Simulation of topological Zak phase in spin-phononic crystal networks

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Topological states of matter are particularly interesting for both fundamental research and practical applications. Simulating topological phases in a quantum system is of great interest due to the ability to explore a plethora of topologically nontrivial phenomena in a controllable fashion. We propose and analyze an efficient scheme for simulating topological Zak phase in two-dimensional spin-phononic crystal networks. We show that through a specially designed periodic driving, one can selectively control and enhance the bipartite silicon-vacancy center arrays, so as to obtain chiral-symmetry-protected spin-spin couplings. More importantly, the Floquet engineering spin-spin interactions support rich quantum phases associated with topological invariants. In momentum space, we analyze and simulate the topological nontrivial properties of the one- and two-dimensional systems. As an application in quantum information processing, we study the robust quantum state transfer via topologically protected edge states. This work opens up new prospects for studying quantum acoustics and offers an experimentally feasible platform for the study of topological phases of matter.

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I. INTRODUCTION

Topological insulators (TIs) possess topologically protected surface or edge states, which can be utilized as robust transmission channels. In condensed-matter physics, topological phenomena such as the quantum Hall effect and the quantum spin Hall effect have been extensively studied [1–3]. With the combination of topology and quantum theory, topological protection has developed some interesting applications in quantum information processing. In photonics, topological edge states can be used to realize one-way transport without breaking time-reversal symmetry [4–9]. In quantum computation, topology was introduced to solve the decoherence problem, in which the non-Abelian topological phases of matter are used to encode and manipulate quantum information [10–12].

The Su-Schrieffer-Heeger (SSH) model, originally derived from the dimerized chain, serves as the simplest example of one-dimensional (1D) topological insulators [13]. So far, the SSH model has been realized in a number of quantum structures. For instance, a recent experiment demonstrated a tunable dimerized model and observed the topological magnon insulator states in a superconducting qubit chain [14]. As for ion-trap or optical lattice systems, an external periodic driving is generally needed to trigger the topological

properties of the system, realizing the Floquet topological insulators in these systems [15–20]. Remarkably, much progress has been achieved in revealing exotic topological phenomena. To investigate the topological properties of high-dimensional quantum systems, several theoretical works generalized the SSH model to the two-dimensional (2D) case [21–24]. On the other hand, higher-order topological insulators (HOTIs) can support unconventional topologically protected corner and hinge states [25–31]. There are a number of works simulating the topological properties, most of which are based on elaborately designed theoretical models [32,33]. However, observation of the bulk topology, both first-order and higher-order topological phases, is still demanding under current experimental conditions. Therefore developing a scalable quantum system is of great significance to the study of topological behaviors of multidimensional structures.

In recent years, quantum acoustics, which mainly studies the coherent interactions between quantized phonon modes and quantum emitters, has aroused growing interest. Mechanical resonators or propagating phonons with low speed of sound in solids offer unique advantages for transmitting quantum information between solid-state quantum systems. To date, experimental and theoretical progress has realized a variety of hybrid mechanical structures involving a large number of different quantum systems, such as solid-state defects [34–43], superconducting circuits [44–47], ultracold atoms [48,49], and quantum dots [50,51]. Among them, due to the excellent coherence properties, defect spins in diamond and silicon carbide have become one of the most promising systems for quantum applications in solid states [52–56]. In particular, the negatively charged silicon-vacancy (SiV) center in diamond serves as an emerging block for hybrid quantum

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FIG. 1. (a) Schematics of the hybrid device studied in this paper. Nanomechanical 2D phonon band-gap setups elaborately designed with high- Q cavities in a patterned diamond membrane. (b) Arrays of SiV color centers are implanted evenly in the phononic waveguide. The lattice constant of the phononic crystal is $a = 100$ nm, the sizes of the ellipse holes are $(b, c) = (30, 76)$ nm, and the thickness is $t = 20$ nm. SiV spins are driven by microwave driving fields. (c) Ground-state energy levels of a single SiV center. Lower and upper states split via an external magnetic field. $g_{n,k}$ describes the coupling strength between the spin and the phonon with the transition $(|g\rangle \leftrightarrow |f\rangle)$, and $\Delta_{BE} = \omega_s - \omega_{BE}$ is the detuning between the spin transition and phononic band-edge frequency. (d) Displacement pattern of the phononic compression mode at the band-edge frequency ω_{BE} .

systems because of high strain susceptibility and remarkable optical properties [57–60].

Previous works have shown that a highly tunable spin-phonon interaction can be achieved near a phononic band gap [61–63]. Phononic crystals, defined as elastic waves propagating in periodic structures, provide a powerful candidate for manipulating the interplay of phonons and other quantum systems. Because of the unique band structure of the phononic crystal, a single phonon bound state emerges within the band gap [64–66], resulting in a stronger and controllable spin-phonon coupling. More importantly, owing to the advantage of the scalable nature of nanofabrication, the spin-phononic crystal setup is experimentally feasible when extending to the high-dimensional cases [63,67–77].

In this paper, we propose an efficient protocol for studying the topological quantum properties in 2D SiV-phononic crystal networks. Driving the SiV color center arrays with periodic microwave fields, we obtain the Floquet engineering spin-spin interactions with some unique properties. We find that it is possible to selectively control the phonon-band-gap-mediated spin-spin couplings by modulating the parameters of the periodic driving. We show that the chiral-symmetry-protected spin-spin interactions are attained in the bipartite SiV center arrays and, more importantly, the Floquet engineering spin-spin interactions support rich quantum topological phases. To investigate the topological nontrivial features of the system, we convert the 1D and 2D Floquet engineering spin-spin Hamiltonian to the momentum space. First, we study the topological invariant winding number and derive the 1D topological Zak phase. In addition to the original definitions,

here we offer a geometrically intuitive way to calculate the topological invariant. Then, we present the analytical and numerical solutions of the 2D topological properties. Finally, we study the topological protected quantum state transfer and discuss the effect of SiV spin dephasing. This work offers an experimentally feasible platform for studying topological nontrivial phenomena in high-dimensional quantum systems.

II. 2D SPIN-PHONONIC CRYSTAL NETWORKS

The 2D spin-phononic crystal setup is depicted in Fig. 1(a), where identical nodes are arranged in a square lattice. The diamond waveguide is perforated with periodic elliptical air holes, which yields the tunable phononic band structures. SiV color centers are evenly located at the nodes of the phononic structure, which are coupled to the acoustic vibrations via lattice strain. The pattern structure of the edge of the diamond membrane is designed to ensure the high- Q phonon band-gap cavities [67].

For the phononic crystal, we first consider a quasi-1D geometry model, which supports acoustic guide modes $\omega_{n,k}$, with n being the band index and k being the wave vector along the waveguide direction. The propagating phonon modes can be described by a displacement field function $\vec{Q}(\vec{r}, t)$. By solving the elastic wave equation, we can derive the mechanical displacement mode profile $\vec{Q}(\vec{r}, t)$ [78]. Analogous to the electromagnetic field in quantum optics, the mechanical displacement field can be quantized, i.e., $H_p = \sum_{n,k} \hbar \omega_{n,k} a_{n,k}^\dagger a_{n,k}$, with $a_{n,k}$ and $a_{n,k}^\dagger$ being the annihilation and creation operators for the phonon modes.

