4D beyond-cohomology topological phase protected by C_2 symmetry and its boundary theoriesSheng-Jie Huang ^{*}*Condensed Matter Theory Center and Joint Quantum Institute, Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA*

(Received 7 February 2020; revised 2 July 2020; accepted 27 July 2020; published 11 August 2020)

We study bosonic symmetry-protected topological (SPT) phases with C_2 rotational symmetry in four spatial dimensions which is not captured by the group cohomology classification. By using the topological crystal approach, we show that the topological crystalline state of this SPT phase is given by placing an E_8 state on the two-dimensional rotational invariant plane, which provides a simple physical picture of this phase. Based on this understanding, we show that a variant of QED in four dimensions (QED_4) with charge-1 and charge-3 Dirac fermions is a field theoretical description of the three-dimensional boundary. We also discuss the connection to a symmetric gapped boundary with topological order and its anomalous signature.

DOI: [10.1103/PhysRevResearch.2.033236](https://doi.org/10.1103/PhysRevResearch.2.033236)

I. INTRODUCTION

Symmetry-protected topological (SPT) phases are gapped phases of matter with a unique ground state which can be adiabatically connected to a trivial product state if the symmetry is broken explicitly [1–12]. For bosonic systems with on-site symmetry, a large class of SPT phases are classified by group cohomology [11]. Although group cohomology captures some of the mathematical structure of SPT phases correctly, it has been recognized that the structure of the classification of SPT phases should be described by generalized cohomology theories [13,14]. In particular, there are many “beyond-cohomology” SPT phases which are not captured by group cohomology. These beyond-cohomology SPT phases are not as well understood as in-cohomology SPT phases.

Besides the development of on-site SPT phases, there has been great progress in SPT phases protected by crystalline symmetry [15–42]. Crystalline SPT phases are known to have a simple physical picture given by the “topological crystals,” which are a special class of states formed by real-space crystalline patterns of lower-dimensional topological states [30,33,35,38–41]. Based on the topological crystal approach, most crystalline SPT (cSPT) phases are easier to understand compared to SPT phases with on-site symmetry. Moreover, it has also been found that the classification of cSPT phases with a spatial symmetry group G is the same as the classification of SPT phases with on-site symmetry G , which is known as the crystalline equivalence principle [32]. Sometimes, it is helpful to think about the crystalline counterpart of an on-site SPT phase since the former is usually easier to understand.

Three-dimensional (3D) beyond-cohomology SPT phases with crystalline symmetry have been classified systematically by using the topological crystal approach [35]. Much less is known about the 4D beyond-cohomology SPT phases. For on-site unitary symmetry, one of the simplest beyond-cohomology SPT phases is given in the case of a 4D bosonic system protected by on-site Z_2 symmetry. The response of this phase has been studied at the field theory level [43–46]. Recently, Ref. [47] constructed an exactly solvable model for this phase and studied its quantized response at the Hamiltonian lattice level. The bulk physical picture of this phase is also revealed by its construction: decorating Z_2 domain walls with 3D Walker-Wang models based on the so-called “3-fermion” topological order, which is a variant of toric-code topological order where all the anyonic excitations are fermionic.

In this paper, we are going to study the crystalline counterpart of the 4D beyond-cohomology SPT phase where the on-site Z_2 symmetry is replaced by C_2 rotation. We are going to see that the beyond-cohomology SPT phase with C_2 rotation is much easier to understand by using the topological crystal approach. We then move on to study its 3D boundary theories. We found that one possible boundary field theory is given by a variant of QED in four dimensions (QED_4) with charge-1 and charge-3 Dirac fermions. This proposal is supported by applying a dimensional reduction argument on the boundary. We also consider how to obtain the anomalous boundary topological order from the field theory we obtained [see Refs. [48–51] for related references on constructing anomalous 3 + 1D topological quantum field theories (TQFTs)]. The anomalous boundary topological order is shown to be a 3D Z_2 gauge theory with a fermionic gauge charge. This is consistent with the finding in Ref. [47] for the case of on-site Z_2 symmetry. We further show that one of the anomalous signatures of the 3D gauge theory is revealed in the core of the loop excitation, in which there are gapless modes that are adiabatically connected to the edge modes of the E_8 state. Another anomalous signature is that the C_2 defect loop carries gapless modes that are equivalent to the $c = 4$ $\text{SO}(8)_1$ chiral conformal field theory (CFT).

