Protecting quantum coherences from static noise and disorder

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Quantum coherences are paramount resources for applications, such as quantum-enhanced light-harvesting or quantum computing, which are fragile against environmental noise. We here derive generalized quantum master equations using perturbation theory in order to describe the effective time evolution of finite-size quantum systems subject to static noise on all time scales. We then analyze the time evolution of the coherences under energy broadening noise in a variety of systems characterized by both short- and long-range interactions and by strongly correlated and fully uncorrelated noise—a single qubit, a lattice model with on-site disorder and a potential ladder, and a Bose-Hubbard dimer with random interaction strength—and show that couplings can partially protect the system from the ensemble-averaging induced loss of coherence. Our work suggests that suitably tuned couplings could be employed to counteract part of the dephasing detrimental to quantum applications. Conversely, tailored noise distributions can be utilized to reach target nonequilibrium quantum states.

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I. INTRODUCTION

For applications such as quantum-enhanced efficient light harvesting [1] or quantum computing [2] coherences are essential resources that in general are fragile against (environmental) noise [3]. Among these, the static noise sources (e.g., disorder, inhomogeneities, or slowly drifting experimental parameters) are typically seen as technological imperfections, whose impact on the effective dynamics of the coherences can, however, be dramatic [4]. At the same time, static noise can be actively exploited, for instance, for random lasing [5], channel coding of a measurement [6], or making novel multicomponent materials with unprecedented properties [7]. A deep understanding of the effects of the static noise on the coherences, and in particular their time evolution, is crucial for the development of quantum technologies [3].

A convenient model for quantum systems with static noise are ensembles of time-independent, random Hamiltonians [4], each one characterizing a realization of the noise. This description can, among others, be applied to disordered quantum systems [4], random unitary maps [4,8], or open quantum systems experiencing noise from the coupling to a time-independent, classical [9] environment [10].

Characterizing the dynamics of such systems requires a statistical approach, as single realizations cannot give reliable

predictions on reproducible features of the system as a whole. In general, one is bound to study and measure ensemble-averaged quantities, even though sometimes also higher-order correlations are experimentally accessible, such as the spatial correlations of a speckle potentials [11]. More often, however, one cannot access a single element at once, such as is the case when performing spectroscopy on an ensemble of molecules [12] or an ensemble of atomic spins [13], and one must consider the ensemble average. Another example is when variations between successive runs of the same measurement [14] to accumulate statistics cannot be resolved, such as when using a photon beam for quantum communication through the (turbulent) atmosphere [15]. A central consequence of the ensemble average is the averaging over the accumulated phases associated with the eigenstates of the underlying random Hamiltonian for each realization of the static noise [4,9,16]. This averaging induces a loss of phase information. Effectively, we thus witness a form of decoherence—dephasing, traditionally also called the T_2 process—in the sense that quantum coherences of the ensemble-averaged state decay [2,17].

In this article we study the evolution and decay of the quantum coherences of the ensemble-averaged state due to static noise, with a particular emphasis on large times and how to prevent complete dephasing. As the theoretical framework we use the generalized (quantum) master equations introduced in Ref. [4], which we review in Sec. II.

In order to access the large-time regime, we use a perturbation expansion on the level of the individual realizations of the noise. The specifics of the expansion are crucial as we shall explain below. Other more straightforward perturbation approaches can only describe very-short times (shorter than the Heisenberg time) [4,10,18], or introduce other strong

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FIG. 1. (Illustration) Static noise in the on-site energies of a chain leads to a decay of coherences $\bar{\rho}_{jk}$ (off-diagonal elements) of the ensemble-averaged density matrix, but long-range hopping $\hat{V}$ can prevent a complete decay in the asymptotic state $\bar{\rho}_{\infty}$.

assumptions such as weak disorder or strict (semigroup) Markovianity (which requires Cauchy-Lorentz noise distributions) [10]. In contrast, while our approach poses rather strong conditions on the properties of the random Hamiltonians (mainly nondegeneracy of coupled eigenstates), it can characterize both Markovian and non-Markovian dynamics, does not otherwise constrain the static noise distributions, and is not at all bound to short time scales. The latter is decisive because in opposition to time-dependent noise, static noise can be seen as having infinite time-correlation. Thus, the statistical properties of static noise strongly impact the ensemble-averaged asymptotic state, and give rise to intricate effects such as Anderson localization [19] that can drive a metal-insulator transition [20], or the enhancement of quantum state distinguishability [6,21] that improves the security of quantum key distribution protocols [22].

As a concrete example, we consider systems with static noise in the energies (diagonal noise) and a weak, noise-independent perturbation potential. We show that the coherences of the levels coupled by the perturbation are protected from complete averaging-induced decoherence (see illustration Fig. 1). We derive explicit perturbative master equations for three cases: a broadened two-level system, a lattice model with on-site disorder, and a Bose-Hubbard dimer with random interaction strength. The resulting dynamics are contrasted to explicit numerical integration.

The article is organized as follows. A review of the quantum master equation framework for systems with static noise is presented in Sec. II. The perturbative model and the derivation of the corresponding perturbative master equation are presented in Sec. III. Section IV focuses on diagonal static noise for which different applications are studied in Sec. V with an emphasis on the effects of the noise on the dynamics of the coherences at long times. Section VI concludes.

II. REVIEW ON RANDOM HAMILTONIAN ENSEMBLES AND QUANTUM MASTER EQUATIONS FOR SYSTEMS WITH STATIC NOISE

In this section we review the general random Hamiltonian model for static noise and the derivation of corresponding master equations introduced in Refs. [4,23].

A. Static noise and ensemble-averaged dynamics

We consider quantum systems described by an ensemble,

$$\{\hat{H}_{\lambda}, p_{\lambda}\}, \quad (1)$$

of finite-size random, d -dimensional, time-independent Hamiltonians $\hat{H}_{\lambda}$ distributed according to the probability density distribution p_{λ} , where the (multi-)index λ labels the different realizations.

This formulation can be used to model a variety of physical situations—generic statistical distributions of parameters usually referred to as disorder or inhomogeneities [19,25,26], slow fluctuations in the experimental system parameters over time [27], or intentional application-driven parameter randomization [21,22]. Furthermore, this model which is inspired by disorder physics, can be mapped into an open quantum system formulation [10] that was related to a measure of classicality [24]. Also, random Hamiltonian ensembles naturally generate random unitary maps, a model for classical noise in quantum information theory [2]. All three approaches generate the same effective dynamics [c.f. Eq. (2) below], that are well characterized in terms of generalized quantum master equations (c.f. Sec. II B below). For convenience, this wider context is summarized in a diagram in Fig. 2.

For each realization $\hat{H}_{\lambda}$, the dynamics are characterized by the unitary time-evolution operator $\hat{U}_{\lambda}(t) = e^{-it/\hbar \hat{H}_{\lambda}}$. The effective ensemble-averaged state $\bar{\rho}(t)$ (ensemble-averaged quantities are denoted with over-bar) is obtained as the weighted average over all realizations,

$$\bar{\rho}(t) = \int d\lambda p_{\lambda} \hat{\rho}_{\lambda}(t) = \int d\lambda p_{\lambda} \hat{U}_{\lambda}(t) \hat{\rho}_0 \hat{U}_{\lambda}^{\dagger}(t). \quad (2)$$

The initial state $\hat{\rho}_0 = \hat{\rho}(t=0)$ is identical for all realizations of the noise. For a given state observable $\hat{A} = \sum_a a|a\rangle\langle a|$, the average probability of measurement outcome a , defined as $\bar{p}(a)(t) = \int d\lambda p_{\lambda} \text{Tr}[\hat{\rho}_{\lambda}(t)|a\rangle\langle a|]$, is obtained directly as

$$\bar{p}(a)(t) = \text{Tr}[\bar{\rho}(t)|a\rangle\langle a|]. \quad (3)$$

Note that we must use the density matrix formalism because the ensemble-averaged dynamics, Eq. (2), are in general nonunitary. Indeed, it can be shown that any nontrivial convex combination of unitary time evolutions is nonunitary [23,28]. As a guidance for intuition, one might think of unitaries as rotations, and a convex combination of rotations is in general not a rotation. For instance, think of an ensemble of spin-1/2 in an inhomogeneous static magnetic field precessing as a consequence with different frequencies in the Bloch sphere. The ensemble averaging amounts to averaging over these frequencies, which results in (nonunitary) dephasing also known as inhomogeneous broadening [13] (given the frequencies are not all commensurate with one another).

FIG. 2. Sketch of the generalized master equation approach to the dynamics of quantum systems with static noise. A convenient representation is given by ensembles of random Hamiltonians (as is done in this article) and gives rise to the effective dynamics $\bar{\rho}(t)$ by ensemble averaging. This representation is equivalent to the random unitary maps from quantum information theory for classical noise [4] and to the interaction of an open system with a classical environment in the theory of open quantum systems [10,24]. The arising effective dynamics $\bar{\rho}(t)$ are obtained by considering a one-parameter (time) family of maps, and by tracing over the environmental degree of freedom, respectively. The generalized quantum master equation characterizes the dynamics in terms of physically meaningful rates and operators, which can be related to the statistical property of the underlying noise model [4].

The ensemble-averaged dynamics, Eq. (2), define a one-parameter family of dynamical maps from the set of density matrices $\mathcal{D}$ onto itself,

$$\Lambda_t : \mathcal{D} \rightarrow \mathcal{D},$$

$$\hat{\rho}(0) \rightarrow \bar{\rho}(t) = \Lambda_t[\hat{\rho}(0)].$$

The dynamical maps Λ_t are completely positive, Hermiticity and trace-preserving, and thus define legitimate quantum dynamics [4,29]. The dynamics are also unital [they preserve the maximally mixed state; c.f. Eq. (2)].

B. Master equation for ensemble-averaged dynamics

Generic nonunitary quantum evolutions such as Eq. (2) can be described in terms of generalized quantum master equations [17,29], which are the natural extension of the von Neumann equation of motion to nonunitary dynamics. In the time-local Lindblad form [30], they read

$$\dot{\bar{\rho}}(t) = -\frac{i}{\hbar}[\mathcal{H}(t), \bar{\rho}(t)] + \sum_k \gamma_k(t) \hat{L}_k(t) \bar{\rho}(t) \hat{L}_k^\dagger(t) - \frac{1}{2} \{ \hat{L}_k^\dagger(t) \hat{L}_k(t), \bar{\rho}(t) \}, \quad (4)$$

where $\mathcal{H}(t)$ is an effective Hamiltonian, $\gamma_k(t)$ decoherence rates, and $\hat{L}_k(t)$ decoherence operators. The curly brackets denote the anticommutator $\{A, B\} = AB + BA$. We remark that possible memory effects are encoded in the time dependence of the rates and operators [23,31].

Let us now review the main steps of the master equation derivation from Ref. [4]. The ensemble-averaged dynamics,

Eq. (2), expressed in an orthonormal basis $\{|j\rangle\}_{j=1}^d$ reads

$$\langle j | \bar{\rho}(t) | k \rangle = \sum_{r,s=1}^d \langle r | \hat{\rho}_0 | s \rangle \int d\lambda p_\lambda \langle j | \hat{U}_\lambda(t) | r \rangle \langle s | \hat{U}_\lambda^\dagger(t) | k \rangle.$$

Adopting a vector/matrix notation [32], we define the $d^2 \times 1$ ensemble-averaged density vector $\bar{\rho}(t)$ and the $d^2 \times d^2$ ensemble-averaged dynamical matrix $\bar{F}$ such that $\bar{\rho}(t) = \bar{F}(t) \cdot \bar{\rho}(0)$ (here $\cdot$ denotes the standard vector product). Componentwise, we have $\bar{\rho}_{jk}(t) = \langle j | \bar{\rho}(t) | k \rangle$ and

$$\bar{F}_{jk,rs}(t) = \int d\lambda p_\lambda \langle j | \hat{U}_\lambda(t) | r \rangle \langle s | \hat{U}_\lambda^\dagger(t) | k \rangle, \quad (5)$$

where (jk) and (rs) are understood as double indices with $j, k, r, s \in \{1, 2, \dots, d\}$. We remark that the dynamical matrix contains the same information as the Choi matrix [33].

