimension at $T = 0$	$d > 1$	$d > 2$	$d > 1$
Vortex structure in $d = 2$	$\ell\varphi(\mathbf{x})$	$\ell\varphi(\mathbf{x}), (p_1x + p_2y)\varphi(\mathbf{x})$	$\ell_1\varphi(\mathbf{x}), px^2\varphi(\mathbf{x})$ $\ell_2\varphi(\mathbf{x}), px^1\varphi(\mathbf{x})$

time-dependent Gross-Pitaevskii equation that governs hydrodynamical behaviors. Furthermore, by taking quantum phase fluctuations into consideration, we find that ODLRO keeps stable in spatial dimensions $d > 2$. In one dimension, the correlation function of the superfluid order parameter exponentially decays at long distances. In two dimensions (2D), it decays in a power-law pattern at zero temperature.

As mentioned above, fractons are just one of many strange particles proposed in fracton topological order. We expect that subdimensional particles can form even more exotic phases of matter as their mobility is partially rather than completely restricted. Along this line, in this work, we consider more general variants of fractonic superfluids where, instead of completely immobile fractons, subdimensional particles meet Mexican hats and thus form a superfluid. For convenience, we introduce a notation $d\text{SF}^n$ (“SF” stands for “superfluid”) which represents a superfluid phase in d spatial dimensions via condensing subdimensional particles of dimension n ($0 \leq n \leq d$). For example, $d\text{SF}^d$ denotes a conventional superfluid where the bosons are free to move in the whole space and the fractonic superfluid phase in a many-fracton model [47] is symbolized as $d\text{SF}^0$.

More specifically, in this work, we take lineons as an example, which leads to a fractonic superfluid phase denoted by $d\text{SF}^1$. The corresponding microscopic second-quantized Hamiltonian model $\mathcal{H}$ contains d components of bosonic lineons. The Hamiltonian has strongly anisotropic quadratic Gaussian terms such that mobility restriction of lineons is correctly encoded. Meanwhile, both angular charge moments and particle numbers of each component are conserved due to the presence of quartic terms that respect a higher-rank symmetry. The candidate Hamiltonian is referred to in Ref. [25] with the second time-derivative terms. Distinguishably, we set about the first-time derivative and apply a Mexican-hat potential to each component. In the coherent-path-integral representation, the Hamiltonian density $\mathcal{H}$ is sent to the Lagrangian density $\mathcal{L} = \sum_{a=1}^d i\phi_a^* \partial_t \phi_a - \mathcal{H}$ after a Wick rotation. Due to the first-order time derivative, we are legitimate to interpret $\phi_a^* \phi_a$ as the particle density of the a^{th} component, which is common in condensed matter and cold-atom systems.

When the chemical potential is turned to a positive value from a negative one, a quantum phase transition occurs from the normal state to the fractonic superfluid phase $d\text{SF}^1$. The normal state has a unique ground state with vanishing momentum. Instead, in $d\text{SF}^1$, ODLRO the ground states are macroscopically degenerate in the classical level and their configurations appear to be a plane wave with a finite momentum and a finite density distribution which establishes a ODLRO. Upon the Hartree-Fock-Bogoliubov mean-field approximation is applied, the boson fields are further split into two components—the normal and the condensed components, towards which we derive a set of nonlinear Bogoliubov–de Gennes equations and Gross-Pitaevskii Hamiltonian, respectively. To deal with gapless Goldstone modes and quantum phase fluctuations (i.e., ODLRO stability at infrared limit), we turn to the framework of an effective field theory. In contrast to the non-Gaussian system in Ref. [47], existence of spatially anisotropic Gaussianity ensures that a superfluid phase or ODLRO can survive against quantum fluctuations in spatial dimensions $d > 1$ at zero temperature and thus this model appears more tractable experimentally. For vortex configurations in 2SF^1 , we point out and apply two statements for the purpose of constructing vortex configurations. The first statement dominates the multivalued part to meet the single valueness of vortex fields and the second controls the smooth part to satisfy a relation between representations of a higher-rank group and a particle number conservation symmetry group. The two statements lead to two types of vortices: the conventional vortex and dipole vortex. The latter carries a dipole charge that is quantized like a momentum. It can be detected by a vorticity from recombination between Noether currents. In fact, the two statements can be applied to point vortex excitations with a general higher-rank symmetry. In Table I, we compare different properties of a conventional superfluid phase 2SF^2 , fractonic superfluid via condensing fractons 2SF^0 in Ref. [47] and fractonic superfluid by condensing lineons 2SF^1 in this work.

The remaining part of this paper is organized as follows. Section II provides a microscopic multicomponent model and Hartree-Fock-Bogoliubov treatment. An effective field theoretical analysis is performed in Sec. III. In Sec. IV, exotic

superfluid vortices are studied. This work is concluded in Sec. V.

II. MICROSCOPIC SYSTEM AND MEAN-FIELD THEORY

In this section, we start with a microscopic model of lineons, which is formulated in the second quantization language with conserved angular charge moments. This conservation is vital to the mobility restriction of lineons. Under the circumstance of condensing lineons, we apply the Hartree-Fock-Bogoliubov (HFB) mean-field theory to derive the Gross-Pitaevskii (GP) equations and Bogoliubov-de Gennes (BdG) Hamiltonian [67]. The former govern the order parameter of the superfluid phase and the latter unifies both gapless phonon and gapped roton modes.

A. A model Hamiltonian

In condensed matter systems, strong anisotropy can constrain particle's propagation. For example, divergence of effective mass localizes particle spatially and a strong electric field allows charged particles to move exclusively along the direction of electric field. We have investigated one microscopic realization of fractons with fully restricted motion in Ref. [47]. As a series of works, here we focus on d -component fields $\hat{\Phi} = (\hat{\Phi}_1, \dots, \hat{\Phi}_d)$ in d spatial dimensions in the Hamiltonian $H = \int d^d \mathbf{x} \mathcal{H}$ where Hamiltonian density $\mathcal{H}$ reads

$$\mathcal{H} = \sum_{a=1}^d \partial_a \hat{\Phi}_a^\dagger \partial_a \hat{\Phi}_a + \sum_{a \neq b} \frac{1}{2} K_{ab} (\hat{\Phi}_a^\dagger \partial_a \hat{\Phi}_b^\dagger + \hat{\Phi}_b^\dagger \partial_b \hat{\Phi}_a^\dagger) \times (\hat{\Phi}_a \partial_a \hat{\Phi}_b + \hat{\Phi}_b \partial_b \hat{\Phi}_a) + V(\hat{\Phi}^\dagger, \hat{\Phi}). \quad (1)$$

The complex fields $\hat{\Phi}_a^\dagger(\mathbf{x})$ and $\hat{\Phi}_b(\mathbf{x})$ create and annihilate an a th-component particle, respectively, and satisfy the bosonic commutation relations,

$$[\hat{\Phi}_a(\mathbf{x}), \hat{\Phi}_b^\dagger(\mathbf{y})] = \delta_{ab} \delta^d(\mathbf{x} - \mathbf{y}), \quad (2)$$

where $\mathbf{x} = (x^1, \dots, x^d)$ is the spatial coordinate. At the quadratic level, each component can only propagate in one certain spatial direction and we set the mass before the quadratic terms to be a unit. For convenience, one can set diagonal terms $K_{aa} = 0$ for $a = 1, \dots, d$ since diagonal terms K_{aa} is absent in Eq. (1). For simplicity, we can take the term $V(\hat{\Phi}^\dagger, \hat{\Phi})$ to be the Mexican-hat potential with component-independent chemical potential μ and interaction coupling constant $g > 0$,

$$V(\hat{\Phi}^\dagger, \hat{\Phi}) = \sum_{a=1}^d -\mu \hat{\Phi}_a^\dagger \hat{\Phi}_a + \frac{g}{2} \hat{\Phi}_a^\dagger \hat{\Phi}_a^\dagger \hat{\Phi}_a \hat{\Phi}_a, \quad (3)$$

