Scaling and diffusion of Dirac composite fermions

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We study the effects of quenched disorder and a dissipative Coulomb interaction on an anyon gas in a periodic potential undergoing a quantum phase transition. We use a $(2+1)$ -dimensional low-energy effective description that involves $N_f = 1$ Dirac fermion coupled to a $U(1)$ Chern-Simons gauge field at level $(\theta - 1/2)$. When $\theta = 1/2$ the anyons are free Dirac fermions that exhibit an integer quantum Hall transition; when $\theta = 1$ the anyons are bosons undergoing a superconductor-insulator transition in the universality class of the three-dimensional XY model. Using the large N_f approximation we perform a renormalization-group analysis. We find the Coulomb interaction to be an irrelevant perturbation of the clean fixed point for any θ . The dissipative Coulomb interaction allows for two classes of IR stable fixed points in the presence of disorder: those with a finite nonzero Coulomb coupling and dynamical critical exponent $z = 1$ and those with an effectively infinite Coulomb coupling and $1 < z < 2$. At $\theta = 1/2$ the clean fixed point is stable to charge-conjugation preserving (random mass) disorder, while a line of diffusive fixed points is obtained when the product of charge-conjugation and time-reversal symmetries is preserved. At $\theta = 1$ we find a finite disorder fixed point with unbroken charge-conjugation symmetry whether or not the Coulomb interaction is present. Other cases result in runaway flows. We comment on the relation of our results to other theoretical studies and the relevancy to experiment.

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I. INTRODUCTION

Delocalization transitions determine the phase diagrams of various electronic systems [1–3]. In three spatial dimensions, such transitions can occur between a diffusive metal and a localized insulator. In two dimensions (and fewer), localization generally relegates $T = 0$ metallic states to isolated critical points. The integer quantum Hall transition (IQHT) and the superconductor-insulator transition (SIT) are prototypical examples of such two-dimensional diffusive quantum critical points, having been well characterized by extensive experimental and numerical work over the past 30 years (see [4–7] and references therein). Nevertheless, our understanding of these quantum states remains incomplete.

Theories of noninteracting electrons have provided valuable insight to the IQHT [8]. As the critical point is approached by tuning the external magnetic field or electron density to criticality $\delta \rightarrow 0$, the localization length is found to diverge as $|\delta|^{-\nu}$ with $\nu = 2.593(5)$ [9], while $\nu_{\text{expt}} \approx 2.38$ experimentally [5,6]. A diverging timescale $\xi_t \sim \xi^{-\nu z}$ is also expected near the quantum critical point. Theories of noninteracting electrons yield a dynamical critical exponent

$z = 2$ [8,10,11]; $z_{\text{expt}} \approx 1$ [4,6] (although see [12]). (For the magnetic field-tuned SIT, $\nu_{\text{expt}} \approx 4/3$ or $\approx 7/3$ and $z_{\text{expt}} \approx 1$ experimentally [13].) The challenge is to develop a framework that combines the effects of electron interactions with those of disorder [14].

Duality is a powerful tool for understanding the behavior of strongly interacting systems. Recent work has uncovered a duality web that relates various $(2+1)$ D relativistic quantum field theories (see [15] and references therein). Included in this set are simple, toy models for integer quantum Hall and superconductor-insulator transitions. In this paper, we study the effects of quenched disorder and a dissipative Coulomb interaction on the critical properties of two such models. The hope is to abstract lessons that may be valid more generally. As we discuss, these theories have a rich set of random critical behaviors.

For the first member of the duality web, consider a system of spinless electrons hopping on a square lattice with a half unit of magnetic flux penetrating each plaquette [16] ([17] may alternatively be considered). An IQHT is obtained as the ratio of the (staggered) chemical potential to next-neighbor hopping is varied. The critical properties of the transition are controlled by a free Dirac fermion Ψ with the Lagrangian:

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi} i \not{D}_A \Psi - M \bar{\Psi} \Psi + \frac{1}{2} \frac{1}{4\pi} A dA, \quad (1.1)$$

where A_μ is a nondynamical (external) $U(1)$ gauge field and the Chern-Simons term $A dA = \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$. (Additional details for the Lagrangians appearing in this section are given

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in Sec. II.) The mass M vanishes at criticality. In the presence of an external magnetic field, (1.1) describes the particle-hole (PH) symmetric limit of the half-filled zeroth/lowest Landau level of Dirac/nonrelativistic electrons [18]. In this paper, we consider vanishing magnetic field. A dual effective theory to (1.1) consists of a Dirac fermion ψ coupled to a dynamical (emergent) $U(1)$ gauge field a_μ :

$$\mathcal{L}_F = \bar{\psi} i \not{D}_a \psi - m \bar{\psi} \psi - \frac{1}{2} \frac{1}{2\pi} adA + \frac{1}{2} \frac{1}{4\pi} AdA - \frac{1}{4} f_{\mu\nu}^2, \quad (1.2)$$

where the mass $m \propto M$ and the field strength $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. (We use “condensed matter” notation when writing these Lagrangians; see [19] for a precise explanation of the meaning of, e.g., Chern-Simons terms with half-integer levels.) Equation (1.2) was first introduced as a dual description of the half-filled Landau level [18] or the gapless surface state of a time-reversal invariant topological insulator [20,21] (when the AdA term is absent) with ψ being the Dirac composite fermion; its inclusion in the duality web was explained in [19,22,23]. When the external magnetic field is zero, the Dirac composite fermion chemical potential sits at the Dirac point.

For the second member of the duality web, consider a collection of repulsive bosons in a periodic potential [24]. For commensurate filling, the system exhibits a superfluid to Mott insulator transition with a charge-conjugation symmetry as the ratio of the boson hopping strength to repulsion is tuned. The long-wavelength critical properties are described by the 3D XY model:

$$\mathcal{L}_{XY} = |D_A \Phi|^2 - M |\Phi|^2 - |\Phi|^4. \quad (1.3)$$

(Broken charge-conjugation symmetry generally results in a term proportional to $\Phi^* i \partial_t \Phi$.) In mean-field theory, the $M < 0$ region is a superfluid, while the $M > 0$ region is an insulator; we will view (1.3) as describing a SIT. A dual effective theory [25–27] to (1.3) is

$$\begin{aligned} \mathcal{L}_B = & \bar{\psi} i \not{D}_a \psi - m \bar{\psi} \psi + \frac{1}{2} \frac{1}{4\pi} ada - \frac{1}{2\pi} adA \\ & + \frac{1}{4\pi} AdA - \frac{1}{4} f_{\mu\nu}^2. \end{aligned} \quad (1.4)$$

The statistics of the particles that (1.2) and (1.4) describe is controlled by the coefficient of the ada term.

Quenched disorder can have a profound effect on the nature of the above critical points and lead to new universality classes. Reference [16] considered the effects of quenched randomness on the free Dirac fermion fixed point in (1.1). While for generic disorder the theory flows to strong coupling, if only a random vector potential $\mathbf{A}(\mathbf{x})$ is present the theory features a line of diffusive fixed points characterized by a continuously variable dynamical exponent z ; the clean fixed point is stable to random mass disorder $M(\mathbf{x})$. Ye and Sachdev [28] and Ye [29] generalized this study to fractional quantum Hall transitions in the presence of an unscreened Coulomb interaction using a model closely related to (1.4). Recently, Goswami, Goldman, and Raghu [30] and Thomson and Sachdev [31] considered the effects of randomness on (1.2) with $2N_f$ fermion flavors. Using the large N_f expan-

sion, we reexamine these works and extend them to include the effects of a dissipative Coulomb interaction (Sec. IIC) and “topological disorder” (Sec. IID), generally confirming prior results that found interacting, diffusive fixed points for certain types of disorder. Related works studying the effects of quenched randomness on theories of Dirac fermions coupled to a fluctuating boson include [32,33].

In contrast to the fermion models, only random mass disorder $M(\mathbf{x})$ has resulted in accessible diffusive fixed points of the XY model. Early work [34–37] studying the $O(2N_f)$ generalization of (1.3) used a double- ϵ expansion to find an interacting, finite disorder fixed point. However, the nature of the renormalization-group flow in the vicinity of the fixed point is peculiar, exhibiting an anomalously long “time” to achieve criticality. Recently, this problem was reexamined within a large N_f expansion by Goldman, Thomson, Nie, and Bi [38], where it was argued that the anomalous renormalization-group trajectories [34–37] are a relic of the double- ϵ expansion. Furthermore, [38] finds remarkable agreement with the critical exponents of the dirty XY model calculated by numerical simulation [39–42]. We consider this analysis from the perspective of the “fermionic dual” of the XY model in (1.4), providing qualitative confirmation of the renormalization-group flow found in [38]. To $\mathcal{O}(1/N_f)$, we find a finite disorder fixed point with critical exponents:

$$\nu^{-1} = 1 \quad \text{and} \quad z = 1 + \frac{1.411}{N_f}; \quad (1.5)$$

$\nu = 1$ and $z = 1 + 0.54/N_f$ is reported in [38]. We also consider other types of disorder that is sourced by the random gauge field $A_\mu(\mathbf{x})$.

The important influence of a Coulomb interaction on the critical properties of the above transitions was stressed long ago [43], where it was argued that an unscreened Coulomb interaction generically results in a dynamical critical exponent $z = 1$. In addition, the observed IQHT and SIT appear to be sensitive to the precise nature of the Coulomb interaction ([44,45] and references therein). For example, a capacitively coupled screening plane has been found to affect the metallic behavior in thin films [46], lifting an anomalous low-temperature metallic regime that intervenes a direct magnetic field-tuned SIT. To investigate such effects, we consider a Coulomb interaction that is screened by a diffusive two-dimensional Fermi gas [47]. The dissipative Coulomb interaction that results allows for two types of fixed points: those with a finite nonzero Coulomb coupling and $z = 1$ and those with an effectively infinite Coulomb interaction and $z \neq 1$ [48]. For the “fermionic dual” of the XY model with random mass disorder, we find critical exponents

$$\nu^{-1} = 1 \quad \text{and} \quad 1 \leq z < 2, \quad (1.6)$$

with z saturating the lower bound for the unscreened Coulomb interaction and varying continuously with an effective dissipation parameter for $z > 1$. In our approach, we are unable to access the “infinite z ” fixed point of the dissipative XY model found in [49]. Our result differs from that of Vishwanath, Moore, and Senthil [47], who studied the effects of a dissipative Coulomb interaction on the dirty XY model using the double- ϵ expansion and found a line of fixed points with $z = 1$ and continuously varying ν . We also consider the effects

of other types of disorder on the theories in (1.4) [and (1.2)] when a dissipative Coulomb interaction is present.

II. MODEL OVERVIEW

In this section, we introduce the effective model that realizes an IQHT/SIT and the critical properties of which we will analyze in Sec. III.

Consider the $(2+1)$ D theory of N_f Dirac fermions ψ_I coupled to a $U(1)$ Chern-Simons gauge field a_μ at level $(\theta - 1/2)$:

$$\mathcal{L}^{(1)} = \sum_{I=1}^{N_f} \bar{\psi}_I (i \not{D}_a - m) \psi_I - \frac{1}{2} \frac{1}{4\pi} ada + \frac{\theta}{4\pi} (a - A) d(a - A) - \frac{1}{4} f_{\mu\nu}^2. \quad (2.1)$$

The notation in (2.1) is as follows: $\bar{\psi} = \psi^\dagger \gamma^0$; $\not{D}_a = (\partial_\mu - ia_\mu) \gamma^\mu$ with $\mu \in \{0, 1, 2\} = \{t, x, y\}$, and a Chern-Simons term $AdA = \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$. For the purpose of discussing the symmetries of (2.1) later in this section, we choose Minkowski signature $\eta^{\mu\nu} = \text{diag}(+1, -1, -1)$ and γ matrices $(\gamma^0, \gamma^1, \gamma^2) = (\sigma^3, i\sigma^1, i\sigma^2)$ where σ^j are the Pauli σ matrices; in the renormalization-group analysis in Sec. III, we will work in Euclidean signature.

When $N_f = 2\theta = 1$, we recover (1.2), the dual of a free Dirac fermion; when $N_f = \theta = 1$, we find the dual (1.4) to the 3D XY model. Reminiscent of conventional flux attachment [50,51], θ^{-1} quantifies the number of attached flux quanta; for general θ , $\mathcal{L}^{(1)}$ is the model for an anyon gas introduced by Chen, Fisher, and Wu [25]. We refer to ψ_I as a Dirac composite fermion. A_μ is a nondynamical $U(1)$ gauge field that we identify with electromagnetism. In Sec. II C we give A_0 dynamics to discuss the Coulomb interaction. In Secs. II A and II B, where we discuss the phase diagram and symmetry of (2.1), we take $N_f = 1$. Otherwise, N_f is an arbitrary parameter that allows for analytic control of the interaction mediated by a_μ as $N_f \rightarrow \infty$.

A. Mean-field phase diagram at $N_f = 1$

For a given θ , the mean-field phase diagram of (2.1) at $N_f = 1$ is parametrized by the Dirac composite fermion mass m . At energies less than $|m|$, we may integrate out the Dirac composite fermion to obtain the effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = \frac{\text{sign}(m) - 1 + 2\theta}{2} \frac{1}{4\pi} ada - \frac{\theta}{2\pi} adA + \frac{\theta}{4\pi} AdA - \frac{1}{4} f_{\mu\nu}^2. \quad (2.2)$$

(By “integrate out,” we refer to path-integral relations of the form $\int \mathcal{D}\phi e^{i \int (\frac{1}{2} \phi K \phi + \phi J)} \propto e^{i \int (-\frac{1}{2} J K^{-1} J)}$, where ϕ and J are real fields and K is some kernel, e.g., a kinetic term for ϕ . Thus, we equate the Lagrangians $\frac{1}{2} \phi K \phi + \phi J = -\frac{1}{2} J K^{-1} J$ upon integrating out ϕ . Such identities follow directly from the ϕ equation of motion when ϕ appears quadratically in the Lagrangian.) Higher-order terms in a_μ can be ignored as $|m| \rightarrow \infty$. The Maxwell term $f_{\mu\nu}^2$ can also be dropped in this long-wavelength analysis.

1. $\theta = 1/2$

Setting $\theta = 1/2$, there are two phases. For $m > 0$ we find the effective Lagrangian for an insulator at zero temperature:

$$\mathcal{L}_{\text{ins}} = \frac{1}{2} \left(\frac{1}{4\pi} ada - \frac{1}{2\pi} adA + \frac{1}{4\pi} AdA \right) = 0, \quad (2.3)$$

where the second equality follows from integrating out a_μ . For $m < 0$ we find the long-wavelength Lagrangian for an integer Hall state:

$$\mathcal{L}_{\text{IQH}} = \frac{1}{2} \left(-\frac{1}{4\pi} ada - \frac{1}{2\pi} adA + \frac{1}{4\pi} AdA \right) = \frac{1}{4\pi} AdA. \quad (2.4)$$

2. $\theta = 1$

Next set $\theta = 1$. We again find the insulator when $m > 0$:

$$\mathcal{L}_{\text{ins}} = \frac{1}{4\pi} ada - \frac{1}{2\pi} adA + \frac{1}{4\pi} AdA = 0. \quad (2.5)$$

To identify the $m < 0$ phase, it is helpful to include the charge $e_* = q$ (measured in units of the electric charge e) carried by the boson Φ in (1.3) by substituting $A_\mu \rightarrow qA_\mu$:

$$\mathcal{L}_{\text{SC}} = -\frac{q}{2\pi} adA + \frac{q^2}{4\pi} AdA. \quad (2.6)$$

Equation (2.6) describes a $\mathbb{Z}/q$ gauge theory, the long-wavelength description of a superconductor with charge- q condensate [52,53].

B. Discrete symmetry at $N_f = 1$

The types of randomness that can be added to (2.1) are characterized by charge-conjugation $\mathcal{C}$ and time-reversal $\mathcal{T}$ symmetries. (Parity, i.e., spatial reflection, is necessarily broken in the presence of quenched disorder.) These symmetries are defined with respect to the electron and boson Lagrangians in Eqs. (1.1) and (1.3). We discuss their implementation [18,19,54–56] in the dual Lagrangian (2.1) at $N_f = 1$ at criticality $m = 0$.

1. $\theta = 1/2$

The free Dirac Lagrangian in (1.1) is invariant under charge conjugation $\mathcal{C}$:

$$\Psi \mapsto \sigma^1 \Psi^*, \quad A_\mu \mapsto -A_\mu. \quad (2.7)$$

The presence of the Chern-Simons term for A_μ reflects the violation of time reversal $\mathcal{T}$:

$$t \mapsto -t, \quad \Psi \mapsto -i\sigma^2 \Psi, \quad (A_0, A_i) \mapsto (A_0, -A_i), \quad (2.8)$$

which is antiunitary ($i \mapsto -i$). On the surface of a time-reversal invariant topological insulator, this Chern-Simons term is absent and so $\mathcal{T}$ can be preserved.

The dual Lagrangian (2.1) at $\theta = 1/2$ is also invariant under $\mathcal{C}$:

$$\psi \mapsto \sigma^1 \psi, \quad a_\mu \mapsto -a_\mu, \quad A_\mu \mapsto -A_\mu. \quad (2.9)$$

Identifying the electromagnetic currents across the duality between (1.1) and (2.1), $\frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta A_\mu} = \frac{\delta \mathcal{L}^{(1)}}{\delta A_\mu}$, we equate

$$\bar{\Psi} \gamma^\mu \Psi = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} \partial_\nu a_\rho. \quad (2.10)$$

Similarly, the a_0 equation of motion relates

$$\frac{1}{4\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho = \bar{\psi} \gamma^\mu \psi. \quad (2.11)$$

Equations (2.10) and (2.11) imply that in (2.1), $\mathcal{CT}$,

$$\begin{aligned} t &\mapsto -t, \quad \psi \mapsto -i\sigma^2 \psi^*, \quad (a_0, a_i) \mapsto (a_0, -a_i), \\ (A_0, A_i) &\mapsto (-A_0, A_i). \end{aligned} \quad (2.12)$$

Thus, $\mathcal{T}$ and $\mathcal{CT}$ are exchanged across the duality: the $\mathcal{T}$ transformations on Ψ and A_μ are identical to the $\mathcal{CT}$ transformations on ψ and a_μ , and vice versa. In the absence of the Chern-Simons term for A_μ , (2.1) is time-reversal invariant.

While the dual Lagrangians in (1.1) and (2.1) violate time-reversal invariance as $(2+1)\text{D}$ theories, they do preserve a “nonlocal” PH transformation. To define this, consider the following transformations of a general Lagrangian $L(A)$ which has a $U(1)$ symmetry current that is coupled to a nondynamical field A_μ [57]:

$$\mathbf{T} : L(A) \mapsto L(A) + \frac{1}{4\pi} A dA; \quad (2.13)$$

$$\mathbf{S} : L(A) \mapsto L(c) + \frac{1}{2\pi} c dA. \quad (2.14)$$

$\mathbf{T}$ shifts the Hall conductivity by a unit; $\mathbf{S}$ converts A_μ into a dynamical $U(1)$ gauge field c_μ and adds a BF term, which couples the field strength dc to a new external field A_μ . Equations (2.13) and (2.14) implement modular transformations on the conductivity tensor of the $U(1)$ symmetry current coupling to A_μ . The PH transformation is defined as $\mathcal{T}$ followed by the modular $\mathbf{T}$ transformation (2.13). Notice that the Dirac masses $\bar{\Psi}\Psi$ and $\bar{\psi}\psi$ are odd under PH symmetry and even under $\mathcal{C}$.

2. $\theta = 1$

The XY model in (1.3) is invariant under charge conjugation $\mathcal{C}$,

$$\Phi \mapsto \Phi^*, \quad A_\mu \mapsto -A_\mu, \quad (2.15)$$

and time reversal $\mathcal{T}$:

$$t \mapsto -t, \quad \Phi \mapsto \Phi, \quad (A_0, A_i) \mapsto (A_0, -A_i). \quad (2.16)$$

The dual Lagrangian in (2.1) at $\theta = 1$ is only invariant under $\mathcal{C}$ defined in (2.9); it is not invariant under $\mathcal{T}$:

$$\begin{aligned} t &\mapsto -t, \quad \psi \mapsto -\sigma^3 \psi^*, \quad (a_0, a_i) \mapsto (-a_0, a_i), \\ (A_0, A_i) &\mapsto (A_0, -A_i), \end{aligned} \quad (2.17)$$

with $i \mapsto -i$. Instead, time reversal is an emergent symmetry of the long-wavelength physics [19,54]. In addition, (2.1) is invariant under a “nonlocal” particle-vortex (PV) transformation:

$$\begin{aligned} t &\mapsto -t, \quad \psi \mapsto -i\sigma^2 \psi, \quad (a_0, a_i) \mapsto (a_0, -a_i), \\ (A_0, A_i) &\mapsto (-A_0, A_i), \end{aligned} \quad (2.18)$$

TABLE I. Charge-conjugation $\mathcal{C}$ and time-reversal $\mathcal{T}$ symmetry assignments of various operators.