A differential equation for $\bar{\rho}(t)$ is obtained as

$$\dot{\bar{\rho}}(t) = \bar{F}(t) \cdot \bar{F}^{-1}(t) \cdot \bar{\rho}(t) = Q(t) \cdot \bar{\rho}(t), \quad (6)$$

where we introduced the master equation matrix $Q(t) := \dot{\bar{F}}(t) \cdot \bar{F}^{-1}(t)$. In terms of the density matrix components we have

$$\dot{\bar{\rho}}(t) = \sum_{j,k=1}^d |j\rangle \langle k| \sum_{r,s=1}^d \bar{Q}_{jk,rs}(t) \langle r | \bar{\rho}(t) | s \rangle. \quad (7)$$

The Lindblad form, Eq. (4), can be obtained by expanding the diadic operators $|j\rangle \langle k|$ and $|s\rangle \langle k|$ in Eq. (7) in a Hermitian operator basis and collecting the terms using the Hermiticity of $\bar{\rho}(t)$ and trace preservation $\text{Tr}[\dot{\bar{\rho}}(t)] = 0$ [17,34].

At this point, the master equation, Eq. (6), is *exact*, i.e., it generates exactly the ensemble-averaged dynamics, Eq. (2). While it is formally possible to derive such a master equation for any noise ensemble of the type of Eq. (1), it is in practice technically not feasible. Analytical results were obtained for pure diagonal disorder (commuting random Hamiltonians) and for random unitary ensembles in Ref. [4]. Furthermore, a short-time approximation was derived in Ref. [4] and applied to the Anderson model in one dimension in Ref. [18]. Finally, by translating the approach to the open system perspective, it was shown in Ref. [10] that the Markovian approximation in general cannot deliver reliable descriptions of the ensemble-averaged dynamics.

III. PERTURBATION MODEL

We now come back to the focus of this article and consider systems that for each realization of the noise are described by a random Hamiltonian $\hat{H}_\lambda$ with an unperturbed part $\hat{H}_\lambda^0$, and a perturbation $\alpha \hat{V}_\lambda$. Here $\alpha \ll 1$ is a dimensionless perturbation parameter which is small in the sense that each $\hat{H}_\lambda$ can be expanded with nondegenerate perturbation theory [35]. The random Hamiltonian ensemble is then characterized by

$$\{\hat{H}_\lambda = \hat{H}_\lambda^0 + \alpha \hat{V}_\lambda, \quad p_\lambda\}. \quad (8)$$

This model is suitable for the description of a variety of experimental scenarios as will be exemplified in Sec. V. For completeness, in this section we focus on the general perturbation ensemble, Eq. (8). In Secs. IV and V, however, we will focus on systems with diagonal noise in order to make concrete predictions.

In order to simplify the reading of the coming section, we assume that the unperturbed Hamiltonians $\hat{H}_\lambda^0$ are diagonal in the same eigenbasis $\{|j\rangle\}_{j=1}^d$ for all realizations of the noise. This assumption is for convenience (i.e., it can be dropped if needed), and serves to identify a specific basis in which to represent the master equation. The random Hamiltonians can be written as

$$\hat{H}_\lambda^0 = \sum_{j=1}^d E_{\lambda,j}^0 |j\rangle\langle j|, \quad \hat{V}_\lambda = \sum_{j,k=1}^d V_{jk}^\lambda |j\rangle\langle k|, \quad (9)$$

where $E_{\lambda,j}^0$ and $|j\rangle$ are the eigenvalues and eigenvectors of $\hat{H}_\lambda^0$, and V_{jk}^λ the matrix elements of $\hat{V}_\lambda$.

A. Derivation of the perturbative master equation

The derivation of the perturbative master equation relies on the perturbative expansion in orders of α of the dynamical matrix $F_\lambda(t)$, with components,

$$F_{jk,rs}^\lambda(t) = \langle j|\hat{U}_\lambda(t)|r\rangle\langle s|\hat{U}_\lambda^\dagger(t)|k\rangle, \quad (10)$$

for each realization of the noise. It is absolutely imperative that the expansion is done *before* the ensemble average is taken. Defining the time-evolution operator for the unperturbed part of the Hamiltonian as $U_\lambda^0(t) = e^{-it/\hbar H_\lambda^0}$, we obtain using standard nondegenerate perturbation

theory [35],

$$\begin{aligned} \hat{U}_\lambda(t) &= \hat{U}_\lambda^0(t) - \left(\frac{i\alpha}{\hbar}\right) \int_0^t dt' \hat{U}_\lambda^0(t-t') \hat{V}_\lambda \hat{U}_\lambda^0(t') \\ &+ \left(-\frac{i\alpha}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' \hat{U}_\lambda^0(t-t') \hat{V}_\lambda \hat{U}_\lambda^0(t'-t'') \\ &\times (t'-t'') \hat{V}_\lambda \hat{U}_\lambda^0(t'') + \dots \end{aligned} \quad (11)$$

After insertion of Eq. (11) into Eq. (10), we can collect the terms order by order and define the corresponding dynamical matrix $F_\lambda^{(n)}(t)$ for each order n in α to obtain the expansion,

$$F_\lambda(t) = F_\lambda^{(0)}(t) + \alpha F_\lambda^{(1)}(t) + \alpha^2 F_\lambda^{(2)}(t) + \dots \quad (12)$$

The matrix elements of each order can be evaluated separately [23].

The ensemble average of the dynamical matrix is obtained by taking the weighted integral over all realizations of the noise of the perturbative dynamical matrix,

$$\begin{aligned} \bar{F}(t) &= \int d\lambda p_\lambda [F_\lambda^{(0)}(t) + \alpha F_\lambda^{(1)}(t) + \alpha^2 F_\lambda^{(2)}(t) + \dots] \\ &= \bar{F}^0(t) + \alpha \bar{F}^1(t) + \alpha^2 \bar{F}^2(t) + \dots \end{aligned} \quad (13)$$

Note that for readability we dropped the parenthesis around the perturbation order in the second line (e.g., $\bar{F}^{(1)} \rightarrow \bar{F}^1$) and will keep this notation from now on. The next step is to compute the master equation matrix $\bar{Q} = \dot{\bar{F}}(t) \cdot \bar{F}^{-1}(t)$ [c.f. Eq. (6)] from the ensemble-averaged perturbative dynamical matrix $\bar{F}(t)$. The time derivative $\dot{\bar{F}}(t)$ is directly obtained by taking the time derivative of Eq. (13). Computing the inverse $\bar{F}^{-1}(t)$ analytically is, however, in general, not possible. We therefore make use of the perturbative nature of the dynamics and express the inverse with the help of the Neumann series [36] as $\bar{F}^{-1}(t) = \sum_{n=0}^{\infty} (\mathbb{1} - \bar{F}^{0-1}(t) \bar{F}(t))^n \bar{F}^{0-1}(t)$ (c.f. Appendix A). We remark for the experts in the theory of open quantum systems that this procedure is analogous to the time-convolution-less (TCL) expansion [17], yet emphasize that the two approaches are not equivalent [23].

The expression for $\bar{Q}(t)$ then yields

$$\begin{aligned} \bar{Q}(t) &= \left(\sum_{n=0}^{\infty} \alpha^n \dot{\bar{F}}^n(t) \right) \\ &\cdot \sum_{m=0}^{\infty} \left(- \sum_{k=1}^{\infty} \alpha^k \bar{F}^{0-1}(t) \bar{F}^k(t) \right)^m \bar{F}^{0-1}(t). \end{aligned} \quad (14)$$

The matrix $\bar{Q}(t)$ fully characterizes the perturbative master equation for the ensemble-averaged dynamics, Eq. (2), and can be truncated at an appropriate order in the perturbation parameter α . If desired, a Lindblad form of the master equation can be obtained by expanding $\bar{Q}(t)$ in a basis of traceless, Hermitian operators [4,17,34] as mentioned in the previous section. Alternatively, one can express the master equation in terms of the density matrix components using Eq. (7) and then collect and rearrange the terms in a suitable way.

We remark that in the whole derivation we never made any assumption of Markovianity, weak disorder, nor rotating-wave approximation. This fundamentally sets apart our approach

from the open quantum system weak coupling derivation of Lindblad master equations [17]. Moreover, we are taking into account the phase information at all times because the perturbation expansion is done on the level of the dynamical matrix, Eq. (13), before taking the ensemble average. Hence, it is not limited to short times as is the perturbation expansion in time used in [4,18]. If not obvious, the latter point should become clear looking at the concrete examples derived in the next two sections.

IV. PERTURBATIVE MASTER EQUATION APPLIED TO DIAGONAL STATIC NOISE/DISORDER

For the rest of this article, we consider d -dimensional systems with diagonal (spectral) static noise and a weaker, noise- and time-independent perturbation, as a particular application of the perturbation model presented in Sec. III. More precisely, the random Hamiltonians read

$$\hat{H}_\lambda = \hat{H}_\lambda^0 + \alpha \hat{V}, \quad (15)$$

where $\alpha \ll |E_{\lambda,j}^0 - E_{\lambda,k}^0|$ for all j, k such that $V_{jk} \neq 0$. In other words, the coupling potential between two unperturbed eigenstates is much smaller than their energy difference. The system can thus be treated with nondegenerate perturbation theory [35]. This model was chosen because it is contrary to the common practice in disorder physics to consider noise as a small perturbation. Hence it allows us to explore rather new territory, and, for instance consider strong disorder. At the same time, the model (15) covers a range of realistic physical situations, from single- to many-particle systems, as illustrated in Sec. V.

Since we assume the perturbation to be independent of the noise ($\hat{V}_\lambda = \hat{V}$), the eigenvalues $E_{\lambda,j}^0$ of the unperturbed Hamiltonian $\hat{H}_\lambda^0$ are the only random variables. Indeed, we remind the reader we assume the eigenvectors $\{|j\rangle\}$ of $\hat{H}_\lambda^0$ to be independent of the noise realization (on the contrary the eigenvectors of $\hat{H}_\lambda$ are obviously modified by the noise). If the eigenvectors of $\hat{H}_\lambda^0$ would also be subject to noise, one would have to expand the dynamical matrix, Eq. (12), in a noise-independent basis before taking the ensemble average. This does not change the derivation procedure but the computations would, in general, become more intricate.

We parametrize the random Hamiltonian ensemble as

$$\{\hat{H}_\lambda = \hat{H}_\lambda^0 + \alpha \hat{V} \quad , \quad p_\lambda\}, \quad (16)$$

with $p_\lambda = p(\lambda_1, \lambda_2, \dots, \lambda_d)$ the joint probability density distribution of the dimensionless variables λ_j that characterize the eigenvalues $E_{\lambda,j}^0 = \varepsilon_j + \hbar\omega_0\lambda_j$ ($j = 1, 2, \dots, d$) of $\hat{H}_\lambda^0$, and where $\hbar\omega_0$ is the reference energy scale and ε_j the noise-free part.

