Efficiency fluctuations of a quantum heat engine

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The efficiency of quantum heat engines is a stochastic quantity owing to the presence of thermal and quantum fluctuations. We here extend the standard two-projective-measurement scheme to derive the efficiency distribution of a quantum Otto engine. We analyze the generic properties of this distribution for scale-invariant driving Hamiltonians which describe a large class of single-particle, many-body, and nonlinear systems. We find that the mean efficiency is equal to the macroscopic efficiency for adiabatic driving. It generally diverges for nonadiabatic driving when no heat is absorbed while work is produced. We further compute the quantum efficiency statistics for an analytically solvable two-level engine and investigate its classical-to-quantum transition.

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I. INTRODUCTION

Efficiency is a key performance measure of thermal machines. For heat engines that cyclically convert heat into useful work, it is defined as the ratio of work output and heat input [1]. For macroscopic systems consisting of a huge number of degrees of freedom, heat, work, and thus efficiency are deterministic quantities. The second law of thermodynamics imposes an upper bound to the efficiency of any heat engine given by the Carnot expression $\eta_{\text{carnot}} = 1 - T_1/T_2$, where $T_{1,2}$ are the respective temperatures of the cold and hot heat reservoirs [1]. By contrast, thermodynamic variables become random at the microscopic scale owing to the presence of non-negligible thermal fluctuations [2,3] and additional quantum fluctuations at low enough temperatures [4,5]. A central question is then to determine their probability distributions in order to assess their stochastic properties.

For classical heat engines, efficiency fluctuations have been theoretically investigated in recent studies [6–16]. In particular, the stochastic efficiency of a Carnot engine has been shown to admit values larger than the Carnot bound [6]. Remarkably, the latter quantity has also been found to be the least likely in the long-time limit [6]. These predictions have been experimentally verified in a stochastic engine based on an optically trapped colloidal particle [17]. In the quantum regime, work distributions of driven oscillators and two-level systems have been analyzed both theoretically [18–23] and experimentally using ion-trap [24], NMR [25], and cold-atom [26] setups. In addition, the efficiency statistics of a quantum thermoelectric junction has been studied using full-counting statistics [27] and that of a SWAP quantum engine using numerical simulations of calorimetric measurements

[28]. However, a general formalism to examine the effects of quantum fluctuations on the stochastic efficiency of quantum heat engines has been missing so far.

In this Rapid Communication, we analytically evaluate the efficiency statistics of a quantum thermal machine and take the quantum Otto engine, a paradigmatic model of a quantum heat engine [29], as an example to illustrate our approach. The quantum Otto cycle, a generalization of the ordinary four-stroke motor, has been extensively studied in the past 30 years [30–39]. The experimental realizations of a quantum Otto spin engine in a NMR system have been reported lately [40,41]. In the following, we determine the respective work and heat probability densities of the different branches of the engine cycle by extending the two-projective-measurement scheme [42] to thermal machines. We use these correlated distributions to derive a general formula for the quantum efficiency statistics and first examine its generic properties for scale-invariant driving Hamiltonians [43–47]. We find that the stochastic efficiency is equal to the deterministic macroscopic efficiency for adiabatic driving, while efficiency values at infinity appear for nonadiabatic driving, when no heat is absorbed although nonzero work is produced. As an application, we treat in detail an analytically solvable model of a two-level engine and obtain an explicit expression for the quantum efficiency distribution. We finally investigate the transition of the nonadiabatic distribution from a regime dominated by thermal fluctuations at high temperatures to a domain characterized by quantum fluctuations at low temperatures.

II. QUANTUM OTTO ENGINE

We consider a generic quantum system with a time-dependent Hamiltonian H_t as the working fluid of a quantum Otto engine. The system is initially thermalized at $t = 0$ by weakly coupling it to a cold heat reservoir at inverse temperature β_1 . The Otto cycle consists of the following four steps (Fig. 1): (1) unitary expansion (AB) during which the Hamiltonian is changed from H_0 to H_{τ_1} in a time τ_1 , consuming an amount of work W_1 , (2) hot isochore (BC)

