Emergent localized states at the interface of a twofold $\mathcal{PT}$ -symmetric latticeJung-Wan Ryu,¹ Nojoon Myoung,² Ara Go,¹ Sungjong Woo,^{1,*} Sang-Jun Choi,^{1,3,†} and Hee Chul Park^{1,‡}¹Center for Theoretical Physics of Complex Systems, Institute for Basic Science (IBS), Daejeon 34126, South Korea²Department of Physics Education, Chosun University, Gwangju 61452, Republic of Korea³Institute for Theoretical Physics and Astrophysics, University of Würzburg, D-97074 Würzburg, Germany

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We consider the role of nontriviality resulting from a non-Hermitian Hamiltonian without band topology that conserves twofold $\mathcal{PT}$ symmetry assembled by interconnections between a $\mathcal{PT}$ -symmetric lattice and its time-reversal partner. Twofold $\mathcal{PT}$ symmetry in the lattice produces additional surface exceptional points that play the role of new critical points, along with the bulk exceptional point. We show that there are two distinct regimes possessing symmetry-protected localized states of which localization lengths are robust against non-Hermiticity parameter. The states are demonstrated by numerical calculation of a quasi-one-dimensional ladder lattice and a two-dimensional bilayered square lattice.

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I. INTRODUCTION

A non-Hermitian system ($H^\dagger \neq H$) with parity-time ($\mathcal{PT}$) symmetry exhibits a phase transition through spontaneous symmetry breaking from an unbroken $\mathcal{PT}$ -symmetric phase with real eigenenergies to a broken phase with pairs of conjugate complex eigenenergies [1]. In a non-Hermitian system, non-Hermiticity is developed from imaginary phase accumulation or from imaginary potential due to an imbalance of particle and/or energy flow and nonlinearity [2–6]. $\mathcal{PT}$ symmetry can be protected in non-Hermitian systems by balancing the energy gain and loss, i.e., $[H, \mathcal{PT}] = 0$. Here, H is the Hamiltonian and $\mathcal{P}$ and $\mathcal{T}$ are the parity and time-reversal operators, respectively. A wide range of $\mathcal{PT}$ -symmetric systems have been explored over several fields, including optics [4,5,7–11], electronic circuits [12], atomic physics [13], and magnetic metamaterials [14]. Phase transitions in these systems occur via exceptional points (EPs), which are degenerate points of eigenenergies in non-Hermitian systems that generate a Möbius strip structure of eigenenergies in parametric space [15,16]. Such a topological structure has been reported in microwave experiments [17,18], optical microcavities [19], and a chaotic exciton-polariton billiard [20].

In non-Hermitian systems, states localized at the edges, interfaces, and defects have recently attracted considerable attention, not only in fundamental studies such as topology and symmetry but also in applications to quantum technologies. Non-Hermitian flat bands generate localized zero

modes with the real part of complex eigenenergies zero [21,22], analogous to Majorana zero modes in condensed-matter physics. Such non-Hermitian zero modes (NHZMs) are protected by non-Hermitian particle-hole (NHPH) symmetry, which is also called charge-conjugate symmetry; $\epsilon_i = -\epsilon_j^*$ with i and j representing NHPH symmetry partners so that $\text{Re}(\epsilon_i) = 0$ if $i = j$ (NHZM) [23,24]. It has been shown that NHPH symmetry generates topological defect modes at the interface between non-Hermitian lattices which, without non-Hermiticity, would be topologically trivial [25,26]. It is also possible to get NHZMs in a non-Hermitian lattice, which is topologically nontrivial even without non-Hermiticity, such as the Su-Schrieffer-Heeger model [27–33]. In addition to NHZMs, non-Hermitian bound states (NHBSs) are protected by $\mathcal{PT}$ symmetry, in which case $\epsilon_i = \epsilon_j^*$ with $\text{Im}(\epsilon_i) = 0$ if $i = j$ (NHBS) [34,35]. Anomalous localized edge states in non-Hermitian lattice models have been attributed to a winding number around an EP in momentum space [36–40]; localized states also exist at the interface between two lattices with the same topological order but with distinct quantum phases, such as unbroken and broken $\mathcal{PT}$ -symmetric phases [41,42].

A $\mathcal{PT}$ -symmetric system can be realized by combining the even parity of the real potential and the odd parity of the imaginary potential with respect to the $\mathcal{PT}$ -symmetric axis. Pairing such a $\mathcal{PT}$ -symmetric system with its time-reversal partner makes the $\mathcal{PT}$ symmetry twofold such that the whole system has two different symmetric axes and guarantees two EPs. Such multiple EPs by multifold $\mathcal{PT}$ symmetry as well as single EPs by simple $\mathcal{PT}$ symmetry have been studied both theoretically and experimentally [43–45]. In this work we introduce a lattice containing twofold $\mathcal{PT}$ symmetry to study how interplay between non-Hermiticity and bulk symmetry affects the wave functions of the lattice. We find two pairs of interface states that decay exponentially in the space distinguished by the bulk states. Particularly, we focus on robust localized states of which localization length is unaffected by variation of external gain/loss in two distinct regimes between

