Entanglement dynamics in dispersive optomechanics: Nonclassicality and revival

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We study entanglement dynamics in a dispersive optomechanical system consisting of two optical modes and a mechanical oscillator inside an optical cavity. The two optical modes interact with the mechanical oscillator, but not directly with each other. The appearance of optical entanglement witnesses nonclassicality of the oscillator. We study the dependence of the entanglement dynamics on the optomechanical coupling, the mean photon number in the cavity, and the oscillator temperature. An experimental realization with ultracold atomic ensembles is proposed.

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I. INTRODUCTION

Entanglement is one of the most striking phenomena of quantum theory [1]. Generating, manipulating, and measuring entanglement in systems with many constituents and with a large number of degrees of freedom is one of the challenges of quantum information and metrology [2] and an interesting frontier in fundamental physics [3]. In particular, entangling massive objects could open the way to interesting tests of quantum theory [4,5] and experiments aimed at probing gravitational effects of quantum mechanical matter [6–11]. Optomechanical systems provide a resourceful platform to this end.

It is well known that entanglement of massive objects can be realized in quantum cavity optomechanical experiments [12]. For instance, a cavity with a moving end mirror can be used to generate entangled “cat states” of both light [13,14] and matter [15–18]. Similar systems have also been proposed as an effective nonlinear medium [19–21], and squeezing [22–24] and optical entanglement [25] have been experimentally demonstrated in a variety of setups such as cavity cold atomic ensembles [26], dispersive dielectric membranes [27], and silicon microresonators [28]. In the linearized regime, particularly, entanglement dynamics [29,30] and stationary entanglement [31–34] have attracted much attention as these systems have a wide applicability ranging from precision force measurements [34,35] to quantum networks [36,37].

Certifying quantumness of optomechanical systems, however, is a far-from-obvious task. Relations among entanglement and nonclassicality measures of quantum states can be

used to probe the quantum nature of inaccessible objects such as a harmonic oscillator in an optical cavity [38]. Recurrence of optical squeezing in a cavity with a moving mirror has also been proposed as a witness of nonclassicality [39], and it has been shown that when two subsystems locally interact with a third one, but not directly with each other, the appearance of entanglement among those subsystems is sufficient to prove nonclassicality of the third party [40]. Building on some of these ideas the present work studies the entanglement dynamics of a dispersive optomechanical system, how to use that dynamics to probe the quantum nature of the oscillator through optical degrees of freedom, and how to optimize the generated optical entanglement by careful choice of the optomechanical coupling and the number of photons in an experiment.

Considering as possible implementations levitated optomechanical systems, such as silica nanospheres or cold atomic ensembles, and a “two-sided” cavity with a moving mirror in the middle, we map how entanglement appears and evolves among the various optical and mechanical subsystems for different optomechanical coupling strengths and optical field intensities. The appearance of mechanically induced optical entanglement and its subsequent death and revival are generic in these systems and thus could be used to probe the quantization of the center-of-mass of the moving object in real experiments. We also point out in a simplified context that under certain circumstances entanglement seems to “flow” through different subsystems, and such dynamics can be used to infer nonclassicality and entanglement among different components of the system. We consider examples of both non-Gaussian and Gaussian initial quantum states, for which we study the dynamics of concurrence and the Duan criteria [41], respectively. An experiment using levitated cold atomic ensembles is proposed.

II. HAMILTONIAN DESCRIPTION

The system we are primarily interested in is shown in Fig. 1(a): two optical cavities, of lengths L_a and L_b , are populated by modes of frequencies ω_a and ω_b and share a common

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FIG. 1. (a) Schematics of coupled optical cavities sharing a “mirror-in-the-middle” under a harmonic potential. No photon transfer from one cavity to the other is allowed. (b) Schematics of a particle trapped by an optical tweezer coupled to two modes of a cavity. The particle can be considered as a silica nanosphere or a cloud of ultracold atoms. When the levitated object is properly positioned, the Hamiltonian describing both systems acquires the same form.

perfect movable mirror of mass m subject to a harmonic potential of frequency ω_m . We refer to this as the “mirror-in-the-middle” configuration. In this system the optical modes never interact directly, except via the dispersive coupling due to the presence of the mechanical mode. Since we are interested in studying entanglement dynamics in optomechanics, we assume the cavities can be initialized in particular states and the laser driving term can be turned off during the course of the experiment. It is also assumed that optical losses are negligible during the time of the experiment, and a discussion of the conditions under which this is true and the experimental feasibility are addressed in the experimental proposal.

The Hamiltonian of the system reads [12]

$$\frac{H}{\hbar} = \omega_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} + \omega_m \hat{c}^\dagger \hat{c} - g_{0,a} \hat{a}^\dagger \hat{a} (\hat{c}^\dagger + \hat{c}) + g_{0,b} \hat{b}^\dagger \hat{b} (\hat{c}^\dagger + \hat{c}), \quad (1)$$

where $g_{0,i} = \omega_i x_{\text{ZPF}}/L_i$ are the optomechanical couplings, with $x_{\text{ZPF}} = \sqrt{\hbar/2m\omega_m}$ being the zero-point fluctuation of the mirror, and $\hat{a}$, $\hat{b}$, and $\hat{c}$ ($\hat{a}^\dagger$, $\hat{b}^\dagger$, and $\hat{c}^\dagger$) are the annihilation (creation) operators of each optical and mechanical modes, denoted by A , B , and C , respectively. Such a Hamiltonian can also be implemented using a cavity with a levitated nanoparticle [42–44] or an ultracold atom cloud [26,45,46] properly positioned along the cavity axis. This is illustrated in Fig. 1(b) (see Appendices A and B for details).

Assuming equal frequencies for the optical modes $\omega_a = \omega_b$ and approximately equal cavity lengths $L_a \sim L_b$, we may simplify the notation and directly write $g_0 \equiv g_{0,a} \sim g_{0,b}$. The unitary evolution operator resulting from Eq. (1) becomes

$$\hat{U}(t) = e^{-i\hat{c}^\dagger \hat{c} t} e^{-ir_a \hat{a}^\dagger \hat{a} t} e^{-ir_b \hat{b}^\dagger \hat{b} t} \times e^{k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})(\hat{c}^\dagger \eta(t) - \hat{c} \eta(t)^*)} e^{-i(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2 B(t)}, \quad (2)$$

where we define the dimensionless optomechanical coupling $k = g_0/\omega_m$, the “normalized frequencies” $r_i = \omega_i/\omega_m$, the scaled time $\omega_m t \rightarrow t$, and the functions $\eta(t) = 1 - e^{-it}$ and $B(t) = -k^2(t - \sin t)$. Note the evolution operator is com-

posed of a *Kerr-like* term, responsible for an effective optical nonlinearity [47], as well as an *optically driven* displacement operator acting on the mechanical mode.