SiV color centers are interstitial point defects wherein a silicon atom is positioned between two adjacent vacancies in the diamond lattice. The negatively charged SiV center can be treated as an effective $S = 1/2$ system. For the electronic ground state of the SiV center, the $|^2E_g\rangle$ states are the combination of a twofold orbital and a twofold spin degeneracy. Considering the spin-orbit interaction and Jahn-Teller effect, the orbital states are separated into a lower branch (LB) and an upper branch (UB) with frequency $\Delta_{GS} = 2\pi \times 46$ GHz. In the presence of an external magnetic field $\vec{B}$, the Zeeman effect will further split the spin degenerate states. The SiV ground-state Hamiltonian can be written as [58]

$$H_{\text{SiV}} = -\hbar\lambda_{SO}L_zS_z + H_{JT} + \hbar f\gamma_L B_z L_z + \hbar\gamma_S \vec{B} \cdot \vec{S}, \quad (1)$$

where λ_{SO} is the strength of the spin-orbit interaction and γ_L and γ_S correspond to the orbital and spin gyromagnetic ratio, respectively. Diagonalizing Eq. (1), we obtain four eigenstates $\{|g\rangle = |e_- \downarrow\rangle, |e\rangle = |e_+ \uparrow\rangle\}$, $\{|f\rangle = |e_+ \downarrow\rangle, |d\rangle = |e_- \uparrow\rangle\}$, where $|e_{\pm}\rangle = (|e_x\rangle \pm i|e_y\rangle)/2$ are eigenstates of the orbital angular momentum operator. The corresponding energy-level diagram is given in Fig. 1(c). As a result, the spin-flip transitions are allowed between the four sublevels with opposite electronic spin components [79,80]. Specifically, the two lowest sublevels ($|g\rangle, |e\rangle$) can be treated as a long-lived qubit and coherently controlled via an optical Raman process. Furthermore, in the high-strain limit, this transition can be directly driven with a microwave field [81].

In the SiV-phonon system, the mechanical lattice vibration modifies the electronic environment of the SiV center, resulting in the coupling of its orbital states $|e_-\rangle$ and $|e_+\rangle$. As for the setup shown in Fig. 1(b), when the transition frequency of the spin state is tuned close to the phononic band edge, we can obtain the strong strain coupling between the SiV center and phononic crystal mode [35]. By utilizing a microwave-assisted Raman process involving the upper state $|f\rangle$, the transition of SiV electronic ground states $|g\rangle$ and $|e\rangle$ can be effectively coupled to the phononic mode. In this case, the spin-phonon interaction can be mapped to the Jaynes-Cummings model, namely,

$$H_{s-p} = \sum_{n,k} \hbar\omega_{n,k} a_{n,k}^\dagger a_{n,k} + \hbar\omega_s \sigma_{ee} + \sum_{n,k} \hbar g_{n,k} (a_{n,k} \sigma_{eg} e^{ikx_0} + \text{H.c.}), \quad (2)$$

where $\sigma_{ij} = |i\rangle\langle j|$, ω_s is the effective spin transition frequency, and $g_{n,k}$ is the coupling strength between the SiV center and the phononic modes. Here, we consider that the defect centers are coupled predominantly to a single band of the phononic crystal, so the index n can be omitted in the following discussion. In Fig. 1(d), we numerically simulate the corresponding displacement pattern of the phononic mode by using the finite-element method (FEM), which is performed with the COMSOL MULTIPHYSICS software.

In a previous work, we proposed the band-gap-engineered spin-phonon interaction. When the spin transition frequency is exactly in a phonon band gap, there will be a phononic bound state. Then we can obtain a much stronger SiV-phononic coupling via tuning the effective acoustic mode volume [61]. In this context, we now study the interaction between the

phononic crystal modes and an array of SiV spins. Here, we assume that the SiV centers are equally coupled to the phononic modes near the band gap. Thus the interaction Hamiltonian of the defect spins and the phonon modes is expressed as

$$H_I = \sum_{j,k} \hbar g (a_k \sigma_{eg}^j e^{i\delta_k t + ikx_j} + \text{H.c.}), \quad (3)$$

with $\delta_k = \omega_s - \omega_k$. Assuming the large-detuning regime, $\delta_k \gg g$, we can obtain an effective spin-spin interaction via adiabatically eliminating the phonon modes [82]. With the band-gap-engineered spin-phononic interaction, we integrate over the phononic modes and obtain the effective Hamiltonian

$$H_{\text{array}} = \sum_{i,j} \hbar J_{i,j} \sigma_{eg}^i \sigma_{ge}^j, \quad (4)$$

where

$$J_{i,j} = \frac{g_c^2}{2\Delta_{BE}} e^{-|x_i - x_j|/L_c} \quad (5)$$

denotes the phononic-band-gap-mediated spin-spin interaction strength and $\Delta_{BE} = \omega_s - \omega_{BE}$ is the detuning between the spin transition and the phononic band-edge frequency. $g_c = g\sqrt{2\pi a/L_c}$ corresponds to the spin-phononic coupling strength, with a being the lattice constant and L_c being the localized length of the phononic wave function. Going back to the two-dimensional setup shown in Fig. 1(a), we consider a phononic network with square lattices on the x - y plane, with $2N \times 2N$ SiV spins located separately at the nodes of the phononic structure. Hence the phononic-mediated spin-spin interactions can be obtained as [62]

$$\begin{aligned} H_{\text{array}}^{(2D)} &= H_{\text{array}}^{(x)} + H_{\text{array}}^{(y)}, \\ H_{\text{array}}^{(x)} &= \sum_{l=1}^{2N} \sum_{i,j=1}^{2N} \hbar (J_{i,j} \sigma_{eg}^{(i,l)} \sigma_{ge}^{(j,l)} + \text{H.c.}), \\ H_{\text{array}}^{(y)} &= \sum_{j=1}^{2N} \sum_{k,l=1}^{2N} \hbar (J_{k,l} \sigma_{eg}^{(j,k)} \sigma_{ge}^{(l,j)} + \text{H.c.}), \end{aligned} \quad (6)$$

where $H_{\text{array}}^{(x)}$ and $H_{\text{array}}^{(y)}$ describe the effective spin-spin interactions in the x and y directions, respectively. $J_{i,j}$ and $J_{k,l}$ are the corresponding phonon-mediated spin-spin hopping rates.

Unlike the conventional dipole-dipole interaction mediated by a mechanical resonator or waveguide, this band-gap-mediated spin-spin interaction decays exponentially with the spin spacing, with a decay length L_c . This form of inter-particle coupling ($J_{i,j} \sim e^{-|x_i - x_j|/\lambda}$) is commonly encountered in several other quantum systems, such as quantum dot and trapped-ion setups [15,17]. In the spin-phononic crystal system, owing to the unique band-gap structures of the phononic crystal, we can get strong and tunable spin-spin interactions by controlling the properties of the mediating phononic modes.