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FIG. 1. (a) Sketch of a $\mathcal{PT}$ -symmetric ladder lattice with gain (red) and loss (blue) on the upper and lower lattices, respectively. (b) Real (black solid line) and imaginary (red dashed line) parts of the complex energy bands of a $\mathcal{PT}$ -symmetric ladder lattice for $\gamma < \gamma_b$ (left), $\gamma = \gamma_b$ (middle), and $\gamma > \gamma_b$ (right). The parameters are $d = 3$ and $t = 1$. (c) Sketch of the intertwined $\mathcal{PT}$ -symmetric ladder lattice considered in this work.

the bulk EP and surface EPs. Furthermore, using these robust localized states, we design a lossless one-dimensional (1D) waveguide whose output power to the external environment vanishes from various geometrical deformations.

II. SINGLE $\mathcal{PT}$ SYMMETRIC LADDER

Let us prepare the $\mathcal{PT}$ -symmetric ladder lattice with a basis of two sites [Fig. 1(a)]. Non-Hermiticity is adopted by respective gain and loss at the two sites in the unit cell, a balanced amount of which preserves $\mathcal{PT}$ symmetry. A series of such unit cells forms a ladder lattice with intercell hopping. The lattice Hamiltonian for this system reads $\hat{H}_0 = \sum_{l=-\infty}^{\infty} \hat{h}_l$, where

$$\hat{h}_l = c_l^\dagger h c_l + c_l^\dagger T^\dagger c_{l+1} + c_{l+1}^\dagger T c_l \quad (1)$$

and $c_l^\dagger \equiv (|l, A\rangle, |l, B\rangle)$. The matrix $h \equiv i\gamma\sigma_z - d\sigma_x$ describes the non-Hermiticity with γ and the intracell hopping with $-d$, while $T \equiv -t\sigma_0$ describes the intercell hopping where $\sigma_{x,z}$ are the Pauli matrices and σ_0 is the 2×2 identity matrix. $\mathcal{PT}$ symmetry is identified by $[\mathcal{PT}, H_0] = 0$, with the combination of parity operator $\mathcal{P} = \sum_{l=-\infty}^{\infty} c_l^\dagger \sigma_x c_l$ and time-reversal operator $\mathcal{T} = \mathcal{K}$, and the NHPH symmetry is expressed by $\{\Xi, H_0\} = 0$ with the antiunitary operator $\Xi = \sum_{l=-\infty}^{\infty} e^{il\pi} c_l^\dagger \sigma_z c_l \mathcal{K}$, where $\mathcal{K}$ is complex conjugation.

The matrix representation of the Hamiltonian for a single ladder lattice in momentum space is written as $H_0(q) = h - 2t \cos q \sigma_0$ through the Bloch wave function $|\psi(q)\rangle \equiv \sum_n e^{inq} c_n^\dagger$, where q is the momentum vector. The (non-normalized) eigenstates are $\psi_\eta(q) = (-i\gamma/2 - \eta\sqrt{d^2 - (\gamma/2)^2}, d)^T$, which are independent of q , where $\eta = \pm 1$ corresponds to the upper and lower bands. The momentum-independent eigenstates imply that the Berry con-

FIG. 2. (a) Real and (b) imaginary parts of the complex eigenenergies of Fig. 1(a) as a function of γ . γ_b is the exceptional point related to the $\mathcal{PT}$ -symmetric phase transition. (c) Real and (d) imaginary parts of the complex eigenenergies of Fig. 1(c) as a function of γ . The gray shaded region represents the bulk-state eigenenergies, and the red and blue lines indicate the eigenenergies of the interface states. γ_b is the bulk exceptional point, and $\gamma_{\pm}$ are the surface exceptional points for the interface states. The parameters are $d = 3$ and $t = 1$, and the number of unit cells is $N = 400$.

nection in the system is zero so that it is trivial from the viewpoint of band topology. It should be noted that bulk states do not have any kind of nontriviality originating from topological invariants, which are well defined in momentum space. The corresponding eigenvalues are $E_\eta(q) = -2t \cos q + \eta\sqrt{d^2 - (\gamma/2)^2}$, which are complex energy bands as $E(k) = E_r(k) + iE_i(k)$ according to different γ in Fig. 1(b). One may notice that there exists a phase transition between unbroken and broken $\mathcal{PT}$ -symmetric phases at the EP, $\gamma = \gamma_b \equiv 2d$. At the EP, two complex energy bands merge into a single energy band, as shown in middle panel of Fig. 1(b). It is important to note that the two bands are separable at any value of γ except the EP since there is no band crossing for any q [46].