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FIG. 1. Quantum Otto cycle consisting of unitary expansion (AB), isochoric heating at inverse temperature β_2 (BC), unitary compression (CD), and isochoric cooling at inverse temperature β_1 (DA). The engine absorbs heat $\langle Q_2 \rangle$ (releases heat $\langle Q_4 \rangle$) and produces the positive work $-(\langle W_1 \rangle + \langle W_3 \rangle)$.

during which the system is put into contact with a heat bath at inverse temperature β_2 to absorb heat Q_2 in a time τ_2 , (3) unitary compression (CD) that drives the isolated system from H_{τ_1} back to H_0 in a time τ_3 , producing an amount of work W_3 , and (4) cold isochore (DA), which closes the cycle by weakly coupling the system to the cold bath at inverse temperature β_1 , thus releasing heat Q_4 in a time τ_4 . Work and heat are positive, when added to the system. We further assume, as in the experimental quantum Otto engine [40], that heating and cooling times $\tau_{2,4}$ are longer than the relaxation time of the system, so that thermalization is achieved after each isochore. This assumption can be weakened by including the relaxation dynamics into the problem.

Quantum work and heat distributions are commonly determined with the help of the so-called two-projective-measurement method [42] or variants thereof [18–26]. In this approach, energy changes of a quantum system during single realizations of a process are associated with the difference of eigenvalues obtained through projective energy measurements at the beginning and at the end of the process. In the quantum Otto cycle, work is performed during the unitary expansion and compression steps, while heat is exchanged during the nonunitary heating and cooling stages. We obtain the corresponding distributions by applying the two-projective-measurement scheme to the respective expansion, hot isochore, and compression branches. The probability distribution of the expansion work W_1 is accordingly [42]

$$P(W_1) = \sum_{n,m} \delta[W_1 - (E_m^{\tau_1} - E_n^0)] P_{n \rightarrow m}^{\tau_1} P_n^0(\beta_1), \quad (1)$$

where E_n^0 and $E_m^{\tau_1}$ are the respective energy eigenvalues at the beginning and at the end of the expansion, $P_n^0(\beta_1) = \exp(-\beta_1 E_n^0)/Z^0$ is the initial thermal occupation probability with partition function Z^0 , and $P_{n \rightarrow m}^{\tau_1} = |\langle n | U_{\text{exp}}(\tau_1) | m \rangle|^2$ the transition probability between the instantaneous eigenstates $|n\rangle$ and $|m\rangle$. The corresponding unitary is denoted by U_{exp} . The

occupation probability $P_n^0(\beta_1)$ describes the effects of thermal fluctuations, whereas the transition probability $P_{n \rightarrow m}^{\tau_1}$ those of quantum fluctuations and quantum dynamics [48].

Similarly, the probability density of the heat Q_2 during the ensuing hot isochore, given the expansion work W_1 , is equal to the conditional distribution [49],

$$P(Q_2|W_1) = \sum_{k,l} \delta[Q_2 - (E_l^{\tau_1} - E_k^{\tau_1})] P_{k \rightarrow l}^{\tau_2} P_k^{\tau_1}, \quad (2)$$

where the occupation probability at time τ_1 is $P_k^{\tau_1} = \delta_{km}$ when the system is in eigenstate $|m\rangle$ after the second projective energy measurement. Noting that the state of the system is thermal with inverse temperature β_2 at the end of the isochore, we further have $P_{k \rightarrow l}^{\tau_2} = P_l^{\tau_1}(\beta_2) = \exp(-\beta_2 E_l^{\tau_1})/Z^{\tau_1}$, with the partition function Z^{τ_1} .

The quantum work distribution for compression, given the expansion work W_1 and the heat Q_2 , is furthermore

$$P(W_3|W_1, Q_2) = \sum_{i,j} \delta[W_3 - (E_j^0 - E_i^{\tau_1})] P_{i \rightarrow j}^{\tau_3} P_i^{\tau_1 + \tau_2}, \quad (3)$$

with the occupation probability $P_i^{\tau_1 + \tau_2} = \delta_{il}$ when the system is in eigenstate $|l\rangle$ after the third projective energy measurement. The transition probability $P_{i \rightarrow j}^{\tau_3} = |\langle i | U_{\text{com}}(\tau_3) | j \rangle|^2$ is fully specified by the unitary time evolution operator for compression U_{com} .