It is expected that a generic separable state will evolve into an entangled one by virtue of the unitary evolution (2). We note that if an initially separable state gives birth to optical entanglement then there will certainly be its entanglement death. This springs from the fact that when $B(t) = 2\pi n$, $n \in \mathbb{N}$, the term in the evolution operator responsible for entangling the optical modes reduces to the identity operator at those times, therefore preserving the separability of the initial state. Analogous arguments show that optomechanical entanglement must also face death when $\eta(t) = 0$.

Not every state will evolve to an entangled one, as can be seen by considering the energy eigenstates of the system

$$\hat{D}_C[k(n_A - m_B)]|n_A, m_B, \ell_C\rangle, \quad (3)$$

where $\{|n_A, m_B, \ell_C\rangle\}$ denotes the number basis and $\hat{D}_C(\alpha)$ the displacement operator acting on the mechanical oscillator, mode C , by a displacement $\alpha \in \mathbb{C}$ (see Appendix C for details).

III. QUBIT STATES

Consider the cavities in Fig. 1(a) initially populated by the state

$$|\Psi(0)\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle. \quad (4)$$

We refer to these as “qubit states” as they are restricted to the vacuum-one-photon subspace. These states can be prepared in quantum optics in an approximate way using photon pair sources and displacement-based detection [48,49] or nonlinear light-matter interactions in cavity quantum electrodynamics [50,51]. In levitated quantum electrodynamics [52], it is also possible to couple two qubits to a nanosphere or rotor according to the interaction Hamiltonian (1), with the qubits assuming the role of the optical fields and the levitated object the role of the mirror-in-the-middle. Among the advantages of this type of scheme is the fact that the optomechanical coupling admits a wide tunability, potentially allowing tests of the optomechanical interaction in novel regimes. Moreover, read-out of the “optical” modes can be achieved through standard qubit read-out techniques [53]. The mirror-in-the-middle is taken to be in the ground state for simplicity; in the next section we consider the effects of a finite-temperature oscillator. Notice the initial state is separable and hence the appearance of entanglement between modes A and B would evidence the nonclassical nature of mode C [40]. Time evolution of (4) in the interaction picture is explicitly given by

$$|\Psi(t)\rangle = \frac{|00\rangle}{2}|0\rangle + e^{iB(t)} \frac{|01\rangle}{2} \hat{D}_C(k\xi(t))|0\rangle + e^{iB(t)} \frac{|10\rangle}{2} \hat{D}_C(-k\xi(t))|0\rangle + \frac{|11\rangle}{2}|0\rangle, \quad (5)$$

where $\xi(t) = e^{it}\eta(t)$. The evolved state (5) exhibits entanglement between modes A and B . This can be promptly seen by noticing that coherent states are nonorthogonal and, in the limit of small coupling k , the state assumes the form

$|\Psi(t)\rangle \simeq |\varphi_{AB}\rangle \otimes |\varphi_C\rangle$, where

$$|\varphi_{AB}\rangle \simeq \frac{|0\rangle}{2}(|0\rangle + e^{iB(t)}|1\rangle) + e^{iB(t)}\frac{|1\rangle}{2}(|0\rangle + e^{-iB(t)}|1\rangle) \quad (6)$$

and $|\varphi_C\rangle \simeq |0\rangle$. For times t such that $|B(t)| \simeq \pi/2 + 2\pi n$, $n \in \mathbb{N}$, the state $|\varphi_{AB}\rangle$ becomes maximally entangled. The origin of this entanglement can be heuristically explained by a simple argument: the ground state of the mechanical oscillator is a Gaussian wave packet in the position basis. Each possible position adds up coherently, introducing correlations in the lengths of the left and right cavities in Fig. 1(a). This imprints correlations in the phases of the corresponding electromagnetic fields in modes A and B , giving rise to entanglement. As long as the dimensionless optomechanical coupling k due to radiation pressure on the middle mirror is sufficiently small, mode C will be approximately unperturbed and therefore, to a good approximation, disentangled from AB . On the other hand, as the coupling strength increases, mode C can become significantly entangled with modes A and B ; the question of entanglement among different subsystems as a function of k is addressed in the next section.

To quantitatively evaluate the entanglement in Eq. (5) we calculate the three-partite density matrix $\rho_{ABC} = |\Psi(t)\rangle\langle\Psi(t)|$, from which we obtain the reduced state $\rho_{AB} = \text{Tr}_C(\rho_{ABC})$:

$$\rho_{AB}(t) = \frac{1}{4} \begin{pmatrix} 1 & e^{C(t)} & e^{C(t)} & 1 \\ e^{C^*(t)} & 1 & e^{-2k^2|\eta(t)|^2} & e^{C^*(t)} \\ e^{C^*(t)} & e^{-2k^2|\eta(t)|^2} & 1 & e^{C^*(t)} \\ 1 & e^{C(t)} & e^{C(t)} & 1 \end{pmatrix}, \quad (7)$$

with $C(t) = iB(t) - k^2|\eta(t)|^2/2$. Notice that for small values of k the mirror is “weakly entangled” with modes A and B and some of the off-diagonal terms of the reduced density matrix acquire exponentials that alternate between periods of decay and periods of growth. This can be seen as an example of a weak form of decoherence and “non-Markovian” evolution for the partitions of the whole system, in which information about the optical modes leak into correlations with the mirror and is later retrieved. The mirror introduces a “memory” in the system [54]. Non-Markovianity springs from the fact that the mirror is part of the system under study and hence its degrees of freedom are under control.