III. TWOFOLD $\mathcal{PT}$ -SYMMETRIC LADDER

We have numerically calculated the complex spectrum for a finite-sized $\mathcal{PT}$ -symmetric ladder lattice with 400 unit cells as a function of γ [Figs. 2(a) and 2(b)]. Plots clearly show a collective phase transition at $\gamma = \gamma_b$; for $\gamma < \gamma_b$, a pair of real energy bands are formed, attracting each other as γ increases up to the EP [Fig. 2(a)]. This is in sharp contrast to the Hermitian case where the real energy bands repel each other. For $\gamma > \gamma_b$, meanwhile, a pair of complex conjugate energy bands are formed, repelling each other as γ increases from the EP [Fig. 2(b)]. We will call γ_b a bulk EP to distinguish it from the additional EPs due to the twofold $\mathcal{PT}$ symmetry.

Now, a twofold $\mathcal{PT}$ -symmetric ladder lattice can be formed by attaching the edge of the $\mathcal{PT}$ -symmetric lattice to the edge of its time-reversal partner, creating an interface [Fig. 1(c)]. The Hamiltonian operator of such an intertwined

FIG. 3. (a) Localization lengths of odd (blue line) and even (red line) interface states using the same parameters as Fig. 2. The NHZMs and NHBSs are robust against γ in the blue and red shaded regimes II and III, respectively. The orange and green lines represent the analytic solutions of localization lengths of the odd and even interface states, respectively. (b), (c) Phase diagrams for the localization lengths of the odd and even states, respectively. Blue solid, black solid, and blue dashed lines in (b) indicate γ_- , γ_b , and γ_{-b} , respectively. Black solid, red solid, and red dashed lines in (c) indicate γ_b , γ_+ , and γ_{+b} , respectively. No localized states exist in the white areas. The black dotted lines indicate the localization lengths shown in (a).

lattice consists of

$$\hat{H} = \hat{H}_L + \hat{H}_R + \hat{H}_c, \quad (2)$$

where $\hat{H}_L = \sum_{l=-\infty}^{-1} \hat{h}_l$ is a Hamiltonian for the left semi-infinite non-Hermitian lattice, and $\hat{H}_R = \mathcal{T} \sum_{l=0}^{\infty} \hat{h}_l \mathcal{T}^{-1}$ is a Hamiltonian for the right semi-infinite lattice with complex conjugate operator $\mathcal{T}$ as a time-reversal operator. A coupling Hamiltonian $\hat{H}_c = c_0^\dagger T c_{-1} + \text{H.c.}$ connects the two time-reversal partners. The additional $\mathcal{PT}$ symmetry from the intertwinement introduces additional EPs while preserving the NHPH symmetry. As shown in Figs. 2(c) and 2(d), the four emergent states (red/blue lines), which are clearly separated from the bulk spectrum, show a $\mathcal{PT}$ -symmetric phase transition, while the bulk spectrum is the same as that of the simple $\mathcal{PT}$ -symmetric lattice.

The emergent states are exponentially localized at the interface, with localization lengths as defined by the exponential decay of the wave functions $\psi \sim e^{-x/\lambda}$ as a function of γ shown in Fig. 3(a). By analyzing the energy eigenstates from our numerical calculations, we have confirmed that there are two types of localized interface states: those that conserve NHPH symmetry as NHZMs and those that conserve $\mathcal{PT}$ symmetry as NHBSs. It is important to note that the localization lengths of the localized NHZMs in the blue-shaded region, $\gamma_- < \gamma < \gamma_b$, and those of the localized NHBSs in the

red-shaded region, $\gamma_b < \gamma < \gamma_+$, are independent of the non-Hermiticity control parameter γ [Fig. 3(a)]. Such robustness of localization length is related to the interplay of the opposite $\mathcal{PT}$ -symmetric phases between interface and bulk states. The localized NHZMs with constant localization length appear when the bulk and interface states have unbroken and broken phases, respectively, while the localized NHBSs with constant localization length appear when the bulk and interface states have broken and unbroken phases, respectively. The rest of the interface states have the same unbroken or broken phases as the bulk states. The phase diagrams in Figs. 3(b) and 3(c) show the localization lengths of the interface states as a function of intracell hopping d and non-Hermiticity parameter γ , respectively. The colored areas correspond to the localized states while the white areas indicate a forbidden regime. It is noted that if $d < 2t$, NHZMs with robust localization lengths are not allowed for any value of γ . We also note that, while invisible in the energy spectra in Figs. 2(c) and 2(d), phase transitions appear for the localized states at the bulk EP, as shown in Fig. 3(a).

The interface states can also be obtained analytically through the Schrödinger equation $H|\Phi\rangle = E|\Phi\rangle$. Let us consider a linear combination of the simple ladder lattice eigenstates to solve the intertwined lattice. The ansatz of the wave function in the system is $|\Phi\rangle = |\Phi\rangle_L + |\Phi\rangle_R = \sum_{\eta=\pm} \{\alpha_\eta |\psi(q_\eta)\rangle_L + \beta_\eta |\psi(-q_\eta)\rangle_R\}$. The matching condition of the counterpropagating wave functions at the interface is

$$2d^2 \sin q_+ \sin q_- + \left(\frac{\gamma}{2}\right)^2 \cos(q_+ + q_-) = \left(\frac{\gamma}{2}\right)^2, \quad (3)$$