Interestingly, and as will become clear in the next section, the first-order correction to the master equation matrix $\bar{Q}(t)$ is enough to capture the main features of the ensemble-averaged dynamics at all times for random ensembles of the form of Eq. (16). The first order in α master equation can be expressed

as (for a detailed derivation see Appendix B)

$$\begin{aligned} \dot{\bar{\rho}}(t) = & \sum_{j,k=1}^d \Upsilon_{jk}(t) \hat{\Pi}_{jj} \bar{\rho}(t) \hat{\Pi}_{kk} - \alpha \frac{i}{\hbar} [\hat{V}, \bar{\rho}(t)] \\ & + \frac{\alpha}{\hbar} \sum_{\substack{j,k=1 \\ j \neq k}}^d \gamma_{jk}(t) V_{jk} [\hat{\Pi}_{jk} \bar{\rho}(t) \hat{\Pi}_{kk} - \hat{\Pi}_{jj} \bar{\rho}(t) \hat{\Pi}_{jk}] \\ & + \frac{\alpha}{\hbar} \left(\sum_{\substack{j,k,r=1 \\ j \neq k \neq r}}^d \Gamma_{jkrj}(t) V_{jr} \hat{\Pi}_{jr} \bar{\rho}(t) \hat{\Pi}_{kk} \right. \\ & \left. + \Gamma_{jkrj}^*(t) V_{rj} \hat{\Pi}_{kk} \bar{\rho}(t) \hat{\Pi}_{rj} \right), \end{aligned} \quad (17)$$

with the diadic operators $\hat{\Pi}_{jk} := |j\rangle\langle k|$. The (time-dependent) rates in the equation above are functions of the phase factors,

$$\varphi_{jk}(t) := e^{-\frac{i}{\hbar}(E_{\lambda,j}^0 - E_{\lambda,k}^0)t} = e^{-\frac{i}{\hbar}(\varepsilon_j - \varepsilon_k)t} e^{-it\omega_0(\lambda_j - \lambda_k)}, \quad (18)$$

and their average $\bar{\varphi}_{jk}(t) = \int d\vec{\lambda} p_\lambda \varphi_{jk}(t)$. The rates are defined as

$$\Upsilon_{jk}(t) := \frac{d}{dt} \ln[\bar{\varphi}_{jk}(t)] = \Upsilon_{kj}^*(t), \quad (19)$$

$$\begin{aligned} \gamma_{jk}(t) &:= i \left(1 - \bar{\varphi}_{jk}(t) + \Upsilon_{jk}(t) \int_0^t dt' \bar{\varphi}_{jk}(t') \right) \\ &= -\gamma_{kj}^*(t), \end{aligned} \quad (20)$$

$$\begin{aligned} \Gamma_{jkrj}(t) &:= i \left[\frac{\dot{\bar{\varphi}}_{jk}(t)}{\bar{\varphi}_{jk}(t) \bar{\varphi}_{rk}(t)} \overline{\varphi_{jk}(t) \int_0^t dt' \varphi_{rj}(t')} \right. \\ &\quad \left. - \frac{1}{\bar{\varphi}_{rk}(t)} \left(\overline{\dot{\varphi}_{jk}(t) \int_0^t dt' \varphi_{rj}(t')} \right) \right] \\ &= -\Gamma_{kjjr}^*(t). \end{aligned} \quad (21)$$

Here the symmetry $\Gamma_{jkrj} = -\Gamma_{kjjr}^*$ is obtained by permuting the first two indices, $jk \rightarrow kj$, and the last two indices, $rj \rightarrow jr$, and reflects the Hermiticity of the density matrix [the dynamical matrix $F(t)$ has the same symmetry as $\Gamma(t)$]. The first order in the α part of the master equation, Eq. (17), is manifestly separated into a fully coherent contribution from the perturbation potential $\hat{V}$ (second term, first line), and additional incoherent terms with decoherence rates $\propto \gamma(t)$ and $\propto \Gamma(t)$ (second, third, and fourth lines).

The master equation (17) is already in ‘second Lindblad form’ [17] with a nondiagonal Kossakowski matrix K [37], i.e., with an incoherent term of the form $\sum_{m,n} K_{mn} (\hat{L}_m \hat{\rho} \hat{L}_n^\dagger - 1/2 \{\hat{L}_n^\dagger \hat{L}_m, \hat{\rho}\})$ [note that the terms of type $\hat{\Pi}_{kk} \hat{\Pi}_{jj}$ in Eq. (17) vanish]. The matrix K is Hermitian, and thus can be diagonalized to put Eq. (17) into diagonal Lindblad form. However, note that the decoherence operators and rates will in general be time dependent. In other words, usually the dynamics do not form a semigroup (pure Markovianity), and can be non-Markovian [4,31].

In order to better understand the dynamics arising from the master equation, Eq. (17), we express it in terms of the density matrix elements. For the coherences (off-diagonal elements $j \neq k$) we have

$$\begin{aligned} \langle j | \dot{\bar{\rho}}(t) | k \rangle &= \Upsilon_{jk}(t) \langle j | \bar{\rho}(t) | k \rangle \\ &- i \frac{\alpha}{\hbar} \sum_{r=1}^d V_{jr} \langle r | \bar{\rho}(t) | k \rangle - \langle j | \bar{\rho}(t) | r \rangle V_{rk} \\ &+ \frac{\alpha}{\hbar} [\gamma_{jk}(t) - i] V_{jk} [\langle k | \bar{\rho}(t) | k \rangle - \langle j | \bar{\rho}(t) | j \rangle] \\ &+ \frac{\alpha}{\hbar} \left(\sum_{\substack{r=1 \\ r \neq j \neq k}}^d [\Gamma_{jkrj}(t) - i] V_{jr} \langle r | \bar{\rho}(t) | k \rangle \right. \\ &\left. - [\Gamma_{jkrj}^*(t) + i] V_{rk} \langle j | \bar{\rho}(t) | r \rangle \right). \end{aligned} \quad (22)$$

The term proportional to $\Upsilon_{jk}(t)$ describes the effects of pure diagonal disorder ($\alpha = 0$) [4], and leads to time-dependent dephasing in the eigenbasis of $\hat{H}^0$. All other terms arise as a consequence of the perturbation potential $\hat{V}$. The rate $\gamma_{jk}(t)$ is associated with the dynamical coupling of the coherences to the population differences, whereas the rate $\Gamma_{jkrj}(t)$ governs the second-order coupling of the coherences to the other off-diagonal terms of the density matrix.

At the same time, for the populations (diagonal terms $j = k$) we have

$$\langle j | \dot{\bar{\rho}}(t) | j \rangle = -i \frac{\alpha}{\hbar} \sum_{\substack{r=1 \\ r \neq j}}^d V_{jr} \langle r | \bar{\rho}(t) | j \rangle - \langle j | \bar{\rho}(t) | r \rangle V_{rj}. \quad (23)$$

Hence the populations do evolve in time (as opposed to the case when $\alpha = 0$), but since Eq. (23) does not contain any noise-dependent decoherence term, their evolution is not directly affected by the noise. They only indirectly feel the noise through their coupling to the coherences via the perturbative potential $\hat{V}$.

The ensemble averaging for ensembles of the type (16) thus leads to complex, in general time-dependent, dephasing (decoherence) processes.

A. Statistical interpretation

The ensemble-averaged phase factors $\bar{\varphi}_{jk}(t)$ that characterize the decoherence rates, Eqs. (19)–(21), can be expressed in terms of the complex conjugate of the characteristic function [38],

$$\phi_{jk}(\omega_0 t) = \mathbb{E}[e^{it\omega_0 \Delta_{jk}}], \quad (24)$$

of the probability density distribution $q_{jk}(\Delta_{jk}) := \int d\tilde{\lambda} p_{\tilde{\lambda}} \delta(\Delta_{jk} - (\lambda_j - \lambda_k))$ of the difference of pairs of

random variables $\Delta_{jk} := \lambda_j - \lambda_k$ as

$$\bar{\varphi}_{jk}(t) = e^{-\frac{it}{\hbar}(\varepsilon_j - \varepsilon_k)} \phi_{jk}^*(\omega_0 t) \quad (25)$$

$$= e^{-\frac{it}{\hbar}(\varepsilon_j - \varepsilon_k)} \int d\Delta_{jk} q_{jk}(\Delta_{jk}) e^{-it\omega_0 \Delta_{jk}}. \quad (26)$$

Hence, the time-dependent properties of the ensemble-averaged dynamics can be traced back to the properties of the characteristic functions of the pairwise eigenvalue difference distributions. We remark that higher order moments (involving more than two eigenvalues) are necessary if one considers contributions beyond the first order in the perturbation.

Conveniently, the characteristic function can be described in terms of its cumulants [39] which yield a direct physical interpretation [4]—the odd cumulants capture the (a)symmetry of the distribution, and contribute to the coherent (Hamiltonian) part of the master equation, while the even cumulants describe the width of the distribution (or strength of the noise), and characterize the decoherence rates Eqs. (27)–(29), i.e., the time dependence and speed of the dephasing process.

Note that while the characteristic function describes the time dependence of the decoherence process, the structure of the Hamiltonian, and in particular the nature of the coupling $\hat{V}$, determines the genre of the decoherence. Indeed, the coupling potential determines the Lindblad operators of the master equation (17), i.e., the type of dephasing/decoherence dynamics. This is consistent with findings in Ref. [4] that the structure of the disorder defines the form of the Lindblad operators.

B. Symmetric noise distributions and decoherence rates

Physical noise distributions $q_{ij}(\Delta_{jk})$ are typically symmetric [40], such as, e.g., Gaussian, Lorentzian, or uniform distributions. The nondegenerate constraint $E_{\lambda,j}^0 - E_{\lambda,k}^0 = (\varepsilon_j - \varepsilon_k) + \hbar\omega_0 \Delta_{jk} \gg \alpha$ for most realization of the noise then implies that the distribution must be sufficiently peaked; if the distribution is too wide, the probability for having degenerate levels could become too high, restricting the validity of the perturbative master equation to short times, c.f. Sec. V D. If the distribution is sufficiently peaked, the decoherence rates Eqs. (19)–(21) can be expressed in terms of the real part (even cumulants) of the characteristic function $\phi_{jk}^*(t)$ and the central values (first cumulant) $\bar{E}_j^0 = \varepsilon_j + \hbar\omega_0 \bar{\lambda}_j$ (c.f. Appendix C),

$$\Upsilon_{jk}(t) = -i[\bar{E}_j^0/\hbar - \bar{E}_k^0/\hbar] + \text{Re} \left[\frac{d}{dt} \ln(\phi_{jk}^*(\omega_0 t)) \right], \quad (27)$$

$$\gamma_{jk}(t) \approx [1 - e^{-\frac{it}{\hbar}(\varepsilon_j - \varepsilon_k)} \phi_{jk}^*(\omega_0 t)] \frac{\text{Re}[\Upsilon_{jk}(t)]}{\bar{E}_j^0/\hbar - \bar{E}_k^0/\hbar}, \quad (28)$$

$$\Gamma_{jkrj}(t) \approx \frac{[\text{Re}[\Upsilon_{jk}(t)] - \text{Re}[\Upsilon_{rk}(t)]]}{\bar{E}_j^0/\hbar - \bar{E}_k^0/\hbar}. \quad (29)$$

The zeroth-order function $\Upsilon_{jk}(t)$ separates into an imaginary coherent (energy shift) contribution and a real-valued, time-dependent decoherence rate. The first-order time-dependent decoherence rate $\Gamma_{jkrj}(t)$ is real valued, whereas $\Gamma_{jk}(t)$ is modulated by a fast decaying, complex oscillating function $1 - e^{-\frac{it}{\hbar}(\varepsilon_j - \varepsilon_k)} \phi_{jk}^*(\omega_0 t)$. Note that by the Riemann-Lebesgue

lemma [41,42] the characteristic function vanishes at large times, i.e., $\phi_{jk}^*(t) \xrightarrow{t \rightarrow \infty} 0$.

We remark that if the distribution q_{jk} is not symmetric, such as, e.g., a general Lévy distribution, one obtains time-dependent energy shifts, i.e., $\text{Im}[\Upsilon_{jk}] = \text{Im}[\Upsilon_{jk}](t)$ [4]. In addition, the energy terms $[\overline{E_j^0}/\hbar - \overline{E_k^0}/\hbar]$ in the decoherence rates, Eqs. (28) and (29), must be replaced by $-\text{Im}[\Upsilon_{jk}](t)$.