^{*}shuang20@umd.edu

II. CLASSIFICATION

In this section, we classify bosonic C_2 SPT phases by using the topological crystal approach. Let (x, y, z, w) be the coordinate of the 4D space. We consider a C_2 rotational symmetry which acts on the spatial coordinates by $C_2 : (x, y, z, w) \rightarrow (x, -y, -z, w)$. Since there is no symmetry away from the rotational invariant plane $(x, 0, 0, w)$, the 4D bulk can be extensively trivialized to a product state except on the rotational invariant plane. On this plane, the C_2 rotational symmetry becomes an on-site $\mathbb{Z}_2$ symmetry. This 2D plane could support two possible short-range entangled states. The first possibility is the Ising SPT state [12]. This is the single nontrivial SPT phase protected by an on-site $\mathbb{Z}_2$ symmetry. The other possibility is the E_8 state, which has a $\mathbb{Z}$ classification in 2D [Fig. 1(a)]. We can take the $\mathbb{Z}_2$ symmetry to act trivially on the E_8 root state. From these root states, we obtain a $\mathbb{Z}_2 \times \mathbb{Z}$ classification of 2D phases on the rotational invariant plane.

To obtain the classification of C_2 SPT phases in 4D, we need to consider block-equivalence relations. Let $|\psi_0\rangle$ denote a state on the rotational invariant plane, and we consider bringing in extra degrees of freedom from the trivial regions in the bulk while maintaining the C_2 rotational symmetry. The state on the rotational invariant plane is then modified into $|L\rangle \otimes |\psi_0\rangle \otimes |R\rangle$, where $|L\rangle$ and $|R\rangle$ each denote a 2D “layer” adjoined to the rotational invariant plane. Since this “adjoined layer” operation is an adiabatic process respecting the symmetry, any two states related by the adjoined layer operation are equivalent. More precisely, we have this equivalence relation,

$$|\psi_0\rangle \sim |L\rangle \otimes |\psi_0\rangle \otimes |R\rangle, \quad (1)$$

where C_2 acts by

$$U_{C_2}|L\rangle = |R\rangle, \quad U_{C_2}|R\rangle = |L\rangle. \quad (2)$$

Now suppose there are n_{E_8} copies of E_8 states on the rotational invariant plane. Since $|L\rangle$ and $|R\rangle$ can be E_8 states of the same chirality, the adjoining layers can change the E_8 index of $|\psi_0\rangle$ by ± 2 [Fig. 1(b)]. Therefore, we found that the E_8 index n_{E_8} should only be well defined modulo 2. This result suggests that a state with $n_{E_8} = 2$ could either be the trivial state, or the Ising SPT state. We show this state is trivial in Sec. IV, by analyzing its boundary topological order. Therefore, we obtain a $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification of bosonic C_2 SPT phases in 4D. This classification is also obtained in Ref. [52]. This result is consistent with the TQFT classification of 4D SPT phases

FIG. 1. (a) The projection of the E_8 root state to the 3D boundary. The red line depicts the edge modes of the E_8 state and the arrow shows the chirality. The E_8 edge modes are pinned on the rotational axis due to C_2 rotational symmetry. (b) Adjoined E_8 layers related by C_2 symmetry change the chirality of the E_8 state on the rotational invariant axis.

protected by the on-site $\mathbb{Z}_2$ symmetry [47] as one would expect from the crystalline equivalence principle [32].

III. BOUNDARY FIELD THEORY

Here, we discuss a boundary field theory of the E_8 root state. Our argument starts by considering the 4D E_8 topological crystalline insulator (TCI) with $U(1) \times C_2$ symmetry, which is a strongly interacting electronic SPT phase. In the dimensional reduction picture, the 4D E_8 TCI is described by placing a neutral bosonic E_8 state on the rotational invariant plane, together with a trivial electronic insulator. By gauging the $U(1)$ symmetry and putting the resulting $U(1)$ gauge theory into a confined phase, we eliminate the trivial fermionic sector and obtain a bosonic state with an E_8 state on the rotational invariant plane. We expect the $U(1)$ gauge theory can be confined since it comes from a trivial electronic insulator so that there is no nontrivial θ angle.