where $\cos q_{\pm} = \frac{-E \pm \sqrt{d^2 - (\gamma/2)^2}}{2}$ is the momentum vector corresponding to each energy band, with all energy scales being dimensionless by t from here. This condition provides exact solutions for the interface states as

$$E_{\zeta,\pm} = \pm \sqrt{\left(1 + \zeta \frac{2}{d}\right) \left[d^2 + \zeta 2d - \left(\frac{\gamma}{2}\right)^2\right]}, \quad (4)$$

which are eigenenergies of the two pairs of states separated from the bulk bands, where the index $\zeta = \pm 1 (ev/od)$ indicates the even/odd interface states as the red/blue curves in Figs. 2(c) and 2(d), respectively. It is easy to see that the two surface EPs are $\gamma_{\pm} = 2\sqrt{d^2 \pm 2d}$ and that the interface states undergo phase transitions from unbroken to broken $\mathcal{PT}$ -symmetric phases. There is no interface state in the Hermitian limit with $\gamma = 0$, i.e., the localization length diverges, and the interface states with real eigenenergies separate from the bulk states as γ increases. The even and odd interface states finally merge into the bulk states at $\gamma_{+b} = 2\sqrt{2}\sqrt{d^2 + d}$ and $\gamma_{-b} = 2\sqrt{2}\sqrt{d^2 - d}$, respectively.

To analyze the characteristics of the emergent interface states, we introduce complex momentum vectors of the states as $q_{\zeta,\pm} = k_{\zeta,\pm} + i\kappa_{\zeta,\pm}$, where $k_{\zeta,\pm}$ and $\kappa_{\zeta,\pm}$ are real. The dispersion relation is generalized by complex momentum $\cos(k + i\kappa) = \cos k \cosh \kappa - i \sin k \sinh \kappa$. By substituting the energies of the interface states as found in Eq. (4) for the energy in the dispersion relation,

$$\cos q_{\zeta,\pm} = \frac{-E_{\zeta,\pm} \pm \sqrt{d^2 - (\gamma/2)^2}}{2}, \quad (5)$$

we can formulate localization lengths as $\lambda^\pm = 1/\kappa_{\zeta,\pm}$ at the interfaces of the intertwined lattice with respect to the distinct regimes of control parameter γ . For the NHZMs, we can evaluate the inverse localization length and wave number for oscillation as follows:

$$\tanh \kappa_{\text{od},\pm} = \sqrt{1 - \frac{2}{d}}, \quad \tan k_{\text{od},\pm} = \sqrt{\frac{2/d}{1 - (\gamma/2d)^2} - 1}. \quad (6)$$

In the case of the NHBSs, the inverse localization length and wave number of the localized states are

$$\coth \kappa_{\text{ev},\pm} = \sqrt{1 + \frac{2}{d}}, \quad \cot k_{\text{ev},\pm} = \sqrt{\frac{2/d}{(\gamma/2d)^2 - 1} - 1}. \quad (7)$$

The localization lengths $1/\kappa_{\zeta,\pm}$ from Eqs. (6) and (7) agree well with numerical results, as shown in Fig. 3(a).

In general, the overlap of wave functions modifies physical properties such as resonant energy and spatial decaying length. The energies of our interface states are determined by matching the boundary conditions with complex momentum $q = k + i\kappa$, of which the real and imaginary parts correspond to the resonant energy and spatial decaying length, respectively. The momentum is determined by Eqs. (4) and (5), which are related to the energy balance between the bulk and interface states. There are two phase transitions of interface states at the bulk and surface EPs through the momentum, like the energy. If an interface state with a $\mathcal{PT}$ -symmetric phase distinct from that of bulk states lies within the bulk-energy gap, the imaginary momentum becomes independent of non-Hermiticity with varying real momentum. This means that the localization length due to wave-function overlap is invariant against the change of an external physical parameter. In contrast, when the phases of both states are the same, the real momentum is fixed to be either $k = 0$ or $k = \pi/2$ with varying the imaginary momentum as the non-Hermiticity parameter changes. This is consistent with the fact that the localization length of a defect state is determined by the uncertainty relation ($\lambda \propto 1/|\Delta E|$), where ΔE is the magnitude of its energy difference from the bulk states. This implies that a $\mathcal{PT}$ -symmetric domain wall introduces phase transitions in the interface states, thereby protecting the minimum localization length under symmetry through the relation between the $\mathcal{PT}$ phases of the bulk and interface states.