C. Asymptotic state

Given the decoherence rates (27)–(29), an analytical formula for the asymptotic state can be derived which corresponds well to the numerical results in Sec. V. The result is expressed in terms of the ensemble-averaged Hamiltonian $\overline{\hat{H}}$. Indeed, the latter is at large times in the kernel of the perturbative master equation: upon insertion of $\overline{\hat{H}} = \overline{\hat{H}}^0 + \alpha V = \sum_j (\varepsilon_j + \hbar\omega_0 \bar{\lambda}_j) |j\rangle\langle j| + \alpha \hat{V}$ into Eq. (17), and using the rates (27)–(29), we see that the first commutator vanishes, the second and third sum cancel each other for $t \gg 0$ because $\phi_{jk}^*(t) \rightarrow 0$, and the fourth sum is of order α^2 [note that the fourth sum vanishes exactly for identically and independently distributed random variables (i.i.d.) because in this case $\Gamma_{jkrj}(t) \equiv 0$].

The ensemble-average Hamiltonian can thus be used to characterize an asymptotic subspace [43] up to first order in α . There is not a unique asymptotic state; rather, as we will show in the examples in Sec. V, the perturbative master equation describes asymptotically a pure dephasing process in the eigenbasis of $\overline{\hat{H}} = \sum_n E_n |n\rangle\langle n|$. Hence, we heuristically deduce that an initial state $\hat{\rho}_0$ will asymptotically be projected onto the basis $\{|n\rangle\}$,

$$\hat{\rho}_0 \rightarrow \overline{\hat{\rho}}_\infty \approx \sum_n |n\rangle\langle n| \hat{\rho}_0 |n\rangle\langle n|. \quad (30)$$

As will become clear with the examples in Sec. V, the coherences (off-diagonal elements $\rho_{0,jk}$) of the initial state expressed in the eigenbasis of $\hat{H}_0$, which are coupled by the perturbation V_{jk} , are thus partially protected from the averaging-induced dephasing. Conversely, coherences in the average energy eigenbasis of $\overline{\hat{H}}$ in general do vanish asymptotically. We remark, however, that certain types of dephasing processes, such as collective dephasing, allow for the existence of true decoherence-free subspaces [28]. Hence, random-Hamiltonian ensembles giving rise to such a dephasing process will also have decoherence-free subspaces in the basis of $\overline{\hat{H}}$. An example is provided at the end of Sec. V A.

Note that while Eq. (30) corresponds well to the numerical results presented in Sec. V, it remains heuristic. It would be interesting in future work to have a more rigorous mathematical derivation for the existence [44] and properties of the asymptotic state [45].

V. APPLICATIONS FOR DIAGONAL STATIC NOISE/DISORDER

As applications of the perturbative master equation for diagonal noise we consider three examples. (A) First, we study the dynamics of a single qubit which allows for a

geometric intuition in terms of Bloch vectors of the dephasing dynamics arising from the ensemble average. This example is illustrative of the effective ensemble picture as one encounters, for instance, in atomic spectroscopy. (B) Second, we consider a one-dimensional lattice model with uncorrelated on-site disorder and study the effect of short- and long-range interactions on the coherences in the asymptotic state. (C) Third, we investigate the effect of strong correlations on the effective dynamics of the coherences in a many-particle boson model. This example illustrates experiments for which parameters are static during the time evolution of the system, but fluctuate between different runs of the same measurement.

A. Geometrical interpretation: Qubit with Gaussian energy distribution

We begin with two-level systems ($d = 2$) with static noise in the energy difference, that physically, for instance, may describe an ensemble of spins $1/2$ precessing in a static, spatially inhomogeneous magnetic field. It is known that this noise leads to the inhomogeneous broadening of the linewidth in spectroscopy experiments, and characterizes the decay of the total magnetization on the time scale T_2^* [13]. While the effect of the ensemble-average decoherence can in this case be canceled out by inverting the spin-precession direction with a spin-echo sequence [13], the latter may also be subject to noise making the inversion incomplete [46]. Thus, understanding the exact dynamical role of the static noise is of relevance.

The static noise ensemble is parametrized as

$$\left\{ \hat{H}_\lambda = \hbar\omega_0 \left(\frac{\lambda}{2} \hat{\sigma}_z + \alpha \hat{\sigma}_x \right) = \frac{\hbar\omega_0}{2} \begin{pmatrix} \lambda & 2\alpha \\ 2\alpha & -\lambda \end{pmatrix}, \quad p_\lambda = p(\lambda) \right\}, \quad (31)$$

with $\hbar\omega_0$ the reference energy, λ a dimensionless random variable, and α a dimensionless perturbation parameter. The unperturbed Hamiltonian $H_\lambda^0 = \hbar\omega_0 \lambda \sigma_z / 2$ is diagonal, with eigenvalues $E_{\lambda,1}^0 = \hbar\omega_0 \lambda / 2$ and $E_{\lambda,2}^0 = -\hbar\omega_0 \lambda / 2$ (here $\varepsilon_{1,2} = 0$), and the perturbed part is $\alpha \hat{V} = \alpha \hbar\omega_0 \sigma_x$. Referring to the physical picture of spins in a static magnetic field, we assume that the field has a static random amplitude along the z axis sampled from the probability distribution $p(\lambda)$, and a small, constant component along the x axis proportional to α . Hence, in total, not only the amplitude of the magnetic field varies from realization to realization, but also the orientation.

We consider a generic Gaussian noise distribution, with average value $\lambda_0 \in \mathbb{R}$ and variance $\sigma > 0$,

$$p_{\text{Ga}}(\lambda) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\lambda - \lambda_0)^2}{2\sigma^2}}, \quad (32)$$

which implies $\overline{E_1^0} = \hbar\omega_0 \lambda_0 / 2$ and $\overline{E_2^0} = -\hbar\omega_0 \lambda_0 / 2$. The non-degeneracy perturbation condition then requires $|\lambda_0| \gtrsim \sigma + \alpha$, i.e., the noisy magnetic field along the z axis is always much larger than the one along the x axis.

Since there are only two levels (i.e., $j, k = 1, 2$) characterized by one single noise parameter λ , the decoherence rates $\Upsilon_{jk}(t)$, $\gamma_{jk}(t)$, and $\Gamma_{jkrj}(t)$ are fully characterized

FIG. 3. Decay of the coherences (x component) of a qubit described by the noisy ensemble Eq. (31) obtained from numerical integration of the perturbative quantum master equation Eqs. (35) and (36) (blue full), from the pure dephasing approximation Eq. (41) (green, dashed), and from direct numerical averaging (purple dots, 10^6 realizations). The parameters are set to $\lambda_0 = 10$, $\sigma = 1$, $\alpha = 1$ and $\rho_0 = 1/2\mathbb{1} + 1/2\sigma_x$. (Inset) The coherences $\text{Re}[\bar{\rho}_{12}]$ do not vanish asymptotically and converge to Eq. (30) (gray horizontal line).

by $\phi_{12}^*(\omega_0 t) = \int d\lambda p(\lambda) e^{-i\omega_0 \lambda} = \exp[-i\omega_0 \lambda_0 t - 1/2\omega_0^2 \sigma^2 t^2]$, Eq. (24). From Eqs. (27)–(29), we obtain

$$\Upsilon_{12}(t) = \Upsilon(t) = -i\omega_0\lambda_0 - \omega_0^2\sigma^2t, \quad (33)$$

$$\gamma_{12}(t) = \gamma(t) \approx -\frac{\omega_0 \sigma^2 t}{\lambda_0} (1 - e^{-i\omega_0 \lambda_0 - \frac{\omega_0^2 \sigma^2}{2} t^2}), \quad (34)$$

and $\Gamma_{jkrj}(t) = 0$ for $d < 3$ because it requires three distinct indices j, k, r . It follows that the coherences, Eq. (22), evolve according to

$$\begin{aligned} \langle 1|\dot{\tilde{\rho}}(t)|2\rangle = & (-i\omega_0\lambda_0 - \omega_0^2\sigma^2t)\langle 1|\tilde{\rho}(t)|2\rangle + \alpha(\gamma(t) - i) \\ & \times [\langle 2|\tilde{\rho}(t)|2\rangle - \langle 1|\tilde{\rho}(t)|1\rangle], \end{aligned} \quad (35)$$

and, by definition, $\langle 2|\dot{\hat{\rho}}(t)|1\rangle = \langle 1|\dot{\hat{\rho}}(t)|2\rangle^*$. The populations, Eq. (23), satisfy the equations,

$$\langle 1 | \dot{\hat{\rho}}(t) | 1 \rangle = i\alpha\omega_0[\langle 1 | \bar{\hat{\rho}}(t) | 2 \rangle - \langle 2 | \bar{\hat{\rho}}(t) | 1 \rangle], \quad (36)$$

and $\langle 2|\dot{\hat{\rho}}(t)|2\rangle = -\langle 1|\dot{\hat{\rho}}(t)|1\rangle$. Thus, the diagonal elements of the ensemble-averaged state evolve coherently under the action of the coupling potential $\hat{V}$, while the off-diagonal elements evolve coherently with $\hat{H}$, and decay incoherently with a time-dependent rate $\omega_0^2 \sigma^2 t$.

In Fig. 3 we show the time evolution of the coherences for an initial state $\rho_0 = 1/2(\mathbb{1} + \sigma_x)$ polarized in the x direction with $\lambda_0 = 10$, $\sigma = 1$, and $\alpha = 1$, which decay to the nonvanishing value obtained from Eq. (30) (see inset of Fig. 3). It is also shown that the first-order master equation well corresponds to the direct numerical averaging of the dynamics.

The effective dynamics, Eqs. (35) and (36), are best understood by expressing the master equation (17) in the eigenbasis of $\hat{H}$ by applying the rotation $\hat{R} = e^{-i\theta/2\hat{\sigma}_y}$ to all operators, i.e., $\hat{O} \rightarrow \hat{O}_r = \hat{R}\hat{O}\hat{R}^\dagger$ with $\theta = \arctan[\lambda_0/(2\alpha)]$ the angle

FIG. 4. Illustration in the Bloch sphere of the ensemble-averaging induced dephasing process (blue line) Eqs.(35) and (36) from static diagonal noise. The asymptotic state is the projection of the initial state onto the average Hamiltonian $\bar{H}$ (green line) and has nonvanishing coherences (x and y components). For visualization purposes the initial state is different from Fig. 3.

between $\bar{H}$ and the z axis. Neglecting the short-time contributions to the rates [i.e., setting $\gamma(t) \approx -\omega_0\sigma^2 t/\lambda_0$] and the terms proportional to $\alpha\sigma^2/\lambda_0$ (since $|\lambda_0| \gg \sigma + \alpha$), we obtain a pure dephasing equation,

$$\begin{aligned} \dot{\hat{\rho}}_r = & -\frac{i}{\hbar} [\hat{H}_r, \hat{\rho}_r] \\ & + \frac{1}{2} \omega_0^2 \sigma^2 t \left(\hat{\sigma}_z^r \hat{\rho}_r \hat{\sigma}_z^r - \frac{1}{2} \{ \hat{\sigma}_z^r \hat{\sigma}_z^r, \hat{\rho}_r \} \right), \end{aligned} \quad (37)$$

with

$$\bar{\hat{\mathbf{H}}}_r = \frac{1}{2} \sqrt{4\alpha^2 + \lambda_0^2} \hat{\sigma}_z. \quad (38)$$

This differential equation can be solved analytically and yields

$$\langle 1 | \hat{\rho}_r(t) | 1 \rangle = \langle 1 | \hat{\rho}_r(0) | 1 \rangle, \quad (39)$$

$$\langle 1 | \bar{\rho}_r(t) | 2 \rangle = e^{-i\omega_0 \sqrt{4\alpha^2 + \lambda_0^2} - \omega_0^2 \sigma^2 t} \langle 1 | \bar{\rho}_r(0) | 2 \rangle. \quad (40)$$

We obtain immediately from Eqs. (39) and (40) that the asymptotic state is

$$\lim_{t \rightarrow \infty} \langle 1 | \bar{\hat{\rho}}_r(t) | 1 \rangle = \langle 1 | \bar{\hat{\rho}}_r(0) | 1 \rangle; \quad \lim_{t \rightarrow \infty} \langle 1 | \bar{\hat{\rho}}_r(t) | 2 \rangle = 0. \quad (41)$$

Applying the inverse transform $R^\dagger \bar{\rho}_r R$, we find that the asymptotic state is the projection of the initial state onto the eigenbasis of the ensemble-averaged Hamiltonian $\bar{\hat{H}}$; c.f., Eq. (30). In other words, in the eigenbasis of $\hat{H}_0$ the ensemble-averaged dynamics results in a dephasing process towards the basis defined by the eigenvectors of $\bar{\hat{H}}$. Thus, initial coherences defined with respect to the eigenbasis of $\hat{H}_0$ do not decay asymptotically as illustrated in Fig. 4.