	$\mathcal{C}$	$\mathcal{T}$
$\bar{\psi}\psi$	+	−
$\bar{\psi}\gamma_0\psi$	−	−
$\bar{\psi}\gamma_j\psi$	−	+
a_0	−	−
a_j	−	+
$b = \partial_x a_y - \partial_y a_x$	−	+
$e_j = \partial_0 a_j - \partial_j a_0$	−	−

followed by the modular $\mathbf{S}$ transformation (2.14). The PV transformation is analogous to the PH transformation of the previous section [58]; it maps the 3D XY model to its scalar quantum electrodynamics dual [59,60], and vice versa.

Duality maps $|\Phi|^2 \leftrightarrow \bar{\psi}\psi$. While it is clear that the Dirac mass is even under $\mathcal{C}$, it is less obvious that perturbation by $\bar{\psi}\psi$ is time-reversal invariant. This can be understood in the following sense: Perturbation of (2.1) by $\bar{\psi}\psi$ and of the time reversal of (2.1) [obtained using (2.17)] by $-\bar{\psi}\psi$ results in identical phases.

3. Symmetry assignment summary

Table I summarizes the transformations of the operators that appear in (2.1) under charge-conjugation $\mathcal{C}$ and time-reversal $\mathcal{T}$ symmetries. We use these transformation assignments to characterize the types of randomness that may be added to $\mathcal{L}^{(1)}$ for general N_f .

C. Dissipative Coulomb interaction

1. Dualizing the Coulomb interaction

The Coulomb interaction between fermions/bosons carrying charge e_* arises from the exchange of a dynamical $(3+1)$ -dimensional electromagnetic scalar potential A_0 . In Fourier space, we consider the action that couples a $(2+1)\text{D}$ charge density $J_0(k_0, k)$ to the scalar potential $A_0(k_0, \vec{k})$:

$$\begin{aligned} & -\frac{1}{2e_*^2} \int d^4k A_0(k_0, \vec{k})(\vec{k}^2) A_0(-k_0, -\vec{k}) \\ & - \int d^3k J_0(k_0, k) \int dk_3 A_0(-k_0, -\vec{k}), \end{aligned} \quad (2.19)$$

where $k = (k_1, k_2)$ and $\vec{k} = (k_1, k_2, k_3)$. $J_0(k_0, k)$ is the Fourier transform of $\bar{\Psi}\gamma^0\Psi(x)$ for the free Dirac fermion (1.1) or $i\Phi^*\partial_0\Phi(x) - i(\partial_0\Phi^*)\Phi(x)$ for the XY model (1.3). The absence of an $A_0 k_0^2 A_0$ term means that A_0 mediates an instantaneous interaction, an approximation valid for particles moving at speeds much less than the photon velocity. Integrating out the A_0 field we find the unscreened Coulomb interaction,

$$S_{\text{unscreened}} = -\frac{\pi}{2} \int d^3k J_0(k_0, k) \frac{e_*^2}{|k|} J_0(-k_0, -k), \quad (2.20)$$

between $(2+1)\text{D}$ particles. It is convenient to interpret $S_{\text{unscreened}}$ as arising from the exchange of a purely $(2+1)\text{D}$

gauge field $\tilde{A}_0$ with kinetic term and coupling to J_0 as

$$S_{\tilde{A}_0} = - \int d^3k \left[\tilde{A}_0(k_0, k) \frac{|k|}{\pi e_*^2} \tilde{A}_0(-k_0, -k) + J_0(k_0, k) \tilde{A}_0(-k_0, -k) \right]. \quad (2.21)$$

The electromagnetic charge density $J_0(x)$ dualizes in (2.1) according to

$$J_0(x) = \frac{\delta \mathcal{L}^{(1)}}{\delta A_0} = -\frac{\theta}{2\pi} \epsilon_{ij} \partial_i a_j, \quad (2.22)$$

for vanishing A_j . Decomposing the gauge field $a_i(k_0, k) = i \frac{k_i}{|k|} a_L(k_0, k) - i \frac{k_j}{|k|} \epsilon_{ji} a_T(k_0, k)$ in terms of its longitudinal and transverse components, the (unscreened) Coulomb interaction becomes a kinetic term for a_T [61],

$$S_{\text{unscreened}} = -\frac{e^2 \theta^2}{8\pi} \int d^3k a_T(k_0, k) |k| a_T(-k_0, -k), \quad (2.23)$$

that dominates a possible Maxwell term for a_μ at long wavelengths. A similar transformation of the Coulomb interaction occurs in nonrelativistic composite fermion theories [62].

2. Dissipation

To model dissipation following [47], we consider an auxiliary system consisting of a parallel two-dimensional electron gas (2DEG) that is coupled to (2.1) through the Coulomb interaction, specifically, through $\tilde{A}_0$. The spatial separation between the system (2.1) and electron gas is ignored. The electron Green's function is assumed to take a diffusive form:

$$G_{2D}^{-1}(ik_0, k) = ik_0 - \left(\frac{k^2}{2m} - \epsilon_F \right) + \frac{i}{2\tau} \text{sign}(k_0). \quad (2.24)$$

The dissipative effects arising from the coupling to the two-dimensional electron gas are encoded in a correction to the $\tilde{A}_0$ kinetic term in $S_{\tilde{A}_0}$ [63,64]:

$$\delta S_{\tilde{A}_0} = \int d^3k \tilde{A}_0(k_0, k) \frac{\sigma_e k^2}{|k_0| + D_e k^2} \tilde{A}_0(-k_0, -k), \quad (2.25)$$

where the Drude conductivity $\tilde{\sigma}_e = q_e^2 N D_e$ with N the density of states at Fermi energy ϵ_F of the two-dimensional electron gas and D_e its diffusivity. Higher-order corrections due to the two-dimensional electron gas will be ignored. Including $\delta S_{\tilde{A}_0}$ we obtain the dissipation-corrected density-density (2.22) interaction upon integrating out $\tilde{A}_0$:

$$S^{(2)} = -\frac{e^2 \theta^2}{8\pi} \int d^3k a_T(k_0, k) \left(\frac{k^2}{|k| + f(k_0, k)} \right) a_T(-k_0, -k), \quad (2.26)$$

where

$$f(k_0, k) = \frac{\sigma_e k^2}{|k_0| + D_e k^2}, \quad \sigma_e = e_*^2 \tilde{\sigma}_e. \quad (2.27)$$

We recover the dual of an unscreened Coulomb interaction when $q_e = 0$, as expected, or as $|k|/|k_0| \rightarrow 0$. The Coulomb interaction is short ranged as $D_e \rightarrow \infty$ at finite density of states N or when $|k_0|/|k| \rightarrow 0$; in either of these limits, we

find a Maxwell-like kinetic term for a_T (albeit with inverted charge $1/e_*$).

D. Quenched randomness

We consider the effects of quenched disorder that is induced by random $A_\mu(\mathbf{x})$ and $M(\mathbf{x})$. In this discussion, we assume the Coulomb interaction has been included via (2.26) and $A_\mu(\mathbf{x})$ is a nondynamical quenched random variable. Since $m(\mathbf{x}) \propto M(\mathbf{x})$, these perturbations readily map across the duality to

$$\delta \mathcal{L} = -m(\mathbf{x}) \bar{\psi} \psi(x) - \frac{\theta}{2\pi} A(\mathbf{x}) da(x), \quad (2.28)$$

where $\mathbf{x} = (x_1, x_2)$ and $x = (x_0, x_1, x_2)$. The second term in Eq. (2.28) is “topological disorder,” i.e., a random source to the field strength or “topological” current da . We have dropped a possible term proportional to $\epsilon_{ij} A_0(\mathbf{x}) \partial_i A_j(\mathbf{x})$ arising from the Chern-Simons term for A_μ in (2.1).

Interactions generate additional operators with random couplings, consistent with the symmetries of $A_\mu(\mathbf{x})$ and $m(\mathbf{x})$. The (quantum) Harris criterion [65] for Gaussian-correlated randomness implies the relevant terms at low energies correspond to operators with scaling dimensions $\Delta \leq z + 1$. At large N_f [25,66,67] the most generic random terms to include are [31]

$$\begin{aligned} \mathcal{L}_{\text{dis}} = & m(\mathbf{x}) \bar{\psi} \psi(x) + i \tilde{a}_0(\mathbf{x}) \bar{\psi} \gamma^0 \psi(x) + i \tilde{a}_j(\mathbf{x}) \bar{\psi} \gamma^j \psi(x) \\ & - A_0(\mathbf{x}) b(x) + \epsilon_{jk} A_j(\mathbf{x}) e_k(x), \end{aligned} \quad (2.29)$$

where $b = \epsilon_{ij} \partial_i a_j$ and $e_k = \partial_0 a_k - \partial_k a_0$. The random couplings are assumed to be independent Gaussian-correlated quenched random variables with zero mean:

$$\begin{aligned} \langle m(\mathbf{x}) m(\mathbf{x}') \rangle_{\text{dis}} &= g_m \delta^{(2)}(\mathbf{x} - \mathbf{x}'), \\ \langle \tilde{a}_0(\mathbf{x}) \tilde{a}_0(\mathbf{x}') \rangle_{\text{dis}} &= g_0 \delta^{(2)}(\mathbf{x} - \mathbf{x}'), \\ \langle \tilde{a}_k(\mathbf{x}) \tilde{a}_k(\mathbf{x}') \rangle_{\text{dis}} &= g_j \delta^{(2)}(\mathbf{x} - \mathbf{x}'), \quad k \in \{x, y\}, \\ \langle A_0(\mathbf{x}) A_0(\mathbf{x}') \rangle_{\text{dis}} &= \Delta_0 \delta^{(2)}(\mathbf{x} - \mathbf{x}'), \\ \langle A_k(\mathbf{x}) A_k(\mathbf{x}') \rangle_{\text{dis}} &= \Delta_j \delta^{(2)}(\mathbf{x} - \mathbf{x}'), \quad k \in \{x, y\}, \end{aligned} \quad (2.30)$$

where $\langle \cdot \rangle_{\text{dis}}$ indicates a disorder average and there is no sum over k . The disorder variances $g_m, g_0, g_j, \Delta_0, \Delta_j$ are positive constants.

We study the effects of the randomness in (2.29) using the replica trick, which enables the calculation of the disorder-averaged free energy and all observables that derive from it. To this end, we introduce n_r replicas $\psi_{\ell, \epsilon}$ and $a_{\mu, \ell}$ with $\ell \in \{1, \dots, n_r\}$ and consider the replicated partition function,

$$Z^{n_r} = \prod_{\ell} \left(\int \mathcal{D}\psi_{\ell} \mathcal{D}\bar{\psi}_{\ell} \mathcal{D}a_{\ell} \right) e^{i \sum_{\ell} (S^{(1)}[\psi_{\ell}, a_{\ell}] + S^{(2)}[a_{\ell}] + S_{\text{dis}}[\psi_{\ell}, a_{\ell}])}, \quad (2.31)$$

where $S^{(1)}[\psi_{\ell}, a_{\ell}] = \int d^3x \mathcal{L}^{(1)}(\psi_{\ell}, a_{\ell})$ with $\mathcal{L}^{(1)}$ given in (2.1), $S^{(2)}[a_{\ell}]$ is given in (2.26), and $S_{\text{dis}}[\psi_{\ell}, a_{\ell}] = \int d^3x \mathcal{L}_{\text{dis}}(\psi_{\ell}, a_{\ell})$ with $\mathcal{L}_{\text{dis}}$ given in (2.29). Using the identity

$$\log Z = \lim_{n_r \rightarrow 0} \frac{Z^{n_r} - 1}{n_r}, \quad (2.32)$$

the disorder-averaged free energy, proportional to $\langle \log Z \rangle_{\text{dis}}$, is found upon disorder averaging. Using (2.30),

$$\langle Z^{n_r} \rangle_{\text{dis}} = \prod_{\ell} \left(\int \mathcal{D}\psi_{\ell} \mathcal{D}\bar{\psi}_{\ell} \mathcal{D}a_{\ell} \right) \times e^{i \sum_{\ell} (S^{(1)}[\psi_{\ell}, a_{\ell}] + S^{(2)}[a_{\ell}] + i S^{(3)}[\psi_{\ell}, a_{\ell}])}, \quad (2.33)$$

where

$$\begin{aligned} S^{(3)}[\psi_{\ell}, a_{\ell}] = & -\frac{1}{2} \sum_k \int dt dt' d^2x [g_m(\bar{\psi}_{\ell} \psi_{\ell})(t)(\bar{\psi}_k \psi_k)(t') \\ & + g_0(\bar{\psi}_{\ell} \gamma^0 \psi_{\ell})(t)(\bar{\psi}_k \gamma^0 \psi_k)(t') \\ & + g_j(\bar{\psi}_{\ell} \gamma^j \psi_{\ell})(t)(\bar{\psi}_k \gamma^j \psi_k)(t') \\ & + \Delta_0 b_{\ell}(t) b_k(t') + \Delta_j \mathbf{e}_{\ell}(t) \cdot \mathbf{e}_k(t')], \end{aligned} \quad (2.34)$$

$$S_E^{(1)} = \int d\tau d^Dx \left\{ \bar{\psi}_{\ell}^{(I)} \left[\gamma_{\tau} \left(\partial_{\tau} + i \frac{g}{\sqrt{N_f}} a_{\tau, \ell} \right) + v \gamma_j \left(\partial_j + i \frac{g}{\sqrt{N_f}} a_{j, \ell} \right) \right] \psi_{\ell}^{(I)} + m \bar{\psi}_{\ell}^{(I)} \psi_{\ell}^{(I)} + \frac{i\kappa}{2} a_{\ell} da_{\ell} \right\}, \quad (3.2)$$

$$S_E^{(2)} = \int d\omega d^Dk \frac{w_x}{2} a_{\ell, T}(\omega, k) \frac{k^2}{|k| + f(\omega, k)} a_{T, \ell}(-\omega, -k), \quad (3.3)$$

$$\begin{aligned} S_E^{(3)} = & -\frac{1}{2} \int d\tau d\tau' d^Dx [g_m(\bar{\psi}_{\ell}^{(I)} \psi_{\ell}^{(I)})(\tau)(\bar{\psi}_k^{(J)} \psi_k^{(J)})(\tau') + g_0(\bar{\psi}_{\ell}^{(I)} \gamma^0 \psi_{\ell}^{(I)})(\tau)(\bar{\psi}_k^{(J)} \gamma^0 \psi_k^{(J)})(\tau') \\ & + g_j(\bar{\psi}_{\ell}^{(I)} \gamma^j \psi_{\ell}^{(I)})(\tau)(\bar{\psi}_k^{(J)} \gamma^j \psi_k^{(J)})(\tau') + \Delta_0 b_{\ell}(\tau) b_k(\tau') + \Delta_j \mathbf{e}_{\ell}(\tau) \cdot \mathbf{e}_k(\tau')]. \end{aligned} \quad (3.4)$$

Replica indices $\ell, k \in \{1, \dots, n_r\}$ and flavor indices $I, J \in \{1, \dots, N_f\}$ with repeated indices summed. In Euclidean signature $(+, +, +)$, the coordinates $(\tau, x_j) = (it, x_j)$, Fourier space variables $(\omega, k_j) = (-ik_0, k_j)$, and the γ matrices $(\gamma_0, \gamma_1, \gamma_2) = (\sigma^3, \sigma^1, \sigma^2)$. We have set the longitudinal component of a_i to zero (Coulomb gauge): $a_i(\omega, k) = i \frac{k_j}{|k|} \epsilon_{ji} a_T(\omega, k)$. The gauge coupling is $g/\sqrt{N_f}$ with g fixed and $N_f \rightarrow \infty$ [68], v is the Dirac composite fermion velocity, and the Dirac mass m vanishes at criticality. The disorder variances g_m, g_0, g_j, Δ_0 , and Δ_j are assumed to scale as $1/N_f$. The Chern-Simons level is controlled by $\kappa = \frac{2\theta-1}{4\pi}$: $\kappa = 0$ gives an IQHT and $\kappa = 1/4\pi$ gives a SIT. $w_x = \frac{e^2}{4\pi}$ parametrizes the strength of the dissipative Coulomb interaction and $f(\omega, k) = \frac{\sigma_c k^2}{|\omega| + D_c k^2}$. The nondynamical (electromagnetic) field $A_{\mu} = 0$. We will often leave replica and flavor indices, as well as the spacetime dependence of fields, implicit.

We regularize UV divergent integrals that appear in our renormalization-group analysis of S_E using *dimensional reduction* [25, 69, 70]. This is the standard approach (e.g., [25, 70–73] and references therein) used in the study of theories of Chern-Simons gauge fields coupled to matter and, in contrast to dimensional regularization, has been shown to preserve gauge invariance at least to two-loop order in the perturbative analysis of $S_E^{(1)}$ [70]. We assume without proof that this regularization procedure maintains gauge invariance in our large N_f study of S_E , which involves three-loop integrals. We consider a slight variation of the conventional dimensional reduction approach. There are two rules to follow. First, all vector, tensor, and spinor algebra is performed in three dimensions; in particular, the antisymmetric symbol $\epsilon^{\mu\nu\rho}$ obeys the usual 3D identities. Second, loop integrals are analytically

$(\bar{\psi}_{\ell}^{(I)} \psi_{\ell}^{(I)})(t)(\bar{\psi}_k^{(J)} \psi_k^{(J)})(t') \equiv \bar{\psi}_{\ell}^{(I)}(x, t) \psi_{\ell}^{(I)}(x, t) \bar{\psi}_k^{(J)}(x, t') \psi_k^{(J)}(x, t')$, and similarly for the other terms appearing in $S_E^{(3)}$.

III. RENORMALIZATION-GROUP ANALYSIS

We now study the critical properties of the model introduced in Sec. II. Details of our calculations are presented in the Appendices.

A. Large N_f expansion and renormalization-group scheme

The Euclidean effective action in $D + 1$ dimensions is

$$S_E = S_E^{(1)} + S_E^{(2)} + S_E^{(3)} \quad (3.1)$$

where

continued to general (Euclidean) spatial dimension $D \leq 2$:

$$\int \frac{d\omega d^Dk}{(2\pi)^3} \rightarrow \mu^{\epsilon} \int \frac{d\omega d^Dk}{(2\pi)^{D+1}}, \quad (3.5)$$

where $\epsilon = 2 - D$ and μ is the renormalization-group scale. (Typically in dimensional reduction the spacetime dimension is analytically continued.) Simple poles proportional to $2/\epsilon$ are identified with logarithmic divergences proportional to $\log(\Lambda^2/\mu^2)$ in a theory with momentum cutoff Λ ; power-law divergences are set to zero.

The large N_f Feynman rules that derive from S_E at $m = 0$ are given in Fig. 1. Note that we have summed once and for all the geometric series of fermion bubble diagrams in Fig. 2 to obtain the effective gauge field propagator:

$$G_{mn} = \begin{pmatrix} \frac{g^2}{16} \frac{k^2}{\sqrt{\omega^2 + v^2 k^2}} & i\kappa |k| \\ i\kappa |k| & w_x \frac{k^2}{|k| + f(\omega, k)} + \frac{g^2}{16} \sqrt{\omega^2 + v^2 k^2} \end{pmatrix}_{mn}^{-1}, \quad (3.6)$$

where $m, n \in \{0, T\}$ correspond to the zeroth and transverse components of a_{μ} . At large N_f this resummation is equivalent to the random-phase approximation. The same effect also leads to a screening of the g_0 and g_j disorders (see Appendix C) [29, 30]. Aside from a few exceptions that we will discuss, we have found disorder screening to be a subleading effect in our analysis.