IV. EXTENSION TO A TWO-DIMENSIONAL (2D) LATTICE

We now propose a two-dimensional (2D) lattice in which a lossless 1D waveguide is realizable by repeating 1D twofold $\mathcal{PT}$ -symmetric ladder lattices. For example, a straight waveguide is considered on the 2D lattice as the interface along the y axis, as shown in the lower inset of Fig. 4(a). This 2D lattice satisfies both translational symmetry along the y axis and twofold $\mathcal{PT}$ symmetry of each 1D ladder lattice. Figure 4(a) plots the imaginary parts of the eigenenergies as a function of γ in the 2D bilayered square lattice with 40×40 unit cells, where it is apparent that the imaginary eigenenergies are broadened when compared to those of the 1D lattice in Fig. 2(d) because of finite-size effects. As shown

FIG. 4. (a) Imaginary parts of the complex eigenenergies as a function of γ in the 2D ladder lattice. The upper inset plots the eigenenergies on the complex plane for $\gamma = 7.6$, and the lower inset illustrates the 2D lattice with a 1D straight interface along the y axis. (b) Intensity distribution of a selected lossless NHBS for $\gamma = 7.6$. (c) Top view of the 2D lattice schematic with an arbitrarily shaped closed interface, with a cut view showing the interface. The upper and lower layers have opposite profiles of gain and loss. (d) Intensity distribution of a selected lossless NHBS for $\gamma = 6.5$. The parameters in this figure are $d = 3$ and $t_x = t_y = 1$, and the number of unit cells is 40×40 .

in the upper inset of Fig. 4(a), the distribution of the complex eigenenergies is similar to that of the 1D lattice, except for the real energy distribution due to the dispersion toward the y axis. The zero-dimensional localized NHBSs, which are robust across a wide range of non-Hermiticity in the 1D lattice, extend into the 1D NHBSs in the 2D lattice. The corresponding eigenstates, which are localized with respect to the x direction and extend along the y direction, are depicted in Fig. 4(b).

Finally, we show the robustness of the lossless 1D NHBSs against geometrical deformations. We design a 2D $\mathcal{PT}$ -symmetric layer composed of square lattices that contain a locally inverted area with an arbitrary shape, as shown in Fig. 4(c). Notably, the geometry presents many abrupt corners where the interface bends but still conserves symmetry. One can see that the NHBSs survive without loss at the interfaces with sharp corners, as shown in Fig. 4(d), in contrast to the fact that there is normally bending loss around a sharp deformation of a waveguide without band topology. Therefore, NHBSs can be implemented, without bending loss, as a waveguide in a 2D lattice while conserving local twofold $\mathcal{PT}$ symmetry. This can be promising in applications to quantum optics. Further, many NHZMs of which the imaginary parts of the complex energies are larger than those of bulk states are also strongly localized at the interface in 2D lattices; these may also be useful in laser applications such as topological insulator lasers [47,48].

V. DISCUSSION

In conclusion, we propose a system in which localized states emerge solely through the nontriviality resulting from non-Hermiticity. Our twofold $\mathcal{PT}$ -symmetric ladder lattice contains symmetry-protected interface states as NHZMs and NHBSs with corresponding phase transitions at the two surface EPs and a bulk EP as a function of γ . The NHZMs protected by NHPH symmetry are localized at the interface between the odd-surface EP and the bulk EP, while the NHBSs protected by $\mathcal{PT}$ symmetry are localized at the interface between the even-surface EP and the bulk EP. Both have constant localization lengths that are unaffected by changes in the non-Hermiticity parameter. We also propose a lossless waveguide using the interface in our non-Hermitian system. We expect our findings regarding the characteristics of these two symmetry-protected interface states to open up a new field of synthetic non-Hermitian systems.

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APPENDIX

1. A ladder lattice

The Hamiltonian of a ladder lattice is given by

$$\begin{aligned} \hat{H} = & \frac{\epsilon}{2} \sum_n (|A_n\rangle\langle A_n| - |B_n\rangle\langle B_n|) \\ & - d \sum_n (|A_n\rangle\langle B_n| + \text{H.c.}) \\ & - t \sum_n (|A_{n+1}\rangle\langle A_n| + |B_{n+1}\rangle\langle B_n| + \text{H.c.}), \quad (\text{A1}) \end{aligned}$$

with a Bloch state $|\psi_k\rangle = \sum_n e^{ink} (a_k|A_n\rangle + b_k|B_n\rangle)$ as an ansatz, which is justified by the translational symmetry, and the Schrödinger equation, $\hat{H}|\psi_k\rangle = E_k|\psi_k\rangle$, gives $\mathcal{H}_k\phi_k = E_k\phi_k$, where $\mathcal{H}_k = -(2t \cos k)I - d\sigma_x + (\epsilon/2)\sigma_z$ and $\phi_k = (a_k, b_k)^T$. Here, σ_i s are Pauli matrices, I is the 2×2 identity matrix, and ϕ_k can be considered as a pseudospinor. Substituting ϵ with $i\gamma$ makes the system $\mathcal{PT}$ symmetric non-Hermitian, whose eigenvalues are $E_k = -2t \cos k \pm \sqrt{d^2 - (\gamma/2)^2}$ with corresponding pseudospinors k independent; $\phi = (a, b)^T$ with $a = -i\gamma/2 \mp \sqrt{d^2 - (\gamma/2)^2}$ and $b = d$ (double signs in the same order). Figure 5 shows the real and imaginary parts of the complex energy bands with $d = 3$ and $t = 1$ when $\gamma = 2.0$, 6.0, and 10.0.