Furthermore, one can deduce the existence of decoherence-free subspaces for the generalized n -qubit case with collective dephasing [28]. For example, consider two qubits with the random Hamiltonian $\hat{H}_\lambda = \hbar\omega_0(\lambda\sigma_z \otimes \sigma_z + \alpha\sigma_x \otimes \mathbb{1})$, with λ drawn from the Gaussian distribution $p_{\text{Ga}}(\lambda)$ [Eq. (32)]. Then the eigenstates of $\hat{H} = \lambda_0\sigma_z \otimes \sigma_z + \alpha\sigma_x \otimes \sigma_x$ span two

FIG. 5. Illustration of a one-dimensional potential ladder with short- and long-range coupling $\hat{V}$, a potential step T , and on-site disorder λ_j [c.f. random Hamiltonians defined in Eq. (42)].

decoherence-free subspaces (one for each of the two double-degenerate eigenvalues).

(a) *Experiment: Nuclear spins.* The model Eq. (31) could be measured in a free-induction-decay (FID) experiment with nuclear spins [13]. After the $\pi/2$ pulse used to flip the magnetization into the transverse plane has been applied, the weak radiofrequency transverse field is not turned off, but only turned off-resonance. Then, in the rotating frame and considering Gaussian distributed spatial inhomogeneity of the polarization field, we obtain a Hamiltonian ensemble as described in Eq. (31). The FID decay of the magnetization M_x on the T_2^* time scale in the rotating frame then is proportional to $\text{Tr}[\hat{\rho}\hat{\sigma}_x] = \text{Re}[\hat{\rho}_{12}]$, and should not decay completely due to the presence of the transverse off-resonance field (c.f. Fig. 3). A similar protection of coherences from dipole-dipole interaction T_2 decay by anomalous resonance conditions for the transverse field was discussed in [47,48].

(b) *Experiment: Optical qubits.* Using as a qubit the horizontal $|H\rangle$ and vertical polarization $|V\rangle$ states of a photon, one can implement with linear quantum optics elements the unitaries $\hat{R}_y(\theta) = e^{-i\theta/2\hat{\sigma}_y}$ and $\hat{R}_z(\chi) = e^{-i\chi/2\hat{\sigma}_z}$. One realization of the noisy dynamics is realized by applying successively the rotations $\hat{R}_y(\theta_\lambda)$, $\hat{R}_z(\chi_\lambda(t))$ and $\hat{R}_y(\theta_\lambda)^{-1}$, with $\theta_\lambda = \arctan[(\lambda_0 + \lambda)/(2\alpha)]$ and $\chi_\lambda(t) = \sqrt{4\alpha^2 + (\lambda_0 + \lambda)^2}t$. The ensemble-average $\text{Tr}[\hat{\rho}\hat{\sigma}_x] = \text{Re}[\hat{\rho}_{12}]$ can be obtained by averaging in post-processing the measurement outcomes for different values of λ sampled from $p(\lambda)$ [6]. Alternatively, one can implement all the realizations of the noise in a single shot using a grating and two orthogonal spatial light modulators [49].

(c) *Experiment: Quantum computer.* On a quantum computer (such as the IBM-Q), one can directly implement the rotations $\hat{R}_y(\theta) = e^{-i\theta/2\hat{\sigma}_y}$ and $\hat{R}_z(\chi) = e^{-i\chi/2\hat{\sigma}_z}$. Thus, the effective dynamics can be simulated analogously as with optical qubits by performing the ensemble averaging in post-processing [21].

B. Effect of long-range couplings: Lattice in the strong bias limit with fully uncorrelated disorder

We consider a one-dimensional tight-binding model with d sites with on-site disorder and a constant potential ladder as illustrated in Fig. 5. The random Hamiltonian ensemble is parametrized as

$$\left\{ \hat{H}_\lambda = \hat{H}_\lambda^s + \alpha \hat{V} + \hat{T}, p_\lambda = \prod_{j=1}^d p_{\text{Ga}}(\lambda_j) \right\}, \quad (42)$$

where λ_j are identically independently distributed (i.i.d.) Gaussian [c.f. Eq. (32)] dimensionless random variables with average value $\bar{\lambda}_j = 0$ and variance σ , and

$$\hat{H}_\lambda^s = \hbar\omega_0 \sum_{j=1}^d \lambda_j |j\rangle\langle j|, \quad \hat{T} = \hbar\omega_0 T \sum_j |j\rangle\langle j|, \quad (43)$$

where $\hbar\omega_0$ is the reference energy, T represents a constant energy shift of the levels, α is a dimensionless perturbation parameter, and $\hat{V}$ (with $\langle j|\hat{V}|j\rangle = 0, \forall j$) is the coupling (or hopping term). To study the effect of short- and long-range couplings, we consider nearest-neighbor (NN) hopping $\hat{V}_{\text{NN}} = \sum_{j=1}^{d-1} |j\rangle\langle j+1| + |j+1\rangle\langle j|$ as well as dipole-type couplings with different exponents,

$$\hat{V}_x = \hbar\omega_0 \sum_{j=1}^d \sum_{k=j+1}^d \frac{1}{|k-j|^x} |j\rangle\langle k| + \text{c.c.}, \quad (44)$$

with $x = 0, 1, 3$.

This model, for instance, describes s -band electron transport in the presence of a strong electrical potential [50,51], and corresponds to an Anderson-type model [19] in the strong bias limit. Experimentally, it can be simulated using atom-optics simulators [52,53] or photonic waveguides [54–56].

The eigenvalues of the unperturbed part $\hat{H}_\lambda^0 = \hat{H}_\lambda^s + \hat{T}$ are $E_{j,\lambda}^0 = \hbar\omega_0(jT + \lambda_j)$ (i.e., $\varepsilon_j = \hbar\omega_0 jT$). As was derived in Sec. IV, the corresponding first order in α perturbative master equation, Eq. (17), is fully characterized by the characteristic function, Eq. (24), $\phi_{jk}^*(\omega_0 t) = \exp[-\omega_0^2 \sigma^2 t^2]$. Using the latter and Eqs. (27)–(29) we obtain for the decoherence rates,

$$\Upsilon_{jk}(t) = -i\omega_0(j-k)T - 2\omega_0^2 \sigma^2 t, \quad (45)$$

$$\gamma_{jk}(t) \approx -\frac{2\omega_0 \sigma^2 t}{(j-k)T}, \quad (46)$$

$$\Gamma_{jkrj}(t) \approx 0, \quad (47)$$

where we neglected the fast decaying contribution to $\gamma_{jk}(t)$. Here $\Gamma_{jkrj}(t) = 0$ because the eigenvalues are i.i.d. Note the factor 2 in Eqs. (45) and (46) as compared to the single qubit case, Eqs. (33) and (34), which is due to the i.i.d. condition.

As a numerical example, we consider a system of $d = 30$ sites with $\sigma = \alpha$ and $T = 10\alpha$, and a broad, centered Gaussian initial state,

$$\hat{\rho}_0 = |G_0\rangle\langle G_0|, \quad |G_0\rangle = \frac{1}{\mathcal{N}} \sum_{j=1}^d p'_{\text{Ga}}(j) |j\rangle, \quad \langle G_0|G_0\rangle = 1, \quad (48)$$

with $\mathcal{N}$ the normalization constant and p'_{Ga} a Gaussian distribution with average $(d+1)/2$ and variance $\sqrt{d}$. This could, for instance, model a photoexcitation in a correlated thin-film transition metal-oxide heterostructure with a strong intrinsic electrical field [51].

In Fig. 6, we show the evolution of the total coherence,

$$c(t) := \sum_{j \neq k} |\rho_{jk}|, \quad (49)$$

for $\hat{V}_{\text{NN}}$, $\hat{V}_x$ ($x = 0, 1, 3$), and $\hat{V} = 0$. The dynamics are obtained by numerical integration of the perturbative master

FIG. 6. Evolution of the total coherence, Eq. (49), for different dipole-type coupling potentials with exponent $x = 0, 1, 3$ [c.f. Eq. (44)], nearest-neighbor coupling (NN) and no coupling ($V = 0$). The longer range the coupling, the more coherences are present in the asymptotic state characterized by Eq. (30) (horizontal gray dashed lines). Numerical integration of the perturbative master equation, Eq. (17) with rates Eq. (45)–(47) (solid lines) agrees with the direct numerical averaging (dots) with 10^6 realizations of the disorder for $x = 0, x = 1$. For $x = 3$, NN and $V = 0$ (only $V = 0$ is shown for visual purposes) the asymptotic state is not reached due to the statistical error $\sim \sqrt{10^6}$. The parameters are $d = 30$, $\sigma = \alpha$, $T = 10\alpha$ and the initial Gaussian state, Eq. (48), has width $\sqrt{30}$ and average value $31/2$.

equation (17) with the “NDSolve” routine in MATHEMATICA 11.0. At short times, the coherences decay, independently of the coupling potential, as $\sim e^{-\sigma^2 \omega_0^2 t^2}$ under the action of the rate Υ_{jk} . At large times, for any nonvanishing coupling $\hat{V}$, the decay eventually stops, and a finite amount of coherence remains in the asymptotic state, which is in agreement with Eq. (30). For long-range couplings the total coherence of the asymptotic state, $c_\infty = \lim_{t \rightarrow \infty} c(t)$, is larger than for short-range couplings. Indeed, for a fully connected network ($x = 0$) the total coherence in the asymptotic state is $c_\infty = 15.1 \times 10^{-3}$, for slowly decreasing coupling ($x = 1$) we obtain $c_\infty = 4.1 \times 10^{-3}$, and for dipole-type coupling ($x = 3$) we have $c_\infty = 1.6 \times 10^{-3}$. Interestingly, the nearest-neighbors coupling converges to $c_\infty = 1.4 \times 10^{-3}$, which is close to the value for the dipole coupling.

As proof of consistency of the perturbative master equation approach, we computed the full dynamics by numerical exact averaging using the numerical solver from the python QUTIP package 4.3.0 [57]. For the long-range interactions $x = 1$ and $x = 0$, we found that 10^6 realizations are sufficient for the convergence of the numerical averaging. The results of the master equation and direct averaging well agree as shown in Fig. 6. For the shorter range couplings ($x = 3$, $\hat{V}_{NN}$, and $\hat{V} = 0$), 10^6 realizations were insufficient. In Fig. 6 we show the deviations for $V = 0$ for which we know that the master equation is exact [4]. Note that on the same computer the numerical integration of the master equation was 10 times faster than the direct numerical averaging with 10^6 realizations. As a further test we verified that the fidelity between the density matrix from the numerical computations and the master equation is larger than 0.999 at all times. Furthermore, we found that the relative purity $\text{Tr}[\rho^2]/\text{Tr}[\rho_{\text{Num}}^2]$ shows deviation of up to

6% in the time range where the decay enters the asymptotic state ($1.5 \lesssim t \lesssim 2.5$), but eventually converges to the same value (for $x = 0, 1$). This deviation likely occurs due to the approximation in the time dependence of the decoherence rates Eqs. (27)–(29).