We use minimal subtraction [74, 75] to renormalize S_E . In this scheme, simple poles in ϵ appear in counterterms $b_{\lambda_a}(\bar{\lambda}^R)/\epsilon$ that relate bare (B) and renormalized (R) couplings:

$$\lambda_a^B \mu^{-\Delta_a(\epsilon)} = \lambda_a^R(\mu, \epsilon) + \frac{b_{\lambda_a}[\bar{\lambda}^R(\mu, \epsilon)]}{\epsilon}, \quad (3.7)$$

FIG. 1. Feynman rules of S_E . The wavy line denotes the effective gauge field propagator and the directed solid line indicates the fermion propagator with $m = 0$. Disorder is represented by a solid line without an arrow and specified by its disorder variance (g_m, g_0, g_j) . Screenings of the disorder (g_0, g_j) and topological disorder (Δ_0, Δ_j) are discussed in Appendix C.

where the vector of coupling constants (either B or R)

$$\vec{\lambda} = \left(\frac{g^2}{N_f}, v, m, \kappa, w_x, \sigma_e, D_e, g_m, g_0, g_j, \Delta_0, \Delta_j \right)^T. \quad (3.8)$$

The renormalized couplings $\lambda_a^R(\mu, \epsilon)$ and residues $b_{\lambda_a}[\vec{\lambda}^R(\mu, \epsilon)]$ are analytic in ϵ . The higher-order poles that generally occur on the right-hand side of Eq. (3.7) can be set to zero. The bare couplings λ_a^B have engineering dimensions equal to $\Delta_{\lambda_a}(\epsilon)$ while the renormalized couplings are dimensionless. Appendix A details the calculation of the counterterms $b_{\lambda_a}(\vec{\lambda}^R)/\epsilon$. The engineering dimensions $\Delta_{\lambda_a}(\epsilon)$ of the bare couplings are given in Table II. These dimensions are determined as follows. Each term in $S_E^{(1)}$ is dimensionless with the assignments:

$$\Delta_\tau = -\Delta_\omega = -z, \quad \Delta_x = -\Delta_k = -1, \quad (3.9)$$

$$\Delta_\psi(\tau, x) = \frac{1}{2} - \frac{\epsilon}{2}, \quad \Delta_{a_0^E}(\tau, x) = z - \eta\epsilon, \quad \Delta_{a_j^E}(\tau, x) = 1 - \eta\epsilon, \quad (3.10)$$

and

$$\Delta_{g^2/N_f} = 2\eta\epsilon, \quad \Delta_v = z - 1, \quad \Delta_m = z, \quad \Delta_\kappa = (2\eta - 1)\epsilon, \quad (3.11)$$

where η is an arbitrary constant. We have introduced the dynamical critical exponent z with a value to be determined later; in the absence of $S_E^{(2)}$ and $S_E^{(3)}$, relativistic symmetry

requires that v be dimensionless and $z = 1$. In the large N_f expansion, g is fixed and we formally take $\Delta_{N_f} = -2\eta\epsilon$. The effective gauge field propagator is consistent with the engineering dimensions $\Delta_{a_0^E}(\omega, k) = -D - \eta\epsilon$ and $\Delta_{a_j^E}(\omega, k) = (1 - z) - D - \eta\epsilon$ if $\eta = 1/2$. The dimensions of the remaining couplings ensure the terms in $S_E^{(2)}$ and $S_E^{(3)}$ are dimensionless.

The beta functions β_{λ_a} at $\epsilon = 0$ are read off from the residues $b_{\lambda_a}(\vec{\lambda}^R)$ using

$$\beta_{\lambda_a}(\vec{\lambda}^R) \equiv -\mu \frac{\partial \lambda_a^R}{\partial \mu} = \bar{\Delta}_{\lambda_a} \lambda_a^R + \rho_{\lambda_a} b_{\lambda_a}(\vec{\lambda}^R) - \sum_c \rho_{\lambda_c} \lambda_c^R \frac{\partial b_{\lambda_a}(\vec{\lambda}^R)}{\partial \lambda_c^R}. \quad (3.12)$$

There is no sum over a in Eq. (3.12). The minus sign in front of $\mu \frac{\partial \lambda_a^R}{\partial \mu}$ means that a relevant/irrelevant coupling has a positive/negative beta function. Notice that only g^2/N_f and the variances g_m, g_0, g_j can contribute to the derivative term on the right-hand side of Eq. (3.12).

We characterize any fixed points $\beta_{\lambda_a}[(\vec{\lambda}^R)^*] = 0$ by the dynamical critical exponent z and correlation length exponent ν , evaluated at the fixed point. The dynamical critical exponent enters the beta functions (3.12) via $\bar{\Delta}_{\lambda_a}$ (see Table II) and we determine its value by the condition of vanishing velocity beta

FIG. 2. The effective gauge field propagator G . The dotted line G_0 represents the bare gauge field propagator and $\Pi_{\mu\nu}$ is the one-loop gauge field self-energy. Each term in this geometric series of diagrams produces an $\mathcal{O}(N_f^0)$ correction to the gauge field propagator, since each fermion loop contributes a factor of N_f and the two vertices associated to each loop contribute an additional factor of g^2/N_f .

TABLE II. Engineering dimension $\Delta_{\lambda_a}(\epsilon) = \bar{\Delta}_{\lambda_a} + \rho_{\lambda_a}\epsilon$ of bare coupling λ_a^B (B superscript omitted in the table), where $\bar{\Delta}_{\lambda_a}$ is independent of $\epsilon = 2 - D$ and ρ_{λ_a} is the constant coefficient of ϵ .

λ_a	$\Delta_{\lambda_a}(\epsilon)$	$\bar{\Delta}_{\lambda_a}$	ρ_{λ_a}
$\frac{g^2}{N_f}$	ϵ	0	1
$\{v, w_x, \sigma_e\}$	$z - 1$	$z - 1$	0
m	z	z	0
$\{\kappa, \Delta_j\}$	0	0	0
D_e	$z - 2$	$z - 2$	0
$\{g_m, g_0, g_j\}$	$2z - D$	$2z - 2$	1
Δ_0	$2z - 2$	$2z - 2$	0

function $\beta_v(\vec{\lambda}^R) = 0$:

$$z = 1 + \frac{1}{v^R} \sum_c \rho_{\lambda_c} \lambda_c^R \frac{\partial b_v[(\vec{\lambda}^R)^*]}{\partial \lambda_c^R}. \quad (3.13)$$

Nonzero β_v implies a quantum correction to the tree-level dynamical exponent, i.e., the engineering dimension $-\Delta_\tau$. This follows from the fermion dispersion relation $|\omega| = v|k|$, e.g., [48]. We have chosen to introduce an arbitrary z in Table II with a value to be determined by vanishing renormalization of the velocity. An equivalent choice is to take engineering dimensions consistent with conformal invariance and infer any correction to the tree-level dynamical scaling from a nonzero velocity beta function.

Since the transitions we consider in this paper are tuned by the Dirac mass, we define the correlation length ξ as the inverse momentum scale μ_0^{-1} at which $v^R(\mu_0)/m^R(\mu_0) = 1$. [The factor of $v^R(\mu_0)$ accounts for possible running of the velocity in the equivalent approach where $z = 1$ is chosen in Table II and the nonzero velocity beta function determines the correction $z - 1$ to dynamical scaling.] We write the mass beta function as

$$\beta_m(\vec{\lambda}^R) = [z - \gamma_{\bar{\psi}\psi}(\vec{\lambda}^R)]m^R, \quad (3.14) \quad \text{and}$$

$$F_w(\overline{w_x}, \kappa, \overline{\sigma_e}) = \frac{1}{4\pi^2 N_f} \int_{-\infty}^{\infty} dy \frac{g_1(-1 + 2y^2)(\sigma_e + |y|) + \overline{w_x}|y|\sqrt{1 + y^2}}{(1 + y^2)^2 [\sqrt{1 + y^2}(g_1^2 + \kappa^2)(\overline{\sigma_e} + |y|) + g_1 \overline{w_x}|y|]}. \quad (3.19)$$

The beta functions $\beta_{\lambda_a} = -\mu \frac{\partial \lambda_a}{\partial \mu}$ for the remaining couplings take the form

$$\beta_{\overline{w_x}} = \overline{w_x}(z - 1), \quad (3.20)$$

$$\beta_{\overline{\sigma_e}} = \overline{\sigma_e}(z - 1), \quad (3.21)$$

$$\beta_{D_e} = D_e(z - 2), \quad (3.22)$$

$$\beta_{\overline{m}} = \overline{m}(z - \gamma_{\bar{\psi}\psi}), \quad (3.23)$$

$$\beta_{\overline{g_m}} = 2\overline{g_m} \left(z - 1 + \frac{2\overline{g_0} \overline{g_j}}{\overline{g_m}} - \gamma_{\bar{\psi}\psi} \right), \quad (3.24)$$

$$\beta_{\overline{g_0}} = 2\overline{g_0} \left(z - 1 + \frac{2\overline{g_j} \overline{g_m}}{\overline{g_0}} \right), \quad (3.25)$$

$$\beta_{\overline{g_j}} = 2\overline{g_j} \left(z - 1 + \frac{\overline{g_m} \overline{g_0}}{\overline{g_j}} - \overline{g_m} - \overline{g_0} - 2\overline{g_j} + F_w(\overline{w_x}, \kappa, \overline{\sigma_e}) \right), \quad (3.26)$$

where the anomalous dimension $\gamma_{\bar{\psi}\psi}$ controls the asymptotic scaling of the correlation function $\langle \bar{\psi}\psi(\tau, x) \bar{\psi}\psi(0) \rangle \sim |v^2 \tau^2 + x^2|^{-(D + \gamma_{\bar{\psi}\psi})}$. Using Eqs. (3.12)–(3.14), we find

$$\xi = \Lambda^{-1} \left(\frac{m_\Lambda / v_\Lambda}{\Lambda} \right)^{-v} \quad (3.15)$$

where Λ is an arbitrary momentum cutoff defining the “initial conditions,” $m^R(\Lambda) = m_\Lambda / \Lambda^z$ and $v^R(\Lambda) = v_\Lambda / \Lambda^{z-1}$, and the inverse correlation length exponent

$$v^{-1} = z - \gamma_{\bar{\psi}\psi}. \quad (3.16)$$

Note that m^R does not enter the residues $b_{\lambda_a}(\vec{\lambda}^R)$ with $\lambda_a^R \neq m^R$ and only appears linearly in $b_m(\vec{\lambda}^R)$.

In the remainder of the main text, we drop the B and R superscripts for notational simplicity; the couplings appearing in the beta functions are renormalized ones.

B. General beta functions

We now present the results of our renormalization-group calculation, which is valid to order $1/N_f$ in the large N_f expansion. See the Appendices for details.

A vanishing velocity beta function determines the dynamical critical exponent to be

$$z = 1 + \overline{g_m} + \overline{g_0} + 2\overline{g_j} - F_w(\overline{w_x}, \kappa, \overline{\sigma_e}), \quad (3.17)$$

where $g_1 = g^2/16$, the rescaled couplings are

$$\begin{aligned} \overline{m} &= \frac{m}{v}, & \overline{w_x} &= \frac{w_x}{v}, & \overline{\sigma_e} &= \frac{\sigma_e}{v}, & \overline{g_m} &= \frac{g_m}{2\pi v^2}, \\ \overline{g_0} &= \frac{g_0}{2\pi v^2}, & \overline{g_j} &= \frac{g_j}{2\pi v^2}, & \overline{\Delta_0} &= \Delta_0, & \overline{\Delta_j} &= \Delta_j v^2, \end{aligned} \quad (3.18)$$

$$\beta_{\overline{\Delta_0}} = \overline{\Delta_0}(2z - 2) + \frac{\overline{g_m}[\overline{\Delta_j}(g_1 + \overline{w_x})^2 + \overline{\Delta_0}\kappa^2]}{64[g_1(g_1 + \overline{w_x})]^2} + \frac{\overline{g_0}\overline{g_m}N_f\pi v^2}{32}, \quad (3.27)$$

$$\beta_{\overline{\Delta_j}} = \frac{\overline{g_m}(g_1^2\overline{\Delta_0} + \kappa^2\overline{\Delta_j})}{128[g_1(g_1 + \overline{w_x}) + \kappa^2]^2} + \frac{\overline{g_j}\overline{g_m}N_f\pi v^4}{64}, \quad (3.28)$$

where z is given in Eq. (3.17), the mass anomalous dimension

$$\gamma_{\bar{\psi}\psi} = 2\overline{g_m} + 2\overline{g_0} - \frac{\overline{g_0}g_1 - \overline{g_j}(g_1 + \overline{w_x})}{g_1^2 + g_1\overline{w_x} + \kappa^2} + F_m(\overline{w_x}, \kappa, \overline{\sigma_e}) - F_w(\overline{w_x}, \kappa, \overline{\sigma_e}), \quad (3.29)$$

and

$$F_m(\overline{w_x}, \kappa, \overline{\sigma_e}) = \frac{1}{4\pi^2 N_f} \int_{-\infty}^{\infty} dy \left[\frac{g_1(\overline{\sigma_e} + |y|)(-2y^2 - 3) - \overline{w_x}\sqrt{1+y^2}|y|}{(1+y^2)^2 [\sqrt{1+y^2}(g_1^2 + \kappa^2)(\overline{\sigma_e} + |y|) + g_1\overline{w_x}|y|]} \right. \\ \left. + \frac{(\overline{\sigma_e} + |y|)[\sqrt{1+y^2}(g_1^2 - \kappa^2)(\overline{\sigma_e} + |y|) + g_1\overline{w_x}|y|]}{2(1+y^2)[\sqrt{1+y^2}(g_1^2 + \kappa^2)(\overline{\sigma_e} + |y|) + g_1\overline{w_x}|y|]^2} \right]. \quad (3.30)$$

To simplify the above expressions, we have ignored terms that arise from the screening of the g_0 and g_j disorders (see Appendix C); our detailed analysis below includes such effects whenever relevant. The gauge coupling $g/\sqrt{N_f}$ is marginal once the large N_f effective gauge field propagator in Fig. 2 is adopted and so its beta function is not included.

Let us make a few additional comments about these expressions.

(1) In general, the above beta functions do not have an IR stable solution at $\overline{m} = 0$, even when disorder screening is included. In the remaining sections, we analyze cases for which we have found fixed points when a symmetry is present.

(2) We have taken the variances to scale as $1/N_f$ for $N_f \rightarrow \infty$. The beta functions have terms that scale as $1/N_f$ and $1/N_f^2$. The “classical” contributions to the beta functions arising from the engineering dimensions of couplings scale as $1/N_f$; the “quantum” corrections generally scale as $1/N_f^2$. The exception to the latter appears in the third term in $\beta_{\overline{\Delta_0}}$ and the second term in $\beta_{\overline{\Delta_j}}$. A possible alternative scaling scheme allows the $\overline{\Delta_0}$ and $\overline{\Delta_j}$ to scale as N^0 with all other variances scaling as $1/N$.

(3) The first three beta functions $\beta_{\overline{w_x}}$, $\beta_{\overline{\sigma_e}}$, and β_{D_e} characterize the dissipative Coulomb interaction. In our analysis, we consider $z < 2$ and so the diffusion constant D_e is an irrelevant parameter that will be set to zero. A nonzero Coulomb interaction allows for two classes of fixed points: (1) a *finite Coulomb interaction* either with $\overline{w_x}, \overline{\sigma_e} \neq 0$ and $z = 1$ or with $\overline{w_x} = \overline{\sigma_e} = 0$ and z determined by Eq. (3.17) and (2) an *infinite Coulomb interaction* with $\overline{w_x} \rightarrow \infty, \overline{\sigma_e} \rightarrow \infty$, and $1 < z < 2$ that is controlled by the dissipation parameter $\overline{\sigma_e}/\overline{w_x}$.

(4) Whenever two of the three disorder variances g_m, g_0 , and g_j are considered, the third variance is radiatively generated. When all three variances g_m, g_0 , and g_j are present, both types of topological disorder Δ_0 and Δ_j are generated. This is consistent with the symmetry assignments in Table I.

C. Finite Coulomb interaction

1. No disorder

In the absence of any disorder, the only nontrivial beta functions are associated to the Coulomb

interaction:

$$\beta_{\overline{w_x}} = -\overline{w_x} F_w(\overline{w_x}, \kappa, \overline{\sigma_e}). \quad (3.31)$$

Since $\frac{1}{\overline{\sigma_e}}\beta_{\overline{\sigma_e}} = \frac{1}{\overline{w_x}}\beta_{\overline{w_x}}$, it is sufficient to consider the behavior of $\overline{w_x}$ when studying a finite Coulomb interaction. The integral that defines F_w in Eq. (3.19) can only be evaluated numerically for general $\overline{\sigma_e}$. We have found that F_w is positive for any κ when $\overline{w_x} \neq 0$. Consequently, the clean fixed point with $\overline{w_x}^* = \overline{\sigma_e}^* = 0$ is perturbatively stable to the addition of a Coulomb interaction and $z = 1$. Two examples for the behavior of $\beta_{\overline{w_x}}$ are displayed in Fig. 3.

In particular, in the limit of $\overline{\sigma_e} = 0$ the beta function for $\overline{w_x}$ is always negative for any Chern-Simons coupling κ . This result should be contrasted with earlier work [28] where a critical value of $|\kappa|$ was reported above which the Coulomb interaction was found to be relevant. The discrepancy seems to arise from assigning the Chern-Simons coupling κ an engineering dimension proportional to $\epsilon = 2 - D$. This choice, which appears to be inconsistent with the scaling of the effective gauge field propagator in the large N_f expansion, results in additional derivatives with respect to κ in the beta function in (3.12). As a check on our calculation, we find the mass anomalous dimension is given by

$$\gamma_{\bar{\psi}\psi} = \frac{128}{3\pi^2 N_f} \frac{1 - 512\kappa^2}{(1 + 256\kappa^2)^2} \quad (3.32)$$

in agreement with [25,67,76,77] for general κ . For the IQHT ($\kappa = 0$), $\nu^{-1} \approx 1 - 4.3/N_f$; for the SIT ($\kappa = 1/4\pi$), $\nu^{-1} \approx 1 + 1.4/N_f$.

2. C symmetry

According to Table I, the Coulomb couplings and random mass disorder (g_m) are allowed when there is charge-conjugation symmetry. The beta functions are

$$\beta_{\overline{w_x}} = \overline{w_x}[\overline{g_m} - F_w(\overline{w_x}, \kappa, \overline{\sigma_e})], \quad (3.33)$$

$$\beta_{\overline{g_m}} = -2\overline{g_m}[\overline{g_m} + F_m(\overline{w_x}, \kappa, \overline{\sigma_e})], \quad (3.34)$$

where F_w and F_m are defined in Eqs. (3.19) and (3.30). The flow of the random mass is controlled by the mass

FIG. 3. $\beta_{\overline{w}_x}$ as a function of $\overline{w}_x$ and κ . (a) $\overline{\sigma}_e = 0$. (b) $\overline{\sigma}_e = 5.0$.

anomalous dimension $\gamma_{\bar{\psi}\psi} = 2\overline{g}_m - F_w(\overline{w}_x, \kappa, \overline{\sigma}_e) + F_m(\overline{w}_x, \kappa, \overline{\sigma}_e)$. Within the large N_f approximation, random mass disorder is only a relevant perturbation to the clean fixed point of the previous section ($\overline{w}_x^* = \overline{\sigma}_e^* = 0$) when $\gamma_{\bar{\psi}\psi} < 0$, i.e., when $1 < 512\kappa^2$ [see Eq. (3.32)], in agreement with [31]. The presence of a Coulomb interaction does not appear to alter this conclusion within our analysis. For $\kappa = 1/4\pi$ (or any $\kappa^2 > 1/512$), there exists a line of fixed points with finite disorder and Coulomb interaction parametrized by $\overline{\sigma}_e$. Figure 4 shows a few examples of this behavior. Since $\beta_{\overline{w}_x} \propto z - 1$ and $\beta_{\overline{g}_m} \propto -\gamma_{\bar{\psi}\psi}$ when $z = 1$, any fixed point with finite disorder and Coulomb interaction has

$$\nu^{-1} = z = 1. \quad (3.35)$$

At the (generally unstable) fixed point with nonzero random mass disorder and vanishing Coulomb interaction,

$$\nu^{-1} = 1 \quad \text{and} \quad z = 1 - \frac{128}{3\pi^2 N_f} \frac{1 - 512\kappa^2}{(1 + 256\kappa^2)^2}, \quad (3.36)$$

where $\kappa^2 > 1/512$. For $\kappa = 1/4\pi$, $z \approx 1 + 1.4/N_f$.

It is interesting to compare our results for the critical exponents with recent analytic and numerical studies of the dirty XY model. In a large N_f expansion [38] report $\nu = 1$

and $z = 1 + 0.5/N_f$; numerics [42] directly probes $N_f = 1$ with the result $\nu = 1.16(5)$ and $z = 1.52(3)$. In [30], a finite disorder fixed point of quantum electrodynamics without the Chern-Simons term was found using an ϵ expansion about $(3+1)\text{D}$. Since our approximation schemes are different, there is no contradiction with our conclusion that random mass disorder is irrelevant when $\kappa = 0$. Nevertheless, it would be interesting to consider this issue further.