III. 1D TOPOLOGICAL PROPERTIES

A. The periodic driving

The periodic driving is generally utilized to derive the effective Hamiltonian, in which specific terms can be

adiabatically eliminated. In particular, the periodic driving can be used to trigger nonequilibrium topological behavior in a trivial setup, which offers an efficient tool to simulate topological phases in quantum systems [83,84]. We consider a periodic driving quantum system with $H(t) = H(t + T)$, characterized by time period $T = 2\pi/\omega$. In this case we can introduce the Floquet theorem to investigate long-time dynamics of the system, as developed in Ref. [85]. With the Floquet-Bloch ansatz, the time-dependent Schrödinger equation is given by

$$i\hbar d_t |\psi_\alpha(t)\rangle = H(t) |\psi_\alpha(t)\rangle, \quad (7)$$

where

$$|\psi_\alpha(t)\rangle = |\phi_\alpha(t)\rangle e^{-i\epsilon_\alpha t/\hbar} = e^{-i\epsilon_\alpha t/\hbar} \sum_m e^{-im\omega t} \phi_m. \quad (8)$$

$|\psi_\alpha(t)\rangle$ is the so-called Floquet eigenstate, and ϵ_α is the quasienergy with band index α . $\phi_\alpha(t) = \phi_\alpha(t + T)$ denotes the time-periodic Floquet eigenmode, which can be constructed by a complete set of orthonormal basis states ϕ_m . With respect to the basis $|\psi_\alpha(t)\rangle$, the system can be expanded in a Fourier series as

$$H_{mn} = \frac{1}{T} \int_0^T dt e^{i(m-n)\omega t} H(t). \quad (9)$$

Note that the Floquet state in time-periodically driven systems is analogous to the Bloch state in spatially periodic systems.

In a recent work, we proposed a periodic driving protocol to simulate topological phases with a color center-phononic crystal system [62]. By applying a standing wave field between the two lowest sublevels ($|g\rangle, |e\rangle$) of the SiV center, we get the Floquet engineering of the spin-spin interactions, resulting in the well-known SSH-type Hamiltonian. Here, we consider a fundamentally different driving protocol, which allows us to selectively control the spin-spin interactions. What is more important, the resulting spin-spin interactions possess chiral symmetry and support rich quantum phases associated with topological invariants. The time-periodic driving has the form [17]

$$H_{\text{driv}}(t) = \sum_j \hbar V_j f(t) \sigma_j^z, \quad (10)$$

where $\sigma_j^z = |e\rangle_j \langle e| - |g\rangle_j \langle g|$ is the Pauli operator component. $f(t)$ denotes the standard square-wave function

$$f(t) = -1 \quad \text{for } t \in \left[0, \frac{T}{2}\right], \\ f(t) = 1 \quad \text{for } t \in \left[\frac{T}{2}, T\right]. \quad (11)$$

V_j denotes the on-site potential

$$V_j = \begin{cases} b_0 + \frac{(a_0+b_0)}{2}(j-1) & j = 1, 3, 5, 7, \dots \\ \frac{(a_0+b_0)}{2}j & j = 2, 4, 6, 8, \dots \end{cases} \quad (12)$$

This stairlike form offers alternating potential difference between two adjacent spins; that is, $V_j - V_{j-1} = a_0$ and $V_{j+1} - V_j = b_0$ are staggered along the spin array.

We first consider the interaction of the periodic driving and the 1D spin array, but the case of 2D will be studied in the next section. Now we transform the total Hamiltonian $H_{\text{1D}} = H_{\text{array}} + H_{\text{driv}}(t)$ into the interaction picture, with

the unitary operator $U(t) = e^{-i \int_0^t d\tau H_{\text{driv}}(\tau)/\hbar}$. After the unitary transformation, we obtain

$$\sigma_{eg}^j \rightarrow e^{i\Delta_j(t)\sigma_j^z} \sigma_{eg}^j e^{-i\Delta_j(t)\sigma_j^z} = \sigma_{eg}^j e^{2i\Delta_j(t)}, \\ \sigma_{ge}^j \rightarrow e^{i\Delta_j(t)\sigma_j^z} \sigma_{ge}^j e^{-i\Delta_j(t)\sigma_j^z} = \sigma_{ge}^j e^{-2i\Delta_j(t)}, \quad (13)$$

with

$$\Delta_j(t) = V_j \int_0^t d\tau f(\tau) \\ = V_j \int_0^t d\tau \left[\sum_{n \neq 0} \frac{1}{n\pi i} (e^{-in\pi} - 1) e^{in\omega\tau} \right], \quad (14)$$

where we expanded $f(t)$ into its Fourier series. In the interaction picture, the total Hamiltonian has the form

$$H_{\text{1D}} = \sum_{i,j} \hbar J_{ij}(t) \sigma_{eg}^i \sigma_{ge}^j, \quad (15)$$

where $J_{ij}(t) = J_{ij} e^{2i[\Delta_i(t) - \Delta_j(t)]}$ is the hopping rate with a temporal periodicity, $J_{ij}(t) = J_{ij}(t + T)$. The Floquet components of the Hamiltonian (15) read

$$H_{\text{1D}}^{mn} = \sum_{i,j} \hbar J_{ij}^{mn} \sigma_{eg}^i \sigma_{ge}^j, \quad (16)$$

$$J_{ij}^{mn} = \frac{1}{T} \int_0^T dt J_{ij}(t) e^{i(m-n)\omega t}. \quad (17)$$

For the time-periodically driven problems, the system Hamiltonian can be effectively expressed by the Floquet-Magnus expansion. In this case, H_{1D} can be described by the static effective Hamiltonian

$$H_{\text{eff}} = H_F^{(0)} + H_F^{(1)} + H_F^{(2)} \dots, \quad (18)$$

where $H_F^{(v)} \sim \omega^{-v}$ describes the v th-order contribution in the Magnus expansion. In the high-frequency regime $\omega \gg \max |J_{i,j}|$, it is a good approximation to truncate the expansion to the zeroth-order term $H_F^{(0)} = \frac{1}{T} \int_0^T dt H_{\text{1D}}$ [85–87]. Then, the leading order term $H_F^{(0)}$ can approximately capture the system dynamics, which is the predominant regime in this paper. As a result, the periodically modulated spin-spin interaction can be written as

$$\mathcal{J}_{ij} = J_{ij} \frac{i\omega}{2\pi(V_i - V_j)} [e^{-2\pi i(V_i - V_j)/\omega} - 1]. \quad (19)$$

As for the SSH model, the interparticle interaction is characterized by staggering hopping amplitudes. Thus the two nearest-neighbor spins can be grouped into a unit cell and classified as odd and even spins, as we proposed in Ref. [62]. Likewise, here we consider the phononic-mediated spin-spin interaction as a bipartite lattice of the form $ABABAB$, with

$$A_n = \sigma_{ge}^j \quad j = 1, 3, 5, 7, \dots, \\ B_n = \sigma_{eg}^j \quad j = 2, 4, 6, 8, \dots \quad (20)$$

Based on this definition, we rewrite the renormalized hopping amplitude $\mathcal{J}_{ij}$. For simplicity, here we introduce n_l to label the spins at the l site of the n th cell, $l = A$ or B . In general, there are two types of interparticle hopping. The even-neighbor

FIG. 2. Schematic diagram of effective spin-spin interactions. (a) Even-neighbor hopping. (b) and (c) illustrate two kinds of odd-neighbor hopping examples, respectively.

hopping, which describes the spin-spin interaction of the same sublattice, is shown in Fig. 2(a). The potential difference is $V_i - V_j = \pm m(a_0 + b_0)$, with $m = 0, 1, 2, \dots$, and plus and minus signs correspond to the left and right hoppings, respectively. In consequence, the even-neighbor spin-spin hopping rate can be written as

$$\mathcal{J}_{n_l, n_l'} = \frac{iJ_{n_l, n_l'}}{\mp 2\pi qm} (e^{\pm i2\pi qm} - 1), \quad (21)$$

where $q = (a_0 + b_0)/\omega$. From Eq. (21), the even-neighbor hopping is always zero if we assign $q = 1, 2, 3, \dots$. Thus we conclude that the even-neighbor hopping can be suppressed by tuning the parameters a_0 and b_0 . Note that the even-neighbor hopping is a detrimental source for the chiral symmetry [17].