Overall we find that the dephasing induced by diagonal disorder does not, in the presence of couplings $\hat{V}$ between the eigenstates, lead to a full decay of the coherences. This is in stark contrast to dynamical diagonal noise such as considered in the Hacken-Strobl model [58] or for homogeneous broadening descriptions [59]. Moreover, the coherences are most efficiently protected from the disorder-induced decoherence by long-range couplings. This can be understood by analogy to the qubit case studied in Sec. V A where we demonstrated that the ensemble averaging leads to an effective dephasing in the eigenbasis $\{|n\rangle\}$ of the ensemble-averaged Hamiltonian $\bar{H}$ (c.f. Fig. 4). For $\hat{V} = 0$, the basis $\{|n\rangle\}$ is equal to the quantization basis $\{|j\rangle\}$, and we thus have a pure dephasing process so that all coherences vanish. However, the stronger the potential $\hat{V}$, the more the eigenstates of $\bar{H}$ will deviate from $|j\rangle$, and thus the more the dephasing basis $\{|n\rangle\}$ differs from the quantization basis. Consequently, a larger amount of the coherences of the initial state do not decay asymptotically.

This result could be of relevance for a better understanding of the interplay between disorder and quantum coherent transport (which relies on the coherences) in strongly connected networks such as the Fenna-Matthew-Olson photosynthesis molecular complex [60], assemblies of ultracold Rydberg atoms [61], strongly correlated materials [50,62], superconducting circuits [63], or photonic circuits [64].

(a) *Experiment: Optical waveguides.* The model Eq. (42) with nearest-neighbor coupling $\hat{V}_{NN}$ could be realized with silicon waveguide lattices, similarly as was shown in Ref. [54] for off-diagonal disorder. The on-site energies $\hbar\omega_0\lambda_j$ are tuned by the waveguide depths, and the coupling $\hat{V}_{NN}$ by the waveguide separation. Another equally deep and evenly spaced waveguide array can generate a delocalized initial state such as Eq. (48). Implementing the energy ladder $\hat{T}$ may be limited by substrate thickness though. Instead, one could use alternating shifts $\hat{T} = \hbar\omega_0 T \sum_j (j \bmod 2) |j\rangle\langle j|$ (with mod the modulo operation). While not identical, the resulting average dynamics should not differ qualitatively from Fig. 6.

(b) *Experiment: Bose-Einstein condensate.* Using plane-wave momentum states of a Bose-Einstein condensate and stimulated Bragg transitions to control the coherent coupling, it is possible to directly simulate the single realization Hamiltonians from the ensemble Eq. (42) [52]. The disorder averaging can be obtained by repeating the measurement with different random values of the on-site energies λ_j and averaging the outcomes in post-processing.

C. Many-body (bosons) dynamics with strongly correlated noise

We consider an asymmetric Bose-Hubbard dimer, i.e., a double-well potential with N interacting bosons (see illustration Fig. 7) described in terms of a tilted Bose-Hubbard model [65–67]. Such a setting was studied, e.g., in the context of quantum chaos [68], superfluidity [69], and distinguishability [70].

FIG. 7. Double-well potential with a tilt T between the wells left (L) and right (R) filled with bosons (blue dots) with on-site interaction U and tunneling barrier J as an illustration of the Bose-Hubbard Hamiltonian (50).

Firstly, we fix the notation and write the Bose-Hubbard Hamiltonian in the absence of noise as

$$\hat{H}_{\text{BH}} = \hat{H}^0 + \alpha \hat{V}, \quad (50)$$

with the on-site potential and interaction,

$$\begin{aligned} \hat{H}^0 &= \hbar\omega_0 T (\hat{N}_L - \hat{N}_R) \\ &+ \hbar\omega_0 \frac{U_0}{2} (\hat{N}_L(\hat{N}_L - 1) + \hat{N}_R(\hat{N}_R - 1)), \end{aligned} \quad (51)$$

and the perturbation,

$$\hat{V} = -\hbar\omega_0 J (\hat{a}_L^\dagger \hat{a}_R + \hat{a}_R^\dagger \hat{a}_L). \quad (52)$$

Here $\hat{a}_L$, $\hat{a}_R$, $\hat{a}_L^\dagger$, $\hat{a}_R^\dagger$ are the bosonic annihilation and creation operators of the left (L) and right (R) wells, respectively, and the number operators are $\hat{N}_L = \hat{a}_L^\dagger \hat{a}_L$, $\hat{N}_R = \hat{a}_R^\dagger \hat{a}_R$. The tunneling rate is denoted as $\hbar\omega_0 J \in \mathbb{R}^+$, the on-site interaction reads $\hbar\omega_0 U \in \mathbb{R}$, and the tilt between the two wells is given by $\hbar\omega_0 T \in \mathbb{R}$. The eigenvalues of the unperturbed Hamiltonian $\hat{H}^0$ can be written as $E_m^0 = \hbar\omega_0(\beta_m U_0 + \chi_m T)$ with the constant factors $\beta_m = 1/2m(m-1) + 1/2(N-m)(N-m-1)$ and $\chi_m = (2m-N)$, $m = 0, \dots, N$ fixed by the number of bosons N in the system and using the Fock basis ordering $|N, 0\rangle, |N-1, 1\rangle, \dots, |0, N\rangle$. For example, for $N = 3$ we have $|E_1\rangle = |3, 0\rangle$, $|E_2\rangle = |2, 1\rangle$, $|E_3\rangle = |1, 2\rangle$, $|E_4\rangle = |0, 3\rangle$ with $\beta_1 = 3$, $\beta_2 = 1$, $\beta_3 = 1$, $\beta_4 = 3$, and $\chi_1 = -3$, $\chi_2 = -1$, $\chi_3 = 1$, $\chi_4 = 3$.

Adding a random noise δU to the interaction, we replace $U_0 \rightarrow U_0 + \delta U$. Consequently, the eigenvalues of the unperturbed random Hamiltonians are given by $E_{m,\lambda}^0 = \hbar\omega_0(\beta_m U_0 + \chi_m T) + \hbar\omega_0 \lambda_m$, with $\lambda_m = \delta U \beta_m$. The noisy eigenenergies are thus strongly correlated as they are characterized by a single random variable δU , i.e., the eigenvalues are *not* independently distributed, as opposed to the previously studied model of a one-dimensional potential ladder with on-site disorder.

Assuming a Gaussian distribution $p_{\text{Ga}}(\delta U)$ of mean value 0 and variance σ , the characteristic function, Eq. (24), evaluates to $\phi_{jk}^*(\omega_0 t) = \exp[-1/2\omega_0^2(\beta_j - \beta_k)^2 \sigma^2 t^2]$, and the decoher-

FIG. 8. Evolution of ensemble-averaged coherences for the Bose-Hubbard Hamiltonian Eq. (50) with $N = 3$ bosons and random on-site interactions for the initial state $1/\sqrt{4} \sum_{j=1}^4 |j\rangle$ obtained from numerical integration of the perturbative master equation. The element $\bar{\rho}_{14}(t)$ (blue line) decays to zero, but slowly because $\beta(1) - \beta(4) = 0$, while $\bar{\rho}_{12}(t)$ (green line) decays fast, but to a nonvanishing value $\sim 4.87 \times 10^{-3}$ (see inset). The parameters are $J = 1$, $U_0 = 1$, $T = 10$, $\alpha = 1$, and $\sigma = 1$.

ence rates Eqs. (27)–(29) read

$$\begin{aligned} \Upsilon_{jk}(t) &= -i\omega_0[(\beta_j - \beta_k)U_0 + (\chi_j - \chi_k)T] \\ &- \omega_0^2(\beta_j - \beta_k)^2 \sigma^2 t, \end{aligned} \quad (53)$$

$$\gamma_{jk}(t) \approx \frac{-i\omega_0(\beta_j - \beta_k)^2 \sigma^2 t}{(\beta_j - \beta_k)U_0 + (\chi_j - \chi_k)T}, \quad (54)$$

$$\Gamma_{jkrj}(t) \approx i \left(\omega_0 \sigma^2 t \frac{(\beta_r - \beta_k)^2 - (\beta_j - \beta_k)^2}{(\beta_j - \beta_k)U_0 + (\chi_j - \chi_k)T} \right). \quad (55)$$

Hence, second-order processes described by $\Gamma_{jkrj}(t)$ now play a role because the eigenvalues are strongly correlated. To guarantee that the eigenvalues $E_{m,\lambda}^0$ of the unperturbed random Hamiltonian remain nondegenerate for most realizations of the noise, we require $|T| \gg \sigma$. For the perturbation to remain small, we further impose the condition $\alpha J < |U_0 + T|$.

Interestingly, for energy levels a, b that are symmetric in their number of bosons in the left/right well (e.g., $E_1 = |3, 0\rangle$ and $E_4 = |0, 3\rangle$), the real parts of the first two decoherence rates, Eqs. (53) and (54), vanish: $\text{Re}[\Upsilon_{ab}(t)] = 0$ and $\text{Re}[\gamma_{ab}(t)] = 0$ (because $\beta_a - \beta_b = 0$). As a consequence, the associated off-diagonal elements $\bar{\rho}_{ab}$ evolve only from the action of the average Hamiltonian $\hat{H}$ and the second-order rates $\Gamma_{abra}(t)$, which results in a slow decay. This is illustrated in Fig. 8 for $N = 3$, the initial state $1/\sqrt{4} \sum_{j=1}^4 |E_j\rangle$, and with parameters $J = 1$, $U_0 = 1$, $T = 10$, $\alpha = 1$, and $\sigma = 1$. Indeed, the coherence $\bar{\rho}_{14}(t)$, for which $\beta_1 - \beta_4 = 0$, decays 10 times slower than $\bar{\rho}_{12}(t)$, for which $\beta_1 - \beta_2 = -2$. However, note that since $\chi_1 - \chi_4 = 0$, there is no direct coupling between the states $|E_1\rangle$ and $|E_4\rangle$ so that $\bar{\rho}_{14}(t)$ asymptotically converges to zero. On the contrary, since $V_{12} = -\sqrt{3}\hbar\omega_0 J$ the fast decaying coherence $\bar{\rho}_{12}(t)$ converges to a finite value $c_\infty \sim 4.87 \times 10^{-3}$ obtained from Eq. (30) (c.f. inset of Fig. 8).

Hence, in addition to coherences being present in the asymptotic state due to the coupling $\hat{V}$, the symmetry of the Hamiltonian gives rise to slowly decaying coherences. This effect could be exploited to generate long-lived coherences of many-body states in systems subject to generic on-site noise.

(a) *Experiment. Bose-Einstein condensates.* The Bose-Hubbard dimer can be realized with Bose-Einstein condensates in an optical lattice [71]. With Feshbach resonances one can control the magnitude of the on-site interaction [72,73] which can be sampled from the distribution P_{Ga} between each measurement run to simulate the static noise. The ensemble average is obtained in post-processing.

D. Range of validity

Before we conclude, let us review the range of validity of the first-order perturbation master equation Eq. (17) for diagonal noise. In the derivation of the general perturbative master equation (c.f. Sec. III A) we assume the convergences of the Neumann series used to expand the inverse of the ensemble-average dynamical matrix $\bar{F}$, i.e., $\lim_{n \rightarrow \infty} (\mathbb{1} - \bar{F}^{-1} \bar{F})^n = 0$. In other words, we assume that nondegenerate perturbation theory applies to most realizations of the noise, which requires in the case of diagonal noise $\alpha V_{jk} \ll E_{\lambda,j}^0 - E_{\lambda,k}^0$ for most Hamiltonians $\hat{H}_\lambda$ in the random ensemble Eq. (16). Hence, the eigenenergy distribution shows level repulsion for all state coupled by $\hat{V}$.

As we have shown with examples in Secs. VA–VC if the above conditions are satisfied, the perturbative master equation, Eq. (17), captures the ensemble-average dynamics not only on transient time scales, but also on asymptotic time scales (up to systematic errors). This can be understood as a consequence of the interplay between the accumulated errors in the phases from the perturbative expansion, and the ensemble-averaging induced dephasing. On the one hand, the finite perturbative expansion leads to an approximation of the exact phases and eigenvectors of the system. In a closed system these errors in the phases accumulate over time, and eventually any phase relation between the perturbative dynamics and the actual dynamics is lost at sufficiently long times. On the other hand, the ensemble-averaging induced dephasing leads to an overall diminution of the phases and keeps the accumulated error at a finite value. Hence, as long as the dephasing process is fast enough to keep the error from perturbation theory finite, the effective nonunitary dynamics arising from the corresponding perturbative master equation only incur a systematic error.