which describes a short-range repulsive interaction via the s -wave scattering. In the following, *no Einstein summation rule is assumed*. The Hamiltonian in Eq. (1) conserves both particle numbers of each component $Q_a \equiv \int d^d x \hat{\rho}_a$ and angular charge moment $Q_{ab} = \int d^d x (\hat{\rho}_a x^b - \hat{\rho}_b x^a)$ with $\hat{\rho}_a = \hat{\Phi}_a^\dagger \hat{\Phi}_a$ being the number operator of a th particles. Accordingly, the symmetry group is composed of transformations $\hat{\Phi}_a \rightarrow e^{i\lambda_a} \hat{\Phi}_a$ for each component and

$$(\hat{\Phi}_a, \hat{\Phi}_b) \rightarrow (\hat{\Phi}_a e^{i\lambda_{ab} x^b}, \hat{\Phi}_b e^{-i\lambda_{ab} x^a}) \quad (4)$$

$$\hat{\Phi} = \begin{pmatrix} \text{red ball} \\ \text{blue ball} \end{pmatrix}$$

FIG. 1. Illustration of the interacting system in two dimensions. The red and blue balls respectively represent two components $a = 1, 2$, which move along distinct orthogonal directions. The spring between two balls represents the interaction due to the K -term in Eq. (1).

for each pair of indices with $\lambda_a, \lambda_{ab} \in \mathbb{R}$. The parameters λ_{ab} are antisymmetric $\lambda_{ab} = -\lambda_{ba}$, thus inducing $\frac{d(d-1)}{2}$ independent conserved angular charge moments Q_{ab} . The transformations in Eq. (4) involve local coordinates $\mathbf{x} = (x^1, x^2, \dots, x^d)$, and they do not form an internal symmetry. We denote the symmetry group as $\mathcal{G}$ which characterizes a higher-rank symmetry [65]. In a periodic boundary condition, the parameters λ_{ab} have the dimension of $[x]^{-1}$ and quantization of the related charges is expected to coincide with a momentum. This symmetry intertwines global and internal symmetries such that strong constraints are imposed on particles' propagations. In two dimensions, conservation of $Q_{12} = \int d^2 x (\hat{\rho}_1 x^2 - \hat{\rho}_2 x^1)$ requires the velocity shall be parallel to a vector $(\hat{\rho}_1, \hat{\rho}_2)$ as a lineon. In three spatial dimensions, we have three angular charge moments Q_{12}, Q_{23}, Q_{13} , such that a particle only propagates in the direction parallel to $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)$. Generally, fundamental particles in dD move with velocity parallel to $(\hat{\rho}_1, \dots, \hat{\rho}_d)$. Figure 1 pictorially shows the interacting system when $d = 2$.

B. Hartree-Fock-Bogoliubov mean-field theory: condensate and rotons

It is well known that a Bose-Einstein condensate consists of a two-component structure: the condensate and the normal components. The HFB mean-field theory allows factorization of fields $\hat{\Phi}_a$ in terms of an appropriate orthonormal single-particle basis,

$$\hat{\Phi}_a(\mathbf{x}) = \phi_{a0}(\mathbf{x}) \hat{c}_{a0} + \sum_{i \neq 0} \phi_{ai}(\mathbf{x}) \hat{c}_{ai} \equiv \phi_{a0}(\mathbf{x}) \hat{c}_{a0} + \hat{\psi}_a(\mathbf{x}), \quad (5)$$

with $\hat{\psi}_a(\mathbf{x}) \equiv \sum_{i \neq 0} \phi_{ai}(\mathbf{x}) \hat{c}_{ai}$ where the operator $\hat{c}_{ai}$ satisfies the bosonic commutation relations $[\hat{c}_{ai}, \hat{c}_{bj}^\dagger] = \delta_{ab} \delta_{ij}$. The wave functions $\phi_{a0}(\mathbf{x})$ signify the condensate component. The noncondensate fields $\hat{c}_{ai} (i \geq 1)$ constitute a branch of gapped quasiparticle excitations and are orthogonal with the condensate component $\phi_{a0}(\mathbf{x})$,

$$\int d^d \mathbf{x} \phi_{a0}(\mathbf{x}) \phi_{bi}^*(\mathbf{x}) = 0 \text{ for } i \geq 1. \quad (6)$$

Here the subscript index “ i ” can be taken as a spectral index of the BdG Hamiltonian [see Eq. (16)], and thus ϕ_{ai} can denote wave functions of low-energy excitations. We take the normal component as a perturbation to the condensate. Substituting Eq. (5) into the Hamiltonian in Eq. (1) leads to a partition into terms with different numbers of field operators $\hat{\psi}_a(\mathbf{x})$. The zeroth-order term is given by

$$H_0 = \sum_{a=1}^d h_{1,a} \hat{N}_{a0} + \sum_{a,b} h_{2,ab} (\hat{N}_{a0} \hat{N}_{b0} - \delta_{ab} \hat{N}_{a0}), \quad (7)$$

where $\hat{N}_{a0} = \hat{c}_{a0}^\dagger \hat{c}_{a0}$ is the number operator for the condensate and $h_1(h_2)$ is the expectation value of the single-particle (two-particle) part of the Hamiltonian,

$$h_{1,a} = \int d^d \mathbf{x} |\partial_a \phi_{a0}|^2 - \mu |\phi_{a0}|^2, \quad (8)$$

$$h_{2,ab} = \int d^d \mathbf{x} \frac{1}{2} K_{ab} |\phi_{a0} \partial_a \phi_{b0} + \phi_{b0} \partial_b \phi_{a0}|^2 + \frac{g \delta_{ab}}{2} |\phi_{a0} \phi_{b0}|^2. \quad (9)$$

Under the particle-number representation $|\{N_{a0}\}\rangle$ with definite condensate particle number $\hat{N}_{b0} |\{N_{a0}\}\rangle = N_{b0} |\{N_{a0}\}\rangle$ for $b = 1, \dots, d$, the ground state energy $E_0(\{N_a\})$ of H_0 in Eq. (7) only depends on the condensate,

$$\begin{aligned} E_0(\{N_a\}) &= \langle \{N_{a0}\} | H_0 | \{N_{a0}\} \rangle \\ &= \sum_a h_{1,a} N_{a0} + \sum_{a,b} h_{2,ab} (N_{a0} N_{b0} - \delta_{ab} N_{a0}). \end{aligned} \quad (10)$$

The next order H_1 has linear dependence on $\hat{\psi}_a(\mathbf{x})$,

$$H_1 = \int d^d \mathbf{x} \sum_{a=1}^d \hat{c}_{a0}^\dagger \phi_{a0}^* \hat{H}_a \hat{\psi}_a + \hat{\psi}_a^\dagger \hat{H}_a \phi_{a0} \hat{c}_{a0}, \quad (11)$$

where

$$\hat{H}_a = -\partial_a^2 - \mu + g |\phi_{a0}|^2 \hat{N}_{a0} + \sum_{a,b} \frac{1}{2} K_{ab} \hat{N}_{b0} \hat{H}_{ab}, \quad (12)$$

with

$$\begin{aligned} \hat{H}_{ab} &= (\partial_a \phi_{b0}^*)(\partial_a \phi_{b0}) + (\partial_a \phi_{b0}^*) \phi_{b0} \partial_b - (\partial_b \phi_{b0}^*)(\partial_a \phi_{b0}) \\ &\quad - \phi_{b0}^*(\partial_a \phi_{b0}) \partial_b - \phi_{b0}^*(\partial_a \partial_b \phi_{b0}) - (\partial_b \phi_{b0}^*) \phi_{b0} \partial_b \\ &\quad - \phi_{b0}^*(\partial_b \phi_{b0}) \partial_b - \phi_{b0}^* \phi_{b0} \partial_b^2. \end{aligned}$$