We are going to first argue that one possible boundary field theory of the E_8 TCI is a $(3+1)$ D Dirac theory with both charge-1 and charge-3 Dirac fermions. Once we obtain this boundary field theory, we gauge the $U(1)$ symmetry and put the resulting $U(1)$ gauge theory into a confined phase to obtain a theory for the bosonic E_8 root state. This theory is a variant of $N_f = 2$ flavors of QED_4 with charge-1 and charge-3 Dirac fermions.

To obtain the boundary field theory, we are going to consider an alternative description of the E_8 TCI. Starting from an E_8 state with $c = -8$ on the rotational invariant plane, we also bring in a $\nu = 8$ integer quantum Hall (IQH) state on the rotational invariant plane. The resulting state is characterized by $c = 0$ and Hall conductance $\nu = 8$. To see that this procedure leaves the system in the same phase, we need to show that putting the $\nu = 8$ IQH state alone on the rotational invariant plane is in a trivial phase. Let us begin by consider two $\nu = 1$ IQH states on the rotational invariant plane formed by fermions c_1 and c_2 . The C_2 symmetry acts trivially. Then we consider adjoined layers with $|L\rangle$ and $|R\rangle$ being the $\nu = -1$ IQH state with fermions d_L and d_R . C_2 symmetry acts on fermions by exchanging d_L and d_R . We take linear combinations $d_{\pm} = d_R \pm d_L$ with eigenvalue ± 1 under C_2 . We can gap the two IQH states with d_+ and c_2 fermions while preserving C_2 symmetry since d_+ and c_2 fermions both have positive eigenvalues under C_2 and have opposite edge chirality. This leaves a nonchiral state, where the c_1 and d_- fermions have opposite eigenvalues under C_2 and form IQH states of opposite edge chirality. This state is precisely the SPT state with $U(1) \times \mathbb{Z}_2$ symmetry considered in Ref. [21]. It was shown that the classification is $\mathbb{Z}_4$ in the presence of interactions. We thus found that the $\nu = 8$ IQH state on the rotational invariant plane is equivalent to four copies of the $U(1) \times \mathbb{Z}_2$ SPT state, which is in the trivial phase.

We see that the E_8 TCI can be described by placing a state with $c = 0$ and Hall conductance $\nu = 8$. This is equivalent to placing a bosonic integer quantum Hall (BIQH) state on the rotational invariant plane, built from charge-2 Cooper pairs together with a trivial electronic insulator. To proceed, we use the “cluston” construction of the 2D BIQH state [53]. The idea is to consider binding three electrons into cluston bound states, and then putting the clustons into a Chern band. The

Cooper pair BIQH state can be obtained by combining a $\nu = 1$ IQH state of clustons with a $\nu = -1$ IQH state of electrons. The resulting state has the desired boundary signature: quantum Hall conductance $\nu = 8$ and central charge $c = 0$. This picture suggests the following $(3 + 1)$ D boundary field theory,

$$\mathcal{L} = -i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\bar{\psi}_c\gamma^\mu\partial_\mu\psi_c, \quad (3)$$

where ψ is a charge-1 fermion and ψ_c is a charge-3 cluston. We use the following convention for the gamma matrices,

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \tau^3, \quad (4)$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = i\sigma^i\tau^2, \quad (5)$$

and

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \tau^1. \quad (6)$$

The $(3 + 1)$ D field theory Eq. (3) is a generalization of the $(2 + 1)$ D field theories discussed in Refs. [53–55]. We are going to argue that Eq. (3) is the boundary field theory of the 4D E_8 TCI with the following unconventional $\tilde{C}_2$ symmetry,

$$\tilde{C}_2 : \psi(\mathbf{r}) \rightarrow -i\sigma^1\tau^1\psi(R\mathbf{r}), \quad (7)$$

$$\tilde{C}_2 : \psi_c(\mathbf{r}) \rightarrow i\sigma^1\tau^1\psi_c(R\mathbf{r}), \quad (8)$$

where $R\mathbf{r} = (x, -y, -z)$. This unconventional $\tilde{C}_2$ symmetry is a combination of the conventional C_2 symmetry in the Dirac theory, the charge $U(1)$, and the axial $U(1)_A$ symmetry. More precisely, let $u_C = e^{i\pi Q/2}$ and $u_A = e^{i\pi\gamma_5/2}$ be the generators of the $\mathbb{Z}_4$ subgroup of the charge $U(1)$ and axial $U(1)_A$ symmetry, respectively, then $\tilde{C}_2 = C_2 u_C u_A$. It is straightforward to see that Eq. (3) cannot be gapped out by adding spatially uniform mass terms. Note that it is important to include the axial $U(1)_A$ symmetry, otherwise Eq. (3) can be gapped out by a uniform mass term.