2. Crossconnected two $\mathcal{PT}$ -symmetric ladder lattices

We investigate numerically a system of crossconnected two $\mathcal{PT}$ -symmetric ladder lattices as in Fig. 6 with finite size of N unit cells each. The Hamiltonian $\mathbf{H}$ in Eq. (A1) in a $2N \times 2N$

FIG. 5. (a)–(c) Real and (d)–(f) imaginary parts of the complex energy bands of $\mathcal{PT}$ -symmetric ladder lattices with (a), (d) $\gamma = 2.0$, (b), (e) $\gamma = 6.0$, and (c), (f) $\gamma = 10.0$.

matrix representation should look like

$$\begin{pmatrix} \ddots & & & & \\ & H_0 & H_1 & & \\ & H_1 & H_0 & H_1 & \\ & & H_1 & H_0^\dagger & H_1 \\ & & & H_1 & H_0^\dagger \\ & & & & \ddots \end{pmatrix}, \quad (\text{A2})$$

where $H_0 = -d\sigma_x + i\gamma/2\sigma_z$ and $H_1 = -tI$. By solving this $2N \times 2N$ matrix, we can obtain $2N$ eigenenergies.

3. Analytic solution of the interface states

The Hamiltonian of an infinite system with a $\mathcal{PT}$ -symmetric crossconnected interface as in Fig. 6 can be written as

$$\begin{aligned} \hat{H} = & \frac{i\gamma}{2} \sum_{n=-\infty}^{-1} (|A_n\rangle\langle A_n| - |B_n\rangle\langle B_n|) \\ & - \frac{i\gamma}{2} \sum_{n=0}^{\infty} (|A_n\rangle\langle A_n| - |B_n\rangle\langle B_n|) \\ & - d \sum_{n=-\infty}^{\infty} (|A_n\rangle\langle B_n| + \text{H.c.}) \\ & - t \sum_{n=-\infty}^{\infty} (|A_{n+1}\rangle\langle A_n| + |B_{n+1}\rangle\langle B_n| + \text{H.c.}). \quad (\text{A3}) \end{aligned}$$

FIG. 6. An intertwined $\mathcal{PT}$ -symmetric ladder lattice we considered.

For a given energy E , let us define four counterpropagating waves for the left and right regions of the interface such as $|\psi_{q_\eta,L}\rangle = \sum_{n=-\infty}^{-1} e^{iq_\eta n} (a_\eta |A_n\rangle + b_\eta |B_n\rangle)$ and $|\psi_{-q_\eta,R}\rangle = \sum_{n=0}^{\infty} e^{-iq_\eta n} (b_\eta |A_n\rangle + a_\eta |B_n\rangle)$ with complex $q_\eta = k_\eta + i\kappa_\eta$ with $\eta = \pm$, where

$$\cos q_\eta = \frac{-E + \eta\sqrt{d^2 - (\gamma/2)^2}}{2t}. \quad (\text{A4})$$

Here, (a_η, b_η) is the aforementioned momentum-independent pseudospinor. Imaginary parts of both q 's are assumed to be positive for the boundary condition at infinity; $e^{\pm iq_\eta} \rightarrow 0$ as $n \rightarrow \pm\infty$. Applying $\hat{H}$ on $|\psi_{q_\eta,L}\rangle$ and $|\psi_{-q_\eta,R}\rangle$ yields

$$\begin{aligned} (\hat{H} - E)|\psi_{q_\eta,L}\rangle &= -te^{-iq_\eta} (a_\eta |A_q\rangle + b_\eta |B_q\rangle) \\ &\quad + t(a_\eta |A_p\rangle + b_\eta |B_p\rangle), \\ (\hat{H} - E)|\psi_{-q_\eta,R}\rangle &= -t(b_\eta |A_p\rangle + a_\eta |B_p\rangle) \\ &\quad + te^{iq_\eta} (b_\eta |A_q\rangle + a_\eta |B_q\rangle). \end{aligned} \quad (\text{A5})$$

Substituting an ansatz, $|\phi\rangle = \sum_{\eta=\pm} (\alpha_\eta |\psi_{q_\eta,L}\rangle + \beta_\eta |\psi_{-q_\eta,R}\rangle)$, into the Schrödinger equation, $(\hat{H} - E)|\phi\rangle = 0$, and rearranging terms in terms of $|A_p\rangle$, $|B_p\rangle$, $|A_q\rangle$, and $|B_q\rangle$ using Eqs. (A5) yields four equations that can be written as a single matrix equation:

$$\begin{pmatrix} -a_+ e^{-iq_+} & -a_- e^{-iq_-} & b_+ e^{iq_+} & b_- e^{iq_-} \\ a_+ & a_- & -b_+ & -b_- \\ -b_+ e^{-iq_+} & -b_- e^{-iq_-} & a_+ e^{iq_+} & a_- e^{iq_-} \\ b_+ & b_- & -a_+ & -a_- \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_- \\ \beta_+ \\ \beta_- \end{pmatrix} = 0. \quad (\text{A6})$$