When the conditions for nondegenerate perturbation theory are not satisfied, the perturbative master equation captures the dynamics at least on short time scales in the sense discussed in Refs. [4,10,18]. Indeed, the first order in time approximation of Eq. (17) yields

$$\begin{aligned} \dot{\bar{\rho}}(t) = & -\frac{i}{\hbar} [\bar{H}, \bar{\rho}(t)] \\ & + 2 \frac{t}{\hbar^2} \sum_{j,k=1}^d (\overline{E_j^0 E_k^0} - \overline{E_j^0} \overline{E_k^0}) \hat{\Pi}_{jj} \bar{\rho}(t) \hat{\Pi}_{kk}. \end{aligned} \quad (56)$$

This equation corresponds to the short-time master equation derived in Refs. [4,18]:

$$\dot{\bar{\rho}} = -\frac{i}{\hbar} [\bar{H}, \bar{\rho}(t)] + \int d\lambda p_\lambda \left(\hat{L}_\lambda \bar{\rho} \hat{L}_\lambda^\dagger - \frac{1}{2} \{ \hat{L}_\lambda^\dagger \hat{L}_\lambda, \bar{\rho} \} \right),$$

with $\hat{L}_\lambda = (\hat{H}_\lambda - \bar{H})/(\hbar\omega_0)$. This equation captures the effective ensemble-averaged dynamics of any random Hamiltonian ensemble at times $t \ll 1/\omega_0$, which corresponds to the Gaussian decay regime at times shorter than the Heisenberg time.

VI. CONCLUSIONS

In this article, we significantly expanded the playground of the master equation description to quantum system with static noise and disorder using a new perturbation approach. In particular, our approximation scheme is the first that captures also the asymptotic time scales of the ensemble-averaged dynamics. We leveraged this property to describe the preservation of coherences by small perturbations in systems with static diagonal noise, which is in contrast to the complete decoherence induced by dynamical diagonal noise [58]. To do so, we started with a brief review of the random Hamiltonian ensemble approach to quantum systems with static noise (which includes disorder, the coupling with a classical environment, or random unitary channels). Then, we derived general perturbative master equations to describe the effective ensemble-averaged dynamics. We showed that the perturbative master equations not only provide a description of the ensemble-averaged dynamics in terms of physically interpretable decoherence operators, energy shifts, and (time-dependent) decoherence rates, but also require much less computational resources to numerically integrate as compared to direct numerical averaging of the dynamics.

As a specific application of the perturbative master equation, we treated in details systems described by Hamiltonians with diagonal static noise, and a noise-free perturbative coupling potential. In the range of parameters where perturbation theory converges for most realizations of the noise, the first-order master equation describes the ensemble-averaged dynamics on all time scales. This is in stark contrast to previous approximation methods such as Markovianity (weak-coupling) [10] or short-time expansion [18], which cannot provide such a description.

The effect of the ensemble averaging can be understood as a dephasing (decoherence) process in the eigenbasis of the average Hamiltonian, which was illustrated in the Bloch sphere for a two-level system with random energy splitting. Thus, the asymptotic state is the projection of the initial state on the eigenbasis of the average Hamiltonian. This was numerically verified using the perturbative master equation and direct numerical averaging for several examples.

The generality of our approach was shown with a one-dimensional tight-binding model with on-site disorder and an asymmetric Bose-Hubbard dimer with on-site interaction noise. In the first example, we showed that the longer range the perturbative potential is, the more coherences are protected from the dephasing and remain in the asymptotic state. In the second example, we showed that strong correlations in the noise distribution result in a slow decay of the coherences,

which are thus partially protected from the averaging-induced decoherence. For all scenarios we discussed various experimental setups to measure these effects.

In all the applications we assumed for simplicity standard Gaussian noise. In many natural systems, however, other distributions such as uniform [19], Cauchy-Lorentz [13], or general Lévy [74] are better models for noise sources. Such distributions in general will lead to a more complex time-dependent behavior of the effective dynamics—for instance, in the uniform distribution case the dynamics are strongly non-Markovian [4]. Nevertheless, the overall dephasing pattern and the asymptotic state are not expected to change because the distributions only affect the rates of the master equation, but not the operators, and thus the type of decoherence [4].

Ultimately, our work suggests that introducing tailored couplings to a system could be used to protect its quantum coherences from static noise. Conversely, static noise could be introduced on purpose to give rise to a desired type of dephasing. One could then easily simulate intricate dynamics such as non-Markovian ones [49], or drive a quantum system into a target nonequilibrium quantum state. This could prove experimentally relevant since it is usually easier to engineer the Hamiltonian of the system rather than its interaction with the environment in order to obtain a desired nonunitary dynamics. In other words, engineering a multitude of Hamiltonians [6,75] might be more feasible than to engineer the environment to obtain a particular nonunitary time evolution or to drive the system into a desired state. In this regards, it would be particularly interesting to further develop the theory to understand whether asymptotic states with exotic properties such as persistent oscillations [45] or many-body localization [76,77] can be reached.

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APPENDIX A: NEUMANN SERIES

The inverse of a given matrix M , which is almost equal to an invertible matrix M_0 , in the sense that

$$\lim_{n \rightarrow \infty} (\mathbb{1} - M_0^{-1}M)^n = 0, \quad (\text{A1})$$

can be written as a Neumann series (see, e.g., [78], p. 75),

$$M^{-1} = \sum_{n=0}^{\infty} (\mathbb{1} - M_0^{-1}M)^n M_0^{-1}. \quad (\text{A2})$$

This result can be applied to the perturbative ensemble-averaged dynamical matrix ($M = \bar{F}$, $M_0 = \bar{F}^0$). We remark that the zeroth-order dynamical matrix F^0 , Eq. (B1), is diagonal in the eigenbasis of the unperturbed Hamiltonians, and, thus, directly invertible. Moreover, $\mathbb{1} - \bar{F}^0{}^{-1}\bar{F} = \bar{F} - \bar{F}^0 \sim \alpha$ and thus condition (A1) is fulfilled and the series converges. The inverse of the ensemble-averaged dynamical matrix is

then given by

$$\begin{aligned} \bar{F}^{-1}(t) &= \sum_{m=0}^{\infty} (\mathbb{1} - \bar{F}^0{}^{-1}(t)\bar{F}(t))^m \bar{F}^0{}^{-1}(t) \\ &= \sum_{m=0}^{\infty} \left(- \sum_{k=1}^{\infty} \alpha^k \bar{F}^0{}^{-1}(t) \bar{F}^k(t) \right)^m \bar{F}^0{}^{-1}(t), \end{aligned}$$

and the master equation matrix, Eq. (14), then reads

$$\begin{aligned} \bar{Q}(t) &= \dot{\bar{F}}(t) \bar{F}^{-1}(t) \\ &= \left(\sum_{n=0}^{\infty} \alpha^n \dot{\bar{F}}^n(t) \right) \\ &\quad \cdot \sum_{m=0}^{\infty} \left(- \sum_{k=1}^{\infty} \alpha^k \bar{F}^0{}^{-1}(t) \bar{F}^k(t) \right)^m \bar{F}^0{}^{-1}(t). \end{aligned}$$

APPENDIX B: FIRST-ORDER PERTURBATION EQUATION

Here we show the computations for deriving the master equation (17). For more details see [10]. In order to ease the notation, let us define the orthonormal basis $\{|jk\rangle\}$ such that $\bar{F}_{jk,rs}(t) = \langle jk | \bar{F}(t) | rs \rangle$.

(a) *Zeroth order* α^0 . The zeroth-order dynamical matrix captures the dynamics of solely the unperturbed part, and describes the evolution of the coherences in the eigenbasis $\{|j\rangle\}$. We recall that $\hat{U}_\lambda^0(t) = e^{-\frac{i}{\hbar} \hat{H}_\lambda^0} = \sum_j e^{-\frac{i}{\hbar} E_{\lambda,j}} |j\rangle \langle j|$ and derive the ensemble-averaged value,

$$\bar{F}_{jk,rs}^0(t) = \overline{\langle j | \hat{U}_\lambda^0(t) | r \rangle \langle s | \hat{U}_\lambda^{0\dagger}(t) | k \rangle} = \delta_{jr} \delta_{sk} \bar{\varphi}_{jk}(t),$$

with the ensemble-averaged phase factors,

$$\bar{\varphi}_{jk}(t) \equiv \int d\vec{\lambda} p_{\vec{\lambda}} e^{-\frac{i}{\hbar} (E_{\lambda,j} - E_{\lambda,k})} = \int d\vec{\lambda} p_{\vec{\lambda}} e^{-it\omega_0(\lambda_j - \lambda_k)}.$$

We thus obtain

$$\dot{\bar{F}}^0 = \sum_{j,k=1}^d \dot{\bar{\varphi}}_{jk}(t) |jk\rangle \langle jk|, \quad (\text{B1})$$

$$\bar{F}^0{}^{-1}(t) = \sum_{j,k=1}^d \frac{1}{\bar{\varphi}_{jk}(t)} |jk\rangle \langle jk|. \quad (\text{B2})$$

Note that one must remain careful here because even though $\varphi_{jk}^{-1}(t) = \varphi_{jk}^*(t)$, in general $\bar{\varphi}_{jk}^{-1}(t) \neq \bar{\varphi}_{jk}^*(t)$.

(b) *First order* α^1 . From Eqs. (11) and (13) we derive

$$\begin{aligned} i\hbar \bar{F}_{jk,rs}^1(t) &= -\frac{i}{\hbar} \int_0^t dt' \overline{\langle j | \hat{U}_\lambda^0(t-t') \hat{V}_\lambda \hat{U}_\lambda^{0\dagger}(t') | r \rangle \langle s | \hat{U}_\lambda^{0\dagger}(t) | k \rangle} \\ &\quad - \frac{i}{\hbar} \int_0^t dt' \overline{\langle j | \hat{U}_\lambda^0(t) | r \rangle \langle s | \hat{U}_\lambda^{0\dagger}(t') \hat{V}_\lambda \hat{U}_\lambda^{0\dagger}(t-t') | k \rangle}. \end{aligned}$$

In Sec. III A, we further assumed that the perturbation is independent of the noise, i.e., $\hat{V}_\lambda \rightarrow \hat{V}$. Then, one obtains in

matrix notation,

$$\begin{aligned}
 i\hbar\overline{F}^{-1}(t) = & t \sum_{j,k=1}^d \overline{\varphi}_{jk}(t)(V_{jj} - V_{kk})|jk\rangle\{jk| \\
 & + \left(\sum_{\substack{j,k=1 \\ j \neq k}}^d f_{jk}(t)V_{kj}(|kk\rangle\{jk| - |jj\rangle\{jk|) \right. \\
 & \left. + f_{jk}(t)V_{jk}(|jk\rangle\{kk| - |jk\rangle\{jj|) \right) \\
 & + \left(\sum_{\substack{j,k,r=1 \\ j \neq r \neq k}}^d f_{jkrj}(t)V_{jr}|jk\rangle\{rk| \right. \\
 & \left. - f_{jkk r}(t)V_{rk}|jk\rangle\{jr| \right), \quad (B3)
 \end{aligned}$$

where we defined for simplicity of reading,

$$f_{jk}(t) := \int_0^t dt' \overline{\varphi}_{jk}(t'), \quad f_{jkrj}(t) := \overline{\varphi}_{jk}(t) \int_0^t dt' \varphi_{rj}(t').$$

For f_{jkrj} the ordering of the indices refers to the ordering of the indices of the phase factors. Note that in the above Eq. (B3), we separated out the terms for which the phase factors are equal to the identity prior to the ensemble averaging because in general $\overline{\varphi}_{jk}(t)\delta_{j,k} \neq \overline{\varphi}_{jj}(t)$.