Our argument is based on a two-step dimensional reduction procedure. We first add the following mass term,

$$\mathcal{L}_m = m(z)\bar{\psi}\psi + m(z)\bar{\psi}_c\psi_c, \quad (9)$$

where $m(z)$ is real and satisfies $m(z) = -m(-z)$, and $m(z) \rightarrow m_0 > 0$ as $z \rightarrow +\infty$. This term preserves all the symmetries. By solving the fermion zero modes on the mass domain wall, we obtain the gapless fermions on the domain wall that are described by the following $(2 + 1)$ D Dirac theory,

$$\mathcal{L}_D = -i\bar{\chi}\tilde{\gamma}^\mu\partial_\mu\chi - i\bar{\chi}_c\tilde{\gamma}^\mu\partial_\mu\chi_c, \quad (10)$$

with $\tilde{C}_2$ acts by

$$\tilde{C}_2 : \chi(x, y) \rightarrow i\tilde{\gamma}^2\chi(x, -y), \quad (11)$$

$$\tilde{C}_2 : \chi_c(x, y) \rightarrow -i\tilde{\gamma}^2\chi_c(x, -y), \quad (12)$$

where $\tilde{\gamma}^2 = i\sigma^1$.

Next, we add the following mass term to Eq. (10),

$$\mathcal{L}_{Dm} = -m(y)\bar{\chi}\chi + m(y)\bar{\chi}_c\chi_c, \quad (13)$$

where $m(y)$ is a real function with $m(y) = -m(-y)$, and $m(y) \rightarrow m_0 > 0$ as $y \rightarrow +\infty$. This results in a pair of counterpropagating chiral modes on the rotational axis, described

by the Hamiltonian density

$$\mathcal{H}_x = iv_F\phi^\dagger\partial_x\phi - iv_F\phi_c^\dagger\partial_x\phi_c, \quad (14)$$

where ϕ carries charge-1 and ϕ_c carries charge-3. $\tilde{C}_2$ acts trivially on ϕ and ϕ_c . This is exactly the edge theory of a Cooper pair BIQH state discussed in Ref. [53]. It follows that the state on the rotational invariant plane has $\nu = 8$ and $c = 0$. Therefore, Eq. (3) is a boundary field theory for the E_8 TCI.

We can now gauge the $U(1)$ symmetry to obtain a boundary field theory for the bosonic E_8 root state. The gauged Lagrangian has the following form,

$$\mathcal{L}_g = -i\bar{\psi}\gamma^\mu(\partial_\mu + ia_\mu)\psi - i\bar{\psi}_c\gamma^\mu(\partial_\mu + i3a_\mu)\psi_c + \dots \quad (15)$$

IV. BOUNDARY TOPOLOGICAL ORDER

Here, we discuss the boundary topological orders of the E_8 root state. One direct route to enter the topologically ordered state is by Higgsing the $U(1)$ gauge symmetry down to $\mathbb{Z}_2$. This can be achieved by adding the appropriate pairing term to Eq. (15) so that the fermions are in the superconducting state while preserving the C_2 symmetry. The resulting 3D topological order we obtain is a $\mathbb{Z}_2$ gauge theory with a fermionic gauge charge, where the gauge charge is identified as the Bogoliubov quasiparticle [56]. This conclusion is consistent with the case of on-site $\mathbb{Z}_2$ symmetry.