Under a basis $|\{N_{a0}\}\rangle$, taking the limit of $N_{a0} \gg 1$, one can recognize $\hat{N}_{a0} \hat{c}_{a0} |\{N_a\}\rangle = \hat{c}_{a0} \hat{N}_{a0} |\{N_a\}\rangle$. If $\phi_{a0}(\mathbf{x})$ are chosen to be eigenstates of the operator $\hat{H}_a$ in Eq. (12),

$$\hat{H}_a(\mathbf{x}) \phi_{a0}(\mathbf{x}) = \epsilon_{a0}(\{N_{a0}\}) \phi_{a0}(\mathbf{x}) \quad a = 1, \dots, d, \quad (13)$$

then H_1 vanishes identically due to orthogonality in Eq. (6) and $N_{a0} \gg 1$, where ground state energy $\epsilon_{a0}(\{N_{a0}\})$ merely depends on the condensate component. This fact establishes the validity of the expansion in Eq. (5). Equation (13) marks a set of the GP equations describing the condensate components where $\hat{H}_a$ in Eq. (12) behaves as a single-particle Hamiltonian. It simply directs us to approximate the original Hamiltonian in Eq. (1) by Eq. (7). With the translational symmetry, the GP equations have a set of very simple solutions. If

the chemical potential μ is negative, the ground state energy E_0 reaches its minimum when the condensate has vanishing density, $N_{a0} = 0$ for $a = 1, \dots, d$. We obtain a normal state. If the chemical potential is switched to a positive value, the ground state energy E_0 reaches maximum with a finite number of the condensate component. In this case, the configurations of ground states can be parametrized by real parameters θ_a and β_{ab} ($\beta_{ab} = -\beta_{ba}$, $a, b = 1, \dots, d$),

$$\phi_{a0}(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{i(\theta_a + \sum_{b=1}^d \beta_{ab} x^b)}, \quad (14)$$

with V as the spatial volume. Remarkably, Eq. (14) depends on the parameters β_{ab} which carry the dimension of “momentum.” In other words, the condensate component can carry finite momentum. Of course, one can include the trap potential that can break a translational symmetry, under which the GP equations may be short of analytical solutions. Casting the solution in Eq. (14) back to H_0 in Eq. (7), we have the ground state energy $E_0(\{N_{a0}\})$,

$$E_0(\{N_{a0}\}) = \sum_{a=1}^d -\mu \frac{N_{a0}}{V} + \frac{g}{2} \left(\frac{N_{a0}}{V} \right)^2. \quad (15)$$

The minimal condition of $E_0(\{N_{a0}\})$ with a positive chemical potential μ requires $\rho_{a0} \equiv \frac{N_{a0}}{V} = \frac{\mu}{g}$, which indicates the ground states for a fractonic superfluid has a macroscopically finite particle density. Hence, we obtain a superfluid phase by condensing lineons, which we dub $d\text{SF}^1$. Therefore, select one ground state in Eq. (14) and we can fix the condensate particle number N_{a0} by simply replacing both $\hat{c}_{a0}$ and $\hat{c}_{a0}^\dagger$ operators by c-number $\sqrt{N_{a0}}$, which indicates occurrence of ODLRO with the condensate density ρ_{a0} .

The next order H_2 goes beyond the GP equation to include the quadratic terms of ψ_a ,

$$\begin{aligned} H_2 &= \sum_{a=1}^d \int d^d \mathbf{x} \hat{\psi}_a^\dagger (-\partial_a^2 + g \rho_{a0}) \hat{\psi}_a \\ &\quad + \sum_{a,b} \frac{1}{2} K_{ab} \rho_{a0} (\partial_a \hat{\psi}_b^\dagger + \partial_b \hat{\psi}_a^\dagger) (\partial_a \hat{\psi}_b + \partial_b \hat{\psi}_a) \\ &\quad + \sum_{a=1}^d \frac{g}{2} \rho_{a0} (\hat{\psi}_a^\dagger \hat{\psi}_a^\dagger + \hat{\psi}_a \hat{\psi}_a), \end{aligned} \quad (16)$$

where we have replaced $\hat{c}_{a0}$ and $\hat{c}_{a0}^\dagger$ with $\sqrt{N_{a0}}$. New quadratic terms emerge from the K_{ab} term which relaxes the restrictions on dynamics. It means that the quasiparticle modes $\hat{\psi}_a$ can propagate along all spatial directions. The mass term $g \rho_{a0} \hat{\psi}_a^\dagger \hat{\psi}_a$ originates from the condensate component. H_2 in Eq. (16) is designated as a BdG Hamiltonian to characterize the noncondensate quasiparticle modes. One can diagonalize H_2 to obtain the canonical quasiparticle modes. For example, in 2D, the spectrum has two branches,

$$\epsilon_{\pm} = \sqrt{\left[\frac{(1 + K \rho_0) |\mathbf{k}|^2}{2} \pm \frac{\Delta(\mathbf{k})}{2} + g \rho_0 \right]^2 - (g \rho_0)^2}. \quad (17)$$

FIG. 2. Spectrum dispersions of quasiparticles from HFB approximation in Eq. (17) in 2D with parameters in (a) and (c) $K = 1$, $\rho_0 = 1$, and (b) and (d) $K = 1.5$, $\rho_0 = 1$. In (a) the two bands are degenerate at two lines $k_1 = 0$ and $k_2 = 0$ which are guided by green dashed lines. The gap between the two bands in (b) keeps finite except at the point $\mathbf{k} = \mathbf{0}$. (c) and (d) show the dispersion relations at the line $k_2 = 0$. Both in (a) and (b), the dispersions are linear at small $\mathbf{k}$ and quadratic at large $\mathbf{k}$.

For small momentum, up to the first order, we have linear gapless dispersion relations,

$$\epsilon_{\pm}(\mathbf{k}) = \sqrt{g\rho_0} \sqrt{(1 + K\rho_0)\mathbf{k}^2 \pm \Delta(\mathbf{k})} \quad \text{small } \mathbf{k}, \quad (18)$$

and they describe two gapless phonon excitations, while at the large momentum, spectrum dispersions depend on momentum quadratically,

$$\epsilon_{\pm} = \frac{(1 + K\rho_0)|\mathbf{k}|^2}{2} \pm \frac{\Delta(\mathbf{k})}{2} \quad \text{large } \mathbf{k}, \quad (19)$$

and instead they correspond to gapped roton modes. Here $\Delta(\mathbf{k}) \equiv \sqrt{(k_1^2 - k_2^2)^2 (K\rho_0 - 1)^2 + 4K^2\rho_0^2 k_1^2 k_2^2}$, $\rho_0 = \frac{\mu}{g}$, and $K \equiv K_{12}$. The smooth change from linear to quartic dispersion is the key feature of the HFB approximation. Although two modes, phonons and rotons, are emphasized, indeed they represent different behaviors at small and high momentum, respectively. The splitting between the two dispersion relations in Eqs. (18) and (19) is controlled by $\Delta(\mathbf{k})$. If $K\rho_0 = 1$ such that $\Delta(\mathbf{k}) = 0$, the two bands are degenerate at two lines $k_1 = 0$ or $k_2 = 0$ and along the two lines, the bands have ill-defined curvatures, which is expected to be detected by thermal Hall effect. Figure 2 depicts the two dispersion relations in Eq. (17) in which the two cases of $K\rho_0 = 1$ or not.

Higher-order terms couple the normal with the condensate part and describe the interaction between phonon modes, which is beyond the scope of this work and we leave it to future work.

III. EFFECTIVE FIELD THEORY

The HFB mean-field method unifies gapless phonon modes and gapped roton modes via a BdG Hamiltonian in Eq. (16) in a fractonic superfluid phase $d\text{SF}^1$. Nevertheless, gapless mode excitations can destroy BEC or ODLRO. In this section, we deal with gapless modes of $d\text{SF}^1$ in the framework of a continuous field theory and discuss stability of $d\text{SF}^1$ against quantum fluctuations.