Since the state $\rho_{AB}(t)$ is restricted to the subspace spanned by $\{|0\rangle, |1\rangle\}$, we can use concurrence as a measure of entanglement. As an example, plots of the concurrence $\mathcal{C}_{AB}(t)$ and the von Neumann entropy $S_{AB}(t)$ of ρ_{AB} are shown in Fig. 2 for the coupling value $k = 0.5$. The optical modes exhibit positive concurrence and hence entanglement as a function of time. Moreover, the system exhibits sudden death and birth of entanglement. It is also possible to see that the entropy, which is initially zero, oscillates as a function of time. This is another indication of the non-Markovian nature of the system. A nonzero entropy of AB signals entanglement among the three-partite system ABC . Moreover, note that the maxima of concurrence (entanglement of AB) coincide with the minima of entropy (entanglement of ABC). This suggests that, after emerging in the system, entanglement “flows” (during the

FIG. 2. Concurrence (blue line) and von Neumann entropy (yellow line) for $\rho_{AB}(t)$ as a function of time. For this plot, $k = 0.5$.

limited period of its lifetime) among different partitions of the system.

IV. CONTINUOUS VARIABLE STATES AND FINITE TEMPERATURE

We now consider a scenario in which initially the optical modes are populated by monochromatic coherent states and the moving object (sphere, cloud of atoms, or mirror) is in a thermal state at temperature T :

$$\rho(0) = |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \otimes \frac{1}{Z} \sum_n e^{-\frac{n\hbar\omega_m}{k_B T}} |n\rangle\langle n|, \quad (8)$$

where $Z = \sum_n e^{-\frac{n\hbar\omega_m}{k_B T}}$ is the thermal partition function. Note that although the initial state here considered is Gaussian, the Hamiltonian (1) has cubic terms in creation and annihilation operators and, therefore, does not preserve Gaussianity [55]. In order to study the dynamics of entanglement for continuous variable states, we turn our attention to the time-dependent Duan criteria [41].

We define D_{ij} as an average of the Einstein-Podolski-Rosen (EPR) variance for modes i and j :

$$D_{ij} \equiv \frac{1}{2}[(\Delta\hat{u}_{ij})^2 + (\Delta\hat{v}_{ij})^2], \quad (9)$$

where

$$\hat{u}_{ij} = \hat{x}_i + \hat{x}_j, \quad (10)$$

$$\hat{v}_{ij} = \hat{p}_i - \hat{p}_j \quad (11)$$

are the EPR operators for different modes, $i, j = A, B, C$, and $(\Delta\hat{o})^2 \equiv \langle\hat{o}^2\rangle - \langle\hat{o}\rangle^2$ denotes the variance of an observable $\hat{o}$. The Duan criteria states that any separable state satisfies

$$D_{ij} \geq 1. \quad (12)$$

Therefore, if at any time t a violation of the above inequality is observed, modes i and j are necessarily entangled at that time. As the Duan criteria is written in terms of field quadratures, it can be promptly measured with homodyne detection techniques readily available in the laboratory [56]. We also note that when $t = 2\pi n/\omega_m$, $n \in \mathbb{N}$, the unitary evolution (2) acts analogous to a $\chi^{(3)}$ interaction in nonlinear optics [57]. In these moments, the optomechanical system behaves as a nonlinear optical source of squeezing, verified numerically,

FIG. 3. Time-dependent EPR variances $D_{ij}(t)$ (yellow shading) for the various bipartitions of the system, with its envelope (blue line) and threshold for the Duan criteria (black dashed line). We use $\omega_a = \omega_b \simeq 2\pi \cdot 1.59 \times 10^{14}$ Hz and the remaining parameters as in Table I.

from which quantum correlations can be read out from the leaking fields of the cavity [58].

Given the time evolution operator in Eq. (2) we are able to find analytical expressions for the EPR variance of every bipartition of the system (see Appendix D for details). For the optical modes we have

$$D_{AB}(t) = 1 + [|\alpha|^2 + 2\alpha\beta \cos[(r_a + r_b)t] + |\beta|^2] - [|\alpha|^2 + 2\alpha\beta \cos[(r_a + r_b)t + 2B(t)] + |\beta|^2] \times e^{-2[|\alpha|^2 + |\beta|^2][1 - \cos(2B(t))]} e^{-k^2|\eta(t)|^2[2\bar{n} + 1]}, \quad (13)$$

where $\bar{n}$ is the thermal occupation number for the mechanical oscillator.

Figure 3 shows the time-evolving EPR variances for different bipartitions of the system: AB (opto-opto), BC (optomechanical), and AC (optomechanical). We once again observe periodic birth of entanglement and, from the discussion in Sec. II, we can assert that there is death and revivals of entanglement for every bipartition of the system. Moreover, as in the qubit case, the appearance of entanglement between optical modes given the initially separable state can be used in experiments to probe the nonclassicality of the mechanical mode [40]. Although the analytical formulas for the EPR variance are rather involved, it is possible to obtain insight into the entanglement dynamics by looking into their periodicity.

For typical optical and mechanical frequencies, the term $\cos[(r_a + r_b)t]$ represents fast oscillations that do not contribute significantly to the overall envelope of the EPR variances. Consequently, when $k \ll 1/\sqrt{2}$, Eq. (13) is dominated by the term $\exp\{-2[|\alpha|^2 + |\beta|^2][1 - \cos(2B(t))]\}$, which has a period of $\tau = \pi/k^2$. We call this the “low-coupling” regime. On the other hand, if $k \gg 1/\sqrt{2}$ the variance is dominated by the term $\exp\{-k^2|\eta(t)|^2[2\bar{n} + 1]\}$, of period $\tau = 2\pi$. We refer to this as the “high-coupling” regime. The periodicity of these functions dictates the overall periodicity of the envelope of the EPR variances. Going back to nonscaled time we make the substitution $\tau \rightarrow \tau/\omega_m$. Then, for values in the low-coupling regime, observation of EPR variance revivals are only possible when $\pi/\omega_m k^2 \ll \kappa^{-1}$, where κ^{-1} is the inverse cavity linewidth, or the approximate photon lifetime in the cavity. This translates into the so-called

FIG. 4. (a) Minimum value of $D_{AB}(t)$ within the photon lifetime inside the cavity, $\tau = \kappa^{-1} \sim 15.6 \mu\text{s}$, as a function of (a) dimensionless coupling k and mechanical oscillator’s temperature T and (b) coherent state amplitudes of α and β . The optimal coherent amplitudes that globally minimize the Duan criteria are found to be $\alpha \sim \beta \sim 0.91$. We use $\omega_a = \omega_b \simeq 2\pi \cdot 1.59 \times 10^{14}$ Hz and every other parameter as in Table I.

photon blockade condition $g_0^2/(\omega_m \kappa) \gg 1$ [45,59]. For the high-coupling regime, observation of full entanglement dynamics is conditioned on satisfying $2\pi/\omega_m \ll \kappa^{-1}$, which translates into the resolved-sideband regime $\omega_m \gg \kappa$ [60].