The odd-neighbor hopping describes the spin-spin interaction of the different sublattices. To better describe the physical picture of the spin-spin interaction, we further classify two kinds of odd-neighbor hopping. We first discuss the case with $n_A \leq n_B$, for which the schematic diagram is shown in Fig. 2(b). If we define $n_B = n_A + r$ ($r = 0, 1, 2, \dots$), the spin-spin interaction can be described by

$$\begin{aligned} \mathcal{J}_{n_A, n_B} &= -\frac{iJ_{n_A, n_B}}{2\pi(qr + \frac{a_0}{\omega})} [e^{2\pi i(qr + \frac{a_0}{\omega})} - 1], \\ \mathcal{J}_{n_B, n_A} &= \frac{iJ_{n_A, n_B}}{2\pi(qr + \frac{a_0}{\omega})} [e^{-2\pi i(qr + \frac{a_0}{\omega})} - 1], \end{aligned} \quad (22)$$

where $\mathcal{J}_{n_A, n_B}$ and $\mathcal{J}_{n_B, n_A}$ describe the forward ($A \rightarrow B$) and backward ($B \rightarrow A$) hoppings, respectively. For the case with $n_A > n_B$, the schematic diagram is shown in Fig. 2(c). If we define $n_B = n_A - r'$ ($r' = 1, 2, 3, \dots$), the spin-spin interaction

FIG. 3. (a) Quasienergy spectrum as a function of the driving parameter a_0 . The corresponding winding number $\mathcal{W}$ is indicated. Here, we assign $|x_j - x_{j+1}| = a$, $L_c = a$. Note that only the first and third neighbor interactions are included. The other parameters are $N = 5$, $\omega = 10$, $q = 1$, and $J_0 = 1$.

tion can be described by

$$\begin{aligned} \mathcal{J}'_{n_A, n_B} &= \frac{iJ_{n_A, n_B}}{2\pi(qr' - \frac{a_0}{\omega})} [e^{-2\pi i(qr' - \frac{a_0}{\omega})} - 1], \\ \mathcal{J}'_{n_B, n_A} &= -\frac{iJ_{n_A, n_B}}{2\pi(qr' - \frac{a_0}{\omega})} [e^{2\pi i(qr' - \frac{a_0}{\omega})} - 1]. \end{aligned} \quad (23)$$

Likewise, $\mathcal{J}'_{n_A, n_B}$ and $\mathcal{J}'_{n_B, n_A}$ represent the backward ($A \rightarrow B$) and forward ($B \rightarrow A$) hoppings, respectively. From Eqs. (22) and (23), we can conclude that

$$\mathcal{J}_{n_A, n_B} = (\mathcal{J}_{n_B, n_A})^*, \quad \mathcal{J}'_{n_A, n_B} = (\mathcal{J}'_{n_B, n_A})^*. \quad (24)$$

Unlike the case of the SSH model, the backward and forward hoppings of the odd-neighbor spin-spin interaction are not equal.

According to this bipartite solution, the Hamiltonian H_{1D} can be rewritten as

$$\begin{aligned} H_{1D} &= \sum_{n, r, r'} \hbar (\mathcal{J}_{n_A, n_B} A_n B_{n+r}^\dagger + \mathcal{J}_{n_B, n_A} A_n^\dagger B_{n+r} \\ &\quad + \mathcal{J}'_{n_A, n_B} A_n B_{n-r'}^\dagger + \mathcal{J}'_{n_B, n_A} A_n^\dagger B_{n-r'}). \end{aligned} \quad (25)$$

Here, we neglected the even-neighbor hopping terms. By applying a particular periodic driving field to the SiV centers, we obtain the Floquet engineering of the spin-spin interactions with unique properties. In this case, the even-neighbor hopping is suppressed by tuning the parameters of the driving field, while the odd-neighbor hopping can be enhanced as needed. This scheme enforces the chiral symmetry which provides topological protection for the spin-spin interaction. In Fig. 3, we numerically calculate the quasienergy spectrum as a function of the driving parameter a_0 . We can see that all the eigenmodes are grouped into chiral-symmetric pairs with opposite energies. For simplicity, here we express the bare spin-spin interaction as

$$J_{i,j} = \frac{g_c^2}{2\Delta_{BE}} e^{-|x_i - x_j|/L_c} = J_0 e^{-|x_i - x_j|/L_c}. \quad (26)$$

Given that the band-gap-mediated spin-spin interaction decays exponentially with the spin spacing, here only the first and third neighbor interactions are included.

B. Topological phases

The periodic driving protocol offers an effective method to investigate the topological character of the spin-phononic crystal system. To explore topological features of the Floquet engineering spin-spin system, we convert H_{1D} to the momentum space. Considering periodic boundary conditions, we can make the Fourier transformation

$$O_n = \frac{1}{\sqrt{N}} \sum_k e^{ink} O_k \quad (O = A, B), \quad (27)$$

where $k = 2\pi m/N$ ($m = 1, 2, \dots, N$) is the wave number in the first Brillouin zone and A_k and B_k are the momentum-space operators. Defining the unitary operator $\psi(k) = (A_k \ B_k)^T$, the Hamiltonian H_{1D} can be expressed as

$$H_{1D} = \sum_k \psi(k)^\dagger H(k) \psi(k). \quad (28)$$

Then we obtain the 2×2 -matrix form of the Hamiltonian in the k space

$$H(k) = \hbar \begin{pmatrix} 0 & f(k) \\ f^*(k) & 0 \end{pmatrix}. \quad (29)$$

with

$$f(k) = \sum_{r,r'} (\mathcal{J}_{n_B, n_A} e^{ikr} + \mathcal{J}'_{n_B, n_A} e^{-ikr'}). \quad (30)$$

Here, $f(k)$ describes the coupling between the A and B spins in momentum space. According to Eq. (29), we can conclude that the bipartite spin arrays possess a chiral symmetry

$$\sigma_z^{-1} H(k) \sigma_z = -H(k), \quad (31)$$

which results from the periodic modulation of the external driving fields. In consequence, the eigenvalues of the system are paired with opposite values ($E, -E$), and the corresponding eigenstates are related by $|-E\rangle = \sigma_z |E\rangle$ [88]. The chiral symmetry enables topologically protected quantum state transfer, of which a detailed discussion is given in Sec. V. Note that in this paper we only keep the leading order term $H_F^{(0)}$ in the effective Floquet Hamiltonian, since the high-frequency regime $\omega \gg \max |J_{i,j}|$ is considered. In this case, the higher-order terms can be safely neglected, and both the chiral symmetry and topological phase transitions are unaffected by these terms.

The dispersion relation can be obtained by solving the eigenvalue equation

$$H(k) \psi(k) = E(k) \psi(k), \quad (32)$$

using the fact that $H^2(k) = E^2(k)I$, with I being the identity operator in the Hilbert space. Then we obtain the energy band structure as

$$E_{\pm}(k) = \pm \hbar |f(k)|, \quad (33)$$

$$\psi(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm e^{-i\vartheta(k)} \end{pmatrix}. \quad (34)$$

$\psi(k)$ corresponds to the eigenfunctions for the lower and upper bands, and $\vartheta(k)$ is defined as the argument of $f(k)$. Figures 4(a1)–4(f1) show the energy spectra for different driving field parameters, which are split into two branches, and there exists a band gap between the lower and upper branches. It should be noted that the band gap will vanish at the critical point of the topological phases.