3. $\mathcal{CT}$ symmetry

According to Table I, the Coulomb coupling, random scalar potential g_0 , and topological disorder Δ_j are allowed by $\mathcal{CT}$ symmetry. Because a nonzero Chern-Simons term is odd under time-reversal symmetry, we only consider $\kappa = 0$ in the next two subsections that study $\mathcal{CT}$ and $\mathcal{T}$ preserving disorder. The beta functions are

$$\beta_{\overline{w}_x} = \overline{w}_x [\overline{g}_0(1 - \phi_1) - F_w(\overline{w}_x, \kappa = 0, \overline{\sigma}_e)], \quad (3.37)$$

$$\beta_{\overline{g}_0} = \frac{2\overline{g}_0}{\overline{w}_x} \beta_{\overline{w}_x}, \quad (3.38)$$

$$\beta_{\Delta_j} = 0, \quad (3.39)$$

FIG. 4. RG flow of $\overline{w}_x$ vs $\overline{g}_m$. In the above plot, we choose $N_f = 4$. (a) $\kappa = 1/4\pi$, $\overline{\sigma}_e = 0$. (b) $\kappa = 1/4\pi$, $\overline{\sigma}_e = 0.1$. (c) $\kappa = 0$, $\overline{\sigma}_e = 0$.

FIG. 5. RG flow of $\overline{w}_x$ vs $\overline{g}_0$. In the above figures, we set $N_f = 4$, $\kappa = 0$, $\overline{\sigma}_e = 0$. (a) Disorder screening ($\phi_1 = 1$). (b) No disorder screening ($\phi_1 = 0$).

where ϕ_1 isolates any terms that arise from the screening of the disorder: $\phi_1 = 0$ means disorder screening is ignored; $\phi_1 = 1$ means that disorder screening is included. While we are unaware of a general reason to exclude disorder screening, we will discuss the behavior of the above beta functions both with and without screening to illustrate its effect.

As mentioned previously, F_w in Eq. (3.19) is positive for any κ when $\overline{w}_x$ is nonzero; in particular, when $\overline{\sigma}_e = 0$, $F_w(\overline{w}_x, 0, 0)$ is a monotonically increasing function that approaches $\frac{8}{N_f \pi^2}$ for $\overline{w}_x \rightarrow \infty$. When disorder screening is ignored ($\phi_1 = 0$), there is a fixed surface defined by $\overline{g}_0 = F_w$, which is parametrized by $(\overline{w}_x^*, \overline{g}_0^*, \overline{\Delta}_j^*)$, in agreement with [31]. This fixed surface is unstable, e.g., consider perturbation to $\overline{g}_0$ at fixed $\overline{w}_x^*$ and $\overline{\Delta}_j^*$. When disorder screening is included ($\phi_1 = 1$), there is a line of *stable* fixed points parametrized by $(\overline{w}_x^*, \overline{g}_0^*, \overline{\Delta}_j^*) = (0, 0, \overline{\Delta}_j^*)$. This result is consistent with [30]. This behavior is illustrated in Fig. 5. The corresponding critical exponents at the stable fixed point ($\phi_1 = 1$) reduce to those of the clean theory without Coulomb interactions. Recall that g_0 and Δ_j disorders are generated by random electrical vector potential $A_j(\mathbf{x})$ in the free Dirac fermion dual at $N_f = 1$ for which a line of diffusive fixed points was found in [16]. It is unclear to what extent the line of fixed points parametrized by Δ_j is related.

If $\mathcal{CT}$ and $\mathcal{T}$ are emergent symmetries of the SIT theory we consider, then the beta functions should only have fixed-point solutions respecting these symmetries at $\kappa = 1/4\pi$. Unfortunately, the leading terms in the large N_f beta functions do not produce any such nontrivial fixed points. Even if $\overline{g}_m$ is initially tuned to zero, the random mass beta function receives a positive correction from disorder screening equal to

$$\delta\beta_{\overline{g}_m} = \frac{8\overline{g}_0^2 \overline{g}_1^2 \kappa^4}{(g_1^2 + g_1 \overline{w}_x + \kappa^2)^4}. \quad (3.40)$$

Nonzero $\overline{g}_0$ and $\overline{g}_m$ then result in the generation of all couplings, for which we find runaway flow.

4. $\mathcal{T}$ symmetry

According to Table I, the Coulomb coupling, random vector potential g_j , and topological disorder Δ_0 are allowed by $\mathcal{T}$ symmetry. The beta functions are

$$\beta_{\overline{w}_x} = \overline{w}_x \left\{ 2\overline{g}_j \left[1 - \phi_1 \frac{g_1 (g_1 + 2\overline{w}_x)}{(g_1 + \overline{w}_x)^2} \right] - F_w(\overline{w}_x, \kappa = 0, \overline{\sigma}_e) \right\}, \quad (3.41)$$

$$\beta_{\overline{g}_j} = 0, \quad (3.42)$$

$$\beta_{\overline{\Delta}_0} = \frac{2\overline{\Delta}_0}{\overline{w}_x} \beta_{\overline{w}_x}, \quad (3.43)$$

where we continue to use ϕ_1 to isolate terms that arise from the screening of the disorder.

If screening is ignored ($\phi_1 = 0$), there is a surface of fixed points defined by $2\overline{g}_j^* = F_w(\overline{w}_x^*, 0, \overline{\sigma}_e^*)$ and parametrized by $(\overline{w}_x^*, \overline{g}_j^*, \overline{\Delta}_0^*)$ with $\overline{g}_j^* < 4/N_f \pi^2$ and non-negative $\overline{\Delta}_0^*$. On this surface $z = 1$ and $v^{-1} = 1 + 2\overline{g}_j^* - \frac{\overline{g}_1^*}{g_1} - F_m(\overline{w}_x^*, 0, \overline{\sigma}_e^*)$; F_m is a monotonically decreasing function of $\overline{w}_x^*$ when $\kappa = \overline{\sigma}_e^* = 0$: $\frac{128}{3\pi^2 N_f} \geq F_m(\overline{w}_x^*) \geq -\frac{8}{\pi^2 N_f}$. For fixed $\overline{g}_j^*$, this surface is stable to small deformation by $\overline{w}_x$ since F_w is an increasing function of $\overline{w}_x$. For $\overline{g}_j^* > 4/N_f \pi^2$, we find runaway flows. This behavior is shown in Fig. 6. If screening is included ($\phi_1 = 1$), the fixed points are determined by the equation

$$\overline{g}_j^* = \frac{F_w(\overline{w}_x^*)}{2[1 - \phi_1 \frac{g_1 (g_1 + 2\overline{w}_x^*)}{(g_1 + \overline{w}_x^*)^2}]} \equiv f_s(\overline{w}_x^*) \quad (3.44)$$

where $f_s(\overline{w}_x^*)$ monotonically decreases from $f_s(\overline{w}_x^* = 0) = \infty$ to $f_s(\overline{w}_x^* \rightarrow \infty) = \frac{4}{N_f \pi^2}$. If $\overline{g}_j^*$ is chosen to be smaller than $\frac{4}{N_f \pi^2}$, then the RG flows to $(\overline{w}_x^*, \overline{\Delta}_0^*) = (0, 0)$ because $\beta_{\overline{w}_x}$ is non-negative and only vanishes when $\overline{w}_x = 0$. If

FIG. 6. RG flow of $\overline{w}_x$ vs $\overline{\Delta}_0$. In the above figures, we take the nonscreening choice ($\phi_1 = 0$) with $\kappa = \sigma_e = 0$, $N_f = 4$, so the critical value $\frac{4}{N_f \pi^2} = 0.101$. (a) $\overline{g}_j = 0$, flowing to a clean fixed point. (b) $\overline{g}_j = 0.04 < \frac{4}{N_f \pi^2}$, there is a fixed line. (c) $\overline{g}_j = 0.2 > \frac{4}{N_f \pi^2}$, running to strong-coupled.

$\overline{g}_j^* > \frac{4}{N_f \pi^2}$, then there exists a finite value of $\overline{w}_x^*$ for which the beta function vanishes; however, the resulting fixed point is IR unstable.

D. Infinite Coulomb interaction

Dissipation has played only a minor role in the above analysis. We will now discuss how dissipation allows for fixed points with $z \neq 1$ in the presence of a Coulomb interaction [48].

The runnings of the Coulomb interaction parameters $\overline{w}_x$ and $\overline{\sigma}_e$ are determined by their engineering dimensions, which are both equal to $z - 1$ [see Eqs. (3.20) and (3.21)], in the

large N_f expansion. Any situation with nonzero Coulomb interaction and $z > 1$ necessarily requires $\overline{w}_x$ and $\overline{\sigma}_e$ individually flowing to strong coupling. Note, however, that it is their dimensionless ratio $\overline{\sigma}_e/\overline{w}_x$ that appears in the action S_E in the limit $\overline{w}_x, \overline{\sigma}_e \rightarrow \infty$. Consequently, we can parametrize this infinite Coulomb interaction limit with the marginal parameter $\alpha = \overline{\sigma}_e/\overline{w}_x$. We refer to α as the dissipation strength. $z > 1$ is required for any fixed point with infinite Coulomb interaction to be IR attractive; treating α as a tuning parameter, we will view any infinite Coulomb interaction fixed point with $0 < z < 1$ as an IR unstable fixed point.

For $\overline{w}_x \rightarrow \infty$, F_w in (3.19) reduces to

$$F_\infty(\kappa, \alpha) \equiv \int_{-\infty}^{\infty} dy \frac{g_1 \sqrt{1+y^2} (-1+2y^2) \alpha + (1+y^2) |y|}{4\pi^2 N_f (1+y^2)^{\frac{5}{2}} [\sqrt{1+y^2} (g_1^2 + \kappa^2) \alpha + g_1 |y|]}. \quad (3.45)$$

For any κ and $\alpha \geq 0$, $0 \leq F_\infty(\kappa, \alpha) \leq \frac{8}{N_f \pi^2}$. In this limit, the dynamical critical exponent at infinite Coulomb coupling is

$$z_\infty = 1 + \overline{g}_m + \overline{g}_0 + 2\overline{g}_j - \phi_1 \overline{g}_0 - F_\infty(\kappa, \alpha), \quad (3.46)$$

where we have explicitly indicated with ϕ_1 how screening appears in z_∞ .

1. C symmetry

At infinite Coulomb coupling and in the presence of charge-conjugation symmetry, there exist nontrivial fixed points for any κ . These occur at small values of $\overline{g}_m$ and are found by solving $\beta_{\overline{g}_m} = 0$ from (3.24) using Eq. (3.46):

$$\begin{aligned} \overline{g}_m^*(\alpha) &= F_m(\alpha, \kappa) \\ &\equiv \int_{-\infty}^{\infty} dy \frac{\sqrt{1+y^2} |y| + g_1 \alpha (3+2y^2)}{4N_f \pi^2 (1+y^2)^2 [g_1 |y| + \alpha \sqrt{1+y^2} (g_1^2 + \kappa^2)]} - \frac{\alpha [g_1 \sqrt{1+y^2} |y| + (1+y^2) (g_1^2 - \kappa^2) \alpha]}{8N_f \pi^2 (1+y^2)^{\frac{3}{2}} [g_1 |y| + \alpha \sqrt{1+y^2} (g_1^2 + \kappa^2)]^2}. \end{aligned} \quad (3.47)$$

Figure 7 shows the behavior of the renormalization-group flows for $\kappa = 0$ and $1/4\pi$.

Note that the beta functions for $\overline{w}_x \rightarrow \infty$ are different from the case of finite $\overline{w}_x$, even when $\alpha = 0$: nontrivial fixed points exist for any value of κ . The correlation length exponent $\nu^{-1} = 1$ because the fixed points are solved from $\beta_{\overline{g}_m} = 0$ [see

Eq. (3.29)]. The dynamical critical exponent z_∞^* is found using (3.46):

$$z_\infty^*(\alpha) = \overline{g}_m^*(\alpha) - F_\infty(\kappa, \alpha). \quad (3.48)$$

To guarantee the irrelevancy of the diffusion constant D_e of the 2DEG bath, $z_\infty^* < 2$ is required. Fortunately, z_∞^* does

FIG. 7. RG flow of α vs $\overline{g}_m$. The dissipation parameter α is exactly marginal, so it is a free parameter that can be tuned. In the above figure, we choose $N_f = 4$. (a) $\kappa = 0$. (b) $\kappa = \frac{1}{4\pi}$.

not exceed 2 for the values of α we have considered—see Fig. 8.

2. $\mathcal{CT}$ symmetry

As with finite Coulomb coupling, we focus on $\kappa = 0$ in this and the next subsection because the Chern-Simons term is odd under $\mathcal{CT}$ and $\mathcal{T}$.

In the $\overline{w}_x \rightarrow \infty$ limit, only $\beta_{\overline{g}_0} = 0$ is nontrivial:

$$2\overline{g}_0^2(-1 + \phi_1) + 2\overline{g}_0 F_\infty(\kappa = 0, \alpha) = 0. \quad (3.49)$$

Including disorder screening ($\phi_1 = 1$), $\overline{g}_0$ is marginally irrelevant. If disorder screening is ignored, an unstable fixed point lies at $\overline{g}_0^* = F_\infty(\kappa = 0, \alpha)$. Perturbation by $\overline{g}_0$ about this fixed point results either in flow (along the $\overline{g}_0$ direction) towards strong coupling or towards the infinite Coulomb interaction clean fixed point with critical exponents:

$$z_\infty(\alpha) = 1 - F_\infty(\kappa = 0, \alpha) < 1, \quad (3.50)$$

$$\nu^{-1} = 1 - F_m(\alpha, \kappa = 0). \quad (3.51)$$

FIG. 8. Dynamical critical exponent z_∞^* vs effective dissipation strength α evaluated on the fixed points when $N_f = 4$. The green line corresponds to $z_\infty^* < 1$, which is an unstable $\overline{w}_x \rightarrow \infty$ fixed point. When $\kappa = 1/4\pi$, $1 < z_\infty^* < 2$ for $\alpha \geq 1.47$.

Since $z_\infty < 1$, these infinite Coulomb coupling fixed points are IR unstable. This renormalization-group flow is shown in Fig. 9.

3. $\mathcal{T}$ symmetry

When time-reversal symmetry is preserved and $\kappa = 0$, the only nontrivial beta function is $\beta_{\overline{\Delta}_0}$. In the limit of strong Coulomb coupling, the disorder screening terms vanish. Solving $\beta_{\overline{\Delta}_0} = 0$ gives the condition

$$\overline{g}_j^* = \frac{1}{2} F_\infty(\alpha, \kappa = 0) \quad (3.52)$$

on the marginal couplings $\overline{g}_j$ and α . The resulting fixed point is IR unstable along the $\overline{\Delta}_0$ direction and, depending on the values of $\overline{g}_j$ and α , flows either to strong coupling or to

FIG. 9. The RG flow of $\overline{g}_0$ with respect to α , where $\kappa = 0$ is used.

FIG. 10. The case of $\overline{w}_x \rightarrow \infty$ with $\mathcal{T}$ symmetry. (a) RG flow of $\overline{\Delta}_0$ vs α for setting $\overline{g}_j = 0.08$. (b) The behavior of $\beta_{\overline{\Delta}_0}$ when adjusting $\overline{g}_j$ and α .

zero when perturbed about this fixed point. This is shown in Fig. 10. Using Eq. (3.46), we see that $z_\infty = 1$ at the fixed point defined by $\beta_{\overline{\Delta}_0} = 0$. The correlation length exponent is given by $\nu^{*-1} = 1 + 2\overline{g}_j^* - F_m(\alpha, \kappa = 0) - \frac{\overline{g}_j^*}{g_1} = 1 - 7F_\infty(\alpha, \kappa = 0) - F_m(\alpha, \kappa = 0)$, where the second equality is obtained after evaluating on the fixed point defined by Eq. (3.52).

IV. DISCUSSION

In this paper, we studied the influence of quenched disorder and a dissipative Coulomb interaction on two different quantum phase transitions: an integer quantum Hall transition and a superconductor-insulator transition. We considered both transitions using effective theories that consist of a Dirac fermion coupled to a $U(1)$ Chern-Simons gauge field at level $(\theta - 1/2)$: $\theta = 1/2$ corresponds to the IQHT, while $\theta = 1$ corresponds to the SIT. We studied the critical properties of these theories using a renormalization-group analysis in which the number of fermion flavors $N_f \rightarrow \infty$. We found both theories to be stable to the addition of a Coulomb interaction. The IQHT was stable to $\mathcal{C}$ preserving disorder and exhibited a line of diffusive fixed points with $\mathcal{CT}$ disorder. ($\mathcal{C}$ is charge-conjugation symmetry and $\mathcal{T}$ is time-reversal symmetry.) The SIT exhibited a line of fixed points parametrized by the strength of dissipation when $\mathcal{C}$ is preserved. Other cases resulted in runaway flow.

Without disorder, the free Dirac fermion in (1.1) has a correlation length exponent $\nu_{\text{Dirac}}^{-1} = 1$, while the 3D XY model in (1.3) has a correlation length exponent $\nu_{\text{XY}}^{-1} \approx 3/2$. In the large N_f expansion, we find $\nu^{-1}(\theta = 1/2) \approx 1 - 4.3/N_f$ and $\nu^{-1}(\theta = 1) \approx 1 + 1.4/N_f$, in agreement with [25,67,76,77]. Evidently the leading-order term in the large N_f expansion provides a poor approximation to the critical exponents of the clean fixed points [25]. Comparing our results ($\nu^{-1} = 1$, $z = 1 + 1.4/N_f$) for the correlation length and dynamical critical exponents with recent numerical ($\nu \approx 1.2$, $z \approx 1.5$) [39–42]

and analytic ($\nu = 1$, $z \approx 1 + 0.5/N_f$) [38] studies (without a Coulomb interaction) suggests this may also be the case for the dirty 3D XY model. Higher-order $\mathcal{O}(1/N_f^2)$ terms may improve the comparison. Interestingly, the free Dirac fermion and 3D XY model admit duals involving a Dirac fermion coupled to a *non-Abelian* $U(N)$ Chern-Simons gauge field for any integer $N > 1$ [78,79]. Such formulations suggest alternative approximation schemes. For instance, without disorder, these theories have a correlation length exponent equal to unity at two-loop order in the planar limit [72]. Importantly, the $U(N)$ generalization does not introduce additional global symmetry, in contrast to the large flavor approximation considered in this paper. Might the planar limit furnish better approximations to such theories, compared with the large N_f expansion?

There are a variety of other observables and generalizations to consider. For example, scaling dimensions of the lowest dimension monopole operators in the Chern-Simons theories we studied should correspond to the η exponents of the Dirac and XY models. The operator mapping constructed in [80] could prove useful for determining this. In $(2+1)$ dimensions, the dc $T \rightarrow 0$ conductivity tensor can be universal [43,81]; it would be interesting to calculate and compare across the duality [82]. Perhaps considering the behavior of a *finite* density of Dirac composite fermions is most pressing, given that the electronic systems inspiring this paper have a finite density of states.

One of the motivations of the current paper was to better understand the emergent symmetries that are found at IQHTs and SITs via electrical transport experiments. For concreteness, consider the magnetic field-tuned SIT at which a “self-duality” with dc $\sigma_{xx} \approx (2e)^2/h$ and $\sigma_{xy} \approx 0$ is found at low temperatures [83]. (“Self-duality” requires that the electrical conductivity tensor of the 3D XY model satisfy $\sigma_{xx}^2 + \sigma_{xy}^2 = (e_*^2/h)^2$, where e_* is the electromagnetic charge of the bosons [84].) It was argued in [27] that PV symmetry (see Sec. II B) of the “fermionic dual” to the XY model in (2.1) results in self-dual transport. How this symmetry might be realized quantum mechanically is unclear [55]. This question is related to the

emergent time-reversal symmetry of this “fermionic dual” at zero Dirac composite fermion density. Perhaps unsurprisingly, the leading-order large N_f beta functions that we studied do not appear to respect the emergent time-reversal symmetry; at least, we have not found nontrivial solutions with an emergent time-reversal invariance at $\kappa = 1/4\pi$. It would be interesting to further understand this apparent shortcoming.

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APPENDIX A: CALCULATION OVERVIEW

In this Appendix we derive the residues $b_{\lambda_a}[\tilde{\lambda}^R(\mu, \epsilon)]$ in Eq. (3.7),

$$\lambda_a^B \mu^{-\Delta_a(\epsilon)} = \lambda_a^R(\mu, \epsilon) + \frac{b_{\lambda_a}[\tilde{\lambda}^R(\mu, \epsilon)]}{\epsilon}, \quad (\text{A1})$$

that determine the beta functions at $\epsilon = 0$ via Eq. (3.12):

$$\beta_{\lambda_a}(\tilde{\lambda}^R) \equiv -\mu \frac{\partial \lambda_a^R}{\partial \mu} = \bar{\Delta}_{\lambda_a} \lambda_a^R + \rho_{\lambda_a} b_{\lambda_a}(\tilde{\lambda}^R) - \sum_c \rho_{\lambda_c} \lambda_c^R \frac{\partial b_{\lambda_a}(\tilde{\lambda}^R)}{\partial \lambda_c^R}. \quad (\text{A2})$$

After establishing notation, we will list the main results used in the main text. Later sections provide algebraic details.