Knowledge of the behavior of the EPR variance as a function of optomechanical coupling and temperature is useful when designing an experiment. We observe that the optical entanglement in modes AB is strongly affected by the dimensionless coupling k in a nontrivial way. Figure 4(a) shows the minimum of the EPR variance for the optical modes within the photon lifetime κ^{-1} inside the cavity as a function of coupling k and oscillator initial temperature T . In the low-coupling regime, where $k \ll 1/\sqrt{2}$, above a threshold coupling of $k \approx 0.1$, $\min(D_{AB}(t))$ generally falls below 1 and hence the system always exhibits entanglement. In contrast, the Duan criteria is very sensitive to k in the high-coupling regime $k \gg 1/\sqrt{2}$, presenting $\min(D_{AB}(t)) \geq 1$ in the vicinity of $2k^2 = N$, where N is a positive integer. Under this condition the criteria becomes greater than 1 and no entanglement can be certified regardless of the oscillator temperature. An immediate implication is that, as k grows, the separation between inconclusive regions becomes smaller. At high-coupling values, small uncertainties in the coupling can then place the system in a parameter region in which the Duan criteria is inconclusive, presenting challenges to an experiment. Therefore, increasing coupling may not be the

TABLE I. Proposed values for the experimental implementation with ultracold atoms.

Parameter	Units	Value
Number of atoms N		5.43×10^5
Trap frequency $\omega_m/2\pi$	kHz	95.49
Coupling $k = \frac{g_0}{\omega_m}$		0.74
Mechanical dissipation $\Gamma/2\pi$	kHz	1
Cavity finesse $\mathcal{F}$		3×10^6
Cavity length L	μm	783
Cavity linewidth $\kappa/2\pi$	kHz	10.19
Temperature T	μK	0.8
Mean photon number $ \alpha ^2 = \beta ^2$		0.25

best strategy for an experiment aimed at verifying optical entanglement.

Moreover we observe that entanglement persists well above the microkelvin temperatures reported in current optomechanical experiments [26,61–63], consistent with the theoretical results for the linearized regime in Ref. [64]. Conversely, from our numerical calculations, we have observed that although the minimum value of $D_{AB}(t)$ remains well below 1, the time spent below this threshold becomes increasingly smaller as the temperature grows higher, making it effectively harder to experimentally verify entanglement with the Duan criteria at microkelvin temperatures.

The mean number of photons in the cavity also plays an important role in optical entanglement generation. Figure 4(b) shows a surface plot of $\min(D_{AB}(t))$ within the photon lifetime as a function of the coherent state amplitudes α and β , taken to be real numbers for simplicity. The Duan criteria can only be conclusive when the energy is approximately evenly distributed among the two optical modes, which happens when $\alpha \sim \beta$. For the parameters used, the optimal coherent amplitudes that minimize $D_{AB}(t)$ are found to be $\alpha \sim \beta \sim 0.91$.

V. EXPERIMENTAL PROPOSAL

With increasing advances in the field of quantum cavity optomechanics [61–63], experiments in the high-coupling and long-coherence-time regimes are expected, although observing entanglement as described in the present work remains challenging. One notable exception and a promising candidate is optomechanics with ultracold atomic ensembles, where a coherent cloud of atoms is trapped within an optical cavity by a standing wave (trap beam) and the collective center-of-mass coordinate effectively behaves as a quantum mechanical oscillator which can be monitored by a second optical mode (probe beam). Couplings as high as $k \approx 10$ have been reported in such ultracold experiments [26], and the system allows wide tunability of the relevant parameters.

In order to observe entanglement dynamics, the lifetime of a photon inside the cavity $1/\kappa$ needs to be longer than the time necessary for optical entanglement to reach its first local maximum. Table I presents values for the relevant experimental parameters adapted from Ref. [26], where the motion of an ultracold gas of ^{87}Rb was studied. With these values the resulting photon lifetime is $\tau_p = 15.7 \times 10^{-6} \text{ s}$ and within this

lifetime the minimum value for the optical EPR variance is found to be $D_{AB}(t) \approx 0.8$. This suggests that in these systems the observation of mechanically induced optical entanglement can be within reach.

Once $D_{AB}(t) < 1$, predicted by Eq. (2) and shown in Fig. 3, the optomechanical interaction can be switched off by acting on the trap beam and moving the oscillator with respect to the nodes of the probe beam (see Appendix B). Quantum correlations can then be read out from the leaking probe field of the cavity by using homodyne detection techniques [58,65].

Moreover, following Ref. [26], we estimate the maximal heating rate of a cloud of atoms due to backaction from each optical mode, $R_c = Ng_0^2/(4\Gamma\kappa)R_{fs}$, and due to spontaneous emission, $R_{fs} = \hbar^2 k_p^2 g_0^2 \bar{n}_{\text{cav}} \Gamma / (m \Delta_{\text{ca}}^2)$, where $k_p \simeq 2\pi/780 \text{ nm}^{-1}$ is the probe wave vector, $m \simeq 1.44 \times 10^{-25} \text{ kg}$ is the atomic mass of ^{87}Rb , $\bar{n}_{\text{cav}}$ is the mean photon number for each optical mode, and $\Delta_{\text{ca}}/(2\pi) \simeq 209.16 \text{ GHz}$ is the detuning between the frequencies for the optical modes and the atomic resonance frequency. Given the proposed parameters, heating is dominated by the backaction heating rate, $R_{fs} \ll R_c \simeq 5.98 \times 10^{-32} \text{ J/s}$. We note that, when compared to the phonon energy, the maximal heating rate is much smaller than phonon and photon loss decoherence rates, $R_c/(\hbar\omega_m) \simeq 9.45 \times 10^{-4} \text{ Hz} \ll \kappa, \Gamma$. Therefore, reheating of the mechanical oscillator should play a negligible role during the course of the experiment.