Requiring a zero determinant of the 4×4 matrix for the existence of nontrivial solutions of $(\alpha_+, \alpha_-, \beta_+, \beta_-)$ and using the expressions of a_η and b_η mentioned in Sec. I yields

$$(\gamma/2)^2 (1 - \cos q_+ \cos q_-) = [2d^2 - (\gamma/2)^2] \sin q_+ \sin q_-. \quad (\text{A7})$$

Using $\cos q_+ \cos q_- = \frac{E^2 - d^2 + (\gamma/2)^2}{4t^2}$ and $\sin^2 q_\eta = \frac{4t^2 - E^2 - d^2 + (\gamma/2)^2 + 2\eta E \sqrt{d^2 - (\gamma/2)^2}}{4t^2}$ as well as Eq. (A4), one can show that there are four energy values that satisfy Eq. (A7):

$$E_{\text{od},\pm} = \pm \sqrt{\left(1 - \frac{2t}{d}\right) \left(d^2 - 2td - \left(\frac{\gamma}{2}\right)^2\right)}, \quad (\text{A8})$$

$$E_{\text{ev},\pm} = \pm \sqrt{\left(1 + \frac{2t}{d}\right) \left(d^2 + 2td - \left(\frac{\gamma}{2}\right)^2\right)}. \quad (\text{A9})$$

Eigenvalues $E_{\text{od},\pm}$ and $E_{\text{ev},\pm}$ respectively correspond to the eigenstates with odd and even parities about the interface. One can get complex $k = \tilde{k} + i\kappa$ for the interface states by substituting these eigenenergies into Eq. (A4) with some arithmetic:

$$\begin{aligned} \cos \tilde{k}_{\text{od},\pm} \cosh \kappa_{\text{od},\pm} - i \sin \tilde{k}_{\text{od},\pm} \sinh \kappa_{\text{od},\pm} \\ = \mp \sqrt{\left(\frac{d}{2t} - 1\right) \left(\frac{d^2 - 2td - (\gamma/2)^2}{2td}\right)} \end{aligned}$$

$$\pm \sqrt{\frac{d}{2t} \left(\frac{d^2 - (\gamma/2)^2}{2td}\right)}, \quad (\text{A10})$$

$$\begin{aligned} \cos \tilde{k}_{\text{ev},\pm} \cosh \kappa_{\text{ev},\pm} - i \sin \tilde{k}_{\text{ev},\pm} \sinh \kappa_{\text{ev},\pm} \\ = \mp \sqrt{\left(\frac{d}{2t} + 1\right) \left(\frac{d^2 + 2td - (\gamma/2)^2}{2td}\right)} \\ \pm \sqrt{\frac{d}{2t} \left(\frac{d^2 - (\gamma/2)^2}{2td}\right)}, \end{aligned} \quad (\text{A11})$$

First, let us consider Eq. (A10). In terms of $k_{\text{od},\pm}$, the system has three distinct phases divided by two critical values of γ : $\gamma_- \equiv 2\sqrt{d^2 - 2td}$ and $\gamma_b \equiv 2d$, with $\gamma_- < \gamma_b$. Specifically, the reciprocal of κ represents the *localization length* of the corresponding interface state. In the case $0 < \gamma < \gamma_-$, both square roots on the right-hand side (RHS) are real so that $\sin \tilde{k}_{\text{od},\pm} \sinh \kappa_{\text{od},\pm} = 0$, or $\tilde{k}_{\text{od},\pm} = 0$ since $\kappa_{\text{od},\pm} \neq 0$, and hence $\cosh \kappa_{\text{od},\pm} = \text{RHS}$. In the case of $\gamma > \gamma_0$, both square roots on the RHS are purely imaginary so that $\cos \tilde{k}_{\text{od},\pm} \cosh \kappa_{\text{od},\pm} = 0$ or $\tilde{k}_{\text{od},\pm} = \pi/2$, and hence $\sinh \kappa_{\text{od},\pm} = i\text{RHS}$. For both cases, κ is a function of γ while $\tilde{k}$ is a constant. For the case of $\gamma_- < \gamma < \gamma_b$, the RHS becomes complex; the first square root is imaginary and the second one is real. By matching the real and imaginary parts of both sides, one can find that

$$\tan \tilde{k}_{\text{od},\pm} = \left(\frac{2t/d}{1 - (\gamma/2d)^2} - 1 \right)^{1/2}, \quad (\text{A12})$$

and

$$\tanh \kappa_{\text{od},\pm} = \left(1 - \frac{2t}{d} \right)^{1/2}. \quad (\text{A13})$$

It is noteworthy that the γ dependence is only on $\tilde{k}$ leaving κ γ -independent or the *localization length* is robust against the change of γ .