A. Euler-Lagrange equation and Noether charge/current

For the coherence and completeness of the present section, we re-derive some quantities from the field-theoretical perspective.

We perform a coherent-state path integral quantization [68] to get the Lagrangian density $\mathcal{L}$,

$$\mathcal{L} = \sum_{a=1}^d i\phi_a^* \partial_t \phi_a - \mathcal{H}(\phi), \quad (20)$$

where $\phi_a(\mathbf{x}, t)$ is an eigenvalue of $\hat{\Phi}_a(\mathbf{x})$ on a coherent state $\hat{\Phi}_a(\mathbf{x})|\phi_a(\mathbf{x}, t)\rangle = \phi_a(\mathbf{x}, t)|\phi_a(\mathbf{x}, t)\rangle$. The first-order derivative in Eq. (20) in nature is determined by the commutation relation in Eq. (2) which can be confirmed through the canonical quantization. For convenience, we apply the Wick's rotation to an imaginary time at zero temperature $T = 0$.

Next, we can derive the Euler-Lagrange equations as well as the Noether currents associated with two types of conserved quantities. The Euler-Lagrange equations can be derived from the formula $\frac{\delta \mathcal{L}}{\delta \phi_a^*} = 0$, explicitly, $i\partial_t \phi_a = \hat{H}_a \phi_a$ ($a = 1, \dots, d$), where $\hat{H}_a$ has the same form as $\mathcal{H}_a$ in Eq. (12),

$$\hat{H}_a = -\partial_a^2 - \mu + g|\phi_a|^2 + \frac{1}{2} \sum_b \mathcal{H}_{ab}. \quad (21)$$

Here $\mathcal{H}_{ab}$ comes from the K_{ab} term,

$$\begin{aligned} \mathcal{H}_{ab} = & K_{ab} \partial_a \phi_b^* \partial_a \phi_b + K_{ab} \partial_a \phi_b^* \phi_b \partial_b - K_{ab} \partial_b \phi_b^* \partial_a \phi_b \\ & - K_{ab} \phi_b^* \partial_a \phi_b \partial_b - K_{ab} \phi_b^* \partial_a \partial_b \phi_b - K_{ab} \partial_b \phi_b^* \phi_b \partial_b \\ & - K_{ab} \phi_b^* \partial_b \phi_b \partial_b - K_{ab} \phi_b^* \phi_b \partial_b^2. \end{aligned} \quad (22)$$

The Euler-Lagrange equations just recover the GP equations. Here ϕ_a plays the same role of representing the condensate component as ϕ_{a0} in Eq. (13).

The Hamiltonian in Eq. (1) stays invariant under transformation in Eq. (4) as well as the particle number conservation symmetry. For the infinitesimal change $\delta \phi_a = i\alpha_a \phi_a$, we have the related Noether charge Q_a with charge density ρ_a and currents J_i^a that read

$$Q^a = \int d^d \mathbf{x} \phi_a^* \phi_a \equiv \int d^d \mathbf{x} \rho_a, \quad (23)$$

$$\begin{aligned} J_i^a = & iK_{ai} \rho_a (\phi_i \partial_a \phi_i^* - \phi_i^* \partial_a \phi_i) \\ & + iK_{ai} \rho_i (\phi_a \partial_i \phi_a^* - \phi_a^* \partial_i \phi_a) \\ & + i(\phi_i \partial_i \phi_i^* - \phi_i^* \partial_i \phi_i) \delta_{ai}, \end{aligned} \quad (24)$$

which satisfies the continuity equations $\partial_t \rho^a + \sum_i \partial_i J_i^a = 0$. Here in Eq. (23), coincidence between $\phi_a^* \phi_a$ and particle density ρ_a arises from the first-order time derivative in the Hamiltonian in Eq. (1). For the transformation $\delta \phi_a =$

$ix_b\phi_a$ and $\delta\phi_b = -ix_a\phi_b$ corresponding to Eq. (4), we have conserved angular moments Q_{ab} (with density ρ_{ab}) and currents D_i^{ab} ,

$$Q_{ab} = \int d^d x (\rho_a x^b - \rho_b x^a) \equiv \int d^d x \rho_{ab}, \quad (25)$$

$$D_i^{ab} = x^b J_i^a - x^a J_i^b, \quad (26)$$

with ρ_a and J_i^a as $U(1)$ charge and current in Eqs. (23) and (24). The continuity equation $\partial_t \rho^{ab} + \sum_{i=1}^d \partial_i D_i^{ab} = 0$ is automatically satisfied as long as the currents J_b^a obey the relations $J_b^a = J_a^b$.

B. Goldstone modes and quantum fluctuations

The HFB mean-field theory in Sec. II B starts with one of the classical field configurations $\phi_a^{\text{cl}} = \sqrt{\rho_0} e^{i(\theta_a + i \sum_{b=1}^d \beta_{ab} x^b)}$ which can be formulated in the second quantization language as

$$|\text{GS}\rangle_{\theta_a}^{\beta_{ab}} = \prod_{a=1}^d \exp[\sqrt{\rho_0} e^{i(\theta_a + i \sum_{b=1}^d \beta_{ab} x^b)} \hat{\Phi}_a^\dagger(\mathbf{x})] |0\rangle, \quad (27)$$

where $\hat{\Phi}_a^\dagger(\mathbf{x})$ ($a = 1, \dots, d$) creates an a th-component lineon with restricted motion. The salience of Eq. (27) features a finite expectation value of operator $\hat{\Phi}_a(\mathbf{x})$,

$$\langle \text{GS} | \hat{\Phi}_a(\mathbf{x}) | \text{GS} \rangle_{\theta_a}^{\beta_{ab}} = \sqrt{\rho_0} \exp \left(i\theta_a + i \sum_b \beta_{ab} x^b \right), \quad (28)$$

thus marking an ODLRO and we obtain a fractonic superfluid phase $d\text{SF}^1$. The expectation value oscillates as a plane wave with fixed momentum $\mathbf{k}_a = (\beta_{a1}, \beta_{a2}, \dots, \beta_{ad})$ for the a -component particle. In this sense, we can rewrite $|\text{GS}\rangle_{\theta_a}^{\beta_{ab}} = \prod_{a=1}^d \exp[\sqrt{\rho_0} e^{i\theta_a} \hat{\Phi}_a^\dagger(\mathbf{k}_a)] |0\rangle$ with $\hat{\Phi}_a^\dagger(\mathbf{k}_a)$ being the Fourier transformation of $\hat{\Phi}_a^\dagger(\mathbf{x})$. These features arise from restricted mobility of condensed particles. As a side note, there is an interesting observation. The ground state in Eq. (27), which carries a finite momentum, looks like an Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state [69,70], if we define the FFLO state as an ordered phase described by an order parameter with a finite momentum. But there are fundamental differences in the microscopic mechanisms. The general FFLO states in the literature are formed by Cooper pairs of *fermions* (i.e., electrons) and the momentum of the center of mass of a Cooper pair is nonzero. In contrast, the superfluid phase in Eq. (27) is directly formed by condensing multicomponent *bosonic* lineons. The degenerate ground-state subspace of fractonic superfluids is generated by group elements of higher-rank symmetry group $\mathcal{G}$ (see Sec. II A), thereby spontaneously breaking the higher-rank symmetry group. In other words, the microscopic Hamiltonian of fractonic superfluids must respect an exotic higher-rank symmetry. Despite the differences, it still will be valuable to explore the possibility of realizing fractonic superfluids from a more microscopic model where “molecules” formed by fermion pairings can be regarded as bosonic lineons.