VI. CONCLUSION

In this article we have studied the entanglement and entropy dynamics of a mirror-in-the-middle optomechanical system. Implementations using levitated particles have been briefly discussed. We have seen that an initially separable quantum state can evolve to an entangled one, exhibiting birth, death, and revivals of entanglement and entropy for qubit and continuous-variable states; moreover, the appearance of entanglement in this setting evidences the nonclassical nature of the mechanical oscillator. Therefore, optical entanglement will arise if and only if the mechanical oscillator is quantum mechanical.

The entanglement dynamics is strongly influenced by the system parameters, notably the dimensionless optomechanical coupling $k = g_0/\omega_m$ and the mean number of photons in the experiment. We have shown the existence of two distinct regimes depending on whether $k < 1/\sqrt{2}$ or $k \geq 1/\sqrt{2}$. In addition, we have observed that optical entanglement is maximized when the energy is evenly distributed in the optical modes and that it persists at microkelvin temperatures of the mechanical mode. These are valuable information when designing an experiment. Optomechanics with ultracold atomic ensembles present an interesting candidate for implementing the studied entanglement dynamics. Although a promising candidate, the dispersive Hamiltonian is not the only available platform to untangle the dynamics of entanglement and information flow in optomechanical systems. Exploring alternatives such as coherent scattering [61,63,66–69] might prove to be a very fruitful approach to observe entanglement and nonclassicality in experimental optomechanical systems.

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APPENDIX A: IMPLEMENTATION USING LEVITATED NANOPARTICLES

Consider the system shown in Fig. 1(b): a nanoparticle of radius r , mass m , and refractive index n_p is trapped in a harmonic trap of frequencies $\omega_{j=x,y,z}$ created by an optical tweezer inside a cavity populated by two optical modes of frequencies, ω_a and ω_b , and annihilation/creation operators, $\hat{a}/\hat{a}^\dagger$ and $\hat{b}/\hat{b}^\dagger$, respectively. The presence of the particle causes a position-dependent shift in the cavity's resonance frequencies, so that the Hamiltonian of this system becomes [44]

$$\frac{\hat{H}}{\hbar} = \omega_a \hat{a}^\dagger \hat{a} - U_{0,a} \sin^2[k_a(x_0 + x)] \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} - U_{0,b} \sin^2[k_b(x_0 + x)] \hat{b}^\dagger \hat{b} + \sum_{j=x,y,z} \omega_j \hat{c}_j^\dagger \hat{c}_j, \quad (\text{A1})$$

where x_0 is the position of the center of the trap, x is the particle's displacement, $U_{0,i} = \omega_i \alpha / (2\epsilon_0 V_i)$ is the frequency shift when the particle is at an intensity maximum of the cavity, $\alpha = 4\pi\epsilon_0 r^3 (n_p^2 - 1)/(n_p^2 + 2)$ is the polarizability of the particle, V_i and k_i are the volume and the wave number of mode i , respectively, and $\hat{c}_j/\hat{c}_j^\dagger$ is the phonon annihilation/creation operators along axis $j = x, y$, and z . We do not consider driving terms since we are interested in the dynamics of an isolated cavity without the influence of external systems such as a driving laser, as explained in the main text.

The interaction terms may yield linear couplings between the optical modes and the sphere if x_0 , k_a , and k_b are properly chosen. Expanding $\sin^2 k_i(x_0 + x)$ around x_0 gives

$$\sin^2[k_i(x_0 + x)] = \sin^2(k_i x_0) + k_i \sin(2k_i x_0)x + 2k_i^2 \cos(2k_i x_0)x^2 + \mathcal{O}(x^3). \quad (\text{A2})$$

Now, consider the particular case in which the frequencies of the optical modes are two consecutive resonance frequencies, and let $L = 2n(\lambda_a/2) = (2n+1)(\lambda_b/2)$ without loss of generality. Then, if the sphere is placed near the center of the cavity at $x_0 = L/2 + \lambda_a/8 \approx L/2 + \lambda_b/8$, we have

$$\sin^2 k_a(x_0 + x) \approx 1/2 + k_a x \quad (\text{A3})$$

and

$$\sin^2 k_b(x_0 + x) \approx 1/2 - k_b x. \quad (\text{A4})$$

Substituting these approximations in Eq. (A1) and disregarding the motion along the y and z axes, we get

$$\frac{\hat{H}}{\hbar} = \omega'_a \hat{a}^\dagger \hat{a} - g_0 \hat{a}^\dagger \hat{a} (\hat{c}_x^\dagger + \hat{c}_x) + \omega'_b \hat{b}^\dagger \hat{b} + g_0 \hat{b}^\dagger \hat{b} (\hat{c}_x^\dagger + \hat{c}_x) + \omega_x \hat{c}_x^\dagger \hat{c}_x,$$

TABLE II. Values reported in Ref. [26].

Parameter	Units	Value
Number of atoms		10^5
Trap frequency ω_m	kHz	$2\pi \times 40$
Coupling $k = \frac{g_0}{\omega_m}$		9.50
Cavity finesse $\mathcal{F}$		5.8×10^5
Cavity length L	μm	194
Cavity mirror's radius R	cm	5
Cavity linewidth κ	MHz	$2\pi \times 0.66$
Temperature T	μK	0.8

where $\omega'_i = \omega_i - U_{0,i}/2$ and $g_0 = U_{0,a} k_a x_{\text{ZPF}} \approx U_{0,b} k_b x_{\text{ZPF}}$ is the intended linear coupling, with $x_{\text{ZPF}} = \sqrt{\hbar/2m\omega_x}$ being the zero-point fluctuation. This Hamiltonian is formally equivalent to that of a double-sided cavity with a moving mirror-in-the-middle analyzed in the main text, with the condition $g_{0,a} \approx g_{0,b}$ being met due to the use of consecutive resonance frequencies. As a final remark, note that the modes A and B are not used for cooling of the particle, which is necessary for observing optical entanglement in the system [40]. Cooling of the center-of-mass motion of the particle can be addressed in a number of ways, such as a feedback acting on the trapping laser [70,71], dispersive coupling with a third optical mode [60], or coherent scattering between the trapping beam and the optical cavity [61,63,66].