The joint probability of having certain values of W_3 , Q_2 , and W_1 during a cycle of the Otto engine readily follows from the chain rule for conditional probabilities, $P(W_3, Q_2, W_1) = P(W_3|Q_2, W_1)P(Q_2|W_1)P(W_1)$ [50]. Using Eqs. (1)–(3), it is explicitly given by

$$\begin{aligned} P(W_1, Q_2, W_3) = & \sum_{n,m,k,l} \delta[W_1 - (E_m^{\tau_1} - E_n^0)] \\ & \times \delta[Q_2 - (E_k^{\tau_1} - E_m^{\tau_1})] \delta[W_3 - (E_l^0 - E_k^{\tau_1})] \\ & \times |\langle n | U_{\text{exp}}(\tau_1) | m \rangle|^2 |\langle k | U_{\text{com}}(\tau_3) | l \rangle|^2 \\ & \times \frac{e^{-\beta_1 E_n^0} e^{-\beta_2 E_k^{\tau_1}}}{Z^0 Z^{\tau_1}}. \end{aligned} \quad (4)$$

Equation (4) is the essential quantity needed to determine the quantum efficiency statistics.

III. QUANTUM EFFICIENCY DISTRIBUTION

The stochastic efficiency of the quantum Otto engine is defined as

$$\eta = -\frac{W_1 + W_3}{Q_2}. \quad (5)$$

It should not be confused with the thermodynamic (macroscopic) efficiency which is defined in terms of the averaged values of work and heat, $\eta_{\text{th}} = -(\langle W_1 \rangle + \langle W_3 \rangle)/\langle Q_2 \rangle$. The distribution $P(\eta)$ is obtained by integrating over all possible values of W_3 , Q_2 , and W_1 via

$$P(\eta) = \int dW_3 dQ_2 dW_1 P(W_1, Q_2, W_3) \delta\left(\eta + \frac{W_1 + W_3}{Q_2}\right). \quad (6)$$

Using Eq. (4) and the properties of the delta, we find

$$P(\eta) = \sum_{n,m,k,l} \delta\left(\eta + \frac{E_m^{\tau_1} - E_n^0 + E_l^0 - E_k^{\tau_1}}{E_k^{\tau_1} - E_m^{\tau_1}}\right) \times \frac{e^{-\beta_1 E_n^0} e^{-\beta_2 E_k^{\tau_1}}}{Z^0 Z^{\tau_1}} |\langle m|U_{\text{exp}}|n\rangle|^2 |\langle k|U_{\text{com}}|l\rangle|^2. \quad (7)$$

Expression (7) is our main result. It shows that the efficiency distribution of the quantum Otto engine is fully determined by the unitary time evolution operators for expansion and compression U_{exp} and U_{com} and by the inverse temperatures β_1 and β_2 of the two reservoirs. It will allow us to analyze the effects of thermal and quantum fluctuations on the efficiency statistics of the engine.

We begin by discussing the generic properties of Eq. (7). For simplicity, we take expansion and compression times to be equal, $\tau_1 = \tau_3 = \tau$. We first note that the mean efficiency $\langle \eta \rangle$ is typically infinite for general (nonadiabatic) driving due to the vanishing denominator in the argument of the Dirac delta, when the eigenvalues E_k^{τ} and E_m^{τ} are identical. In this case, no heat is absorbed during the hot isochore, while the total work is nonzero. However, $\langle \eta \rangle$ is finite for adiabatic (transitionless) driving and equal to the adiabatic macroscopic efficiency $\eta_{\text{th}}^{\text{ad}}$. To see this, we consider scale-invariant driving Hamiltonians of the form $H_{\tau} = \mathbf{p}^2/2m + U(\mathbf{x}, \gamma_{\tau})$, with $U(\mathbf{x}, \gamma_{\tau}) = U_0(\mathbf{x}/\gamma_{\tau})/\gamma_{\tau}^2$ and scaling parameter γ_{τ} . These Hamiltonians describe a wide class of single-particle, many-body, and nonlinear systems with scale-invariant spectra, $E_j^{\tau} = E_j^0/\gamma_{\tau}^2$ [43–47]. For instance, for a one-dimensional oscillator, $U_0(x) = m\omega_{\tau}^2 x^2/2$, the scaling parameter is simply related to the ratio of initial and final frequencies, $\gamma_{\tau}^2 = \omega_0/\omega_{\tau}$. For adiabatic expansion and compression, $|\langle m|U_{\text{exp}}|n\rangle|^2 = \delta_{nm}$ and $|\langle k|U_{\text{com}}|l\rangle|^2 = \delta_{kl}$, the distribution (7) then becomes