After condensation, the Noether currents in Eqs. (24) and (26) reduce to simpler forms by expanding the field $\phi_a = \phi_a^{\text{cl}} e^{i\theta_a}$ where ϕ_a^{cl} denotes classical configurations and θ_a are

the quantum phase fluctuations,

$$\rho_a = \rho_0, J_i^a = 2K\rho_0^2(\partial_i\theta_a + \partial_a\theta_i) + 2\rho_0\partial_i\theta_a\delta_{ai}, \quad (29)$$

and

$$\rho_{ab} = \rho_0(x^b - x^a), D_i^{ab} = x^b J_i^a - x^a J_i^b. \quad (30)$$

To derive the effective theory for quantum fluctuations or the gapless Goldstone modes, we expand the fields around a selected classical configuration $\phi_a(\mathbf{x}, t) = \sqrt{\rho_0 + \rho_a(\mathbf{x}, t)} e^{i\theta_a(\mathbf{x}, t)}$ where ρ_a and θ_a denote density and phase fluctuations, respectively. Up to the second order, we have

$$\begin{aligned} \mathcal{L} = & \sum_{a=1}^d -\rho_a \partial_t \theta_a - \rho_0 (\partial_a \theta_a)^2 - \frac{1}{4\rho_0} (\partial_a \rho_a)^2 - \frac{g}{2} \rho_a^2 \\ & - \sum_{a,b} \frac{1}{2} K_{ab} \rho_0^2 (\partial_a \theta_b + \partial_b \theta_a)^2 \\ & - \frac{1}{8} \sum_{a,b} K_{ab} (\partial_a \rho_b + \partial_b \rho_a)^2. \end{aligned} \quad (31)$$

The density fluctuation fields ρ_a ($a = 1, \dots, d$) work as auxiliary fields that are subject to the constraint equations,

$$\begin{aligned} \partial_t \theta_a = & -g\rho_a - \left(\frac{1}{2\rho_0} \partial_a^2 + \frac{1}{2} \sum_b K_{ab} \partial_b^2 \right) \rho_a \\ & - \frac{1}{2} \sum_b K_{ab} \partial_a \partial_b \rho_b. \end{aligned} \quad (32)$$

Since we are only interested in the low-energy physics, the momentum $\mathbf{k}$ has an upper bound $|\mathbf{k}| \leq 2\pi\xi_c^{-1}$ where the coherent length ξ_c can be estimated as

$$g = 4\pi^2 \xi_c^{-2} \left(\frac{1}{2\rho_0} + \sum_b K_{ab} \right), \quad (33)$$

as such the last two terms in Eq. (32) can be neglected. In the low-energy limit, we obtain the solutions $\rho_a = -\frac{1}{g} \partial_t \theta_a$. Cast it back to Eq. (31) and we arrive at an effective theory of quantum fluctuations by excluding the higher derivative terms of θ_a ,

$$\begin{aligned} \mathcal{L} = & \sum_a \frac{1}{2g} (\partial_t \theta_a)^2 - \rho_0 (\partial_a \theta_a)^2 \\ & - \sum_{a,b} \frac{1}{2} K_{ab} \rho_0^2 (\partial_a \theta_b + \partial_b \theta_a)^2. \end{aligned} \quad (34)$$

The effective theory in Eq. (34) stays invariant under the transformation $\theta_a \rightarrow \theta_a + \lambda_a + \sum_b \lambda_{ab} x^b$ with $\lambda_{ab} = -\lambda_{ba}$. It describes d gapless Goldstone modes θ_a ($a = 1, \dots, d$) [71] that have entangled motions arising from the K_{ab} term.

To get a deeper insight, we concentrate ourselves on the two-dimensional case. Introduce the canonical modes $\Theta_+(\mathbf{k})$ and $\Theta_-(\mathbf{k})$,

$$\Theta_+(\mathbf{k}) = \cos \frac{\varphi_{\mathbf{k}}}{2} \theta_1(\mathbf{k}) + \sin \frac{\varphi_{\mathbf{k}}}{2} \theta_2(\mathbf{k}), \quad (35)$$

$$\Theta_-(\mathbf{k}) = -\sin \frac{\varphi_{\mathbf{k}}}{2} \theta_1(\mathbf{k}) + \cos \frac{\varphi_{\mathbf{k}}}{2} \theta_2(\mathbf{k}), \quad (36)$$

where $\tan \frac{\varphi_{\mathbf{k}}}{2} = \frac{2K\rho_0 k_1 k_2}{\Delta(\mathbf{k})^2}$ with denoting $K_{12} = K$ and their dispersion relations take the form $\epsilon_{\pm}(\mathbf{k}) = \sqrt{g\rho_0[(1 + K\rho_0)\mathbf{k}^2 \pm \Delta(\mathbf{k})]}$, which is identical to Eq. (18).

Stability of a superfluid phase is determined by the long-distance behavior of the correlator of the order parameter under the influence of quantum fluctuations,

$$\begin{aligned} & \langle \text{GS} | \Phi_a^\dagger(\mathbf{x}) \Phi_b(\mathbf{0}) | \text{GS} \rangle_{\theta_a^b}^{\beta_{ab}} \\ &= \rho_0 \exp \left[i \sum_c (\beta_{bc} - \beta_{ac}) x^c \right] \langle e^{-i\Theta_a(\mathbf{x})} e^{i\Theta_b(\mathbf{0})} \rangle. \end{aligned} \quad (37)$$

We need to calculate equal-time correlators of the canonical modes $\langle e^{-i\Theta_{\pm}(\mathbf{x})} e^{i\Theta_{\pm}(\mathbf{0})} \rangle = e^{-\frac{1}{2} \langle [\Theta_{\pm}(\mathbf{x}) - \Theta_{\pm}(\mathbf{0})]^2 \rangle}$. Explicitly, in two spatial dimensions, we have

$$\begin{aligned} \langle \Theta_{\pm}(\mathbf{x}) \Theta_{\pm}(\mathbf{0}) \rangle &= \int \frac{d\omega d^2k}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{\omega^2 - \omega_{\pm}(\mathbf{k})} \\ &= \int \frac{dk d\theta}{(2\pi)^2} \frac{e^{ik|\mathbf{x}| \cos \theta}}{\sqrt{g\rho_0[(1 + K\rho_0) \pm \bar{\Delta}(\theta)]}} \\ &< \int \frac{dk d\theta}{(2\pi)^2} \frac{e^{ik|\mathbf{x}| \cos \theta}}{\sqrt{g\rho_0 c_{\pm}}} = \frac{1}{2\pi|\mathbf{x}|} \frac{1}{\sqrt{g\rho_0 c_{\pm}}}, \end{aligned} \quad (38)$$

where $\bar{\Delta}(\theta) = \sqrt{(1 - 2K\rho_0) \cos^2 2\theta + K^2 \rho_0^2}$ and $c_{\pm}$ denotes the minimum value of $(1 + K\rho_0) \pm \bar{\Delta}(\theta)$.

At the long distance $|\mathbf{x}| \rightarrow \infty$, the correlator $\langle \Theta_{\pm}(\mathbf{x}) \Theta_{\pm}(\mathbf{0}) \rangle$ vanishes. Thus, $\langle \Phi_a^\dagger(\mathbf{x}) \Phi_b(\mathbf{0}) \rangle = \rho_0 \exp[i \sum_c (\beta_{bc} - \beta_{ac}) x^c]$ has a finite value modulated by a plane wave. It confirms a true long-range order that survives against quantum fluctuations when both the two lineons condensate simultaneously in zero temperature. Since quantum fluctuations are weaker in higher dimensions, a fractonic superfluid phase $d\text{SF}^1$ stays stable in two spatial dimensions $d = 2$ and higher $d > 2$. In Appendix A, the component-dependent Mexican-hat potential in 2D is considered, which leads to condensation of only one component of lineons. The uncondensed component gets released from mobility restriction and the Goldstone mode behaves still as a subdimensional particle. Thus, the superfluid there obeys an algebraic order.