APPENDIX B: IMPLEMENTATION USING ULTRACOLD ATOMIC ENSEMBLES

In ultracold atom optomechanical experiments, an atomic ensemble is trapped inside an optical cavity. Collective center-of-mass motion of the atoms alters the cavity resonance frequency. It is similar to the dispersive optomechanical experiments with levitated spheres described in the previous section, with the cloud of atoms playing the role of the levitated nanoparticle.

The optomechanical coupling between an ultracold atomic cloud and a cavity optical mode with wavelength λ_a is [26]

$$g_{0,a} = k_a N \frac{\alpha_0^2}{\Delta_{\text{ca}}} \sin(2k_a z_0) \sqrt{\frac{\hbar}{2Nm\omega_m}}, \quad (\text{B1})$$

where k_a is the wave number, N is the number of atoms, Δ_{ca} is the atom-cavity detuning, m is the mass of a single atom, ω_m is the mechanical frequency, and $\alpha_0 = \sqrt{d^2 \omega_c / (2\hbar\epsilon_0 V_c)}$ is the atom-single photon coupling rate, with d being the dipole moment for the transition between the relevant atomic levels and V_c being the cavity mode volume. Reported values from Ref. [26] for these quantities are shown in Table II. As in the previous case of a levitated nanoparticle, we chose $\sin(2k_a z_0) = 1$. The wavelength λ_b should be chosen so that $\sin(2k_b z_0) = -1$, so as to provide $g_{0,a} = g_{0,b}$.

As discussed in the main text, for the entanglement dynamics experiment to be feasible, the photon lifetime τ_p must be greater than the entanglement period τ_e . One of the main constraints in fulfilling this condition is set by the finesse of the cavity. Therefore, we look for the minimum value of τ_e/τ_p by varying the cavity length L , the number of atoms N , and

TABLE III. Optimal parameters for different cavity mirrors' radii.

R (cm)	L (μm)	N (10^5)	ω_m (kHz)	$\mathcal{F}$ (10^6)
1	1211	3.85	$2\pi \times 95$	1.30
2.5	1035	5.64	$2\pi \times 92$	1.57
5	783	5.43	$2\pi \times 95$	2.01
10	669	5.80	$2\pi \times 91$	2.47

the mechanical frequency ω_m around the values in Table II, and then we calculate the minimum finesse necessary to make $\tau_e/\tau_p < 1$ given the optimal values of L , N , and ω_m .

In doing these calculations, it is necessary to account for the changes in the mode volume, given by $V_c = \pi w_a^2 L$, where

$$w_a = \sqrt{\frac{\lambda_a}{2\pi}} \sqrt{L(2R - L)} \quad (\text{B2})$$

is the mode's waist, and the changes in the cavity linewidth

$$\kappa = \frac{\mathcal{F}}{\nu_{\text{FSR}}}, \quad (\text{B3})$$

with $\nu_{\text{FSR}} = c/2L$ being the cavity's free spectral range. Finally, it is important to make sure that $k = g_0/\omega_m \neq \sqrt{n/2}$, where n a positive integer, when the Duan criteria is inconclusive. We find that for $L = 783 \mu\text{m}$, $N = 5.43 \times 10^5$, and $\omega_m = 2\pi \times 95 \text{ kHz} \approx 2\pi \times 95.49 \text{ kHz}$, the dimensionless optomechanical coupling is $k = 0.743$ and the entanglement period to photon lifetime ratio is $\tau_e/\tau_p = 3.46$. The finesse should then be increased to 2.01×10^6 , so that $\tau_e/\tau_p \simeq 1$ and entanglement becomes measurable. In the main text, the proposed finesse is about 1.5 times larger, so that $\tau_e/\tau_p = 0.669$.

Another parameter that could be varied is the radii of the cavity mirrors. Considering 1, 2.5, 5, and 10 cm as possible radii, we find the values presented in Table III. As we can see, the finesse constraint can be relaxed provided that a smaller radius is used. Overall, the necessary values for N , ω_m , and $\mathcal{F}$ differ by less than 1 order of magnitude from reported values in the literature.

APPENDIX C: THE UNITARY EVOLUTION

The mirror-in-the-middle Hamiltonian can be written as

$$\frac{\hat{H}}{\hbar} = \omega_m \hat{c}^\dagger \hat{c} + \omega_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} - g_0 (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) (\hat{c} + \hat{c}^\dagger), \quad (\text{C1})$$

where $\hat{a}$, $\hat{b}$, and $\hat{c}$ denote annihilation operators for the optical mode in the left cavity (A), the right cavity (B), and the mechanical oscillator (C), respectively. In what follows we work with the rescaled time $\omega_m t$, denoted henceforth by t , and introduce the dimensionless variables $r_a = \omega_a/\omega_m$, $r_b = \omega_b/\omega_m$, and $k = g_0/\omega_m$.

Define the unitary operator [15]

$$\hat{E}(k) = \exp[k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})(\hat{c}^\dagger - \hat{c})]. \quad (\text{C2})$$

The operator $\hat{E}(k)$ commutes with $\hat{a}^\dagger \hat{a}$ and $\hat{b}^\dagger \hat{b}$, but not with $\hat{c}$,

$$\hat{E}(k)^\dagger \hat{c} \hat{E}(k) = \hat{c} + k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \quad (\text{C3})$$

which is calculated through the general identity

$$e^{-\hat{A}} \hat{B} e^{+\hat{A}} = \hat{B} + [\hat{B}, \hat{A}] + \frac{1}{2!} [[\hat{B}, \hat{A}], \hat{A}] + \dots \quad (\text{C4})$$

Using Eq. (C3) and its adjoint, we have

$$\hat{E}(k)^\dagger \frac{\hat{H}}{\hbar \omega_m} \hat{E}(k) = \hat{c}^\dagger \hat{c} + r_a \hat{a}^\dagger \hat{a} + r_b \hat{b}^\dagger \hat{b} - k^2 (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2. \quad (\text{C5})$$

