Reply to “Comment on ‘Nondispersive analytical solutions to the Dirac equation’ ”

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(Received 18 April 2020; accepted 12 June 2020; published 6 July 2020)

A new method was developed to solve the Dirac equation and used to find analytical solutions for time-dependent electromagnetic fields that translate a given Dirac spinor along *any desired trajectory* in the x - y plane without distortion. As is well known, negative energy solutions describing antiparticles play a prominent role in the Dirac theory of the electron, and a separation of the part of the wave function describing only electrons from the full solutions often needs to be done. Here, we clarify how the aforementioned separation is done, which is not clear in our recent publication [Campos and Cabrera, *Phys. Rev. Research* 2, 013051 (2020)].

DOI: [10.1103/PhysRevResearch.2.038002](https://doi.org/10.1103/PhysRevResearch.2.038002)

In the recent Comment, the authors state that “physical interpretation of the results (given in Refs. [1,2]) is incomplete. . .” The reason stated by the authors is that “The Dirac equation in the presence of any time-dependent electromagnetic field is not an equation describing just the evolution of a wave packet of an electron. In order to use a solution of the Dirac equation to describe the motion of the *electron wave packets*, one must be sure that the part describing the motion of electrons has been separated from the full solution. The authors of Ref. [1] have not done this.”

By *full solutions* the authors mean a part describing positrons. Here, we want to clarify that the aforementioned separation has in fact been done. The reason is that in all solutions given in Refs. [1,2] the angle β appearing in the general form of the matrix spinor, which modulates the positive and negative energy components of the wave function [3], is either zero or some function of space. In fact, as discussed in Ref. [4], the matrix i that multiplies β in Eq. (4) of Ref. [1], when multiplying the spinor, leads to a spacetime reflection. Hence, in the examples in which the spinor is composed solely of boost, rotations, and dilatations (given by the ρ), negative energy states will never occur since these Lorentz transformations, along with space reflection, never mix up positive and negative energy states [5].

In order to better illustrate that the separation between electronic and positronic states has been done, we numerically solve the Dirac equation using an adapted version of the code given in Ref. [6] (see Ref. [7]) for the vector potential given by Eqs. (17) of Ref. [1] for the case of an elliptical trajectory with an initial state given by Eq. (15) of Ref. [1] for $t = 0$, which has the form of a spin-down, positive energy spinor.

However, due to the Gaussian envelope there will be a small fraction of negative energy states (see, for instance, p. 156 of Ref. [5]) which in our case amounts to 5%, remaining constant throughout the entire time evolution, another indication that the Lorentz transformations composing the spinor are not mixing up positive and negative energy states. As is well known (see Ref. [8]), the appearance of negative energy states during the time evolution of the Dirac spinor leads to a distortion of the electron’s density. However, such a distortion is not seen in our simulations as shown in Fig. 1. The initial electron density depicted in Fig. 1(a) keeps the same shape even after completing more than a full revolution along the elliptical path as shown in Fig. 1(b). Moreover, we can see from the expectation values of $\langle x \rangle = \langle \psi_b | x | \psi_b \rangle$ and $\langle y \rangle = \langle \psi_b | y | \psi_b \rangle$ depicted in Fig. 2 that the center of the wave packet exactly traces the desired elliptical path. Any interference with negative energy states would necessarily lead to deviations from the prescribed trajectory.

In conclusion, we numerically solve the Dirac equation with the vector potential given in Ref. [1] in order to show that the proposed dynamics is fully consistent with our choice of spinor parametrization in which negative energy states are absent. It should be noted, though, that while the separation between positive and negative energy states has been done, our solutions would lose their physical meaning in the strong field regime where radiation reaction effects become important. Moreover, the interpretation of the Dirac equation at large energies is still a matter of research (see, for instance, Refs. [9,10]), but the results of Ref. [1] can be nevertheless applied to the low relativistic regime where we present our examples. The same cannot be said for hydrogenlike ions in strong laser fields, as discussed in Refs. [9,10], since in this case β is a rather complicated function of position [11]. Finally, although the relation between β and negative energy states was recognized long ago [12,13], its role in the solutions of the time-dependent Dirac equation has not yet been thoroughly investigated. We appreciate the authors’ interest in our work, whose relevant Comment highlights an important point that was not clearly stated in Refs. [1,2].

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FIG. 1. Dispersionless motion. Time snapshot of the state evolution by numerically solving the Dirac equation with vector potential (17) from Ref. [1], (a) at the beginning of the translation $\omega t = 0$ and (b) after one and a half full revolutions at $\omega t = 15$. The red diffused circle represents the electron probability density moving along the solid blue curve with frequency ω without changing its shape. The values of the parameters are $B = 0.35$ T, $a_1 = 2 \mu\text{m}$, $a_2 = 1 \mu\text{m}$, and $\omega = 0.5 \text{ ns}^{-1}$.

FIG. 2. Expectation values of $\langle x \rangle = \langle \psi_b | x | \psi_b \rangle$ and $\langle y \rangle = \langle \psi_b | y | \psi_b \rangle$ calculated from the numerical evolved spinor ψ_b . The values of the parameters are $B = 0.35$ T, $a_1 = 2 \mu\text{m}$, $a_2 = 1 \mu\text{m}$, and $\omega = 0.5 \text{ ns}^{-1}$.

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