The band-gap structures are generally associated with topological properties of bulk-boundary correspondence. For the 1D Floquet engineering spin-spin system, we introduce the topological Zak phase [89]

$$\begin{aligned} \mathcal{Z} &= -i \sum_{j=1}^{\text{occ.}} \int_0^{2\pi} dk \psi^\dagger(k) \partial_k \psi(k) \\ &= N_{\text{occ.}} \frac{1}{2} \int_0^{2\pi} dk \frac{d}{dk} \vartheta(k) \\ &= \mathcal{W} \pi, \end{aligned} \quad (35)$$

where $\mathcal{W}$ is the topological winding number and $N_{\text{occ.}}$ describes the number of occupied energy bands. Now we need to investigate the winding number of the system. Alternatively, $H(k)$ can be expressed in the form

$$H(k) = \hbar d(k) \cdot \sigma, \quad (36)$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli matrix and $d(k)$ denotes a three-dimensional vector field

$$\begin{aligned} d_x(k) &= \frac{1}{2} \sum_{r,r'} (\mathcal{J}_{n_B, n_A} e^{ikr} + \mathcal{J}'_{n_B, n_A} e^{-ikr'} + \text{c.c.}), \\ d_y(k) &= \frac{i}{2} \sum_{r,r'} (\mathcal{J}_{n_B, n_A} e^{ikr} + \mathcal{J}'_{n_B, n_A} e^{-ikr'} - \text{c.c.}), \\ d_z(k) &= 0. \end{aligned} \quad (37)$$

For general two-band topological insulators, owing to the periodicity of the momentum-space Hamiltonian, the path of the endpoint of $d(k)$ is a closed loop in the auxiliary space (d_x, d_y) [90]. The topology of this loop can be characterized by an integer, the winding number

$$\mathcal{W} = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{n} \times \partial_k \mathbf{n} dk, \quad (38)$$

where $\mathbf{n} = (n_x, n_y) = (d_x, d_y)/\sqrt{d_x^2 + d_y^2}$ is the normalized vector. Here, the winding number $\mathcal{W}$ counts the number of times the loop winds around the origin of the dx - dy plane. Figures 4(a2)–4(f2) present the path of the endpoint of $d(k)$ on the dx - dy plane. For different values of a_0 , the winding number of the system exhibits four possible values: $-1, 0, 1$, and 2 . According to Eq. (35), we can derive the relevant topological Zak phase directly. Furthermore, one can implement the topological phase transition in this SiV-phononic system by modulating the periodic driving.

From the numerical simulation results, we derive rich quantum phases related to the topological winding number. As for the generalized SSH model, a prototypical example to investigate topological properties in a trivial system, the 1D Zak phase has only two possible values, 0 or π . This work offers an effective scheme for studying topological phases induced by periodic driving. The distinct feature is that it

FIG. 4. (a1)–(f1) show the dispersion relations for various parameter settings of periodic driving: (a1) $a_0 = -16$, (b1) $a_0 = -11$, (c1) $a_0 = 4$, (d1) $a_0 = 19$, (e1) $a_0 = 21$, and (f1) $a_0 = 26$. As the wave number k runs through the Brillouin zone $(-\pi, \pi)$, the energy spectrum splits into two branches, and there exists a band gap between the lower and higher branches. (a2)–(f2) correspond to the winding configuration of (d_x, d_y) around the origin (red star), and the relevant winding number $\mathcal{W}$ is explicitly shown. In (a2) and (c2), the loop wind avoids the origin, and then $\mathcal{W} = 0$. In (b2), (d2), and (f2), the endpoint of $d(k)$ encircles the origin once, but these are topologically inequivalent. For (d2) and (f2), the endpoint of $d(k)$ is a closed loop in the counterclockwise direction, and $\mathcal{W} = 1$. In contrast, in (b2), the endpoint of $d(k)$ is along the clockwise direction, and $\mathcal{W} = -1$. In (e2), the endpoint of $d(k)$ encircles the origin two times, and $\mathcal{W} = 2$. Other parameters are the same as those in Fig. 3.

allows for simulating a larger winding number and related Zak phase transitions in topological trivial systems.

C. Edge states

In the following, we proceed to simulate topological edge states in the 1D spin-phononic system. The core step is to look for the zero-energy eigenstates. Here, we introduce the single-excited state

$$\psi = \sum_n (a_n A_n^\dagger + b_n B_n^\dagger) |0\rangle, \quad (39)$$

where a_n and b_n are the amplitudes of the occupying probability in the n th cell. $|0\rangle = |ggg \dots\rangle$ is the vacuum state, which describes that all spins stay in the ground state $|g\rangle$. In the single-excited state subspace, we can get the zero-energy eigenstates by solving

$$H_{1D} \sum_n (a_n A_n^\dagger + b_n B_n^\dagger) |0\rangle = 0. \quad (40)$$

There will be $2N$ equations for the amplitudes a_n and b_n . Considering the boundary conditions, $b_0 = a_{N+1} = 0$. We can analytically derive the left and right zero-energy edge states, respectively.

To verify the model, we numerically calculate the energy spectrum and zero-energy eigenstates of the system. Figures 5(a1)–5(d1) show the eigenvalues for various parameter settings of the periodic driving. As for the nontopological regime, $\mathcal{W} = 0$, there will be an energy band gap, but no gapless modes appear. For the cases with $\mathcal{W} = 1$ and $\mathcal{W} = -1$, there are two zero-energy eigenvalues. For the case with $\mathcal{W} = 2$, there are four zero-energy eigenvalues. Correspondingly, we plot the zero-energy edge states in Figs. 5(a2)–5(d2). We see that the wave functions are located in the vicinity of the array boundaries, which are the so-called topological

edge states. In addition, the edge states are only distributed at certain (odd or even) sites, which is determined by the chiral symmetry of the system.

IV. 2D TOPOLOGICAL PROPERTIES

A. The periodic driving

Now we generalize the above 1D results to 2D spin-phononic crystal networks. Here, we consider adding two mutually perpendicular microwave fields to the color center arrays [91]. The first one is a time-dependent microwave driving of frequency ω_x in the x direction. The other is a time-dependent driving of frequency ω_y in the y direction. These two periodic driving fields have the form

$$\begin{aligned} H_{\text{driv}}^{(x)} &= \sum_{l=1}^{2N} \sum_{j=1}^{2N} \hbar V_{j,l} f_x(t) \sigma_{j,l}^z, \\ H_{\text{driv}}^{(y)} &= \sum_{j=1}^{2N} \sum_{l=1}^{2N} \hbar V_{j,l} f_y(t) \sigma_{j,l}^z. \end{aligned} \quad (41)$$

$V_{j,l} = (V_j, V_l)$ describes the on-site potential in the 2D phononic network, the two components of which are in the form of Eq. (12). $f_x(t)$ and $f_y(t)$ denote the square-wave function in the x and y directions, respectively.