Considering the number basis $\{|n, m, \ell\rangle\}$, we see by the equation above that $\hat{E}(k)|n, m, \ell\rangle$ are the energy eigenstates of the system. Those are of the form

$$\hat{E}(k)|n, m, \ell\rangle = \hat{\mathcal{D}}_C[k(n-m)]|n, m, \ell\rangle, \quad (\text{C6})$$

where we denote the displacement operator of the mechanical oscillator by $\hat{\mathcal{D}}_C(\kappa) = \exp(\kappa \hat{c}^\dagger - \kappa^* \hat{c})$, where κ is a complex number. The energies corresponding to the eigenstates above are

$$E_{n,m,\ell} = \hbar \omega_m \ell + \hbar \omega_a n + \hbar \omega_b m - \hbar \omega_m k^2 (n-m)^2. \quad (\text{C7})$$

By exponentiation of Eq. (C5), the unitary evolution operator is found to be

$$\hat{U}(t) = \hat{E}(k) e^{-i[\hat{c}^\dagger \hat{c} + r_a \hat{a}^\dagger \hat{a} + r_b \hat{b}^\dagger \hat{b} - k^2 (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2] t} \hat{E}(k)^\dagger. \quad (\text{C8})$$

Next, we recall that $e^{i\hat{c}^\dagger \hat{c} t} \hat{c}^\dagger e^{-i\hat{c}^\dagger \hat{c} t} = e^{it} \hat{c}^\dagger$ and $e^{i\hat{c}^\dagger \hat{c} t} \hat{c} e^{-i\hat{c}^\dagger \hat{c} t} = e^{-it} \hat{c}$, from which we derive the following identity:

$$\begin{aligned} e^{i\hat{c}^\dagger \hat{c} t} \hat{A} (\hat{c}^\dagger - \hat{c}) e^{-i\hat{c}^\dagger \hat{c} t} &= \hat{A} (e^{it} \hat{c}^\dagger - e^{-it} \hat{c}) \\ \Rightarrow e^{i\hat{c}^\dagger \hat{c} t} e^{\hat{A}(\hat{c}^\dagger - \hat{c})} e^{-i\hat{c}^\dagger \hat{c} t} &= e^{\hat{A}(e^{it} \hat{c}^\dagger - e^{-it} \hat{c})}, \end{aligned} \quad (\text{C9})$$

where $\hat{A}$ is any operator that commutes with both $\hat{c}$ and $\hat{c}^\dagger$.

Letting $\hat{A} = k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$ in the identity above, we conclude that

$$\begin{aligned} \hat{E}(k) e^{-i\hat{c}^\dagger \hat{c} t} &= e^{-i\hat{c}^\dagger \hat{c} t} e^{i\hat{c}^\dagger \hat{c} t} \hat{E}(k) e^{-i\hat{c}^\dagger \hat{c} t} \\ &= e^{-i\hat{c}^\dagger \hat{c} t} \exp[k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})(\hat{c}^\dagger e^{it} - \hat{c} e^{-it})], \end{aligned} \quad (\text{C10})$$

hence

$$\begin{aligned} \hat{E}(k) e^{-i\hat{c}^\dagger \hat{c} t} \hat{E}(k)^\dagger &= e^{-i\hat{c}^\dagger \hat{c} t} \times \exp[k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})(\hat{c}^\dagger e^{it} - \hat{c} e^{-it})] \\ &\times \exp[-k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})(\hat{c}^\dagger - \hat{c})]. \end{aligned} \quad (\text{C11})$$

From the commutator

$$\frac{1}{2} [\hat{c}^\dagger e^{it} - \hat{c} e^{-it}, \hat{c}^\dagger - \hat{c}] = i \sin(t), \quad (\text{C12})$$

it becomes straightforward to compute

$$\begin{aligned} [k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})(\hat{c}^\dagger e^{it} - \hat{c} e^{-it}), k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})(\hat{c}^\dagger - \hat{c})] \\ = 2ik^2 (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2 \sin(t). \end{aligned} \quad (\text{C13})$$

Since this commutes with both $(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})(\hat{c}^\dagger e^{it} - \hat{c} e^{-it})$ and $(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})(\hat{c}^\dagger - \hat{c})$, Eq. (C11) can be simplified using the particular form of the Baker-Campbell-Hausdorff formula,

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2} [\hat{A}, \hat{B}]}, \quad (\text{C14})$$

valid when operators $\hat{A}$ and $\hat{B}$ commute with $[\hat{A}, \hat{B}]$. Thus, the expression,

$$\hat{U}(t) = e^{-i\hat{c}^\dagger \hat{c} t} e^{-ir_a \hat{a}^\dagger \hat{a} t} e^{-ir_b \hat{b}^\dagger \hat{b} t} \times e^{k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})[\hat{c}\eta(t) - \hat{c}^\dagger \eta(t)^*]} e^{-i(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2 B(t)} \quad (\text{C15})$$

holds for the unitary evolution operator $\hat{U}(t)$, where $\eta(t) = 1 - e^{-it}$ and $B(t) = -k^2[t - \sin(t)]$. It is often convenient to work in the interaction picture, where we evolve states by the unitary operator

$$\hat{U}_{\text{I.P.}}(t) = e^{k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})[\hat{c}\eta(t) - \hat{c}^\dagger \eta(t)^*]} e^{-i(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2 B(t)}. \quad (\text{C16})$$

The Heisenberg evolution of the annihilation operators $\hat{a}$, $\hat{b}$, and $\hat{c}$ are then derived as follows:

$$\begin{aligned} \hat{a}(t) &= U^\dagger(t) \hat{a} U(t) \\ &= e^{-i[r_a t + B(t)]} e^{2iB(t) \hat{b}^\dagger \hat{b}} e^{-2iB(t) \hat{a}^\dagger \hat{a}} \hat{D}_C(+k\xi(t)) \hat{a}, \end{aligned} \quad (\text{C17})$$

$$\begin{aligned} \hat{b}(t) &= U^\dagger(t) \hat{b} U(t) \\ &= e^{-i[r_b t + B(t)]} e^{-2iB(t) \hat{b}^\dagger \hat{b}} e^{2iB(t) \hat{a}^\dagger \hat{a}} \hat{D}_C(-k\xi(t)) \hat{b}, \end{aligned} \quad (\text{C18})$$