The anomalous nature of this $\mathbb{Z}_2$ gauge theory is hidden in the structure of the loop excitation. To see this, we first notice that the same $\mathbb{Z}_2$ gauge theory can be obtained from Eq. (3) by using the “vortex condensation” argument [57,58]. We start by introducing the same pairing term into the cluston theory. Now the $U(1)$ symmetry is broken and the boundary is in the superconducting state. We would like to restore the $U(1)$ symmetry by proliferating vortices while maintaining the pairing gap. To do so, we need to understand the structure of the vortices in the superconducting state. One possible way to proceed is to solve the vortices explicitly. However, this is not necessary since, in Sec. III, we have shown that Eq. (3) can be dimensionally reduced to a pair of counterpropagating chiral modes, described by Eq. (14), on the rotational axis, and the procedure for constructing a vortex in the superconducting state is essentially the same as a procedure for the two-step dimensional reduction. We expect that the results of two different dimensional reduction procedures are adiabatically connected. Therefore, the gapless modes in the core of the vortex must be equivalent to the charge-1–charge-3 helical modes described by Eq. (14) in order to match the bulk invariant. We see that there is an obstruction to enter a symmetry-preserving gapped state by proliferating vortices due to the presence of these helical gapless modes in the cores of vortices.

The $\mathbb{Z}_2$ classification of the E_8 root state would imply that the gapless modes in the twofold vortex loops can be gapped out while preserving the symmetry. To see the classification is indeed $\mathbb{Z}_2$, we use the following trick. We start from the 3D boundary of the E_8 root state with the chiral edge modes of the E_8 state on the x axis. Then, we compress the system in the z direction to obtain a 2D system and the C_2 rotation becomes reflection symmetry. This 2D system is now essentially the same as the 2D surface of a 3D reflection SPT state with

an E_8 on the mirror plane. It has been shown in Ref. [30] that stacking two copies of such surfaces results in a trivially gapped state. This suggests that the classification is indeed $\mathbb{Z}_2$. We can therefore condense twofold vortices to produce an insulating state and restore the $U(1)$ symmetry. The resulting insulating state is a Z_2 gauge theory with a fermionic gauge charge tensor with a trivial fermion, where the $U(1)$ symmetry only acts on the trivial fermion. We can now gauge the $U(1)$ symmetry and put the resulting gauge theory into a $U(1)$ confined phase such that the trivial fermions are excluded from the excitation spectrum and only the Z_2 gauge theory remains. We expect such confinement can be achieved since there is no nontrivial θ term after gauging the $U(1)$ symmetry. Following the above reasoning, we see that there are helical gapless modes, consisting of charge-1 and charge-3 fermions described by Eq. (14), that are equivalent to the gapless modes of the E_8 edge state in the core of the loop excitation, which is the descendant of the fundamental vortex. This is one of the anomalous signatures of this Z_2 gauge theory.

From Eqs. (7) and (8) we see that C_2 squares to -1 on the gauge charge since it is identified as the Bogoliubov quasiparticle. This property of the gauge charge can also be seen from the compression argument. Again, we compress the system in the z direction to obtain a 2D system, on which the C_2 rotation becomes the reflection symmetry, and we obtain a surface of a 3D reflection SPT state with an E_8 on the mirror plane. Reference [30] also shows that the surface topological order of a 3D reflection SPT state built from an E_8 state is a 3-fermion Z_2 gauge theory with an $e_f P m_f P$ symmetry fractionalization pattern. This means that the reflection symmetry $M^2 = -1$ on the gauge charge and gauge flux. When we compress our 3D Z_2 gauge theory, the resulting 2D Z_2 gauge theory is precisely the $e_f P m_f P$ state. In particular, the gauge charge in the 3D Z_2 gauge theory just becomes the gauge charge in the $e_f P m_f P$ state. Therefore, the gauge charge in our 3D Z_2 gauge theory must carry half C_2 charge. In fact, this compression argument also shows that the 3D Z_2 gauge theory must be anomalous since the resulting state of compression—the $e_f P m_f P$ state—is anomalous.