Let us discuss the two directions separately. For the periodic driving spin arrays along the x direction, the total Hamiltonian can be written as

$$H_{2D}^{(x)} = H_{\text{array}}^{(x)} + H_{\text{driv}}^{(x)}. \quad (42)$$

In the interaction picture, we introduce the unitary operator $U_x(t) = e^{-i \int_0^t d\tau H_{\text{driv}}^{(x)} / \hbar}$. After the unitary transformation, we obtain

$$\sigma_{eg}^{(j,l)} \rightarrow \sigma_{eg}^{(j,l)} e^{2i\Delta_j(t)}, \quad \sigma_{ge}^{(j,l)} \rightarrow \sigma_{ge}^{(j,l)} e^{-2i\Delta_j(t)}, \quad (43)$$

FIG. 5. (a1)–(d1) show the energy spectrum in the single-excited state subspace for various parameter settings of periodic driving: (a1) $a_0 = -11$, (b1) $a_0 = 19$, (c1) $a_0 = 21$, and (d1) $a_0 = 26$. For the cases with $\mathcal{W} = 1$ and $\mathcal{W} = -1$, there are two zero-energy eigenvalues. For the case with $\mathcal{W} = 2$, there are four zero-energy eigenvalues. The zero-energy eigenvalues (the red points) correspond to the topological edge states. The rest of the eigenvalues (the blue points) correspond to the bulk states of the system. (a2)–(d2) show the related eigenfunction of the gapless modes. Here, we consider $N = 50$. Other parameters are the same as those in Fig. 3.

with

$$\Delta_j(t) = V_j \int_0^t d\tau f(\tau), \quad (44)$$

while for the spin arrays along the y direction, the total Hamiltonian is given by

$$H_{2D}^{(y)} = H_{\text{array}}^{(y)} + H_{\text{driv}}^{(y)}. \quad (45)$$

Similarly, we introduce the unitary operator $U_y(t) = e^{-i \int_0^t d\tau H_{\text{driv}}^{(y)}/\hbar}$. After the unitary transformation, we obtain

$$\sigma_{eg}^{(j,l)} \rightarrow \sigma_{eg}^{(j,l)} e^{2i\Delta_l(t)}, \quad \sigma_{ge}^{(j,l)} \rightarrow \sigma_{ge}^{(j,l)} e^{-2i\Delta_l(t)}, \quad (46)$$

with

$$\Delta_l(t) = V_l \int_0^t d\tau f(\tau). \quad (47)$$

In the following, analogous to the 1D case, we consider the bipartite interaction in both x and y directions. Then we get a 2D system with $N \times N$ unit cells. As depicted in Fig. 6, there are four spins in each unit cell, which are labeled as $\{A, B, C, D\}$, respectively. In the regime $\omega \gg J_{i,j}, J_{k,l}$, we derive the Floquet engineering spin-spin interactions along the x and y directions

$$H_{2D}^{(x)} = \sum_m \sum_{n,r,r'} \hbar [\mathcal{J}_{n,n+r} (A_{n,m} B_{n+r,m}^\dagger + C_{n,m} D_{n+r,m}^\dagger) + \mathcal{J}'_{n,n-r} (A_{n,m} B_{n-r,m}^\dagger + C_{n,m} D_{n-r,m}^\dagger) + \text{H.c.}],$$

$$H_{2D}^{(y)} = \sum_n \sum_{m,r,r'} \hbar [\mathcal{J}_{m,m+r} (A_{n,m} C_{n,m+r}^\dagger + B_{n,m} D_{n,m+r}^\dagger) + \mathcal{J}'_{m,m-r} (A_{n,m} C_{n,m-r}^\dagger + B_{n,m} D_{n,m-r}^\dagger) + \text{H.c.}]. \quad (48)$$

For simplicity, we introduce (n, m) to describe the position of each unit cell in the 2D spin-spin networks, with $n, m = 1, 2, \dots, N$.

FIG. 6. Schematic diagram of the 2D Floquet engineering spin-spin interaction. There are four spins in a unit cell, which are labeled as $\{A, B, C, D\}$, respectively. Here, we introduce (n, m) to describe the position of each unit cell in the 2D spin-spin networks. For simplicity, only the nearest-neighbor interactions are illustrated.

FIG. 7. (a) Band structure of the 2D Floquet engineering spin-spin interaction in the $\mathbf{k}$ space, $a_0 = 4$. The energy spectrum has four branches. The lowest and highest bands are isolated, while the two middle bands are connected at the edges of the Brillouin zone: $(0,0)$, $(\pm\pi, \pm\pi)$, and $(\mp\pi, \pm\pi)$. (b) Dispersion relation along the path $M \rightarrow \Gamma \rightarrow X \rightarrow M$ in the first Brillouin zone, $a_0 = 21$. The parities of each band at high-symmetry points are marked by the plus and minus signs. (c) The bulk spectra for $a_0 = 5$, which describes a topological phase transition. Other parameters are the same as those in Fig. 3.

To simplify the model, here we suppose that the spin spacing $d_x = d_y$ and the periodic driving frequencies $\omega_x = \omega_y$. In this case, we can derive

$$\begin{aligned} \mathcal{J}_{n,n+r} &= \mathcal{J}_{m,m+r}, \\ \mathcal{J}'_{n,n-r'} &= \mathcal{J}'_{m,m-r'}. \end{aligned} \quad (49)$$

Thus we can define $s = n$ or m , and the two-dimensional Hamiltonian can be further integrated as

$$\begin{aligned} H_{2D} = \sum_{r,r'} \sum_{n,m} \hbar [&\mathcal{J}_{s,s+r} (A_{n,m} B_{n+r,m}^\dagger + C_{n,m} D_{n+r,m}^\dagger \\ &+ A_{n,m} C_{n,m+r}^\dagger + B_{n,m} D_{n,m+r}^\dagger) \\ &+ \mathcal{J}'_{s,s-r'} (A_{n,m} B_{n-r',m}^\dagger + C_{n,m} D_{n-r',m}^\dagger \\ &+ A_{n,m} C_{n,m-r'}^\dagger + B_{n,m} D_{n,m-r'}^\dagger) + \text{H.c.}]. \end{aligned} \quad (50)$$

B. Topological phases

To investigate the topological features in the 2D Floquet engineering spin-spin system, we convert the Hamiltonian H_{2D} to the momentum space. Here, we consider periodic boundary conditions along both the x and y directions. Then we apply the Fourier transformation to the four spins in a unit cell

$$O_{n,m} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i(k_x n + k_y m)} O_{\mathbf{k}} \quad (O = A, B, C, D), \quad (51)$$

where $\mathbf{k} = (k_x, k_y)$ is the wave number in the first Brillouin zone. If we define the unitary operator $\psi(\mathbf{k}) = (A_{\mathbf{k}} \ B_{\mathbf{k}} \ C_{\mathbf{k}} \ D_{\mathbf{k}})^T$, the two-dimensional Hamiltonian can be rewritten as

$$H_{2D} = \sum_{\mathbf{k}} \psi^\dagger(\mathbf{k}) H(\mathbf{k}) \psi(\mathbf{k}). \quad (52)$$

Along with that we get the 4×4 -matrix form of the Hamiltonian in the $\mathbf{k}$ space

$$H(\mathbf{k}) = \hbar \begin{pmatrix} 0 & f(k_x) & f(k_y) & 0 \\ f^*(k_x) & 0 & 0 & f(k_y) \\ f^*(k_y) & 0 & 0 & f(k_x) \\ 0 & f^*(k_y) & f^*(k_x) & 0 \end{pmatrix}. \quad (53)$$

$f(k_x)$ and $f(k_y)$ have the same form as Eq. (30) and describe the spin-spin couplings in the x and y directions, respectively.