Setup

Identify S_E in Eqs. (3.2)–(3.4) with the bare action S_B by endowing all fields and couplings with bare (B) subscripts/superscripts. To simplify notation, we will leave replica and flavor indices implicit. Define renormalized (R) fields and couplings,

$$\begin{aligned} \psi_B &= Z_f^{1/2} \psi_R, & a_B^0 &= Z_{a,0}^{1/2} a_R^0, & a_B^j &= Z_{a,j}^{1/2} a_R^j, \\ \lambda_c^B &= Z_c^{-1/2} \lambda_c^R \end{aligned} \quad (\text{A3})$$

where the vector of couplings (either B or R)

$$\tilde{\lambda} = \left(\frac{g^2}{N_f}, v, m, \kappa, w_x, \sigma_e, D_e, g_m, g_0, g_j, \Delta_0, \Delta_j \right)^T. \quad (\text{A4})$$

Separate S_B into physical and counterterm actions:

$$S_B = S_{\text{phys}}^{(1)} + S_{\text{phys}}^{(2)} + S_{\text{phys}}^{(3)} + S_{\text{CT}}^{(1)} + S_{\text{CT}}^{(2)} + S_{\text{CT}}^{(3)} \quad (\text{A5})$$

with

$$S_{\text{phys}}^{(1)} = \int d\tau d^D x \left\{ \bar{\psi}_R \left[\gamma_\tau \left(\partial^\tau + i \frac{g_R \mu^{\epsilon/2}}{\sqrt{N_f}} a_R^\tau \right) + v_R \mu^{z-1} \gamma_j \left(\partial^j + i \frac{g_R \mu^{\epsilon/2}}{\sqrt{N_f}} a_R^j \right) \right] \psi_R + \mu^z m_R \bar{\psi}_R \psi_R + \frac{i\kappa_R}{2} a_R da_R \right\}, \quad (\text{A6})$$

$$S_{\text{phys}}^{(2)} = \int d\omega d^D k \frac{w_R \mu^{z-1}}{2} a_R^T(-\omega, -k) \frac{k^2}{|k| + f_R(\omega, k)} a_R^T(\omega, k), \quad (\text{A7})$$

$$\begin{aligned} S_{\text{phys}}^{(3)} &= -\frac{1}{2} \int d\tau d\tau' d^D x \left[(g_m)_R \mu^{2z-D} (\bar{\psi}_R \psi_R)(\bar{\psi}_R \psi_R) + (g_0)_R \mu^{2z-D} (\bar{\psi}_R \gamma^0 \psi_R)(\bar{\psi}_R \gamma^0 \psi_R) \right. \\ &\quad \left. + (g_j)_R \mu^{2z-D} (\bar{\psi}_R \gamma^j \psi_R)(\bar{\psi}_R \gamma^j \psi_R) + (\Delta_0)_R \mu^{2z-2} b_R b_R + (\Delta_j)_R \mathbf{e}_R \cdot \mathbf{e}_R \right], \end{aligned} \quad (\text{A8})$$

$$S_{\text{CT}}^{(1)} = \int d\tau d^D x \left\{ \bar{\psi}_R \left[\gamma_\tau \left(\partial^\tau + i \frac{g_R \mu^{\epsilon/2}}{\sqrt{N_f}} a_R^\tau \right) \delta_1 + v_R \mu^{z-1} \gamma_j \left(\partial^j + i \frac{g_R \mu^{\epsilon/2}}{\sqrt{N_f}} a_R^j \right) \delta_2 \right] \psi_R + \mu^z m_R \bar{\psi}_R \psi_R \delta_m + \frac{i\kappa_R}{2} a_R da_R \delta_\kappa \right\}, \quad (\text{A9})$$

$$S_{\text{CT}}^{(2)} = \int d\omega d^D k \frac{w_R \mu^{z-1} \delta_w}{2} a_R^T(-\omega, -k) \frac{k^2}{|k| + f_R(\omega, k)} a_R^T(\omega, k), \quad (\text{A10})$$

$$\begin{aligned} S_{\text{CT}}^{(3)} &= -\frac{1}{2} \int d\tau d\tau' d^D x \left[(g_m)_R \mu^{2z-D} \delta_{g_m} (\bar{\psi}_R \psi_R)(\bar{\psi}_R \psi_R) + (g_0)_R \mu^{2z-D} \delta_{g_0} (\bar{\psi}_R \gamma^0 \psi_R)(\bar{\psi}_R \gamma^0 \psi_R) \right. \\ &\quad \left. + (g_j)_R \mu^{2z-D} \delta_{g_j} (\bar{\psi}_R \gamma^j \psi_R)(\bar{\psi}_R \gamma^j \psi_R) + (\Delta_0)_R \mu^{2z-2} \delta_{\Delta_0} b_R b_R + (\Delta_j)_R \delta_{\Delta_j} \mathbf{e}_R \cdot \mathbf{e}_R \right], \end{aligned} \quad (\text{A11})$$

$$f_R = \frac{e_*^2(\sigma_e)_R \mu^{z-1}}{|\omega| + (D_e)_R \mu^{z-2} k^2}, \quad (\text{A12})$$

and the renormalization-group scale μ enters in accord with the engineering dimensions listed in Table II. The counterterms δ_X have poles in ϵ with coefficients determined by the requirement that correlation functions of physical fields have no divergences as $\epsilon \rightarrow 0$. We focus exclusively on the terms in δ_X proportional to $1/\epsilon$. Using Eq. (A3) to impose Eq. (A5), we relate the bare and renormalized couplings:

$$v^B \mu^{1-z} = v^R (1 + \delta_2 - \delta_1), \quad (\text{A13})$$

$$m^B \mu^{-z} = m^R (1 + \delta_m - \delta_1), \quad (\text{A14})$$

$$\kappa^B = \kappa^R (1 + \delta_\kappa), \quad (\text{A15})$$

$$w_x^B \mu^{1-z} = w_x^R (1 + \delta_w), \quad (\text{A16})$$

$$g_X^B \mu^{D-2z} = g_X^R (1 - 2\delta_1 + \delta_{g_X}), \quad g_X \in \{g_m, g_0, g_j\}, \quad (\text{A17})$$

$$\Delta_0^B \mu^{2-2z} = \Delta_0^R (1 + \delta_{\Delta_0}), \quad (\text{A18})$$

$$\Delta_j^B = \Delta_j^R (1 + \delta_{\Delta_j}), \quad (\text{A19})$$

$$\sigma_e^B \mu^{1-z} = \sigma_e^R (1 + \delta_{\sigma_e}), \quad (\text{A20})$$

$$D_e^B = D_e^R (1 + \delta_D). \quad (\text{A21})$$

Thus, we can read off the residues:

$$b_v = v^R (\delta_2 - \delta_1) \epsilon, \quad (\text{A22})$$

$$b_m = m^R (\delta_m - \delta_1) \epsilon, \quad (\text{A23})$$

$$b_\kappa = \kappa^R \delta_\kappa \epsilon, \quad (\text{A24})$$

$$b_w = w_x^R \delta_w \epsilon, \quad (\text{A25})$$

$$b_{g_X} = -g_X^R (2\delta_1 - \delta_{g_X}) \epsilon, \quad g_X \in \{g_m, g_0, g_j\}, \quad (\text{A26})$$

$$b_{\Delta_X} = \Delta_X^R \delta_{\Delta_X} \epsilon, \quad \Delta_X \in \{\Delta_0, \Delta_j\}, \quad (\text{A27})$$

$$b_{\sigma_e} = \sigma_e^R \delta_{\sigma_e} \epsilon, \quad (\text{A28})$$

$$b_{D_e} = D_e^R \delta_D \epsilon. \quad (\text{A29})$$

APPENDIX B: COUNTERTERMS

As discussed in the main text, we choose the dynamical critical exponent z in such a way that the fermion velocity v does not run, i.e., the velocity beta function is zero. In the expressions below, it is convenient to redefine couplings to absorb the velocity dependence as follows:

$$\begin{aligned} \overline{g_m} &= \frac{g_m}{2\pi v^2}, \quad \overline{g_0} = \frac{g_0}{2\pi v^2}, \quad \overline{g_j} = \frac{g_j}{2\pi v^2}, \quad \overline{\Delta_0} = \Delta_0, \\ \overline{\Delta_j} &= \Delta_j v^2, \quad \overline{w_x} = \frac{w_x}{v}, \quad \overline{\sigma_e} = \frac{\sigma_e}{v}, \quad \int_{-\infty}^{\infty} dz F(z) \\ &= \int_{-\infty}^{\infty} v dy F\left(y = \frac{z}{v}\right), \end{aligned} \quad (\text{B1})$$

where the function $F(z)$ is introduced below. We also define $g_1 = g^2/16$ and $\overline{m} = m/v$.

Let us make a few remarks about the expressions below.

(1) We use ϕ_1 to parametrize the screening of the disorder described in Appendix C. $\phi_1 = 0$ means the screening is ignored; $\phi_1 = 1$ means the screening is included.

(2) Terms proportional to ξ are divergent. However, this is an unphysical divergence due to our gauge choice: This divergence does not appear in physical quantities such as critical exponents.

(3) The Ward identity guarantees that the gauge field corrections to δ_1 and δ_{g_0} cancel; the ones in δ_2 and δ_{g_j} likewise cancel. In the absence of the Coulomb interaction, the equality of the gauge corrections in δ_1 and δ_2 is a coincidence, which makes β_{g_0} and β_{g_j} independent of the gauge corrections. When the Coulomb interaction is included, β_{g_0} receives corrections from the gauge field, while β_{g_j} does not.

FIG. 11. Diagrams contributing to δ_1 and δ_2 .

1. δ_κ , δ_w , δ_σ , and δ_D counterterms

Quantization of the Chern-Simons level and finiteness of the gauge field self-energy in three dimensions implies

$$\delta_\kappa = \delta_w = \delta_\sigma = \delta_D = 0. \quad (\text{B2})$$

Consequently, renormalizations of κ , w_x , σ_e , and D_e are controlled by their engineering dimensions.

2. δ_1 counterterm

The diagrams that contribute to δ_1 are given by taking the temporal component of Fig. 11:

$$\begin{aligned} \delta_1 \epsilon = & -(\overline{g_m} + \overline{g_0} + 2\overline{g_j}) + \phi_1 \left[\frac{g_1^2 \overline{w_x} (g_1 \overline{g_j} - g_1 \overline{g_0} - \overline{g_0} \overline{w_x})}{(g_1^2 + g_1 \overline{w_x} + \kappa^2)^2} + \frac{g_1 (g_1 \overline{g_j} + g_1 \overline{g_0} + 2\overline{g_0} \overline{w_x})}{g_1^2 + g_1 \overline{w_x} + \kappa^2} \right] \\ & + \int_{-\infty}^{\infty} \frac{1}{4\pi^2 N_f} \frac{(1-y^2)(g_1 \overline{\sigma_e} y^2 + \overline{w_x} \sqrt{1+y^2} |y|) + g_1 |y| y^2 (1-y^2 \xi)}{(1+y^2)^2 [\sqrt{1+y^2} (g_1^2 + \kappa^2) (\overline{\sigma_e} + |y|) + g_1 \overline{w_x} |y|]}. \end{aligned} \quad (\text{B3})$$

3. δ_2 counterterm

The diagrams that contribute to δ_2 are given by taking the spatial component of Fig. 11:

$$\begin{aligned} \delta_2 \epsilon = & -\phi_1 \frac{g_1^2 [g_1^2 \overline{g_j} + 2g_1 \overline{g_j} \overline{w_x} - (\overline{g_0} - 2\overline{g_j}) \kappa^2]}{(g_1^2 + g_1 \overline{w_x} + \kappa^2)^2} \\ & + \int_{-\infty}^{\infty} dy \frac{1}{4\pi^2 N_f} \frac{g_1 (1-y^2-y^4) \overline{\sigma_e} - (\overline{w_x} y^2 \sqrt{1+y^2}) |y| + g_1 (1-y^2-y^4 \xi) |y|}{(1+y^2)^2 [\sqrt{1+y^2} (g_1^2 + \kappa^2) (\overline{\sigma_e} + |y|) + g_1 \overline{w_x} |y|]}. \end{aligned} \quad (\text{B4})$$

4. δ_{g_m} counterterm

δ_{g_m} is extracted from the diagrams in Figs. 12 and 13:

$$\begin{aligned} \delta_{g_m} \epsilon = & \left[\frac{2(\overline{g_0} + \overline{g_m})(\overline{g_m} - 2\overline{g_j})}{\overline{g_m}} + 2 \frac{g_1(\overline{g_j} - \overline{g_0}) + \overline{g_j} \overline{w_x}}{g_1^2 + g_1 \overline{w_x} + \kappa^2} \right] \\ & + \phi_1 \left[4g_1 \frac{g_1 \kappa^2 (-\overline{g_0}^2 - \overline{g_j}^2) + \overline{g_j} \overline{g_0} [2g_1^3 + 4g_1^2 \overline{w_x} + 2\overline{w_x} \kappa^2 + g_1 (\overline{w_x}^2 + 4\kappa^2)]}{\overline{g_m} [g_1^2 + g_1 \overline{w_x} + \kappa^2]^2} \right] \\ & + \phi_1 \left[\frac{g_1^5 (\overline{g_0} - \overline{g_j}) + g_1^4 \overline{w_x} (2\overline{g_0} - 3\overline{g_j}) + g_1^3 (\overline{g_0} \overline{w_x}^2 - 2\overline{g_j} \overline{w_x}^2 + 3\overline{g_0} \kappa^2 - 3\overline{g_j} \kappa^2)}{[g_1^2 + g_1 \overline{w_x} + \kappa^2]^3} + \frac{g_1^2 \kappa^2 (3\overline{g_0} \overline{w_x}) + 2g_1 \kappa^4 (\overline{g_j} - \overline{g_0})}{[g_1^2 + g_1 \overline{w_x} + \kappa^2]^3} \right] \\ & + \phi_1 \left[\frac{2g_1^4 (\overline{g_j} - \overline{g_0}) + 4g_1^3 \overline{w_x} (\overline{g_j} - \overline{g_0}) + g_1^2 (-2\overline{g_0} \overline{w_x}^2 - 6\overline{g_0} \kappa^2 + 6\overline{g_j} \kappa^2) - 4g_1 \overline{g_0} \overline{w_x} \kappa^2}{[g_1^2 + g_1 \overline{w_x} + \kappa^2]^2} \right] \\ & + \phi_1^2 \left[\frac{4g_1^2 \overline{g_j}^2 \kappa^2 [g_1^2 (g_1^2 + 2g_1 \overline{w_x} - \overline{w_x}^2) + 2g_1 (g_1 - \overline{w_x}) \kappa^2 - \kappa^4]}{\overline{g_m} [g_1^2 + g_1 \overline{w_x} + \kappa^2]^4} \right. \\ & \left. + \frac{4g_1^2 \overline{g_0}^2 \kappa^2 [g_1^2 (g_1^2 + 2g_1 \overline{w_x} + \overline{w_x}^2) + 2g_1 (g_1 + \overline{w_x}) \kappa^2 - \kappa^4]}{\overline{g_m} [g_1^2 + g_1 \overline{w_x} + \kappa^2]^4} \right] \end{aligned}$$

FIG. 12. Two particle-irreducible (2-PI) diagrams contributing to δ_{g_m} , δ_{g_0} , and δ_{g_j} .

$$\begin{aligned}
 & + \frac{4\bar{g}_0 \bar{g}_j g_1^2 \left[-g_1^3 (g_1 + \bar{w}_x)^2 (g_1 + 2\bar{w}_x) - 2g_1^2 (g_1 + \bar{w}_x) (2g_1 + 3\bar{w}_x) \kappa^2 - g_1 (5g_1 + 2\bar{w}_x) \kappa^4 + 2\kappa^6 \right]}{\bar{g}_m [g_1^2 + g_1 \bar{w}_x + \kappa^2]^4} \Big] \\
 & - \int_{-\infty}^{\infty} dy \frac{g_1 (y^2 + 2) (1 + y^2) \bar{\sigma}_e + [\bar{w}_x (1 + y^2) \sqrt{1 + y^2} + g_1 (2 + 3y^2 + y^4 \xi)] |y|}{2\pi^2 N_f (1 + y^2)^2 [\sqrt{1 + y^2} (g_1^2 + \kappa^2) (\bar{\sigma}_e + |y|) + g_1 \bar{w}_x |y|]} \\
 & + \int_{-\infty}^{\infty} dy \frac{(\bar{\sigma}_e + |y|) [(1 + y^2) (g_1^2 - \kappa^2) (\bar{\sigma}_e + |y|) + (g_1 \bar{w}_x \sqrt{1 + y^2}) |y|]}{4\pi^2 N_f (1 + y^2)^{\frac{3}{2}} [\sqrt{1 + y^2} (g_1^2 + \kappa^2) (\bar{\sigma}_e + |y|) + g_1 \bar{w}_x |y|]^2}.
 \end{aligned} \tag{B5}$$

5. δ_m counterterm

δ_m is extracted from the diagrams in Fig. 13:

$$\begin{aligned}
 m \in \delta_m &= \frac{1}{2} g_m \in (\delta_{g_m} - \text{2-PI box}) \\
 &= \bar{g}_0 - 2\bar{g}_j + \bar{g}_m - \frac{g_1 (\bar{g}_0 - \bar{g}_j) - \bar{g}_j \bar{w}_x}{2(g_1^2 + g_1 \bar{w}_x + \kappa^2)} + \phi_1 \left(-\frac{g_1 (\bar{g}_0 - \bar{g}_j) + 3g_1 (\bar{g}_0 - \bar{g}_j) + 2\bar{g}_0 \bar{w}_x}{g_1^2 + g_1 \bar{w}_x + \kappa^2} \right. \\
 &\quad \left. - g_1^2 \frac{\bar{g}_j [4g_1^2 + 4\bar{w}_x + g_1 (7 + 2\bar{w}_x)] - \bar{g}_0 [4g_1^2 + \bar{w}_x (7 + 2\bar{w}_x) + g_1 (7 + 6\bar{w}_x)]}{2(g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} \right. \\
 &\quad \left. - \frac{-2g_1^3 (g_1 + \bar{w}_x) [-g_1 \bar{g}_j + \bar{g}_0 (g_1 + \bar{w}_x)]}{(g_1^2 + g_1 \bar{w}_x + \kappa^2)^3} \right) \\
 &\quad - \int_{-\infty}^{\infty} dy \frac{g_1 (y^2 + 2) (1 + y^2) \bar{\sigma}_e + (\bar{w}_x (1 + y^2) \sqrt{1 + y^2} + g_1 (2 + 3y^2 + y^4 \xi)) |y|}{4\pi^2 N_f (1 + y^2)^2 [\sqrt{1 + y^2} (g_1^2 + \kappa^2) (\bar{\sigma}_e + |y|) + g_1 \bar{w}_x |y|]} \\
 &\quad + \int_{-\infty}^{\infty} dy \frac{(\bar{\sigma}_e + |y|) [(1 + y^2) (g_1^2 - \kappa^2) (\bar{\sigma}_e + |y|) + (g_1 \bar{w}_x \sqrt{1 + y^2}) |y|]}{8\pi^2 N_f (1 + y^2)^{\frac{3}{2}} [\sqrt{1 + y^2} (g_1^2 + \kappa^2) (\bar{\sigma}_e + |y|) + g_1 \bar{w}_x |y|]^2},
 \end{aligned} \tag{B6}$$

where [two particle-irreducible (2-PI) boxes] refers to the contributions to δ_{g_m} from the diagrams in Fig. 12.

FIG. 13. The mass vertex contributions to δ_{g_m} .

FIG. 14. $\gamma_{\mu=0}$ vertex component contributions to δ_{g_0} . FIG. 15. $\gamma_{\mu=j}$ vertex component contributions to δ_{g_j} .