The fact that the gauge charge carries half C_2 charge has a dramatic consequence on the C_2 symmetry-twisted defect loop Ω , which is as a C_2 disclination in a solid. The defining property of a C_2 defect loop is that, when a gauge charge braids with the defect loop, the gauge charge is transformed by the C_2 rotation. Now consider a gauge charge braids with a two-defect loop $\Omega \times \Omega$, and this process implements a C_2^2 transformation on the gauge charge. Since C_2 squares to -1 on the gauge charge, we see that the result of braiding between the gauge charge e and two defect loops $\Omega \times \Omega$ is a pure phase, $\Theta_{e, \Omega \times \Omega} = -1$, which is the same as the braiding phase between the gauge charge e and the gauge flux loop m , $\Theta_{e, m} = -1$. From the principle of remote detectability, we conclude that the defect loop Ω must satisfy the fusion rule $\Omega \times \Omega = m$, where m represents the flux loop excitation in the gauge theory. Since we have shown that the loop excitation m carries gapless modes that are equivalent to the E_8 edges, the C_2 defect loop Ω should carry gapless modes that are equivalent to a $c = 4$ $SO(8)_1$ chiral CFT. This is another anomalous signature of this Z_2 gauge theory.

V. DISCUSSION

In this paper, we classified 4D bosonic C_2 SPT phases by using the topological crystal approach. The classification is found to be $\mathbb{Z}_2 \times \mathbb{Z}_2$. This classification is consistent with the case of on-site Z_2 symmetry as expected from the crystalline equivalence principal. One of the $\mathbb{Z}_2$ root states is understood as having a Ising SPT state on the 2D C_2 invariant plane. The other $\mathbb{Z}_2$ root state is given by having an E_8 state on the rotational invariant plane. This state is beyond cohomology since the building block itself is not an SPT state classified by group cohomology.

Focusing on this E_8 root state, we consider its boundary field theories. We found a variant of QED_4 with single charge-1 and single charge-3 Dirac fermions with the C_2 symmetry action defined in Eqs. (7) and (8) is a candidate boundary field theory by using the dimensional reduction argument. This field theory is inspired by its $(2+1)$ D version based on the “cluston” construction, introduced in Ref. [53] for various SPT phases with time-reversal symmetry in three spatial dimensions. We show that its $(3+1)$ D generalization can describe the boundary of 4D bosonic C_2 SPT phases built by placing an E_8 state on the rotational invariant plane.

We further consider how to obtain a gapped topologically ordered state from the field theory. The topological order we obtain is a $(3+1)$ D Z_2 gauge theory with a fermionic gauge charge. One of the anomalous signatures of this Z_2 gauge theory is shown in the core of the loop excitation—there are gapless modes that carry the same anomaly as the edge modes of the E_8 state in the core of the loop excitation. Another anomalous signature is that the C_2 defect loop carries gapless modes that are equivalent to a $c = 4$ $SO(8)_1$ chiral CFT.

Although the 4D SPT phases themselves are unrealistic, it is crucial for understanding the anomalies of $(3+1)$ D field theories. The anomaly of the beyond-cohomology state has been studied from field-theoretic perspectives [48,49,59,60]. It is plausible that the anomalous $(3+1)$ D topological orders of this state can be constructed by the symmetry-extension method [61]. The QED_4 with a charge-1 and a charge-3 Dirac fermions might also describe the beyond-cohomology state with on-site Z_2 symmetry. It would be interesting to show this explicitly by using a more traditional field theory analysis. In general, it would be interesting to have a more detailed understanding of 4D SPT phases with on-site and/or crystalline symmetry.

It is straightforward to see that the same E_8 root state exists if the symmetry is C_n rotation since there is still a 2D C_n invariant plane which can support the E_8 state. Indeed, if we consider the case with on-site Z_2 symmetry, the beyond-cohomology state is the descendant of the one with $U(1)$ symmetry. The similar beyond-cohomology phases exist if the symmetry is broken down to its Z_n subgroup. The classification was found to be $\mathbb{Z}$ for $U(1)$ symmetry and is $\mathbb{Z}_n$ for Z_n symmetry in Ref. [44]. We expect the same classification for the SPT phases with C_n rotation based on the crystalline equivalence principal. The boundary field theories are presumably similar to what we discussed in this paper while the existence of symmetric boundary topologically ordered states is less clear. We leave detailed considerations of these problems to future work.

ACKNOWLEDGMENTS

I am grateful to Meng Cheng for illuminating discussions on the property of the C_2 defect loop in the anomalous Z_2 gauge theory, to Chong Wang for discussions on the gapless modes in the vortex core, and to Maissam Barkeshli, Lukasz Fidkowski, and Cenke Xu for useful correspondence. I acknowledge support from a JQI postdoctoral fellowship and the Laboratory for Physical Sciences.