IV. SUPERFLUID VORTICES OF 2SF^1

Besides the gapless Goldstone modes and gapped roton modes, thermal vortices are fundamental to a superfluid phase as an effect of compactness of phase fields θ_a . The existence of symmetry in Eq. (4) admits a complicated structure in 2SF^1 . We present two guiding statements on construction of point thermal vortices in 2D and then give the two types of vortices in 2SF^1 .

A. Two statements on construction

A superfluid vortex is an excitation as a consequence of compactness of a phase field and mathematically one can represent compactness by a multivalued function. Given a phase field θ_a with component a , we can always decompose it as $\theta_a(\mathbf{x}) = \theta_a^v(\mathbf{x}) + \theta_a^s(\mathbf{x})$ where $\theta_a^s(\mathbf{x})$ denotes the smooth

component. In general, the multivalued component $\theta_a^v(\mathbf{x})$ can be formulated as

$$\theta_a^v(\mathbf{x}) = f_a(\mathbf{x})\varphi(\mathbf{x}), \quad (39)$$

where $\varphi(\mathbf{x})$ defined mod 2π is the angle of site $\mathbf{x}$ relative to vortex core and $f_a(\mathbf{x})$ is a single-valued function. Equation (39) sets an equivalent relation $\theta_a^v(\mathbf{x}) \sim \theta_a^v(\mathbf{x}) + 2\pi f_a(\mathbf{x})$. Subtly, $f_a(\mathbf{x})$ should be understood under a lattice regularization to protect single valuedness of field ϕ_a , where spatial coordinates $\mathbf{x}$ are regarded as $\mathbf{x} = (x^1, x^2) = \mathbf{n}a$ with $\mathbf{n} = (n_1, n_2)$ being a pair of integers and a being the lattice constant. The equivalence relation in Eq. (39) resembles a gauge freedom. Whether we start with $\theta_a^v(\mathbf{x})$ or $\theta_a^v(\mathbf{x}) + 2\pi f_a(\mathbf{x})$ should cause no physical effects. Therefore, we arrive at Statement 1 below.

Statement 1. The physical Hamiltonian density should be single valued even in the presence of multivalued vortex configurations.

Statement 1 clarifies that the Hamiltonian density $\mathcal{H}[\theta_a(\mathbf{x})]$ is invariant when $\theta_a(\mathbf{x})$ is shifted by $2\pi f_a(\mathbf{x})$, $\mathcal{H}[\theta_a(\mathbf{x}) + 2\pi f_a(\mathbf{x})] = \mathcal{H}[\theta_a(\mathbf{x})]$, which determines the most singular part of a vortex. Take a conventional superfluid 2SF^2 as an example with Hamiltonian density $\mathcal{H} = \frac{1}{2}[(\partial_1 \theta(\mathbf{x}))^2 + (\partial_2 \theta(\mathbf{x}))^2]$. With the assumption $\theta^v(\mathbf{x}) = f(\mathbf{x})\varphi(\mathbf{x})$, the constraint imposed by Statement 1 on shifting $\theta(\mathbf{x})$ by $2\pi f(\mathbf{x})$ gives the equations $\partial_1 f(\mathbf{x}) = 0$, $\partial_2 f(\mathbf{x}) = 0$ towards which we have the solution $f(\mathbf{x}) = \ell$ with $\ell \in \mathbb{Z}$. Thus we recover vortex configurations in a conventional superfluid.

The second statement to be introduced below controls the smooth component after we obtain the multivalued component from Statement 1. A higher-rank symmetry group contains not only conventional $U(1)$ charges that induce a global $U(1)$ phase shift, but also charges that generate a phase shift depending on local coordinates. For convenience, we call conventional $U(1)$ charges as rank-0 while the others are higher-rank charges. Statement 2 below establishes the relations between higher-rank and rank-0 charges.

Statement 2. The action of a higher-rank symmetry group on some bound states of operators charged in the higher-rank symmetry group is equivalent to an action of a global $U(1)$ symmetry with appropriate rank-0 charges.

Statement 2 allows us to construct a set of bound states such that the higher-rank group only induces a global phase shift. Explicitly, given vortices carrying higher-rank charges, Statement 2 claims that some bound state of these vortices is proportional to $\varphi(\mathbf{x})$ as a conventional vortex, that is, the smooth component vanishes. Thus, the essence is to find the structures of bound states which are significantly determined by relations between higher-rank charges and rank-0 charges. For example, we consider a higher-rank symmetry [47] which shifts $\theta(\mathbf{x})$ by $\theta(\mathbf{x}) \rightarrow \theta(\mathbf{x}) + \lambda + \sum_a \lambda_a x^a$. Then the group action on bound states like $\hat{\mathcal{O}}_{\mathbf{d}} = e^{-i\theta(\mathbf{x})} e^{i\theta(\mathbf{x}-\mathbf{d})}$ with a constant vector $\mathbf{d}$ generates a pure global phase, $\hat{\mathcal{O}}_{\mathbf{d}} \rightarrow \hat{\mathcal{O}}_{\mathbf{d}} e^{-i \sum_a \lambda_a d^a}$. Thus, on these bound states, the higher-rank group is equivalent to group $U(1)$ and we shall expect that $\hat{\mathcal{O}}_{\mathbf{d}}$ takes the form of a conventional vortex whose smooth part can be set to vanish. In Appendix B, we present a detailed derivation on vortices in a higher-rank superfluid phase in Ref. [47].

B. Vortex structure

The two statements are more generally applicable for a system of a higher-rank symmetry. At present, we specialize our attention to the case of 2SF^1 . Statement 1 leads to an assumption for the multivalued component,

$$\theta_1^v(\mathbf{x}) = f_1(\mathbf{x})\varphi(\mathbf{x}), \quad \theta_2^v(\mathbf{x}) = f_2(\mathbf{x})\varphi(\mathbf{x}), \quad (40)$$

and $f_{1,2}(\mathbf{x})$ should satisfy the following equations:

$$\partial_1 f_1(\mathbf{x}) = 0, \quad \partial_2 f_2(\mathbf{x}) = 0, \quad \partial_2 f_1(\mathbf{x}) + \partial_1 f_2(\mathbf{x}) = 0. \quad (41)$$

The solutions generally can be parametrized by three parameters,

$$f_1(\mathbf{x}) = px^2 + \ell_1, \quad f_2(\mathbf{x}) = -px^1 + \ell_2. \quad (42)$$

Here, p has the dimension $[x]^{-1}$, which we dub a dipole charge, while ℓ_1 and ℓ_2 are dimensionless. Under the lattice regularization, pa and ℓ_1, ℓ_2 are all integers.

The parameters ℓ_1 and ℓ_2 describe conventional vortices and they are interpreted as the winding numbers. To obtain vortices carrying a dipole charge p , we apply Statement 2. The essence of Statement 2 is to recognize bound states. In 2D, the higher-rank group $\mathcal{G}$ is parametrized by λ_1, λ_2 , and λ_{12} . We denote the group element with $\lambda_1 = \lambda_2 = 0, \lambda_{12} = 1$ as G_e . Given a vortex operator $\hat{O}_1(\mathbf{x}) = e^{i\theta_1(\mathbf{x})}$ with the charge $-q_{12}$, which is transformed by G_e in $\mathcal{G}$ as $\hat{O}_1(\mathbf{x}) \rightarrow \hat{O}_1(\mathbf{x})e^{iq_{12}x^2}$, then a “particle-hole” bound state $\hat{O}_1^\dagger(\mathbf{x})\hat{O}_1(\mathbf{x} - \mathbf{d})$ with a constant vector $\mathbf{d} = (0, d)$ is transformed by G_e as

$$\hat{O}_1^\dagger(\mathbf{x})\hat{O}_1(\mathbf{x} - \mathbf{d}) \rightarrow \hat{O}_1^\dagger(\mathbf{x})\hat{O}_1(\mathbf{x} - \mathbf{d})e^{-iq_{12}d}. \quad (43)$$