6. δ_{g_0} counterterm

δ_{g_0} is extracted from the diagrams in Figs. 12 and 14:

$$\begin{aligned} \delta_{g_0}\epsilon = & \frac{-2(\overline{g_0} + 2\overline{g_j})(\overline{g_0} + \overline{g_m})}{\overline{g_0}} + \phi_1 \left[\frac{4\overline{g_1}^2 \overline{g_j} \overline{g_m} (g_1^2 + 2g_1 \overline{w_x} + 2\kappa^2)}{\overline{g_0} [g_1^2 + g_1 \overline{w_x} + \kappa^2]^2} + \frac{2\overline{g_1}^2 [g_1^2 \overline{g_j} + 2g_1 \overline{g_j} \overline{w_x} + (\overline{g_j} - 2\overline{g_m}) \kappa^2]}{[g_1^2 + g_1 \overline{w_x} + \kappa^2]^2} \right. \\ & \left. + \frac{2\overline{g_0} g_1 (g_1^3 + 2g_1^2 \overline{w_x} + 2\kappa^2 \overline{w_x} + g_1 \overline{w_x}^2 + g_1 \kappa^2)}{[g_1^2 + g_1 \overline{w_x} + \kappa^2]^2} \right] \\ & + \int_{-\infty}^{\infty} dy \frac{g_1 y^2 (1 - y^2) \overline{\sigma_e} + g_1 y^2 (1 - y^2 \xi) |y| + (1 - y^2) \overline{w_x} \sqrt{1 + y^2} |y|}{2\pi^2 N_f (1 + y^2)^2 [\sqrt{1 + y^2} (g_1^2 + \kappa^2) (\overline{\sigma_e} + |y|) + g_1 \overline{w_x} |y|]}. \end{aligned} \quad (B7)$$

7. δ_{g_j} counterterm

δ_{g_j} is extracted from the diagrams in Figs. 12 and 15:

$$\begin{aligned} \delta_{g_j}\epsilon = & \frac{-2\overline{g_0} \overline{g_m}}{\overline{g_j}} + \phi_1 \left[\frac{2\overline{g_1}^2 (\overline{g_0} - \overline{g_m}) \kappa^2}{[g_1^2 + g_1 \overline{w_x} + \kappa^2]^2} \right] + \phi_1 \left[\frac{2\overline{g_1} \overline{g_0} \overline{g_m} (g_1^3 + 2g_1^2 \overline{w_x} + 2\overline{w_x} \kappa^2 + g_1 \overline{w_x}^2 + 2g_1 \kappa^2)}{[g_1^2 + g_1 \overline{w_x} + \kappa^2]^2} \right] \\ & + \int_{-\infty}^{\infty} dy \frac{1}{2\pi^2 N_f} \frac{g_1 (1 - y^2 - y^4) \overline{\sigma_e} - (\overline{w_x} y^2 \sqrt{1 + y^2}) |y| + g_1 (1 - y^2 - y^4 \xi) |y|}{(1 + y^2)^2 [\sqrt{1 + y^2} (g_1^2 + \kappa^2) (\overline{\sigma_e} + |y|) + g_1 \overline{w_x} |y|]}. \end{aligned} \quad (B8)$$

8. δ_{Δ_0} counterterm

δ_{Δ_0} is extracted from the diagram in Fig. 16:

$$\begin{aligned} \delta_{\Delta_0}\epsilon = & \frac{-\overline{g_m} (g_1^2 \overline{\Delta_j} + 2g_1 \overline{w_x} \overline{\Delta_j} + \overline{w_x}^2 \overline{\Delta_j} + \overline{\Delta_0} \kappa^2)}{64\overline{\Delta_0} (g_1^2 + g_1 \overline{w_x} + \kappa^2)^2} \\ & + \left[\frac{-\overline{g_0} \overline{g_m} N_f \pi v^2}{32\overline{\Delta_0}} + \phi_1 \frac{g_1 \overline{g_m} N_f \pi v^2 [-g_1 \overline{g_j} \kappa^2 + \overline{g_0} (g_1 + \overline{w_x}) (g_1^2 + g_1 \overline{w_x} + 2\kappa^2)]}{32\overline{\Delta_0} (g_1^2 + g_1 \overline{w_x} + \kappa^2)^2} \right]. \end{aligned} \quad (B9)$$

9. δ_{Δ_j} counterterm

δ_{Δ_j} is extracted from the diagram in Fig. 16:

$$\delta_{\Delta_j}\epsilon = \frac{-\overline{g_m} (g_1^2 \overline{\Delta_0} + \overline{\Delta_j} \kappa^2)}{128\overline{\Delta_j} (g_1^2 + g_1 \overline{w_x} + \kappa^2)^2} + \left[\frac{-\overline{g_j} \overline{g_m} N_f \pi v^2}{64\overline{\Delta_j}} + \phi_1 \frac{g_1^2 \overline{g_m} N_f \pi v^2 [g_1^2 \overline{g_j} + 2g_1 \overline{g_j} \overline{w_x} - (\overline{g_0} - 2\overline{g_j}) \kappa^2]}{64\overline{\Delta_j} (g_1^2 + g_1 \overline{w_x} + \kappa^2)^2} \right]. \quad (B10)$$

APPENDIX C: FEYNMAN RULES FOR DISORDER AND SCREENING

1. Feynman rules for disorder vertices

From the action (A8), we can read the Feynman rules for the various types of disorder: (1) the four-fermion mass vertex,

$$\frac{1}{2}(\bar{\psi}\psi)(\bar{\psi}\psi) \Rightarrow \boxed{+g_m 2\pi\delta(\omega=0)}; \quad (C1)$$

FIG. 16. Diagrams contributing to δ_{Δ_0} and δ_{Δ_j} .

(2) the four-fermion density vertex,

$$\frac{1}{2}(\bar{\psi}\gamma^0\psi)(\bar{\psi}\gamma^0\psi) \Rightarrow \boxed{(g_0)(\gamma^0)(\gamma^0)2\pi\delta(\omega=0)}; \quad (\text{C2})$$

(3) the four-fermion current vertex along the k direction $k = x$ or y ,

$$\frac{1}{2}(\bar{\psi}i\gamma^k\psi)(\bar{\psi}i\gamma^k\psi) \Rightarrow \boxed{(+g_j)(i\gamma^k)(i\gamma^k)2\pi\delta(\omega=0)}; \quad (\text{C3})$$

(4) the $b(\tau)b(\tau')$ disordered two-point vertex rule,

$$\frac{1}{2}b_z b_z \Rightarrow \boxed{\Delta_0(\delta_{ij}\mathbf{k}^2 - k_i k_j)2\pi\delta(\omega)a_i(\mathbf{k})a_j(-\mathbf{k})}; \quad (\text{C4})$$

and (5) the $e_j(\tau)e_j(\tau')$ disordered two-point vertex,

$$\frac{1}{2}(e_x e_x + e_y e_y) \Rightarrow \boxed{(-\Delta_j c^2)(k_x^2 + k_y^2)2\pi\delta(\omega)a_0(\mathbf{k})a_0(-\mathbf{k})}. \quad (\text{C5})$$

The factor of $\frac{1}{2}$ is canceled by a symmetry factor equal to 2. The 2π factor always cancels with the $1/2\pi$ factor that accompanies any frequency integral $\int \frac{d\omega}{2\pi}$.

2. Gauge propagator

a. Vacuum polarization tensor

$$\Pi_{\mu\nu}(k_0, \mathbf{k}) = \left(\frac{-ig}{\sqrt{N_f}}\right)^2 \left(\frac{v}{c}\right)^{2-\delta_{\mu 0}-\delta_{\nu 0}} \times N_f \times (-1) \int \frac{d^2 p_E}{(2\pi)^3} \text{Tr}[\gamma_\mu S_F(k+p)\gamma_\nu S_F(p)] \quad (\text{C6})$$

$$= (+g^2)(-i)^2 \left(\frac{v}{c}\right)^{2-\delta_{\mu 0}-\delta_{\nu 0}} \frac{1}{v^2} \int \frac{d^2 \bar{p}_E d p_0}{(2\pi)^3} \text{Tr}[\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta] \frac{(\bar{k} + \bar{p})_\alpha (\bar{p})_\beta}{[x(\bar{k} + \bar{p})^2 + (1-x)\bar{p}^2]^2}, \quad (\text{C7})$$

$$\Pi_{\mu\nu}(k_0, \mathbf{k}) = \frac{-1}{16} \frac{g^2}{v^2} \left(\frac{v}{c}\right)^{2-\delta_{\mu 0}-\delta_{\nu 0}} \frac{1}{|\bar{k}|} [\delta_{\mu\nu} \bar{k}^2 - \bar{k}_\mu \bar{k}_\nu], \quad (\text{C8})$$

$$\bar{k} = (\omega, v\mathbf{k}), \quad |\bar{k}| = \sqrt{\omega^2 + v^2 \mathbf{k}^2}.$$

The minus sign comes from the fermion loop. The ratio v/c can be set to v in future equations.

b. “ $I-\mu$ ” vacuum polarization vector

$$\Pi_\mu(k_0, \mathbf{k}) = \sqrt{g_m} \left(\frac{-ig}{\sqrt{N_f}}\right) \left(\frac{v}{c}\right)^{1-\delta_{\mu 0}} \times N_f \times (-1) \int \frac{d^2 p_E}{(2\pi)^3} \text{Tr}[\gamma_\mu S_F(k+p) \mathbf{1} S_F(p)] \quad (\text{C9})$$

$$= \sqrt{g_m} (+ig)(-i)^2 \left(\frac{v}{c}\right)^{1-\delta_{\mu 0}} \frac{1}{v^2} \int \frac{d^2 \bar{p}_E d p_0}{(2\pi)^3} \text{Tr}[\gamma_\mu \gamma_\alpha \gamma_\beta] \frac{(\bar{k} + \bar{p})_\alpha (\bar{p})_\beta}{[x(\bar{k} + \bar{p})^2 + (1-x)\bar{p}^2]^2} = 0. \quad (\text{C10})$$

The momentum part is proportional to $\delta_{\alpha\beta} p^2$, while the trace is proportional to the $\epsilon_{\mu\alpha\beta}$ tensor, and so it vanishes.

c. “1-1” vacuum polarization scalar

$$\Pi_m(k_0, \mathbf{k}) = (\sqrt{g_m})^2 N_f \times (-1) \int \frac{d^2 p_E d p_0}{(2\pi)^3} \text{Tr}[\mathbf{1} S_F(k+p) \mathbf{1} S_F(p)] \quad (\text{C11})$$

$$= g_m \frac{1}{v^2} \int \frac{d^2 \bar{p}_E d p_0}{(2\pi)^3} \text{Tr}[\gamma_\alpha \gamma_\beta] \frac{(\bar{k} + \bar{p})_\alpha (\bar{p})_\beta}{[x(\bar{k} + \bar{p})^2 + (1-x)\bar{p}^2]^2} = \frac{-|k| g_m}{8v^2}. \quad (\text{C12})$$

Although nonzero, when connecting external fermion lines, the resulting diagram would be proportional to the number of replicas n_R and vanish in the $n_r \rightarrow 0$ limit.

d. Effective gauge propagator

In Coulomb gauge (longitudinal component $a_L = 0$), the kinetic term for the gauge field is

$$S_{\text{gauge}} = \frac{1}{2} \int dk^2 d\omega (a_0 \quad a_T) \begin{pmatrix} \frac{g^2}{16} \frac{k^2}{\sqrt{\omega^2 + v^2 k^2}} & i\kappa |k| \\ i\kappa |k| & w_x \frac{k^2}{|k| + f(\omega, k)} + \frac{g^2}{16} \sqrt{\omega^2 + v^2 k^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_T \end{pmatrix}, \quad (\text{C13})$$

where $(k^2 \equiv |\mathbf{k}|^2)$. Recall that $g_1 \equiv \frac{g^2}{16} = \frac{1}{16}$, the effective Coulomb coupling $w_x \equiv \frac{+e^2}{4\pi^2}$, and $f(k, \omega) \equiv \frac{\sigma_e k^2}{|\omega| + D_e k^2}$. The transverse component of the gauge field is $a_T(k, \omega) \equiv i\hat{k}_x a_y(k, \omega) - i\hat{k}_y a_x(k, \omega)$, where $\hat{k}_j = k_j/|k|$.

When dealing with the gamma matrix contraction in Feynman diagram calculations, we have to write the effective gauge propagator obtained from S_{gauge} in the a_0 , a_x , and a_y basis ($i, j = x, y$):

$$G_{00} = \frac{1}{\mathbf{k}^2} \frac{g_1 \sqrt{\omega^2 + v^2 \mathbf{k}^2} + F(k, \omega)}{g_1^2 + \kappa^2 + \frac{g_1 F(k, \omega)}{\sqrt{\omega^2 + v^2 \mathbf{k}^2}}}, \quad (\text{C14})$$

$$G_{0i} = \frac{\kappa}{\mathbf{k}^2} \frac{-\epsilon_{ij} k_j}{g_1^2 + \kappa^2 + \frac{g_1 F(k, \omega)}{\sqrt{\omega^2 + v^2 \mathbf{k}^2}}},$$

$$G_{i0} = -G_{0i} = \frac{\kappa}{\mathbf{k}^2} \frac{+\epsilon_{ij} k_j}{g_1^2 + \kappa^2 + \frac{g_1 F(k, \omega)}{\sqrt{\omega^2 + v^2 \mathbf{k}^2}}}, \quad (\text{C15})$$

$$G_{ij} = \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \right) \frac{\frac{g_1}{\sqrt{\omega^2 + v^2 \mathbf{k}^2}}}{\left[g_1^2 + \kappa^2 + \frac{g_1 F(k, \omega)}{\sqrt{\omega^2 + v^2 \mathbf{k}^2}} \right]} \quad (\text{C16})$$

$$\text{where } F(k, \omega) \equiv \frac{+e^2}{4\pi^2} \frac{|\mathbf{k}|^2}{|\mathbf{k}| + f(k, \omega)} = \frac{+e^2}{4\pi^2} \frac{|\mathbf{k}|^2}{|\mathbf{k}| + \frac{\sigma_e k^2}{|\omega| + D_e k^2}}$$

$$\xrightarrow{\text{critical limit, } D_e=0} \frac{+e^2}{4\pi^2} \frac{|\mathbf{k}|^2}{|\mathbf{k}| + \frac{\sigma_e k^2}{|\omega|}}. \quad (\text{C17})$$

3. Screened disorder $W_{\mu\nu}$

The disorders g_0 and g_j are screened by the fermion polarization. The Feynman rules in (C2) and (C3) have to be adjusted to account for this screening:

$$W_{\mu\nu} = W_{\mu\nu}^{(0)} + \phi_1 W_{\mu\nu}^{(\text{sc})}, \quad (\text{C18})$$

where $W_{\mu\nu}^{(0)} = \text{diag}(g_0, i^2 g_j, i^2 g_j)$ is the bare part in (C2) and (C3) and $W_{\mu\nu}^{(\text{sc})}$ is the screening part from the summation of fermion bubbles.

The prefactor ϕ_1 isolates the screened and unscreened contributions: $\phi_1 = 0$ means that disorder screening is ignored; $\phi_1 = 1$ means that disorder screening is included. When disorder connects with the gauge propagator, we should set $\sigma_e = 0$ before setting $\omega = 0$ [due to the presence of the $\delta(\omega)$] factor. Otherwise, there is no disorder screening. Note that the vertex factors are included in $W_{\mu\nu}$, so when applying the Feynman rules we only need to multiply by γ_μ without any constant or velocity factor.

We separate the screening part into symmetric and antisymmetric components:

$$W^{(\text{sc})} = W^{\text{sym}} + W^{\text{as}}, \quad (\text{C19})$$

$$W_{00}^{\text{sym}} = g_1 \frac{-g_1 g_j \kappa^2 + g_0(g_1 + \bar{w}_x)(g_1^2 + g_1 \bar{w}_x + 2\kappa^2)}{(g_1^2 + g_1 \bar{w}_x + \kappa^2)^2}, \quad (\text{C20})$$

$$W_{ij}^{\text{sym}} = \frac{g_1^2 [g_1^2 g_j + 2g_1 g_j \bar{w}_x - (g_0 - 2g_j)\kappa^2]}{(g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} \frac{1}{k^2} (k^2 \delta_{ij} - k_i k_j), \quad (\text{C21})$$

$$W_{0i}^{\text{as}} = \frac{g_1 \kappa [g_1 g_j \bar{w}_x + (-g_0 + g_j)\kappa^2]}{[g_1^2 + g_1 \bar{w}_x + \kappa^2]^2} \frac{-\epsilon_{ij} k_j}{k}. \quad (\text{C22})$$

Other components of $W^{(\text{sc})}$ not included above vanish.

4. Effective gauge disorder $D_{\mu\nu}^{\text{dis}}$

The expressions in (C4) and (C5) are two-point vertex rules: each side of the vertex connects with the dressed propagator found in (C14)–(C16). The effective gauge disorder is defined by

$$D_{\mu\nu}^{\text{dis}} = G_{\mu\alpha} D_{\alpha\beta}^{0,\text{dis}} G_{\beta\nu}, \quad (\text{C23})$$

where $D_{00}^{0,\text{dis}} = -\Delta_j k^2$ defined in (C5), $D_{ij}^{0,\text{dis}} = \Delta_0 (\delta_{ij} k^2 - k_i k_j)$ defined in (C4), and $D_{0i}^{0,\text{dis}} = D_{i0}^{0,\text{dis}} = 0$. We decompose $D_{\mu\nu}^{\text{dis}}$ into symmetric and antisymmetric components:

$$D_{\mu\nu}^{\text{dis}} = D_{\mu\nu}^{\text{S}} + D_{\mu\nu}^{\text{AS}}, \quad (\text{C24})$$

$$D_{00}^{\text{S}} = -\frac{g_1^2 v^2 \Delta_j + 2g_1 v^2 \bar{w}_x \Delta_j + v^2 \bar{w}_x^2 \Delta_j + \Delta_0 \kappa^2}{(g_1^2 + g_1 \bar{w}_x + \kappa^2)^2}, \quad (\text{C25})$$

$$D_{ij}^{\text{S}} = \frac{(g_1^2 \Delta_0 + v^2 \Delta_j k^2)}{v^2 (g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} \frac{k^2 \delta_{ij} - k_i k_j}{k^2}, \quad (\text{C26})$$

$$D_{0i}^{\text{AS}} = \frac{\kappa (-g_1 \Delta_0 + g_1 v^2 \Delta_j + v^2 \bar{w}_x \Delta_j)}{v (g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} \frac{\epsilon_{ij} k_j}{k}. \quad (\text{C27})$$

Components of D^{dis} not listed above are zero.

Since $G_{\mu\nu}$ is constructed by the random-phase-approximation sum of fermion loops, $G_{\mu\nu}$ can no longer connect any more fermion loops. Consequently, $D_{\mu\nu}^{\text{dis}}$ does not include any fermion loops. Note that $D_{\mu\nu}^{\text{dis}}$ generates $\frac{\Delta_{0,j}}{N_f}$. This disorder renormalizes Δ_0 and Δ_j at three-loop order.