Using the zeroth-order Eqs. (B1) and (B2), and the first-order Eq. (B3) for the dynamical matrix, we can compute the first-order expansion of $\overline{Q}$, Eq. (14), which fully characterizes the perturbative master equation. To zeroth order we obtain

$$\begin{aligned}
 \overline{Q}^0 = & \overline{F^0 F^0}^{-1} = \sum_{j,k=1}^d \frac{\dot{\overline{\varphi}}_{jk}(t)}{\overline{\varphi}_{jk}(t)} |jk\rangle\{jk| \\
 := & \sum_{\substack{j,k=1 \\ j \neq k}}^d \Upsilon_{jk}(t) |jk\rangle\{jk|, \quad (B4)
 \end{aligned}$$

where we defined

$$\Upsilon_{jk}(t) := \frac{d}{dt} \ln[\overline{\varphi}_{jk}(t)]. \quad (B5)$$

The first order yields

$$\begin{aligned}
 \overline{Q}^{-1}(t) = & \overline{F^{-1}(t) F^0}^{-1}(t) - \overline{F^0(t) F^0}^{-1}(t) \overline{F^{-1}(t) F^0}^{-1}(t) \\
 = & -\frac{i}{\hbar} \sum_{j,k=1}^d (V_{jj} - V_{kk}) |jk\rangle\{jk| \\
 & - \frac{i}{\hbar} \sum_{\substack{j,k=1 \\ j \neq k}}^d V_{kj} (|kk\rangle\{jk| - |jj\rangle\{jk|)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\hbar} \sum_{\substack{j,k=1 \\ j \neq k}}^d \tilde{\gamma}_{jk}(t) V_{jk} (|jk\rangle\{kk| - |jk\rangle\{jj|) \\
 & + \frac{1}{\hbar} \left(\sum_{\substack{j,k,r=1 \\ j \neq r \neq k}}^d \tilde{\Gamma}_{jkrj}(t) V_{jr} |jk\rangle\{rk| \right. \\
 & \left. - \tilde{\Gamma}_{jkrj}^*(t) V_{rj} |kj\rangle\{kr| \right), \quad (B6)
 \end{aligned}$$

where we defined the anti-Hermitian rate matrix,

$$\begin{aligned}
 \tilde{\gamma}_{jk}(t) & := -i(\dot{f}_{jk}(t) - \Upsilon_{jk}(t)f_{jk}(t)) \\
 & = -i \left[\overline{\varphi}_{jk}(t) - \Upsilon_{jk}(t) \int_0^t dt' \overline{\varphi}_{jk}(t') \right] = -\tilde{\gamma}_{kj}^*(t), \quad (B7)
 \end{aligned}$$

and the matrix,

$$\begin{aligned}
 \tilde{\Gamma}_{jkrj}(t) & := -i \left[\frac{\dot{f}_{jkrj}(t)}{\overline{\varphi}_{rk}(t)} - \Upsilon_{jk}(t) \frac{f_{jkrj}(t)}{\overline{\varphi}_{rk}(t)} \right] \\
 & = -i - \frac{i}{\overline{\varphi}_{rk}(t)} \left(\dot{\varphi}_{jk}(t) \int_0^t dt' \varphi_{rj}(t') \right) \\
 & + i \frac{\dot{\overline{\varphi}}_{jk}(t)}{\overline{\varphi}_{jk}(t)\overline{\varphi}_{rk}(t)} \overline{\varphi}_{jk}(t) \int_0^t dt' \varphi_{rj}(t') = -\tilde{\Gamma}_{kjjr}^*(t). \quad (B8)
 \end{aligned}$$

The symmetry $\tilde{\Gamma}_{jkrj} = -\tilde{\Gamma}_{kjjr}^*$ is obtained by permuting the first two indices, $jk \rightarrow kj$, and the last two indices, $rj \rightarrow jr$, and reflects the Hermiticity of the density matrix.

In order to derive a master equation in the Lindblad form following the general method described in [4], we should now expand $\overline{Q}^{-1}(t)$ in a basis of Hermitian traceless operators and collect the terms for the coherent and incoherent parts. However, this proves to be, in general, technically rather involved. We prefer to go back to the Hilbert space representation using the identity,

$$\dot{\tilde{\rho}}(t) = \sum_{j,k,r,s=1}^d \overline{Q}_{jk,rs}(t) |j\rangle\langle r| \tilde{\rho} |s\rangle\langle k|,$$

in order to obtain the evolution equation in terms of the matrix elements of the ensemble-averaged density matrix. A nice form of the equation is obtained by defining

$$\gamma_{jk}(t) := \tilde{\gamma}_{jk}(t) + i, \quad \Gamma_{jkrj}(t) := \tilde{\Gamma}_{jkrj}(t) + i, \quad (B9)$$

thereby isolating the time-independent parts $\gamma(0) = -i$ and $\Gamma(0) = -i$, and with both $\gamma(t)$ and $\Gamma(t)$ being at least of second order in time. Then, the master equation can be expressed

as

$$\begin{aligned} \dot{\bar{\rho}}(t) = & \sum_{j,k=1}^d \Upsilon_{jk}(t) \hat{\Pi}_{jj} \bar{\rho}(t) \hat{\Pi}_{kk} - \alpha \frac{i}{\hbar} [\hat{V}, \bar{\rho}(t)] \\ & + \frac{\alpha}{\hbar} \sum_{\substack{j,k=1 \\ j \neq k}}^d \gamma_{jk}(t) V_{jk} [\hat{\Pi}_{jk} \bar{\rho}(t) \hat{\Pi}_{kk} - \hat{\Pi}_{jj} \bar{\rho}(t) \hat{\Pi}_{jk}] \\ & + \frac{\alpha}{\hbar} \left(\sum_{\substack{j,k,r=1 \\ j \neq k \neq r}}^d \Gamma_{jkrj}(t) V_{jr} \hat{\Pi}_{jr} \bar{\rho}(t) \hat{\Pi}_{kk} \right. \\ & \left. + \Gamma_{jkrj}^*(t) V_{rj} \hat{\Pi}_{kk} \bar{\rho}(t) \hat{\Pi}_{rj} + O(\alpha^2) \right), \end{aligned} \quad (\text{B10})$$

with the diadic operators $\hat{\Pi}_{jk} := |j\rangle\langle k|$.

APPENDIX C: SYMMETRIC DISTRIBUTIONS

Let us derive the decoherence rates $\gamma_{jk}(t)$, Eq. (20), and $\Gamma_{jkrj}(t)$, Eq. (21), in the case that the joint-noise-distribution $q_{jk}(\lambda_j - \lambda_k)$ for any pair of variables λ_j, λ_k has vanishing odd cumulants (except the central value), or in other words, the probability density functions q_{jk} are symmetric around their central value. In this case, considering furthermore that the probability distributions shall be continuous and smooth, we know that the associated characteristic function vanishes in the limit of large times,

$$\phi_{jk}(t) = \int d\lambda_j \int \lambda_k q_{jk}(\lambda_j - \lambda_k) e^{-i(\lambda_j - \lambda_k)t} \xrightarrow{t \rightarrow \infty} 0. \quad (\text{C1})$$

We begin with the rate,

$$\gamma_{jk}(t) = i - i\bar{\varphi}_{jk}(t) + i\Upsilon_{jk}(t) \int_0^t dt' \bar{\varphi}_{jk}(t'), \quad (\text{C2})$$

and first evaluate the integral,

$$\begin{aligned} \int_0^t dt' \bar{\varphi}_{jk}(t') &= \int_0^t dt' e^{-\frac{it'}{\hbar}(\varepsilon_j - \varepsilon_k)} \phi_{jk}^*(\omega_0 t') \\ &= \int d\Delta_{jk} q_{jk}(\Delta_{jk}) \int_0^t dt' e^{-i\omega_0 t'(\lambda_j - \lambda_k)} e^{-\frac{it'}{\hbar}(\varepsilon_j - \varepsilon_k)} \\ &= \int d\Delta_{jk} q_{jk}(\Delta_{jk}) i \frac{e^{-i\omega_0(\lambda_j - \lambda_k)t} - 1}{\omega_0(\lambda_j - \lambda_k) + (\varepsilon_j/\hbar - \varepsilon_k/\hbar)}. \end{aligned}$$

Since we are working in the nondegenerate perturbation regime, i.e., $E_{\lambda,j}^0 - E_{\lambda,k}^0 = (\varepsilon_j/\hbar + \omega_0\lambda_j) - (\varepsilon_k/\hbar - \omega_0\lambda_k) \gg \alpha$ for most λ , the probability of degenerate levels must be sufficiently small. Thus, we can approximate the integral as

$$\begin{aligned} \int_0^t dt' \bar{\varphi}_{jk}(t') &\approx \int d\Delta_{jk} q_{jk}(\Delta_{jk}) i \frac{e^{-i\omega_0(\lambda_j - \lambda_k)t} - 1}{\omega_0(\bar{\lambda}_j - \bar{\lambda}_k) + (\varepsilon_j/\hbar - \varepsilon_k/\hbar)} \\ &= \frac{i}{E_j^0/\hbar - E_k^0/\hbar} (\bar{\varphi}_{jk}(t) - 1) \\ &= \frac{\bar{\varphi}_{jk}(t) - 1}{i\text{Im}[\Upsilon_{jk}(t)]}. \end{aligned}$$

Inserting these results into the definition Eq. (C2) we obtain

$$\begin{aligned} \gamma_{jk}(t) &\approx i - i\bar{\varphi}_{jk}(t) + i\Upsilon_{jk}(t) \frac{(\bar{\varphi}_{jk}(t) - 1)}{i\text{Im}[\Upsilon_{jk}(t)]} \\ &= (\bar{\varphi}_{jk}(t) - 1) \frac{\text{Re}[\Upsilon_{jk}(t)]}{\text{Im}[\Upsilon_{jk}(t)]}. \end{aligned}$$

For the second rate,

$$\Gamma_{jkrj}(t) = i \left[\frac{\dot{\bar{\varphi}}_{jk}(t)}{\bar{\varphi}_{jk}(t)\bar{\varphi}_{rk}(t)} \varphi_{jk}(t) \int_0^t dt' \varphi_{rj}(t') - \frac{1}{\bar{\varphi}_{rk}(t)} \left(\dot{\varphi}_{jk}(t) \int_0^t dt' \varphi_{rj}(t') \right) \right],$$

we proceed analogously. First, we obtain

$$\overline{\varphi_{jk}(t) \int_0^t dt' \varphi_{rj}(t')} \approx i \frac{\bar{\varphi}_{rk}(t) - \bar{\varphi}_{jk}}{E_j^0/\hbar - E_k^0/\hbar},$$

where we again made use of the fact that levels coupled by the perturbation $\hat{V}$ must have a vanishing probability to be degenerate. Furthermore,

$$\begin{aligned} \overline{\dot{\varphi}_{jk}(t) \int_0^t dt' \varphi_{rj}(t')} &= -\overline{\varphi}_{rk}(t) + \frac{d}{dt} \left(\overline{\varphi}_{jk}(t) \int_0^t dt' \varphi_{rj}(t') \right) \\ &= -\overline{\varphi}_{rk}(t) + i \frac{\dot{\overline{\varphi}}_{rk}(t) - \dot{\overline{\varphi}}_{jk}}{E_j^0/\hbar - E_k^0/\hbar}. \end{aligned}$$

We then obtain

$$\Gamma_{jkrj}(t) \approx \frac{\Upsilon_{jk}(t) - \Upsilon_{rk}(t)}{E_j^0/\hbar - E_k^0/\hbar} + i = \frac{\text{Re}[\Upsilon_{jk}(t)] - \text{Re}[\Upsilon_{rk}(t)]}{E_j^0/\hbar - E_k^0/\hbar}.$$

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