If the particle-hole bound state is attributed with a $U(1)$ charge $q_{12}d$, we find the action in Eq. (43) can be re-explained as the action of $U(1)$ symmetry. Statement 2 asserts that the bound state reduces to a conventional vortex, which imposes a constraint $\theta_1(\mathbf{x}) - \theta_1(\mathbf{x} - \mathbf{d}) = d\partial_2\theta_1(\mathbf{x}) = q_{12}d\varphi(\mathbf{x})$ for small $\mathbf{d}$. We have

$$pd\varphi(\mathbf{x}) - q_{12}d\varphi(\mathbf{x}) = 0, \quad (44)$$

$$\partial_2\theta_1^s(\mathbf{x}) + f_1(\mathbf{x})\partial_2\varphi(\mathbf{x}) = 0. \quad (45)$$

Equation (44) shows $p = q_{12}$ which indicates the dipole charge p in $f_1(\mathbf{x})$ represents a higher-rank charge of group $\mathcal{G}$. In fact, the bound state with $\mathbf{d} = (d, 0)$ is invariant under G_e and thus it requires $\hat{O}_1^\dagger(\mathbf{x})\hat{O}_1(\mathbf{x} - \mathbf{d})$ to be single valued, which is satisfied since $f_1(\mathbf{x})$ is independent of x^1 . We consider vortex operator object $\hat{O}_2(\mathbf{x}) = e^{i\theta_2}$ carrying a charge q_{12} with a transformation by G_e as $\hat{O}_2(\mathbf{x}) \rightarrow \hat{O}_2(\mathbf{x})e^{-iq_{12}x^1}$. Then G_e induces a global phase shift on the bound state $\hat{O}_2(\mathbf{x})^\dagger\hat{O}_2(\mathbf{x} - \mathbf{d}) \rightarrow \hat{O}_2(\mathbf{x})^\dagger\hat{O}_2(\mathbf{x} - \mathbf{d})e^{iq_{12}d}$ with $\mathbf{d} = (d, 0)$. Thus, we are allowed to re-interpret action of $\mathcal{G}$ on a bound state $\hat{O}_2(\mathbf{x})^\dagger\hat{O}_2(\mathbf{x} - \mathbf{d})$ as an action of the $U(1)$ group on a charged $-q_{12}d$ operator. Therefore, according to Statement 2, we have $\theta_2(\mathbf{x}) - \theta_2(\mathbf{x} - \mathbf{d}) = -q_{12}d\varphi(\mathbf{x})$ for small d . Equivalently, we have

$$-pd\varphi(\mathbf{x}) + q_{12}d\varphi(\mathbf{x}) = 0, \quad (46)$$

$$\partial_1\theta_2^s(\mathbf{x}) + f_2(\mathbf{x})\partial_1\varphi(\mathbf{x}) = 0. \quad (47)$$

Here the dipole charge denotes a higher-rank charge of group $\mathcal{G}$.

Thus, we can obtain two types of vortices. The first one is the conventional vortex characterized by winding numbers ℓ_1, ℓ_2 ,

$$\theta_1(\mathbf{x}) = \ell_1\varphi(\mathbf{x}), \quad \theta_2(\mathbf{x}) = \ell_2\varphi(\mathbf{x}). \quad (48)$$

And the second one with a configuration,

$$\theta_1(\mathbf{x}) = -px^1 \log |\mathbf{x}| + px^2\varphi(\mathbf{x}), \quad (49)$$

$$\theta_2(\mathbf{x}) = -px^2 \log |\mathbf{x}| - px^1\varphi(\mathbf{x}), \quad (50)$$

carries a higher-rank charge p of symmetry $\mathcal{G}$. We emphasize again that the charge p should be regularized as $p = \ell a^{-1}$ ($\ell \in \mathbb{Z}$) to ensure $f_{1,2}(\mathbf{x})$ in Eq. (40) to be integer valued. When we circle around the vortex core, the vortex configuration gets an extra phase $\delta\theta_1 = 2\pi px^2 = 2\pi \ell n_2$ and $\delta\theta_2 = -2\pi px = -2\pi \ell n_1$ with $\mathbf{x} = (n_1, n_2)a$, which keeps in consistence with compactness of θ_a . Different from a conventional vortex, here $\partial_1\theta_2$ and $\partial_2\theta_1$ are still multivalued while $\partial_1\theta_1$ and $\partial_2\theta_2$ are single valued.

We can define the vorticity for the dipole charge by recombination of Noether currents. After condensation, in two spatial dimensions the Noether currents can be formulated as

$$J_1^1 = 2\rho_0\partial_1\theta_1, \quad J_2^2 = 2\rho_0\partial_2\theta_2, \quad J_1^2 = 2K\rho_0^2(\partial_1\theta_2 + \partial_2\theta_1). \quad (51)$$

As indicated by Statement 2, a “particle-hole” bound state of vortices behaves as a vortex in 2SF^2 and it only encodes the dipole charge. Above all, the density ρ_{dipole} of such a bound state can be written as

$$\rho_{\text{dipole}} = \sum_{a,b=1}^2 \frac{1}{2} \epsilon_{ab} \partial_a \partial_b (\partial_2\theta_1 - \partial_1\theta_2), \quad (52)$$

where ϵ_{ab} is an antisymmetric tensor $\epsilon_{12} = -\epsilon_{21} = 1$. Actually we have a relation,

$$\rho_{\text{dipole}} = \sum_{a,b=1}^2 \epsilon_{ab} \partial_a \partial_b \partial_2\theta_1 = \sum_{a,b=1}^2 -\epsilon_{ab} \partial_a \partial_b \partial_1\theta_2, \quad (53)$$

since θ_1 and θ_2 take the same dipole charge. Following the lesson we learned for vortices in superfluid phase 2SF^2 , we can construct the currents $\mathbf{J}_{\text{dipole}}$ based on condensed currents in Eq. (51) with components,

$$\mathbf{J}_{\text{dipole}}^1 = -\frac{1}{2}aK^{-1}\rho_0^{-2}\partial_1J_1^2 + a\rho_0^{-1}\partial_2J_1^1, \quad (54)$$

$$\mathbf{J}_{\text{dipole}}^2 = \frac{1}{2}aK^{-1}\rho_0^{-2}\partial_2J_1^2 - a\rho_0^{-1}\partial_1J_2^2, \quad (55)$$

where the cutoff a is introduced to make up the dimension of $\mathbf{J}_{\text{dipole}}$. Then the vorticity will give the dipole charge,

$$\ell_{\text{dipole}} = \frac{1}{2\pi} \oint_C d\mathbf{x} \cdot \mathbf{J}_{\text{dipole}} = \ell, \quad (56)$$

where C is a closed path encircling the vortex core and $p = \ell a^{-1}$.

V. CONCLUDING REMARKS

As a series of the work [47], we have further explored more possibilities of exotic states of matter formed by particles with restricted mobility. We have discussed a fractonic superfluid phase $d\text{SF}^1$ in a microscopic model by condensing subdimensional particles. This model is invariant under

a higher-rank symmetry such that its fundamental particles are lineons. We use the HFB mean-field theory to derive a set of highly nonlinear GP equations and a BdG Hamiltonian which characterize the condensed and the norm components, respectively. In the framework of a continuous field theory, we construct macroscopic degeneracies of ground states with finite momentum and derive an effective theory for gapless Goldstone modes. At zero temperature, a phase $d\text{SF}^1$ stays stable in two spatial dimensions and higher. We emphasize two guiding statements to construct vortex excitations in two spatial dimensions. Explicitly, there are two types of vortices in 2SF^1 . Besides conventional vortices, the other type carries a dipole charge. The two guiding statements are more generally applicable (see Appendix B).