Let us study the 2D dispersion relation by solving the eigenvalue equation

$$H(\mathbf{k})\psi(\mathbf{k}) = E(\mathbf{k})\psi(\mathbf{k}); \quad (54)$$

then we obtain

$$E(\mathbf{k}) = \epsilon_x \hbar |f(k_x)| + \epsilon_y \hbar |f(k_y)|, \quad (55)$$

$$\psi(\mathbf{k}) = \frac{1}{2} \begin{pmatrix} 1 \\ \epsilon_x e^{-i\vartheta_x(k_x)} \\ \epsilon_y e^{-i\vartheta_y(k_y)} \\ \epsilon_x \epsilon_y e^{-i[\vartheta_x(k_x) + \vartheta_y(k_y)]} \end{pmatrix}, \quad (56)$$

where $\epsilon_i = \pm 1$ and $\vartheta_i(k_i) = \arg[f(k_i)]$, with $i = x, y$. In Fig. 7(a), we numerically calculate the 2D energy spectrum in the momentum space. There are four energy bands since there are four spins in a unit cell. The lowest and highest bands are isolated, while the two middle bands are joined at the edges of the Brillouin zone: $(0,0)$, $(\pm\pi, \pm\pi)$, and $(\mp\pi, \pm\pi)$. According to Eq. (55), there exist two equal energy band gaps. When assigning suitable values of a_0 , these four bands will be joined together, and the band gaps will vanish. This is a signature of the topological phase transition.

In the following, we discuss the extended Zak phase in the 2D spin-phononic structure. As shown in Fig. 7(b), we simulate the dispersion relation in the first Brillouin zone (BZ). There are four energy bands, which can be grouped as two symmetric pairs with opposite energies, and the middle pair degenerate at C_{4v} invariant points. Figure 7(c) shows the energy band structures at a critical point of topological phase transition. We see that the four energy bands touch at the Γ and X points, and the band gaps are closed. To characterize the bulk topology of the 2D system, we introduce the vectored Zak phase $\mathbf{Z} = (Z_x, Z_y)$ [23,24], which can be expressed as

$$\mathbf{Z} = \int_{\text{BZ}} d^2\mathbf{k} \text{Tr}[\mathcal{A}_m(\mathbf{k})] = 2\pi \mathbf{P}, \quad (57)$$

where $\mathcal{A}_m(\mathbf{k}) = i\psi_m^\dagger(\mathbf{k})\partial_{\mathbf{k}}\psi_m(\mathbf{k})$ is the non-Abelian Berry connection and $\psi_m(\mathbf{k})$ denotes the periodic Bloch function for the m th energy band. Here, $\mathbf{P}$ denotes the topological invariant

based on the electric polarization. For a system with inversion symmetry, the charge polarization can be related to the parity of the wave function [92],

$$\mathcal{P}_i = \frac{1}{2}(q_i \bmod 2), \quad (-1)^{q_i} = \prod_{m \in \text{occ.}} \frac{\zeta_m(X_i)}{\zeta_m(\Gamma)}, \quad (58)$$

where $i = x, y$ and ζ_m denotes the parity of the m th Bloch eigenstate. Note that the Zak phase is only summed over those bands that lie below the Fermi surface. Because of the C_{4v} point-group symmetry (PGS), we can derive $\mathcal{P}_x = \mathcal{P}_y$ in the $\mathbf{k}$ space. In Fig. 7(b), we label the parity of the energy bands and therefore obtain $\mathbf{P} = (1/2, 1/2)$. In this case, the system is identified by the fractional topological invariant and is in a topological nontrivial insulating phase. For the topological trivial case, the parities of each band will experience a π rotation at the X point, and the wave polarization becomes $\mathbf{P} = (0, 0)$. It is worth noting that although the 2D Zak phase is totally determined by the bulk property, the bulk-edge correspondence associated with the topological phase is manifested.

C. Edge states

After discussing the topological invariants, we are now in a position to study topological edge states in the 2D spin-phononic system. To show the behavior of the edge states, here we consider a 2D strip structure with the periodic boundary condition in the y direction and N_x unit cells in the x direction [93,94]. In this case, the Floquet engineering spin-spin interaction is translationally invariant only along the y direction.

Taking Fourier transformation in the y direction, the two-dimensional strip can be reduced to a set of 1D spin-spin interactions indexed by a continuous parameter k_y . The two-dimensional Hamiltonian can be rewritten as

$$\begin{aligned} H_{2D}(k_y) = \sum_{n,r,r'} \hbar [& \mathcal{J}_{n,n+r}(A_n B_{n+r}^\dagger + C_n D_{n+r}^\dagger \\ & + A_n C_n^\dagger e^{-ik_y r} + B_n D_n^\dagger e^{-ik_y r}) \\ & + \mathcal{J}'_{n,n-r'}(A_n B_{n-r'}^\dagger + C_n D_{n-r'}^\dagger \\ & + A_n C_n^\dagger e^{ik_y r'} + B_n D_n^\dagger e^{ik_y r'}) + \text{H.c.}], \end{aligned} \quad (59)$$

with $n = 1, 2, \dots, N_x$. In the single-excited state subspace,

$$\psi(k_y) = \sum_n (a_n A_n^\dagger + b_n B_n^\dagger + c_n C_n^\dagger + d_n D_n^\dagger) |0\rangle, \quad (60)$$

where a_n , b_n , c_n , and d_n denote the amplitudes of the occupying probability in the n th cell, respectively. Substituting Eqs. (59) and (60) into the eigenvalue equation

$$H_{2D}(k_y) \psi(k_y) = E \psi(k_y), \quad (61)$$

we obtain the following set of equations of motion

$$\begin{aligned} E a_n &= f^*(k_y) c_n + \mathcal{J}_{n,n+r} b_{n+r} + \mathcal{J}'_{n,n-r'} b_{n-r'}, \\ E b_n &= f^*(k_y) d_n + \mathcal{J}_{n,n-r} a_{n-r} + \mathcal{J}'_{n,n+r'} a_{n+r'}, \\ E c_n &= f(k_y) a_n + \mathcal{J}_{n,n+r} d_{n+r} + \mathcal{J}'_{n,n-r'} d_{n-r'}, \\ E d_n &= f(k_y) b_n + \mathcal{J}_{n,n-r} c_{n-r} + \mathcal{J}'_{n,n+r'} c_{n+r'}. \end{aligned} \quad (62)$$

For the open boundaries $x = 0, N_x + 1$, the following amplitudes will vanish:

$$\begin{aligned} b_0 &= d_0 = 0, \\ a_{N_x+1} &= c_{N_x+1} = 0. \end{aligned} \quad (63)$$

In this way, we can analytically drive the edge modes.

In Figs. 8(a)–8(c), we numerically calculate the resulting projected band structures with $N_x = 11$. From the energy spectrum in the k_y direction, we also verify the existence of edge states. The number of projected bands is determined by N_x . For the trivial case with $a_0 = 4$, there exist only the bulk modes (blue curves); no gapless modes emerge. For the topological nontrivial case with $a_0 = -11$ and 21 , we see that the edge modes (red dashed curves) appear inside the energy band gaps. When $a_0 = 21$, the topological invariant winding number $\mathcal{W} = 2$, and there are four zero-energy eigenstates, two of which are degenerated. In addition, we can also notice that the energy spectrum is symmetric with respect to the $E = 0$, which is related to the chiral symmetry of the system.

V. ROBUST QUANTUM STATE TRANSFER

Topological nontrivial spin-spin interactions host zero-energy bound states at both ends. In the following, we show that the topological edge states can be employed as a quantum channel between distant qubits. Since quantum information can be transferred directly between the boundary spins, the intermediate spins are virtually excited during the process, which ensures robust quantum state transfer [63,95].