$$P(\eta) = \delta[\eta - (1 - \gamma_{\tau}^2)], \quad (8)$$

where the indeterminate expression $0/0$ appearing in the argument of the first Dirac delta has been identified with the adiabatic Otto efficiency $\eta_{\text{th}}^{\text{ad}} = 1 - \gamma_{\tau}^2$ [51]. The efficiency is therefore delta distributed in the absence of quantum fluctuations and equal to the adiabatic thermodynamic efficiency. Noting that $\langle W/Q_2 \rangle = \langle W \rangle / \langle Q_2 \rangle - \text{cov}(W/Q_2, Q_2) / \langle Q_2 \rangle$, where $W = W_1 + W_3$ is the total work and $\text{cov}(W/Q_2, Q_2)$ denotes the covariance between the ratio W/Q_2 and Q_2 [52], the equality $\langle \eta \rangle = \eta_{\text{th}}^{\text{ad}}$ implies that stochastic efficiency and absorbed heat are uncorrelated, $\text{cov}(\eta, Q_2) = 0$, in this case.

IV. EXAMPLE OF A SPIN HEAT ENGINE

Formula (7) is valid for any working fluid and driving. As an illustration, we now study the properties of the stochastic efficiency (5) for an analytical solvable two-level heat engine. This choice is motivated by the recent realization of a spin quantum Otto engine in a NMR setup [40]. Compression and expansion are implemented by driving a spin-1/2 with

a constant magnetic field with strength $\omega/2$ along the z axis and a rotating magnetic field with varying strength $\lambda(t)$ in the (x - y) plane. The expansion Hamiltonian is

$$H_{\text{exp}}(t) = \lambda(t)(\cos \omega t \sigma_x + \sin \omega t \sigma_y) + \frac{\omega}{2} \sigma_z, \quad (9)$$

where σ_i , $i = (x, y, z)$, are the usual Pauli operators (with $\hbar = 1$) [53]. The rotation frequency is chosen to be $\omega = \pi/2\tau$ to ensure a complete rotation from the x axis to the y axis during the expansion step of duration τ . The amplitude of the rotating field, $\lambda(t) = \lambda_1(1 - t/\tau) + \lambda_2(t/\tau)$, is increased from λ_1 at time zero to λ_2 at time τ . This driving leads to a widening of the energy spacing of the two-level system from $2\nu^0 = \sqrt{4\lambda(0)^2 + \omega^2}$ to $2\nu^{\tau} = \sqrt{4\lambda(\tau)^2 + \omega^2}$. The compression stroke is simply obtained from the time-reversed process, $H_{\text{com}}(t) = -H_{\text{exp}}(\tau - t)$. The corresponding expansion time evolution operator reads [51]

$$U_{\text{exp}} = \begin{pmatrix} e^{-i\omega t/2} \cos I & ie^{-i\omega t/2} \sin I \\ ie^{i\omega t/2} \sin I & e^{i\omega t/2} \cos I \end{pmatrix}, \quad (10)$$

where $I = -\int_0^t dt' \lambda(t')$ is the integral over the increasing strength of the rotating magnetic field. The operator U_{com} follows from U_{exp} by replacing t with $\tau - t$. The probability of no level transition during expansion or compression steps may be inferred from (10) as [51]

$$u = u_{\text{exp}} = u_{\text{com}} = \cos^2 I. \quad (11)$$

The probability of a (nonadiabatic) level transition during either driving phases is accordingly $v = 1 - u$.

In order to operate as an engine, the mean heat absorbed during (BC) should be positive, $\langle Q_2 \rangle \geq 0$, as well as the total mean work output during the cycle, $-(\langle W_1 \rangle + \langle W_3 \rangle) \geq 0$. Using the above results, we obtain the thermodynamic efficiency [51],

$$\eta_{\text{th}} = 1 + \frac{\nu^0 \tanh(\beta_2 \nu^{\tau}) A^* + \tanh(\beta_1 \nu^0)}{\nu^{\tau} \tanh(\beta_2 \nu^{\tau}) + \tanh(\beta_1 \nu^0) A^*}. \quad (12)$$

It reduces to the known adiabatic Otto efficiency $\eta_{\text{th}}^{\text{ad}} = 1 - \nu^0/\nu^{\tau}$ for $A^* = 1 - 2u = -1$, as expected [29–39].