-
- [1] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, *Phys. Rev. B* **78**, 195125 (2008).
- [2] A. Kitaev, Periodic table for topological insulators and superconductors, in *Advances in Theoretical Physics: Landau Memorial Conference*, edited by V. Lebedev and M. Feigel'man, AIP Conf. Proc. Vol. 1134 (AIP, Melville, NY, 2009), p. 22.
- [3] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, *New J. Phys.* **12**, 065010 (2010).
- [4] Z.-C. Gu and X.-G. Wen, *Phys. Rev. B* **80**, 155131 (2009).
- [5] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, *Phys. Rev. B* **81**, 064439 (2010).
- [6] L. Fidkowski and A. Kitaev, *Phys. Rev. B* **83**, 075103 (2011).
- [7] A. M. Turner, F. Pollmann, and E. Berg, *Phys. Rev. B* **83**, 075102 (2011).
- [8] X. Chen, Z.-C. Gu, and X.-G. Wen, *Phys. Rev. B* **83**, 035107 (2011).
- [9] X. Chen, Z.-C. Gu, and X.-G. Wen, *Phys. Rev. B* **84**, 235128 (2011).
- [10] N. Schuch, D. Pérez-García, and I. Cirac, *Phys. Rev. B* **84**, 165139 (2011).
- [11] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Phys. Rev. B* **87**, 155114 (2013).
- [12] M. Levin and Z.-C. Gu, *Phys. Rev. B* **86**, 115109 (2012).
- [13] C. Z. Xiong, *J. Phys. A: Math. Theor.* **51**, 445001 (2018).
- [14] D. Gaiotto and T. Johnson-Freyd, *J. High Energy Phys.* **05** (2019) 007.
- [15] L. Fu, *Phys. Rev. Lett.* **106**, 106802 (2011).
- [16] T. H. Hsieh, H. Lin, J. Liu, W. Duan, A. Bansil, and L. Fu, *Nat. Commun.* **3**, 982 (2012).
- [17] Y. Ando and L. Fu, *Annu. Rev. Condens. Matter Phys.* **6**, 361 (2015).
- [18] C.-K. Chiu, H. Yao, and S. Ryu, *Phys. Rev. B* **88**, 075142 (2013).
- [19] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, *Rev. Mod. Phys.* **88**, 035005 (2016).
- [20] K. Shiozaki and M. Sato, *Phys. Rev. B* **90**, 165114 (2014).
- [21] H. Isobe and L. Fu, *Phys. Rev. B* **92**, 081304(R) (2015).
- [22] C. Fang and L. Fu, *Phys. Rev. B* **91**, 161105(R) (2015).
- [23] K. Shiozaki, M. Sato, and K. Gomi, *Phys. Rev. B* **91**, 155120 (2015).
- [24] T. Yoshida, T. Morimoto, and A. Furusaki, *Phys. Rev. B* **92**, 245122 (2015).
- [25] Z. Wang, A. Alexandradinata, R. J. Cava, and B. A. Bernevig, *Nature (London)* **532**, 189 (2016).
- [26] C. Fang and L. Fu, *Sci. Adv.* **5**, eaat2374 (2019).
- [27] H. C. Po, A. Vishwanath, and H. Watanabe, *Nat. Commun.* **8**, 50 (2017).
- [28] B. Bradlyn, L. Elcoro, J. Cano, M. G. Vergniory, Z. Wang, C. Felser, M. I. Aroyo, and B. A. Bernevig, *Nature (London)* **547**, 298 (2017).
- [29] J. Kruthoff, J. de Boer, J. van Wezel, C. L. Kane, and R.-J. Slager, *Phys. Rev. X* **7**, 041069 (2017).
- [30] H. Song, S.-J. Huang, L. Fu, and M. Hermele, *Phys. Rev. X* **7**, 011020 (2017).
- [31] S. Jiang and Y. Ran, *Phys. Rev. B* **95**, 125107 (2017).
- [32] R. Thorngren and D. V. Else, *Phys. Rev. X* **8**, 011040 (2018).
- [33] S.-J. Huang, H. Song, Y.-P. Huang, and M. Hermele, *Phys. Rev. B* **96**, 205106 (2017).
- [34] S.-J. Huang and M. Hermele, *Phys. Rev. B* **97**, 075145 (2018).
- [35] H. Song, C. Zhaoxi Xiong, and S.-J. Huang, *Phys. Rev. B* **101**, 165129 (2020).
- [36] M. Cheng and C. Wang, *arXiv:1810.12308*.