Taking into account the coupling of the system with the environment in the Markovian approximation, the evolution of the system follows the master equation

$$\dot{\rho} = -\frac{i}{\hbar} [H_{1D}, \rho] + \sum_{j=1}^{2N} \gamma_s \mathcal{D}[\sigma_j^z] \rho, \quad (64)$$

with $\sigma_j^z = |e\rangle_j \langle e| - |g\rangle_j \langle g|$, γ_s being the spin dephasing rate of the single SiV centers, and $\mathcal{D}[O]\rho = O\rho O^\dagger - \frac{1}{2}\rho O^\dagger O - \frac{1}{2}O^\dagger O\rho$ for a given operator O .

To verify the theoretical results, we perform numerical calculations by using the Quantum Toolbox in Python (QUTIP) library for the 1D spin array with six spins, i.e., $N = 3$. Here, we take the excited left end spin as the initial condition. As illustrated in Fig. 9(a), we obtain significant Rabi oscillation of the left end spin. This implies that there is indeed quantum state transfer between the two ends of the spin array. However, for the nontopological condition, no direct quantum state transfer can be seen, as shown in Fig. 9(c), since in the topological trivial regime, the eigenstates are the superposition of entire spin arrays. In addition, we simulate the excitation dynamics for different parameters of the periodic driving. Comparing Fig. 9(a) with Fig. 9(b), we see that the localization of the edge states is more obvious when setting $a_0 = 12$. For the case with $a_0 = 24$, it takes a shorter time to accomplish quantum state transfer. Finally, we also consider the effect of spin dephasing on quantum state transfer. As shown in Fig. 9(a), when setting the dephasing rate $\gamma_s = 1 \times 10^{-4} J_0$, which is close to the practical experimental conditions, the fidelity can reach 0.9. The numerical results

FIG. 8. Projected band structures for various parameter settings of periodic driving: (a) $a_0 = 4$, (b) $a_0 = -11$, and (c) $a_0 = 21$. The blue and red curves denote the bulk and edge modes, respectively. Here, we consider $N_x = 11$. Other parameters are the same as those in Fig. 3.

can be optimized by adjusting the parameters of the periodic driving field.

VI. EXPERIMENTAL FEASIBILITY

We consider a 2D spin-phononic crystal network with SiV centers individually embedded in the nodes of a phononic crystal with square geometry. With state-of-the-art nanofabrication techniques, color center arrays can be readily realized via ion implantation [96]. Additionally, SiV centers can be treated as spin-qubit carriers and can be individually addressed with optical or microwave driving fields [79–81]. Floquet engineering can be achieved by applying periodic control pulse sequences to SiV center arrays [97]. Nanoscale phononic structures such as diamond crystals have been realized experimentally [67,68,98]. Compared with photonic

crystals, phononic crystals exhibit extremely low thermoelastic mechanical dissipation, providing a more promising platform for quantum technology. Owing to the advantage of the scalable nature of nanofabrication, the extension of phononic crystal structures to high dimensions is experimentally feasible, and extensive research has been conducted [71–76,99,100].

For the diamond phononic crystal illustrated in Fig. 1(a), the material properties are $E = 1050$ GPa, $\nu = 0.2$, and $\rho = 3539$ kg/m³. The lattice constant and cross section of the phononic crystal are $a = 100$ nm and $A = 100 \times 20$ nm², and the sizes of the elliptical holes are $(b, c) = (30, 76)$ nm. With these well-designed parameters, we obtain a unique band-gap structure, and the corresponding phononic band-edge frequency is $\omega_{BE} = 2\pi \times 44.933$ GHz. As for SiV color centers, the ground-state transition frequency is $\omega_s \simeq 2\pi \times 46$ GHz, which is exactly located within the phononic band gap [61]. In this case, the interplay between the SiV center and phononic crystal mode k is described by $g_k = \frac{d}{v_l} \sqrt{\frac{\hbar \omega_{BE}}{4\pi \rho a A}} \xi(\vec{r})$ [59]. Here, $d/2\pi \sim 1$ PHz is the strain sensitivity, $v_l = 1.71 \times 10^4$ m/s is the speed of sound in diamond, and $\xi(\vec{r})$ is the dimensionless strain distribution at the position of the SiV center $\vec{r}$ [76]. Then, we can obtain the effective SiV-phononic coupling rate as $g_k/2\pi \simeq 100$ MHz. In the large-detuning regime, $g \sim 0.1g_k$, the band-gap-engineered spin-phononic coupling rate is given by $g_c = g\sqrt{2\pi a/L_c} \simeq 2\pi \times 25$ MHz [61].

In addition, we should consider the decoherence of the SiV-phononic crystal setup. For the SiV color center in diamond, at millikelvin temperatures, the spin dephasing time is about $\gamma_s = 2\pi \times 100$ Hz [79]. As for phononic crystals, the mechanical quality factor can reach $Q \sim 10^7$ by using 2D phononic crystal shields [67]. In this case, we derive the mechanical damping rate $\gamma_m = \omega_{BE}/Q = 2\pi \times 4.5$ kHz. From the calculation above, we achieve the strong-coupling regime $g_c \gg \{\gamma_s, \gamma_m\}$, which enables coherent interactions between the SiV centers and phonon crystal modes. For the nearest-neighbor spins with $d_0 = a$, the bare spin-spin interaction $J_0 = \frac{g_c^2}{2\Delta_{BE}} \simeq 2\pi \times 4.1$ MHz. Since the band-gap-mediated spin interaction decays exponentially with the spin spacing, we conclude that the interaction between any two spins satisfies $J_{i,j} \leq J_0$. To simplify the effective Hamiltonian in Eq. (18), we assume the high-frequency driving regime $\omega \gg \max |J_{i,j}|$. Considering the experimental implementation, we can use the pulse sequences with a period of $t_p \sim$

FIG. 9. Excitation dynamics of the left end spin for various parameter settings of periodic driving: (a) $a_0 = 12$, (b) $a_0 = 24$, and (c) $a_0 = -2$. In (a), we also add the result in the case of $\gamma_s = 1 \times 10^{-4} J_0$. Here, we consider $N = 3$. Other parameters are the same as those in Fig. 3.

30 ns [97], and then one can readily derive $J_{i,j}t_p \ll 1$. As for the dynamic process simulated in Fig. 9(a), the period of quantum state transfer is $\mathcal{T} = 900/J_0 \simeq 35 \mu\text{s}$, which is much shorter than the SiV spin coherence time [101]. Therefore, with practical experimental conditions, this proposal can be implemented to achieve high-fidelity quantum state transfer.

VII. CONCLUSION

To conclude, we explore the topological quantum properties in two-dimensional SiV-phononic crystal networks. Applying a special periodic drive to the SiV centers, the phononic-band-gap-mediated spin-spin interactions exhibit a topologically protected chiral symmetry. Then, we study the topological properties of the 1D and 2D Floquet engineering SiV center arrays, respectively. For the periodic driving with properly chosen parameters, we analyze and simulate the corresponding topological invariants.

In contrast to the SSH model, the present Floquet engineering spin-spin interaction can be selectively controlled by modulating the periodic driving, which is essential for generating the necessary symmetries for topological protection. More interestingly, we present rich topological winding numbers in this paper. Owing to the highly controllable and tunable nature of the periodic driving, it is feasible to investigate the topological properties of the trimer case in SiV-phononic crystal systems. Looking forward, this proposal can be enriched to study chiral quantum acoustics, topological quantum computing, and the implementation of hybrid quantum networks.

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