The quantum efficiency distribution (7) may be further evaluated analytically as [51]

$$P(\eta) = \frac{2}{Z^0 Z^{\tau}} \left\{ [u^2 (\cosh(\beta_1 \nu^0 + \beta_2 \nu^{\tau}) + \cosh(\beta_1 \nu^0 - \beta_2 \nu^{\tau})) + v^2 \cosh(\beta_1 \nu^0 - \beta_2 \nu^{\tau})] \delta\left[\eta - \left(1 - \frac{\nu^0}{\nu^{\tau}}\right)\right] + v^2 \cosh(\beta_1 \nu^0 + \beta_2 \nu^{\tau}) \delta\left[\eta - \left(1 + \frac{\nu^0}{\nu^{\tau}}\right)\right] + uv \cosh(\beta_2 \nu^{\tau}) [e^{\beta_1 \nu^0} \delta(\eta + \infty) + e^{-\beta_1 \nu^0} \delta(\eta - \infty)] \right\} + uv \delta(\eta - 1). \quad (13)$$

This distribution is normalized to one, as it should. We observe that the stochastic efficiency η can take five different discrete values as seen in Fig. 2. The values at infinity that

FIG. 2. Distribution $P(\eta)$, Eq. (13), of the quantum stochastic efficiency (5) for adiabatic (blue squares) and nonadiabatic (red dots) driving. The efficiency is delta distributed for adiabatic driving. We further observe the appearance of peaks above unit efficiency, in particular at infinity when the absorbed heat vanishes, in the nonadiabatic regime. Parameters are $\beta_1 = 2$, $\beta_2 = 0.1$ and $\lambda_1 = 0.5$, $\lambda_2 = 3$. Adiabatic and nonadiabatic driving times are $\tau_{\text{ad}} = 7.18$ and $\tau_{\text{nad}} = 2.39$.

occur when the heat Q_2 is zero are notable, since the efficiency is not defined at these points.

The efficiency statistics (13) depends on the driving time τ (Fig. 2). For adiabatic (transitionless) driving, $u = 1$ ($v = 0$) (blue squares), reached exactly for $I = n\pi$, the stochastic efficiency takes only one value given by the adiabatic macroscopic efficiency $\eta_{\text{th}}^{\text{ad}}$. All other peaks, including those at infinity, do not appear. The efficiency is thus delta distributed in the absence of quantum fluctuations, as already noted above. By contrast, for nonadiabatic driving, $v > 0$ ($u < 1$) (red dots), three peaks at and above unit efficiency are visible, including one at infinity. As a result, an average efficiency is not defined in this case. Specifically, Eq. (13) reveals that the three values at one as well as at plus and minus infinity disappear when $uv = 0$, that is, for certain events ($v = 1$).

A comparison of the mean efficiency $\langle \eta \rangle$ in the adiabatic regime, $u = 1$, with the thermodynamic efficiency η_{th} , Eq. (12), is shown in Fig. 3 as a function of the driving time τ . As expected, the mean efficiency is always larger than the macroscopic efficiency, $\langle \eta \rangle = \eta_{\text{th}}^{\text{ad}} \geq \eta_{\text{th}}$. Since the driving is periodic, the upper bound is reached exactly and not just asymptotically for large driving times. The shaded area represents the regime where the quantum Otto machine operates as a heat engine.