- [37] J.-H. Zhang, Q.-R. Wang, S. Yang, Y. Qi, and Z.-C. Gu, *Phys. Rev. B* **101**, 100501 (2020).
- [38] K. Shiozaki, C. Zhaoxi Xiong, and K. Gomi, *arXiv:1810.00801*.
- [39] D. V. Else and R. Thorngren, *Phys. Rev. B* **99**, 115116 (2019).
- [40] Z. Song, C. Fang, and Y. Qi, *arXiv:1810.11013*.
- [41] Z. Song, S.-J. Huang, Y. Qi, C. Fang, and M. Hermele, *Sci. Adv.* **5**, eaax2007 (2019).
- [42] M. Guo, K. Ohmori, P. Putrov, Z. Wan, and J. Wang, *Commun. Math. Phys.* **376**, 1073 (2020).
- [43] A. Kapustin, *arXiv:1403.1467*.
- [44] X.-G. Wen, *Phys. Rev. B* **91**, 205101 (2015).
- [45] J. C. Wang, Z.-C. Gu, and X.-G. Wen, *Phys. Rev. Lett.* **114**, 031601 (2015).
- [46] D. S. Freed and M. J. Hopkins, *arXiv:1604.06527*.
- [47] L. Fidkowski, J. Haah, and M. B. Hastings, *Phys. Rev. B* **101**, 155124 (2020).
- [48] C. Cordova and T. T. Dumitrescu, *arXiv:1806.09592*.
- [49] Z. Wan and J. Wang, *Phys. Rev. D* **99**, 065013 (2019).
- [50] Z. Wan, J. Wang, and Y. Zheng, *Phys. Rev. D* **100**, 085012 (2019).
- [51] C. Cordova and K. Ohmori, *arXiv:1910.04962*.
- [52] R. Kobayashi and K. Shiozaki, *arXiv:1901.06195*.
- [53] C. Wang, *Phys. Rev. B* **91**, 245124 (2015).
- [54] N. Seiberg and E. Witten, *Prog. Theor. Exp. Phys.* **2016**, 12C101 (2016).
- [55] M. Cheng and C. Xu, *Phys. Rev. B* **94**, 214415 (2016).
- [56] Here is another heuristic argument to see why the gauge charge should be fermionic. This argument is similar in spirit to the argument in Ref. [47] for the case with on-site Z_2 symmetry. We start from the E_8 root state with the chiral edge modes of the E_8 state running on the rotational axis, which we choose to be the x axis. Notice that, on the x - y plane, the C_2 rotation becomes the reflection symmetry. Now we introduce a pair of 3-fermion topological orders on the x - y plane, related by C_2 rotation. We choose the edge chirality of the 3-fermion topological order edges to be opposite to the E_8 edge. The result in Ref. [30] implies that, on the surface of a 3D reflection SPT state, the

chiral edge modes of the 3-fermion topological order and the chiral edge modes of the E_8 state with opposite chirality can be gapped out while preserving the reflection symmetry. Focusing on the x - y plane with C_2 symmetry, we have essentially the same setting. Therefore, we obtain a state with a 3-fermion topological order on the x - y plane. To produce an isotropic 3D topologically ordered state, we suppose that the 3D bulk is the Z_2 gauge theory with a fermionic gauge charge. The 3-fermion topological order on the x - y plane can then be made trivial by condensing the bound state of the fermionic gauge charge and one of the fermionic anyons in the 3-fermion

topological order. This results in the desired Z_2 gauge theory with a fermionic gauge charge in the bulk.

- [57] C. Wang, A. C. Potter, and T. Senthil, [Phys. Rev. B](#) **88**, 115137 (2013).
- [58] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, [Phys. Rev. B](#) **92**, 125111 (2015).
- [59] R. Thorngren, [J. High Energy Phys.](#) **02** (2015) 152.
- [60] J. Wang, X.-G. Wen, and E. Witten, [J. Math. Phys.](#) **60**, 052301 (2019).
- [61] J. Wang, X.-G. Wen, and E. Witten, [Phys. Rev. X](#) **8**, 031048 (2018).