Towards a complete understanding of a fractonic superfluid phase, we have to deal with more questions. Tightly related to the present paper, vortex excitations form a hierarchy which is dominated by the two statements, and then interactions between vortices and BKT transitions, should also inherit such a hierarchy. A natural question is to investigate a superfluid phase by condensing other spatially extended excitations [27]. On the other hand, by viewing the model in Eq. (1) as coupled Luttinger liquids, we can derive an effective theory by bosonization. Then, it is straightforward to obtain universal properties of fractonic superfluids such as conductivity and general thermodynamic properties [72,73]. In three spatial dimensions, more exotic vortex line excitations can be excited and their construction needs further investigation. If we condense these defects to recover the symmetry as the scheme to construct a symmetry-protected topological phase [74–78], what phase can be obtained? Besides, what is the universal class of the phase transition between a high-rank superfluid phase and a normal state? Experimentally, we expect that the Hamiltonian in Eq. (1) can be realized in the cold atomic gas subjected to an optical lattice by tuning a two-particle state [79], which opens a new horizon to search exotic phases of matter.

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APPENDIX A: COMPONENT-DEPENDENT POTENTIAL

In the main text, a Mexican-hat potential is chosen to be component independent and the two components of lineons are condensed simultaneously. Here, we briefly discuss a component-dependent Mexican-hat potential in two spatial dimensions with only one of the two components condensed. In the framework of coherent-path integral representation, we introduce a Mexican-hat potential only for ϕ_2 ,

$$V = -\mu_2 |\phi_2|^2 + \frac{g_2}{2} |\phi_2|^4, \quad (\text{A1})$$

with $g_2 > 0$. When $\mu_2 < 0$, ϕ_2 is in an insulating state. When $\mu_2 > 0$, ϕ_2 picks up a finite particle density, $\rho_{20} \equiv \frac{\mu_2}{g_2}$ to minimize the potential in Eq. (A1). In the classical level, the configurations of ϕ_2 take the form as

$$\phi_2^{\text{cl}} = \sqrt{\rho_{20}} e^{i\theta_2 + i\beta_{12}x^2}. \quad (\text{A2})$$

We can expand the field ϕ_2 around ϕ_2^{cl} in Eq. (A2). The obtained Hamiltonian explicitly depends on the parameter β_{12} . This dependence can be removed by a transformation,

$$\phi_1(\mathbf{x}) \rightarrow \phi_1(\mathbf{x}) e^{-i\beta_{12}x^1}. \quad (\text{A3})$$

Following the procedure in Sec. III and up to second order, we obtain an effective theory,

$$\mathcal{L}[\phi_1] = i\phi_1^* \partial_t \phi_1 - |\partial_1 \phi_1|^2 - K \rho_{20} |\partial_2 \phi_1|^2, \quad (\text{A4})$$

$$\mathcal{L}[\theta_2] = \frac{1}{2g_2} (\partial_t \theta_2)^2 - \rho_{20} (\partial_2 \theta_2)^2, \quad (\text{A5})$$

where $K = K_{12}$. The symmetry of ϕ_1 in Eq. (A4) reduces to a conventional particle conservation $U(1)$ symmetry after ϕ_2 condensation, and thus the field ϕ_1 gets liberated from the mobility restriction due to the last term in Eq. (A4). However, the Goldstone mode θ_2 still lacks full mobility and can only propagate in one direction. The stability can be inferred from the long-range behaviors of order parameter correlator $\langle e^{i\theta_2(\mathbf{x})} e^{-i\theta_2(\mathbf{0})} \rangle = e^{-\frac{1}{2} \langle (\theta_2(\mathbf{x}) - \theta_2(\mathbf{0}))^2 \rangle}$, which is determined by

$$\langle \theta_2(\mathbf{x}) \theta_2(\mathbf{0}) \rangle = \delta(x^1) \int \frac{d\omega dk_2}{(2\pi)^3} e^{-ik_2 x^2} \frac{g_2}{\omega^2 - \mu_2 k_2^2}. \quad (\text{A6})$$

If $x^1 \neq 0$, the correlator in Eq. (A6) vanishes as a natural consequence of mobility constraint. When $x^1 = 0$, we have

$$\langle \theta_2(\mathbf{x}) \theta_2(\mathbf{0}) \rangle = -\frac{2\pi}{\xi} \frac{\pi g_2}{\sqrt{\mu_2}} \log \frac{4e^{-\gamma} (x^2)^2}{\mu_2}, \quad (\text{A7})$$

such that $\langle e^{i\theta_2(\mathbf{x})} e^{-i\theta_2(\mathbf{0})} \rangle$ decays in a power-law pattern where ξ is the coherent length and γ is the Euler constant. Therefore, the superfluid phase here is algebraically ordered.

APPENDIX B: APPLICATION OF THE TWO STATEMENTS

In the main body, we put forward two statements to construct vortex excitations and indicate they are generally applicable. Here, we apply them to a fracton model that is considered in Ref. [47]. The Hamiltonian density for the Goldstone modes reads

$$\mathcal{H} = \frac{1}{2} [(\partial_1^2 \theta)^2 + 2(\partial_1 \partial_2 \theta)^2 + (\partial_2^2 \theta)^2]. \quad (\text{B1})$$

The vortex excitations arise when θ is multivalued. We decompose the vortex field as $\theta = \theta^v + \theta^s$, where the multivalued component can be formulated as

$$\theta^v(\mathbf{x}) = f(\mathbf{x}) \varphi(\mathbf{x}), \quad (\text{B2})$$

with $\varphi(\mathbf{x})$ the angle of site $\mathbf{x} = (x^1, x^2)$ relative to the vortex core. Statement 1 requires Hamiltonian density $\mathcal{H}$ be single valued for the field configuration in Eq. (B2), which induces the restrictions on $f(\mathbf{x})$,

$$\partial_1 f(\mathbf{x}) = 0, \quad \partial_2 f(\mathbf{x}) = 0. \quad (\text{B3})$$

It is easy to find the solutions,

$$f(\mathbf{x}) = \ell, \quad (\text{B4})$$

or

$$f(\mathbf{x}) = p_1 x^1, \quad f(\mathbf{x}) = p_2 x^2. \quad (\text{B5})$$

The first solution in Eq. (B4) marks a conventional vortex solution with $\theta = \ell\varphi(\mathbf{x})$ where $\ell \in \mathbb{Z}$ represents the winding number. The second solution in Eq. (B5) takes charges p_1, p_2 with dimension $[x^{-1}]$. The charges are expected to be quantized as a momentum. The full solution should take into consideration the smooth component $\theta^s(\mathbf{x})$, which is instructed by Statement 2. For $f(\mathbf{x}) = p_1 x^1$, we consider a bound state of a vortex-anti-vortex pair with distance $\mathbf{d} =$

$(d, 0)$, on which the vortex bound state reduces to a conventional vortex with winding number dq_1 , i.e.,

$$d\partial_1[p_1 x^1 \varphi(\mathbf{x}) + \theta^s(\mathbf{x})] = dq_1 \varphi(\mathbf{x}). \quad (\text{B6})$$

So $\theta^s(\mathbf{x}) = q_1 x^2 \log |\mathbf{x}|$. Similarly, consider a bound state of a vortex-anti-vortex pair with distance $\mathbf{d} = (0, d)$ and we can obtain the smooth component for $f(\mathbf{x}) = -q_2 x \log |\mathbf{x}|$. In summary, vortices for the model in Eq. (B1) have the configurations as

$$\theta = \ell\varphi(\mathbf{x}), \quad (\text{B7})$$

$$\theta = p_1 x^1 \varphi(\mathbf{x}) + p_1 y \log |\mathbf{x}|, \quad (\text{B8})$$

$$\theta = p_2 x^2 \varphi(\mathbf{x}) - p_2 x \log |\mathbf{x}|. \quad (\text{B9})$$

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