One can also follow the same procedure for the even-parity solution, Eq. (A11), with different critical values of γ : $\gamma_b \equiv 2d$ and $\gamma_+ \equiv 2\sqrt{d^2 + 2td}$ with $\gamma_b < \gamma_+$. In case $\gamma < \gamma_b$, both square roots on the RHS are real so that $\tilde{k}_{\text{ev},\pm} = 0$ and $\cosh \kappa_{\text{ev},\pm} = \text{RHS}$. In the case $\gamma > \gamma_+$, both square roots on the RHS are purely imaginary so that $\tilde{k}_{\text{ev},\pm} = \pi/2$ and $\sinh \kappa_{\text{ev},\pm} = i\text{RHS}$. For the case of $\gamma_b < \gamma < \gamma_+$, the RHS becomes complex; the first square root is now real and the second one is imaginary. By matching the real and imaginary parts of both sides, one can find that

$$\tan \tilde{k}_{\text{ev},\pm} = \left(\frac{2t/d}{(\gamma/2d)^2 - 1} - 1 \right)^{-1/2}, \quad (\text{A14})$$

and

$$\tanh \kappa_{\text{ev},\pm} = \left(1 + \frac{2t}{d} \right)^{-1/2}. \quad (\text{A15})$$

Again, the γ dependence is only on $\tilde{k}$, leaving κ γ -independent or the *localization length* is robust against γ .

FIG. 7. Random disorders are added to the real parts of on-site energy δ (a)–(c) and to the imaginary parts of on-site energy γ (d)–(f). Random disorders are added to the intracell hopping d (g)–(i) and the intercell hopping t (j)–(l). (a), (d), (g), (j) and (b), (e), (h), (k) are the real and imaginary parts of the complex eigenenergies as a function of γ , respectively. The black region represents the bulk-state eigenenergies, and the blue and red lines indicate the eigenenergies of the interface states. (c), (f), (i), (l) are localization lengths of odd (blue line) and even (red line) interface states. The orange and green lines represent the analytic solutions of localization lengths of the odd and even interface states, respectively. The parameters are $d = 3$ and $t = 1$, and the number of unit cells is $N = 100$. The disorder strength $a = 0.1$.

4. Hamiltonian of a 2D ladder lattice

The Hamiltonian of a 2D ladder lattice is given by

$$\begin{aligned} \hat{H} = & \frac{\epsilon}{2} \sum_{n,m} (|A_{n,m}\rangle \langle A_{n,m}| - |B_{n,m}\rangle \langle B_{n,m}|) \\ & - d \sum_{n,m} (|A_{n,m}\rangle \langle B_{n,m}| + \text{H.c.}) \\ & - t \sum_{n,m} \{ |A_{n+1,m}\rangle \langle A_{n,m}| + |B_{n+1,m}\rangle \langle B_{n,m}| \\ & + |A_{n,m+1}\rangle \langle A_{n,m}| + |B_{n,m+1}\rangle \langle B_{n,m}| + \text{H.c.} \}. \end{aligned}$$

In a similar procedure as in the 1D case with a Bloch state $|\psi_{k_x, k_y}\rangle = \sum_{n_x, n_y} e^{i(n_x k_x + n_y k_y)} (a|A_n\rangle + b|B_n\rangle)$ as an ansatz, one can find the Bloch Hamiltonian as

$$\mathcal{H}_{k_x, k_y} = -2t(\cos k_x + \cos k_y)I - d\sigma_x + \frac{\epsilon}{2}\sigma_z. \quad (\text{A16})$$

The corresponding pseudospinor $\phi = (a, b)^T$ is again independent of k_x and k_y , since the k dependence in the Bloch Hamiltonian is associated only with the identity matrix.

FIG. 8. Real on-site energies of (a) upper and (b) lower layers of the 2D lattice with a linear interface including spatial defects. (c) Imaginary on-site energies of upper layers of the lattice. Red and blue colors represent $\gamma/2$ and $-\gamma/2$, respectively. The upper and lower layers have opposite profiles of gain and loss. (d) Intensity distribution of a selected localized interface state for $\gamma = 7.6$.

5. Robustness to disorder and spatial defect

For numerical verification of the robustness to disorder, we additionally introduce random disorders to Eq. (A2) through four different channels: real part δ of on-site energy, imaginary part γ of on-site energy, intracell hopping d , and intercell hopping t . We introduce random disorders $\delta_i = a_i \xi$, $\gamma_i = \gamma + a \xi$, $d_i = d + a_i \xi$, and $t_i = t + a_i \xi$, where ξ is uniformly distributed between -1 and 1 and a is the disorder strength.

We can find the interface states and their localization lengths are robust against various kinds of random disorder, even though the random disorders break twofold PT symmetry and symmetry-protected states no longer preserve their symmetry. When the random disorders preserve twofold PT symmetry, the results are also qualitatively the same as Fig. 7. If the disorder strength does not exceed the band-gap size, while the interface states near EPs and symmetry-protected states in a twofold PT-symmetric lattice without disorder are quite robust to disorder as shown in Fig. 7, the interface states near bulk states when γ is small or large are not robust.

Next, we consider spatial defects on a designed 1D waveguide in the 2D ladder lattice. We add random disorder to the real on-site energies of eight sites near the interface as shown in Figs. 8(a) and 8(b). The parameters are $d = 3$ and $t_x = t_y = 1$, and the number of unit cells is 40×40 . The disorder strength $a = 0.2$. In spite of spatial defects near the interface, we can also find the interface states are robust against spatial disorders.

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