V. HIGH-TEMPERATURE AND LOW-TEMPERATURE DISTRIBUTIONS

In order to analyze the effects of quantum fluctuations on the efficiency statistics, we next examine the distribution (13) in the high-temperature and low-temperature regimes. In the high-temperature limit, $\beta_i v^j \ll 1$, $i = (1, 2)$, and $j = (0, \tau)$, we find with the variables $x = \beta_1 v^0$ and

FIG. 3. Mean adiabatic efficiency $\langle \eta \rangle = \eta_{\text{th}}^{\text{ad}}$ (red dots) and macroscopic efficiency η_{th} , Eq. (12) (blue solid), as a function of the driving time τ . The shaded area represents the regime where the quantum Otto machine operates as a heat engine. Same parameters as in Fig. 2.

$$y = \beta_2 v^\tau,$$

$$P_{\text{high}}(\eta) = \frac{1}{2 + x^2 + y^2} \left\{ \begin{aligned} &u^2(2 + x^2 + y^2) \\ &+ v^2 \left(1 + \frac{x^2}{2} + \frac{y^2}{2} - xy \right) \delta \left[\eta - \left(1 - \frac{v^0}{v^\tau} \right) \right] \\ &+ v^2 \left(1 + \frac{x^2}{2} + \frac{y^2}{2} + xy \right) \delta \left[\eta - \left(1 + \frac{v^0}{v^\tau} \right) \right] \\ &+ uv \left(1 + x + \frac{x^2}{2} + \frac{y^2}{2} \right) \delta(\eta + \infty) \\ &+ uv \left(1 - x + \frac{x^2}{2} + \frac{y^2}{2} \right) \delta(\eta - \infty) \end{aligned} \right\} + uv\delta(\eta - 1). \quad (14)$$

On the other hand, in the low-temperature limit, $\beta_i v^j \gg 1$, we have with normalization $N = 1 - uv(e^{-2x} + e^{-2y})$,

$$P_{\text{low}}(\eta) = \frac{1}{N} \left\{ \begin{aligned} &[u^2(1 + e^{-2x} + e^{-2y}) + v^2(e^{-2x} + e^{-2y})] \\ &\times \delta \left[\eta - \left(1 - \frac{x^2}{2} \right) \right] \\ &+ v^2 \delta \left[\eta - \left(1 + \frac{v^0}{v^\tau} \right) \right] + uv\delta(\eta - 1) \\ &+ uv(1 + e^{-2y})\delta(\eta + \infty) \\ &+ uve^{-2x}\delta(\eta - \infty) \end{aligned} \right\}. \quad (15)$$

Both efficiency distributions are shown in Fig. 4. In the high-temperature limit, $\beta_i v^j \ll 1$ (orange diamonds), we observe that the values at plus and minus infinity are equally likely, while the highest peak is located at $1 - v^0/v^\tau \leq 1$, which corresponds to the adiabatic macroscopic efficiency $\eta_{\text{th}}^{\text{ad}}$. On the other hand, in the low-temperature limit, $\beta_i v^j \gg 1$ (green triangles), the peak at minus infinity is enhanced at the

FIG. 4. Distribution $P(\eta)$, Eq. (13), in the high-temperature limit (orange diamonds) ($\beta_1 = 0.02$ and $\beta_2 = 0.01$), and in the low-temperature limit (green triangles) ($\beta_1 = 20$ and $\beta_2 = 1$). The crossover from a regime dominated by thermal fluctuations at high temperatures to a domain characterized by quantum fluctuations at low temperatures is clearly visible. Same parameters as in Fig. 2.

expense of the peak at plus infinity, which practically disappears. The most probable value is now found at $1 + v^0/v^\tau \geq 1$, which is above unit efficiency. Interestingly, the peak at unit efficiency remains unaffected in both limits. The transition from a regime dominated by thermal fluctuations at high temperatures to a domain characterized by quantum fluctuations at low temperatures is thus clearly visible. In particular,

the increase of the peak at minus infinity is counterbalanced by an amplification of the efficiency values above one, when thermal fluctuations are replaced by smaller quantum fluctuations.

VI. CONCLUSIONS

We have developed a general framework to evaluate the efficiency distribution of a quantum heat engine based on the two-projective-measurement approach. It allows the investigation of the effects of quantum fluctuations on the performance of the quantum motor. We have established that it is fully determined by the time evolution operators of the two isothermal compression and expansion steps, and by the two bath temperatures for an Otto cycle with complete thermalization. We have shown that the adiabatic efficiency is deterministic and equal to the macroscopic efficiency. We have furthermore found that the average efficiency diverges and is thus not defined for nonadiabatic driving. We expect these findings to become a valuable tool to study the performance of small quantum thermal machines of any kind, from heat engines and heat pumps to refrigerators.

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