APPENDIX D: FERMION SELF-ENERGY

1. Self-energy: Screened disorder correction $W_{\mu\nu}$

$$\begin{aligned} \Sigma_d(p_0, \mathbf{p}) &= \int \frac{d^2 \mathbf{k}}{(2\pi)^2} (\mathbf{1}_{2 \times 2}) S_F(p_0 - 0, \mathbf{p} - \mathbf{k}) (\mathbf{1}_{2 \times 2}) g_m + \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \gamma_\nu S_F(p_0 - 0, \mathbf{p} - \mathbf{k}) \gamma_\mu W_{\mu\nu}(\mathbf{k}) \\ &= \int \frac{d^2 \mathbf{k}}{(2\pi)^2} (\mathbf{1}_{2 \times 2}) S_F(p_0 - 0, \mathbf{p} - \mathbf{k}) (\mathbf{1}_{2 \times 2}) g_m + \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \gamma_\nu \left[(+i) \frac{\gamma_0 p_0 + v(\mathbf{p} - \mathbf{k})_i \gamma_i}{p_0^2 + v^2(\mathbf{p} - \mathbf{k})^2} \right] \gamma_\mu W_{\mu\nu}(\mathbf{k}) \\ &= \frac{+i p_0 \gamma_0}{v^2 2\pi\epsilon} [g_m] + \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \gamma_\nu \left[(+i) \frac{\gamma_0 p_0 + v(\mathbf{p} - \mathbf{k})_i \gamma_i}{p_0^2 + v^2(\mathbf{p} - \mathbf{k})^2} \right] \gamma_\mu W_{\mu\nu}(\mathbf{k}) \quad (\text{D1}) \end{aligned}$$

$$\begin{aligned} &= \frac{+i p_0 \gamma_0}{2\pi\epsilon v^2} g_m + \frac{+i p_0 \gamma_0}{2\pi\epsilon v^2} (g_0 + 2g_j) + \phi_1 \left[\frac{g_1^2 \bar{w}_x (g_1 \bar{g}_j - g_1 \bar{g}_0 - \bar{g}_0 \bar{w}_x)}{(g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} \right. \\ &\quad \left. + \frac{g_1 (g_1 \bar{g}_j + g_1 \bar{g}_0 + 2\bar{g}_0 \bar{w}_x)}{g_1^2 + g_1 \bar{w}_x + \kappa^2} \right] \frac{-i}{\epsilon} p_0 \gamma_0 + \phi_1 \frac{(-1) g_1^2 [g_1^2 \bar{g}_j + 2g_1 \bar{g}_j \bar{w}_x - (\bar{g}_0 - 2\bar{g}_j) \kappa^2]}{(g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} \frac{-i}{\epsilon} v p_j \gamma_j. \quad (\text{D2}) \end{aligned}$$

2. Self-energy: Gauge correction

Only the symmetric part of the gauge propagator produces a divergence at $\mathcal{O}(\frac{1}{N_f})$:

$$\Sigma_g(p_0, \mathbf{p}) = \left(\frac{-ig}{\sqrt{N_f}} \right)^2 \left(\frac{v}{c} \right)^{2-\delta_{\mu 0}-\delta_{\nu 0}} \int \frac{d^3 k}{(2\pi)^3} \gamma_\nu S_F(p-k) \gamma_\mu G_{\mu\nu}(k_0, \mathbf{k}) \quad (\text{D3})$$

$$= \frac{-g^2}{N_f} (+i) \left(\frac{v}{c} \right)^{2-\delta_{\mu 0}-\delta_{\nu 0}} \frac{1}{v^2} \int \frac{d^2 \bar{k} dk_0}{(2\pi)^3} \left[\gamma_\nu \frac{(p_0 - k_0)\gamma_0 + (\bar{p} - \bar{k})_a \gamma_a}{(p_0 - k_0)^2 + (\bar{p} - \bar{k})^2} \gamma_\mu \right] G_{\mu\nu}(k_0, \bar{k}). \quad (\text{D4})$$

Carrying out the momentum integral and setting $c = 1$,

$$\begin{aligned} -\Sigma_g(p_0, \mathbf{p}) = & \int_{-\infty}^{\infty} dz \frac{ig^2 \{ (v^2 - z^2) g_1 z^2 \sigma_e + [g_1 z^2 v^2 - g_1 z^4 \times \xi + w_x \sqrt{v^2 + z^2} (v^2 - z^2)] |z| \} (p_0 \gamma_0)}{4 \epsilon N_f \pi^2 (v^2 + z^2)^2 [\sqrt{v^2 + z^2} (g_1^2 + \kappa^2) \sigma_e + |z| ((g_1^2 + \kappa^2) \sqrt{v^2 + z^2} + g_1 w_x)]} \\ & + \int_{-\infty}^{\infty} dz \frac{ig^2 \{ g_1 (v^4 - v^2 z^2 - z^4) \sigma_e - [w_x z^2 \sqrt{v^2 + z^2} + g_1 (-v^4 + v^2 z^2 + z^4 \times \xi)] |z| \} (v p_j \gamma_j)}{4 \epsilon N_f \pi^2 (v^2 + z^2)^2 [\sqrt{v^2 + z^2} (g_1^2 + \kappa^2) \sigma_e + |z| ((g_1^2 + \kappa^2) \sqrt{v^2 + z^2} + g_1 w_x)]}. \end{aligned} \quad (\text{D5})$$

To obtain the above expression, we first perform a gradient expansion of $\Sigma(p_0, \mathbf{p})$ around $p_0 = \mathbf{p} = 0$. Next, focus on the linear term in p_0 and $\mathbf{p}$ and replace the frequency integral $k_0 \rightarrow z |\mathbf{k}|$. When the 2D spatial momentum integral is done, the result is the expression shown above. The above expression is integrable only at $\sigma_e = 0$. The ξ term is a divergent integral that arises from the choice of Coulomb gauge. Physical observables are free from any ξ dependence.

3. Self-energy: Effective gauge disorder $D_{\mu\nu}^{\text{dis}}, \mathcal{O}(\frac{\Delta_X}{N_f})$

$$\Sigma_b(p_0, \mathbf{p}) = \left(\frac{-ig}{\sqrt{N_f}} \right)^2 \left(\frac{v}{c} \right)^{2-\delta_{\mu 0}-\delta_{\nu 0}} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \gamma_\nu S_F(p_0 - 0, \mathbf{p} - \mathbf{k}) \gamma_\mu D_{\mu\nu}^{\text{dis}}(\mathbf{k}) \quad (\text{D6})$$

$$= \frac{+i(\Delta_0 + v^2 \Delta_j)}{2N_f \pi v^2 \epsilon (g_1^2 + \kappa^2)} \gamma_0 p_0 + \frac{-i(g_1^2 \Delta_0 + \kappa^2 v^2 \Delta_j)}{2N_f \pi v^2 \epsilon (g_1^2 + \kappa^2)^2} (v \gamma_j p_j). \quad (\text{D7})$$

APPENDIX E: THREE-POINT VERTEX $\bar{u}(\mathbf{q}) \delta\Gamma^\mu u(\mathbf{p})$

1. Γ^μ : Gauge correction

$$\Gamma_1^\mu = \left(\frac{-ig}{\sqrt{N_f}} \right)^3 \left(\frac{v}{c} \right)^{3-\delta_{\alpha 0}-\delta_{\beta 0}-\delta_{\mu 0}} \int \frac{d^2 \mathbf{k} d\omega}{(2\pi)^3} \gamma_\alpha S_F(q-k) \gamma_\mu S_F(p-k) \gamma_\beta G_{\beta\alpha}(k, \omega) \quad (\text{E1})$$

$$= \left(\frac{-ig}{\sqrt{N_f}} \right)^3 \left(\frac{v}{c} \right)^{3-\delta_{\alpha 0}-\delta_{\beta 0}-\delta_{\mu 0}} \frac{1}{v^2} \int \frac{d^2 \bar{k} d\omega}{(2\pi)^3} \gamma_\alpha (+i) \frac{\gamma_0(q_0 - k_0) + \gamma_c(\bar{q} - \bar{k})_c}{(q_0 - k_0)^2 + (\bar{q} - \bar{k})^2} \gamma_\mu (+i) \frac{\gamma_0(p_0 - k_0) + \gamma_d(\bar{p} - \bar{k})_d}{(p_0 - k_0)^2 + (\bar{p} - \bar{k})^2} \gamma_\beta G_{\beta\alpha}(k, \omega). \quad (\text{E2})$$

To isolate the divergent part, one can set the external momentum $p = q = 0$. Following the same steps we used in the self-energy diagram evaluation, we obtain

$$\begin{aligned} \Gamma_1^\mu = & \int_{-\infty}^{\infty} dz \frac{-ig}{\sqrt{N_f}} \frac{(-g^2) \{ (v^2 - z^2) g_1 z^2 \sigma_e + [g_1 z^2 v^2 - g_1 z^4 \times \xi + w_x \sqrt{v^2 + z^2} (v^2 - z^2)] |z| \}}{4 \epsilon N_f \pi^2 (v^2 + z^2)^2 [\sqrt{v^2 + z^2} (g_1^2 + \kappa^2) \sigma_e + |z| ((g_1^2 + \kappa^2) \sqrt{v^2 + z^2} + g_1 w_x)]} (\gamma_0) \\ & + \int_{-\infty}^{\infty} dz \frac{-ig}{\sqrt{N_f}} \frac{v (-g^2) (g_1 (v^4 - v^2 z^2 - z^4) \sigma_e - [w_x z^2 \sqrt{v^2 + z^2} + g_1 (-v^4 + v^2 z^2 + z^4 \times \xi)] |z|)}{4 \epsilon N_f \pi^2 (v^2 + z^2)^2 [\sqrt{v^2 + z^2} (g_1^2 + \kappa^2) \sigma_e + |z| ((g_1^2 + \kappa^2) \sqrt{v^2 + z^2} + g_1 w_x)]} (\gamma_j). \end{aligned} \quad (E3)$$

As before, ξ labels the divergent part. Gauge invariance is easy to check by comparing with Eq. (D5): $\Gamma_1^t = \frac{-g}{\sqrt{N_f}} \frac{\partial \Sigma_g}{\partial p_0}$ and $\Gamma_1^j = \frac{-g}{\sqrt{N_f}} \frac{\partial \Sigma_g}{\partial p_j}$.

2. Γ^μ : Effective gauge disorder correction

$$\Gamma_2^\mu = \left(\frac{-ig}{\sqrt{N_f}} \right)^3 \left(\frac{v}{c} \right)^{3-\delta_{\alpha 0}-\delta_{\beta 0}-\delta_{\mu 0}} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \gamma_\alpha S_F(q-k) \gamma_\mu S_F(p-k) \gamma_\beta D_{\beta\alpha}^{\text{dis}}(\mathbf{k}) \quad (E4)$$

$$= \frac{-i(\Delta_0 + v^2 \Delta_j)}{2N_f^{3/2} \pi v^2 \epsilon (g_1^2 + \kappa^2)} \gamma^0 + \frac{i(g_1^2 \Delta_0 + v^2 \kappa^2 \Delta_j)}{2N_f^{3/2} \pi v \epsilon (g_1^2 + \kappa^2)^2} \gamma^j. \quad (E5)$$

3. Γ^μ : Screened disorder correction $W_{\mu\nu}$

$$\Gamma_3^\mu = \left(\frac{-ig}{\sqrt{N_f}} \right) \left(\frac{v}{c} \right)^{1-\delta_{\mu 0}} \times \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \gamma_\alpha S_F(q-k) \gamma_\mu S_F(p-k) \gamma_\beta W_{\beta\alpha} \quad (E6)$$

$$\begin{aligned} = & \frac{-ig}{\sqrt{N_f}} \frac{1}{\epsilon} \left(\frac{g_0 + 2g_j}{2\pi v^2} + \phi_1 \left[\frac{g_1^2 \bar{w}_x (-g_1 \bar{g}_j + g_1 \bar{g}_0 + \bar{g}_0 \bar{w}_x)}{(g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} - \frac{g_1 (g_1 \bar{g}_j + g_1 \bar{g}_0 + 2\bar{g}_0 \bar{w}_x)}{g_1^2 + g_1 \bar{w}_x + \kappa^2} \right] \right) \gamma^0 \\ & + \frac{-ig}{\sqrt{N_f}} \phi_1 \frac{1}{\epsilon} \left(\frac{g_1^2 [g_1^2 \bar{g}_j + 2g_1 \bar{g}_j \bar{w}_x - (\bar{g}_0 - 2\bar{g}_j) \kappa^2]}{(g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} \right) \gamma^j. \end{aligned} \quad (E7)$$

4. Γ^μ : Random mass correction g_m

$$\Gamma_4^\mu = \left(\frac{-ig}{\sqrt{N_f}} \right) \left(\frac{v}{c} \right)^{1-\delta_{\mu 0}} \times \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \mathbf{1} S_F(q-k) \gamma_\mu S_F(p-k) \mathbf{1} g_m = \frac{-ig}{\sqrt{N_f}} \frac{g_m}{2\pi v^2 \epsilon} \gamma^0 + 0 \gamma^j. \quad (E8)$$

APPENDIX F: THREE-POINT VERTEX $\bar{u}(\mathbf{q}) \mathbf{1}_{2 \times 2} u(\mathbf{p})$

1. $\mathbf{1}_{2 \times 2}$: Gauge correction

$$\Gamma_1^m = \left(\frac{-ig}{\sqrt{N_f}} \right)^2 \left(\frac{v}{c} \right)^{2-\delta_{\alpha 0}-\delta_{\beta 0}} \frac{1}{v^2} (+i)^2 \int \frac{d^2 \bar{k} d\omega}{(2\pi)^3} \gamma_\alpha \frac{\gamma_0(q_0 - \omega) + \gamma_c(\bar{q} - \bar{k})_c}{(q_0 - \omega)^2 + (\bar{q} - \bar{k})^2} \mathbf{1} \frac{\gamma_0(p_0 - \omega) + \gamma_d(\bar{p} - \bar{k})_d}{(p_0 - \omega)^2 + (\bar{p} - \bar{k})^2} \gamma_\beta G_{\alpha\beta}, \quad (F1)$$

$$\begin{aligned}\Gamma_1^m &= \int_{-\infty}^{\infty} dz \frac{(v^2 + z^2) g^2 [g_1(2v^2 + z^2)\sigma_e + w_x \sqrt{v^2 + z^2} + g_1(2v^2 + z^2) |z|]}{4 \epsilon N_f \pi^2 (v^2 + z^2)^2 [\sqrt{v^2 + z^2} (g_1^2 + \kappa^2) \sigma_e + |z| ((g_1^2 + \kappa^2) \sqrt{v^2 + z^2} + g_1 w_x)]} \mathbf{1}_{2 \times 2} \\ &= \int_{-\infty}^{\infty} dz \frac{g^2 [g_1(2v^2 + z^2)(v^2 + z^2) \sigma_e + w_x (v^2 + z^2) \sqrt{v^2 + z^2} + g_1(2v^4 + 3z^2 v^2 + \xi z^4) |z|]}{4 \epsilon N_f \pi^2 (v^2 + z^2)^2 [\sqrt{v^2 + z^2} (g_1^2 + \kappa^2) \sigma_e + |z| ((g_1^2 + \kappa^2) \sqrt{v^2 + z^2} + g_1 w_x)]} \mathbf{1}_{2 \times 2}.\end{aligned}\quad (\text{F2})$$

2. $\mathbf{1}_{2 \times 2}$: Effective gauge disorder correction

$$\begin{aligned}\Gamma_2^m &= \left(\frac{-ig}{\sqrt{N_f}} \right)^2 \left(\frac{v}{c} \right)^{2-\delta_{\alpha 0}-\delta_{\beta 0}} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \gamma_\alpha S_F(q-k) \mathbf{1} S_F(p-k) \gamma_\beta D_{\beta\alpha}^{\text{dis}}(\mathbf{k}) \\ &= \frac{(\Delta_0 - v^2 \Delta_j)(g_1^2 - c^4 \kappa^2)}{2\pi N_f v^2 \epsilon (g_1^2 + c^4 \kappa^2)^2} \mathbf{1}_{2 \times 2}.\end{aligned}\quad (\text{F3})$$

3. $\mathbf{1}_{2 \times 2}$: Screened disorder correction $W_{\mu\nu}$

$$\begin{aligned}\Gamma_3^m &= \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \gamma_\alpha S_F(q-k) \mathbf{1} S_F(p-k) \gamma_\beta W_{\beta\alpha} \\ &= \left[\frac{-g_0 + 2g_j}{2\pi v^2 \epsilon} + \phi_1 \frac{1}{\epsilon} \left(-\frac{g_1^2(2g_1 + \overline{w}_x)(-g_1 g_j + \overline{g}_0 g_1 + \overline{g}_0 \overline{w}_x)}{(g_1^2 + g_1 \overline{w}_x + \kappa^2)^2} + \frac{g_1(3\overline{g}_0 g_1 - 3g_1 \overline{g}_j + 2\overline{g}_0 \overline{w}_x)}{g_1^2 + g_1 \overline{w}_x + \kappa^2} \right) \right] \mathbf{1}_{2 \times 2}.\end{aligned}\quad (\text{F4})$$

4. $\mathbf{1}_{2 \times 2}$: Random mass correction g_m

$$\Gamma_4^m = \left(\frac{-ig}{\sqrt{N_f}} \right) \times \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \mathbf{1} S_F(q-k) \mathbf{1} S_F(p-k) \mathbf{1} g_m = \frac{-g_m}{2\pi v^2 \epsilon} \mathbf{1}_{2 \times 2}.\quad (\text{F5})$$

APPENDIX G: FOUR-POINT FERMION-FERMION INTERACTION

Define

$$H_{\mu\nu} \equiv D_{\mu\nu}^{\text{dis}} \left(\frac{-ig}{\sqrt{N_f}} \right)^2 \left(\frac{v}{c} \right)^{2-\delta_{\mu 0}-\delta_{\nu 0}}.\quad (\text{G1})$$

Note that $W_{\mu\nu} \sim \mathcal{O}(\frac{1}{N_f^0})$ and $H_{\mu\nu} \sim \mathcal{O}(\frac{1}{N_f})$. Take the external three momenta to be p_1, p_2, p_3, p_4 , where $p_i = (\omega_i, \mathbf{p}_i)$. Schematically, the interaction has the form $[\overline{\psi}(p_3) \dots \psi(p_1)] [\overline{\psi}(p_4) \dots \psi(p_2)]$. Define

$$\begin{aligned}\Gamma_A &\equiv (\gamma_7, \gamma_0, +\gamma_x, +\gamma_y), \quad \gamma_7 \equiv \mathbf{1}_{2 \times 2} \\ T^{A_1} &= (\mathbf{1} g_m, W_{\mu\nu}, H_{\mu\nu}), \quad \tilde{T}^{A_1} = (\mathbf{1} g_m, W_{\mu\nu}^{(0)}, H_{\mu\nu}),\end{aligned}\quad (\text{G2})$$

$$W_{\mu\nu}^{(0)} = \text{diag}(g_0, i^2 g_j, i^2 g_j).\quad (\text{G3})$$

We use $A, B, C, D = \{1, 2, 3, 4\}$ indices to label $\mathbf{1}, \gamma_0, \gamma_x, \gamma_y$ and number subscripts, e.g., A_1 and A_2 , to label which interaction we choose: $A_1 = 1$ for the g_m interaction; $A_1 = 2$ for the $W_{\mu\nu}$ interaction; $A_1 = 3$ for the $H_{\mu\nu}$ interaction.

The diagrams below correspond to the following expressions:

$$B_1 = \int \frac{d^2 k}{(2\pi)^2} \overline{\psi}(p_3) \Gamma_B S_F(p_1 + k) \Gamma_A \psi(p_1) \overline{\psi}(p_4) \Gamma_D S_F(p_2 - k) \Gamma_C \psi(p_2) T_{CA}^{A_1}(\mathbf{k}, \omega = 0) T_{BD}^{A_2}(p_1 + k - p_3, \omega = 0), \quad (\text{G4})$$

$$B_2 = \int \frac{d^2 k}{(2\pi)^2} \overline{\psi}(p_3) \Gamma_B S_F(p_1 + k) \Gamma_A \psi(p_1) \overline{\psi}(p_4) \Gamma_C S_F(p_4 + k) \Gamma_D \psi(p_2) T_{CA}^{A_1}(\mathbf{k}, \omega = 0) T_{BD}^{A_2}(p_1 - p_3 + k, \omega = 0), \quad (\text{G5})$$

$$B_3 = \int \frac{d^2 k}{(2\pi)^2} \overline{\psi}(p_3) \Gamma_D S_F(p_3 - k) \Gamma_A S_F(p_1 - k) \Gamma_B \psi(p_1) T_{BD}^{A_2}(\mathbf{k}, \omega = 0), \overline{\psi}(p_4) \Gamma_C \psi(p_2) \tilde{T}_{AC}^{A_1}(p_3 - p_1, \omega = 0), \quad (\text{G6})$$

$$B_4 = \int \frac{d^2 k}{(2\pi)^2} \overline{\psi}(p_3) \Gamma_A \psi(p_1) \tilde{T}_{AC}^{A_1}(p_3 - p_1, \omega = 0) \overline{\psi}(p_4) \Gamma_D S_F(p_4 - k) \Gamma_C S_F(p_2 - k) \Gamma_B \psi(p_2) T_{BD}^{A_2}(k, \omega = 0). \quad (\text{G7})$$

For diagrams B_3 and B_4 , the T_{A1} vertex is undressed, i.e., $W_{\mu\nu}^0$, which is directly related to the random coupling being renormalized.

1. Four-point interaction: Boxes B_3 and B_4

Diagrams of type B_3 and B_4 can be directly obtained from the three-point vertex corrections in Appendices E and F with symmetry factor 2 (counting upper or lower vertices), so we do not have to recompute them here. The terms in Γ^μ renormalize g_0 and g_j and the terms in Γ^m renormalize g_m .

2. Four-point fermion interaction: Boxes B_1 and B_2

Diagrams for boxes B_1 and B_2 are presented below.

The W - H diagrams are $\mathcal{O}(\frac{\Delta_X g_X}{N_f})$. The H - H diagrams are $\mathcal{O}(\frac{\Delta_X^2}{N_f^2})$.