$$\hat{c}(t) = U^\dagger(t) \hat{c} U(t) = \hat{c} e^{-it} + k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) \eta(t), \quad (\text{C19})$$

where $\xi(t) = e^{it} \eta(t) = e^{it} - 1 = -\eta(t)^*$. In deriving these expressions we made use of the relations

$$e^{i\hat{A}\hat{a}^\dagger \hat{a}} \hat{a} e^{-i\hat{A}\hat{a}^\dagger \hat{a}} = e^{-i\hat{A}} \hat{a}, \quad (\text{C20})$$

$$e^{i\hat{A}(\hat{a}^\dagger \hat{a})^2} \hat{a} e^{-i\hat{A}(\hat{a}^\dagger \hat{a})^2} = e^{-i\hat{A}(2\hat{a}^\dagger \hat{a} + I)} \hat{a}, \quad (\text{C21})$$

applicable whenever $\hat{A}$ commutes with both $\hat{a}$ and $\hat{a}^\dagger$. For instance, Eq. (C19) is derived as

$$\begin{aligned} U^\dagger(t) \hat{c} U(t) &= e^{-k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})[\hat{c}\eta(t) - \hat{c}^\dagger \eta(t)^*]} \\ &\times e^{+i\hat{c}^\dagger \hat{c} t} \hat{c} e^{-i\hat{c}^\dagger \hat{c} t} e^{k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})[\hat{c}\eta(t) - \hat{c}^\dagger \eta(t)^*]} \\ &= e^{-it} e^{-k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})[\hat{c}\eta(t) - \hat{c}^\dagger \eta(t)^*]} \\ &\times \hat{c} e^{k(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})[\hat{c}\eta(t) - \hat{c}^\dagger \eta(t)^*]} \\ &= \hat{c} e^{-it} - k\eta(t)^* e^{-it} (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \end{aligned} \quad (\text{C22})$$

which leads to the desired result after noting that $-\eta(t)^* e^{-it} = \eta(t)$.

APPENDIX D: DUAN CRITERIA

A general sufficient condition for nonseparability of a bipartite continuous-variable system is provided by the Duan criteria [41]. For any bipartite quantum system with observables $\hat{X}_1$, $\hat{P}_1$, $\hat{X}_2$, and $\hat{P}_2$ satisfying the canonical commutation

relations

$$[\hat{X}_j, \hat{P}_k] = i\delta_{jk}, \quad (\text{D1})$$

separability of the state ρ implies that

$$D = \frac{[\Delta(\hat{X}_1 + \hat{X}_2)]^2}{2} + \frac{[\Delta(\hat{P}_1 - \hat{P}_2)]^2}{2} \geq 1. \quad (\text{D2})$$

In other words, if we verify that $D < 1$, then we may conclude that the modes are in an entangled state. We stress that $D \geq 1$ provides no information: this inequality does not imply that the state is separable.

If the operators above are expressed in terms of annihilation and creation operators, defined by

$$\hat{X}_j = \frac{1}{\sqrt{2}}(\hat{a}_j^\dagger + \hat{a}_j), \quad \hat{P}_j = \frac{i}{\sqrt{2}}(\hat{a}_j^\dagger - \hat{a}_j), \quad (\text{D3})$$

then the following expression is useful:

$$\begin{aligned} D &= \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle + \langle \hat{a}_1 \hat{a}_2 \rangle \\ &\quad - (\langle \hat{a}_1^\dagger \rangle + \langle \hat{a}_2 \rangle)(\langle \hat{a}_2^\dagger \rangle + \langle \hat{a}_1 \rangle) + 1. \end{aligned} \quad (\text{D4})$$

By simple algebraic manipulations, we find

$$\begin{aligned} D &= \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + 2\text{Re}(\langle \hat{a}_1 \hat{a}_2 \rangle - \langle \hat{a}_1 \rangle \langle \hat{a}_2 \rangle) \\ &\quad - |\langle \hat{a}_1 \rangle|^2 - |\langle \hat{a}_2 \rangle|^2 + 1. \end{aligned} \quad (\text{D5})$$

For the optical modes, $\hat{a}_1 \rightarrow \hat{a}$ and $\hat{a}_2 \rightarrow \hat{b}$, this gives

$$\begin{aligned} D_{AB}(t) &= 1 + [|\alpha|^2 + 2\alpha\beta \cos[(r_a + r_b)t] + |\beta|^2] \\ &\quad - [|\alpha|^2 + 2\alpha\beta \cos[(r_a + r_b)t + 2B(t)] + |\beta|^2] \\ &\quad \times e^{-2[|\alpha|^2 + |\beta|^2][1 - \cos(2B(t))]} e^{-k^2|\eta(t)|^2[2\bar{n} + 1]}, \end{aligned} \quad (\text{D6})$$

assuming $\alpha, \beta \in \mathbb{R}$ for simplicity. The corresponding quantity between an optical mode and a mechanical mode, say $\hat{a}_1 \rightarrow \hat{a}$ and $\hat{a}_2 \rightarrow \hat{c}$, is given by

$$\begin{aligned} D_{AC}(t) &= 1 + |\alpha|^2 + \bar{n} + 2\text{Re}(R(\alpha, t)) \\ &\quad + |\eta(t)|^2[k^2(|\alpha|^2 + |\beta|^2) + (|\alpha|^2 - |\beta|^2)^2] \\ &\quad - k[|\alpha|^2 - |\beta|^2] \\ &\quad - |\alpha|^2 e^{-2[|\alpha|^2 + |\beta|^2][1 - \cos(2B(t))]} e^{-k^2|\eta(t)|^2[2\bar{n} + 1]}, \end{aligned} \quad (\text{D7})$$

where

$$\begin{aligned} R(\alpha, t) &\equiv \alpha e^{-|\alpha|^2(1 - e^{-2iB(t)}) - |\beta|^2(1 - e^{2iB(t)})} \\ &\quad \times [\bar{n} + 1 - |\alpha|^2(1 - e^{-2iB(t)}) + |\beta|^2(1 - e^{2iB(t)})] \\ &\quad \times k\eta(t) e^{-i[r_a t + B(t)]} e^{-k^2|\eta(t)|^2/2} e^{-\bar{n}k^2|\eta(t)|^2}. \end{aligned} \quad (\text{D8})$$

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