For each interaction, $(\bar{\psi}\psi)(\bar{\psi}\psi)$, $(\bar{\psi}\gamma_0\psi)(\bar{\psi}\gamma_0\psi)$, $(\bar{\psi}i\gamma_j\psi)(\bar{\psi}i\gamma_j\psi)$, we sum all these diagrams with the help of computer software to $\mathcal{O}(g_X^2, \frac{g_X}{N_f})$. The contributions from diagrams B_1 and B_2 are the following:

$$\begin{aligned} \text{Box}_{11} = & \left[\frac{2g_0g_j}{\pi v^2 \epsilon g_m} + \phi_1(-4g_1) \frac{g_1 \bar{g}_0 \bar{g}_j (2g_1^2 + 4g_1 \bar{w}_x + \bar{w}_x^2) - \kappa^2 [\bar{g}_0^2 g_1 + g_1 \bar{g}_j^2 - 2\bar{g}_0 \bar{g}_j (2g_1 + \bar{w}_x)]}{\epsilon (g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} \right. \\ & - \frac{\phi_1^2}{\epsilon} \left(\frac{4g_1^2 \bar{g}_j^2 \kappa^2 [g_1^2 (g_1^2 + 2g_1 \bar{w}_x - \bar{w}_x^2) + 2g_1 (g_1 - \bar{w}_x) \kappa^2 - \kappa^4]}{\bar{g}_m [g_1^2 + g_1 \bar{w}_x + \kappa^2]^4} \right. \\ & \left. \left. + \frac{4g_1^2 \bar{g}_0^2 \kappa^2 [g_1^2 (g_1^2 + 2g_1 \bar{w}_x + \bar{w}_x^2) + 2g_1 (g_1 + \bar{w}_x) \kappa^2 - \kappa^4]}{\bar{g}_m [g_1^2 + g_1 \bar{w}_x + \kappa^2]^4} \right) \right] \end{aligned}$$

$$+ \frac{4\bar{g}_0 \bar{g}_j g_1^2 \left[-g_1^3 (g_1 + \bar{w}_x)^2 (g_1 + 2\bar{w}_x) - 2g_1^2 (g_1 + \bar{w}_x) (2g_1 + 3\bar{w}_x) \kappa^2 - g_1 (5g_1 + 2\bar{w}_x) \kappa^4 + 2\kappa^6 \right]}{\bar{g}_m [g_1^2 + g_1 \bar{w}_x + \kappa^2]^4} \Bigg) \Bigg] \\ \times (\bar{\psi} \mathbf{1} \psi) (\bar{\psi} \mathbf{1} \psi), \quad (\text{G8})$$

$$\text{Box}_{\gamma_0 \gamma_0} = \left[\frac{2g_j g_m}{\pi v^2 \epsilon g_0} - \phi_1 \frac{4g_1^2 \bar{g}_m [g_1^2 \bar{g}_j + 2g_1 \bar{g}_j \bar{w}_x - (\bar{g}_0 - 2\bar{g}_j) \kappa^2]}{\bar{g}_0 \epsilon [g_1^2 + g_1 \bar{w}_x + \kappa^2]^2} \right] (\bar{\psi} \gamma_0 \psi) (\bar{\psi} \gamma_0 \psi), \quad (\text{G9})$$

$$\text{Box}_{\gamma_j \gamma_j} = \left[\frac{2g_0 g_m}{\pi v^2 \epsilon g_j} - \phi_1 \frac{g_1 \bar{g}_m [-g_1 \bar{g}_j \kappa^2 + \bar{g}_0 (g_1^3 + 2g_1^2 \bar{w}_x + 2\bar{w}_x \kappa^2) + g_1 (\bar{w}_x^2 + 2\kappa^2)]}{\bar{g}_j \epsilon [g_1^2 + g_1 \bar{w}_x + \kappa^2]^2} \right] (\bar{\psi} i \gamma_j \psi) (\bar{\psi} i \gamma_j \psi). \quad (\text{G10})$$

As mentioned before, the index $j = x$ or y ; there is no index sum here. And we assume the random current disorder variance $g_x = g_y \equiv g_j$ (isotropic).

APPENDIX H: TWO-LOOP VERTEX CORRECTIONS

At leading order, the generic two-loop diagram has the form pictured below.

The interaction legs X_1 and X_2 can be chosen to be the gauge propagator $G_{\mu\nu}$ or disorder $E_{\mu\nu} \in \{W_{\mu\nu}, g_m\}$. In principal there are four possible choices: $(X_1, X_2) = (G, G), (E, G), (G, E),$ or (E, E) . In the replica limit $n_r \rightarrow 0$, the (E, E) diagram vanishes because the fermion bubble is proportional to n_R . Also, (E, G) and (G, E) are the same diagrams so we only need to compute one of them. The top vertex can be either γ^μ or $\mathbf{1}_{2 \times 2}$. However, we will see below that diagrams using the γ^μ vertex are zero.

1. Mass vertex $\bar{u} \mathbf{1}_{2 \times 2} u$: One leg gauge, one leg disorder

$$\Gamma_{X_1} = \frac{1}{v^2} \int \frac{d^2 \bar{q} d q_0}{(2\pi)^3} \frac{1}{v^2} \int \frac{d^2 \bar{k}}{(2\pi)^2} \bar{u}(p_3) \left(\mathbf{1} \right)_{\gamma_\beta} S_F(p_1 - k) \frac{-ig}{\sqrt{N_f}} \left(\frac{v}{c} \right)^{1-\delta_{\mu 0}} \gamma_\mu u(p_1) G_{\mu\nu}(k) (-1) \\ \times N_f \text{Tr} \left[\frac{-ig}{\sqrt{N_f}} \left(\frac{v}{c} \right)^{1-\delta_{\nu 0}} \gamma_\nu S_F(q - p_1 + p_3) \mathbf{1} S_F(q) \left(\mathbf{1} \right)_{\gamma_\alpha} S_F(k + q - p_1 + p_3) \right] \left(W_{\alpha\beta}(k - p_1 + p_3) \right)^{g_m}. \quad (\text{H1})$$

$$\tilde{\Gamma}_{X_1} = \frac{1}{v^2} \int \frac{d^2 \bar{q} d q_0}{(2\pi)^3} \frac{1}{v^2} \int \frac{d^2 \bar{k}}{(2\pi)^2} \bar{u}(p_3) \left(\mathbf{1} \right)_{\gamma_\beta} S_F(p_1 - k) \frac{-ig}{\sqrt{N_f}} \left(\frac{v}{c} \right)^{1-\delta_{\mu 0}} \gamma_\mu u(p_1) G_{\mu\nu}(k) (-1) \\ \times N_f \text{Tr} \left[\frac{-ig}{\sqrt{N_f}} \left(\frac{v}{c} \right)^{1-\delta_{\nu 0}} \gamma_\nu S_F(q - k) \left(\mathbf{1} \right)_{\gamma_\alpha} S_F(q - p_1 + p_3) \mathbf{1} S_F(q) \right] \left(W_{\alpha\beta}(k - p_1 + p_3) \right)^{g_m}. \quad (\text{H2})$$

The direction of the fermionic loop momenta is different in Γ and $\tilde{\Gamma}$. We use the upper/lower components to distinguish the diagrams that arise from either $g_m/W_{\mu\nu}$.

To extract the UV divergence, we can set $p_1 = p_3 = 0$. For g_m , the divergences in Γ_{X_1} and $\tilde{\Gamma}_{X_1}$ cancel (upon changing variables $q \rightarrow -q$ in $\tilde{\Gamma}_{X_1}$ and using basic properties of the trace). For $W_{\alpha\beta}$, Γ_{X_1} and $\tilde{\Gamma}_{X_1}$ have identical esdivergences:

$$\Gamma_{X_1} = \frac{-g^2}{1} \left(\frac{v}{c}\right)^{2-\delta_{\mu 0}-\delta_{\nu 0}} \int \frac{d^3 k}{(2\pi)^3} \bar{u}(p_3) \left[\gamma_\beta \frac{k_0 \gamma_0 + v k_c \gamma_c}{k_0^2 + v^2 k^2} \gamma_\mu \right] u(p_1) \\ \times G_{\mu\nu}(k) \times W_{\alpha\beta}(k) \times \int \frac{d^3 q}{(2\pi)^3} \text{Tr} \left[\gamma_\nu \frac{q_0 \gamma_0 + v q_d \gamma_d}{q_0^2 + v^2 q^2} \mathbf{1} \frac{q_0 \gamma_0 + v q_e \gamma_e}{q_0^2 + v^2 q^2} \gamma_\alpha \frac{(k_0 + q_0) \gamma_0 + v (k_f + q_f) \gamma_f}{(k_0 + q_0)^2 + v^2 (k + q)^2} \right]. \quad (\text{H3})$$

Refer to the calculations in (H16) to compute

$$\int \frac{d^3 q}{(2\pi)^3} \text{Tr} \left[\gamma_\nu \frac{q_0 \gamma_0 + v q_d \gamma_d}{q_0^2 + v^2 q^2} \frac{q_0 \gamma_0 + v q_e \gamma_e}{q_0^2 + v^2 q^2} \gamma_\alpha \frac{(k_0 + q_0) \gamma_0 + v (k_f + q_f) \gamma_f}{(k_0 + q_0)^2 + v^2 (k + q)^2} \right] = \frac{i \epsilon_{\nu\alpha\sigma} (k_0, v \mathbf{k})_\sigma}{8v^2 \sqrt{k_0^2 + v^2 \mathbf{k}^2}}. \quad (\text{H4})$$

After setting $g = c = 1$,

$$\Gamma_{X_1} = \frac{-g^2}{1} \left(\frac{v}{c}\right)^{2-\delta_{\mu 0}-\delta_{\nu 0}} \int \frac{d^3 k}{(2\pi)^3} \bar{u}(p_3) \left[\gamma_\beta \frac{k_0 \gamma_0 + v k_c \gamma_c}{k_0^2 + v^2 k^2} \gamma_\mu \right] u(p_1) \times G_{\mu\nu}(k) \times W_{\alpha\beta}(k) \times \frac{i \epsilon_{\nu\alpha\sigma} (k_0, v \mathbf{k})_\sigma}{8v^2 \sqrt{k_0^2 + v^2 \mathbf{k}^2}} \\ = \frac{(g_0 g_1 - g_j g_1 - g_j \bar{w}_x)}{4\pi v^2 \epsilon (g_1^2 + g_1 \bar{w}_x + \kappa^2)} + \phi_1 \left[\frac{2g_1^3 (g_1 + \bar{w}_x) (-g_1 \bar{g}_j + g_1 \bar{g}_0 + \bar{g}_0 \bar{w}_x)}{\epsilon (g_1^2 + g_1 \bar{w}_x + \kappa^2)^3} \right. \\ \left. - \frac{g_1^2 [7\bar{g}_0 (g_1 + \bar{w}_x) - \bar{g}_j (7g_1 + 4\bar{w}_x)]}{2\epsilon (g_1^2 + g_1 \bar{w}_x + \kappa^2)^2} + \frac{g_1 (\bar{g}_0 - \bar{g}_j)}{\epsilon (g_1^2 + g_1 \bar{w}_x + \kappa^2)} \right]. \quad (\text{H5})$$

In total, we need to multiply by a factor of 4 to count the clockwise/counterclockwise fermion loops and the exchange of $W \leftrightarrow G$ in the diagrams:

$$\Gamma_{X_1} + \tilde{\Gamma}_{X_1} + (\Gamma_{X_1} + \tilde{\Gamma}_{X_1})_{W \leftrightarrow G} = 4\Gamma_{X_1}. \quad (\text{H6})$$

2. Vector vertex $\bar{u} \gamma^\rho u$: One leg gauge, one leg disorder

Replace the mass-vertex expressions $\mathbf{1}$ by γ_ρ :

$$\Gamma_{X_\rho} = \frac{1}{v^2} \int \frac{d^2 \bar{q} d q_0}{(2\pi)^3} \frac{1}{v^2} \int \frac{d^2 \bar{k}}{(2\pi)^2} \bar{u}(p_3) \left(\mathbf{1} \right)_{\gamma_\beta} S_F(p_1 - k) \frac{-ig}{\sqrt{N_f}} \left(\frac{v}{c}\right)^{1-\delta_{\mu 0}} \gamma_\mu u(p_1) G_{\mu\nu}(k) (-1) \times N_f \\ \times \text{Tr} \left[\frac{-ig}{\sqrt{N_f}} \left(\frac{v}{c}\right)^{1-\delta_{\nu 0}} \gamma_\nu S_F(q - p_1 + p_3) \gamma_\rho S_F(q) \left(\mathbf{1} \right)_{\gamma_\alpha} S_F(k + q - p_1 + p_3) \right] \left(W_{\alpha\beta}(k - p_1 + p_3) \right), \quad (\text{H7})$$

$$\tilde{\Gamma}_{X_\rho} = \frac{1}{v^2} \int \frac{d^2 \bar{q} d q_0}{(2\pi)^3} \frac{1}{v^2} \int \frac{d^2 \bar{k}}{(2\pi)^2} \bar{u}(p_3) \left(\mathbf{1} \right)_{\gamma_\beta} S_F(p_1 - k) \frac{-ig}{\sqrt{N_f}} \left(\frac{v}{c}\right)^{1-\delta_{\mu 0}} \gamma_\mu u(p_1) G_{\mu\nu}(k) (-1) \times N_f \\ \times \text{Tr} \left[\frac{-ig}{\sqrt{N_f}} \left(\frac{v}{c}\right)^{1-\delta_{\nu 0}} \gamma_\nu S_F(q - k) \left(\mathbf{1} \right)_{\gamma_\alpha} S_F(q - p_1 + p_3) \gamma_\rho S_F(q) \right] \left(W_{\alpha\beta}(k - p_1 + p_3) \right). \quad (\text{H8})$$

By similar argument, the term with an even number of γ 's in the trace would cancel between Γ and $\tilde{\Gamma}$, so in this case we only need to compute upper component (g_m). Setting $p_1 = p_3 = 0$, straightforward calculation gives

$$\Gamma_{X_\rho} = (-N_f)(-i)^4 \left(\frac{-ig}{\sqrt{N_f}} \right)^2 \left(\frac{v}{c} \right)^{2-\delta_{\mu 0}-\delta_{\nu 0}} \frac{1}{v^2} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{v^2} \int \frac{d^2 k}{(2\pi)^2} \times \left[\bar{u}(p_3) \frac{(-k)^c}{k^2} \gamma_c \gamma_\mu u(p_1) \right] \text{Tr}[\gamma_\nu q \gamma_\rho q(q+k)] \frac{1}{q^2 q^2 (k+q)^2} G_{\mu\nu}(\mathbf{k}, \omega=0) g_m = 0. \quad (\text{H9})$$

So there is no contribution from Γ_{X_ρ} , $\tilde{\Gamma}_{X_\rho}$

3. Mass vertex $\bar{u} \mathbf{1}_{2 \times 2} u$: Both legs are gauge propagators

$$\Gamma_{Z_1} = \left(\frac{-ig}{\sqrt{N_f}} \right)^4 \left(\frac{v}{c} \right)^{4-\delta_{\mu 0}-\delta_{\nu 0}-\delta_{\alpha 0}-\delta_{\beta 0}} \times (-1) \times (N_f) \int \frac{d^3 k d^3 q}{(2\pi)^3 (2\pi)^3} \bar{u}(p_3) \gamma_\beta S_F(p_1 - k) \gamma_\mu u(p_1) \times G_{\mu\nu}(k) \text{Tr}[\gamma_\nu S_F(q - p_1 + p_3) \mathbf{1} S_F(q) \gamma_\alpha S_F(k + q - p_1 + p_3)] G_{\alpha\beta}(k - p_1 + p_3), \quad (\text{H10})$$

$$\tilde{\Gamma}_{Z_1} = \left(\frac{-ig}{\sqrt{N_f}} \right)^4 \left(\frac{v}{c} \right)^{4-\delta_{\mu 0}-\delta_{\nu 0}-\delta_{\alpha 0}-\delta_{\beta 0}} \times (-1) \times (N_f) \int \frac{d^3 k d^3 q}{(2\pi)^3 (2\pi)^3} \bar{u}(p_3) \gamma_\beta S_F(p_1 - k) \gamma_\mu u(p_1) \times G_{\mu\nu}(k) \text{Tr}[\gamma_\nu S_F(-k + q + p_1 - p_3) \gamma_\alpha S_F(q) \mathbf{1} S_F(q + p_1 - p_3)] G_{\alpha\beta}(k - p_1 + p_3). \quad (\text{H11})$$

Upon taking the external momenta to zero,

$$\Gamma_{Z_1} = \frac{g^4}{N_f} \left(\frac{v}{c} \right)^{4-\delta_{\mu 0}-\delta_{\nu 0}-\delta_{\alpha 0}-\delta_{\beta 0}} \int \frac{d^3 k}{(2\pi)^3} \bar{u}(p_3) \left[\gamma_\beta \frac{k_0 \gamma_0 + v k_c \gamma_c}{k_0^2 + v^2 k^2} \gamma_\mu \right] u(p_1) \times G_{\mu\nu}(k) G_{\alpha\beta}(k) \times \int \frac{d^3 q}{(2\pi)^3} \text{Tr} \left[\gamma_\nu \frac{q_0 \gamma_0 + v q_d \gamma_d}{q_0^2 + v^2 q^2} \mathbf{1} \frac{q_0 \gamma_0 + v q_e \gamma_e}{q_0^2 + v^2 q^2} \gamma_\alpha \frac{(k_0 + q_0) \gamma_0 + v (k_f + q_f) \gamma_f}{(k_0 + q_0)^2 + v^2 (k + q)^2} \right]. \quad (\text{H12})$$

Perform the q integral first:

$$F_{\Gamma_z}(k) \equiv \int \frac{d^3 q}{(2\pi)^3} \text{Tr} \left[\gamma_\nu \frac{q_0 \gamma_0 + v q_d \gamma_d}{q_0^2 + v^2 q^2} \frac{q_0 \gamma_0 + v q_e \gamma_e}{q_0^2 + v^2 q^2} \gamma_\alpha \frac{(k_0 + q_0) \gamma_0 + v (k_f + q_f) \gamma_f}{(k_0 + q_0)^2 + v^2 (k + q)^2} \right] \quad (\text{H13})$$

$$= \frac{1}{v^2} \int \frac{d^3 Q}{(2\pi)^3} \text{Tr} \left[\gamma_\nu \frac{Q_\lambda \gamma_\lambda}{Q^2} \frac{Q_\rho \gamma_\rho}{Q^2} \gamma_\alpha \frac{(K + Q)_\sigma \gamma_\sigma}{(K + Q)^2} \right]. \quad (\text{H14})$$

Here we define $Q \equiv (q_0, v \mathbf{q})$, $d^3 Q \equiv dq_0 d^2(v \mathbf{q})$, $K \equiv (k_0, v \mathbf{k})$. Standard Feynman tricks give

$$F_{\Gamma_z}(k) = \frac{i \epsilon_{\nu\alpha\sigma} (k_0, v \mathbf{k})_\sigma}{8v^2 \sqrt{k_0^2 + v^2 \mathbf{k}^2}}. \quad (\text{H15})$$

So we have ($k_0 = \omega$)

$$\Gamma_{Z_1} = \frac{g^4}{N_f} \left(\frac{v}{c} \right)^{4-\delta_{\mu 0}-\delta_{\nu 0}-\delta_{\alpha 0}-\delta_{\beta 0}} \int \frac{d^3 k}{(2\pi)^3} \bar{u}(p_3) \left[\gamma_\beta \frac{k_0 \gamma_0 + v k_c \gamma_c}{k_0^2 + v^2 k^2} \gamma_\mu \right] u(p_1) G_{\mu\nu}(k) G_{\alpha\beta}(k) \left(\frac{i \epsilon_{\nu\alpha\sigma} (k_0, v \mathbf{k})_\sigma}{8v^2 \sqrt{k_0^2 + v^2 \mathbf{k}^2}} \right) = \bar{u}(p_3) \mathbf{1}_{2 \times 2} u(p_1) \int_{-\infty}^{\infty} dz \frac{-g^4 v^2 (\sigma_e + |z|) [(v^2 + z^2)(g_1^2 - \kappa^2) \sigma_e + |z| (g_1 w_x \sqrt{v^2 + z^2} + (g_1^2 - \kappa^2) (v^2 + z^2))]}{16 \epsilon N_f \pi^2 (v^2 + z^2)^{\frac{3}{2}} [\sqrt{v^2 + z^2} (g_1^2 + \kappa^2) \sigma_e + |z| (g_1^2 \sqrt{v^2 + z^2} + (g_1 w_x + \sqrt{v^2 + z^2} \kappa^2))]}^2. \quad (\text{H16})$$

The same manipulations are used in the computations of δ_1 and δ_2 . Note that unlike the case of δ_1 and δ_2 this term renormalizes g_m without any divergent integration, labeled by ξ .

Taking the limit $\sigma_e = w_x = 0$, the expression reduces to

$$\Gamma_{Z_1} \Big|_{\sigma_e=w_x=0} = \bar{u}(p_3) \mathbf{1}_{2 \times 2} u(p_1) \times \frac{-g^4 (g_1^2 - \kappa^2)}{8 \epsilon N_f \pi^2 (g_1^2 + \kappa^2)^2}, \quad (\text{H17})$$

which agrees with the result in [25] Unlike the case of Γ_{X_1} , the two legs are identical so the symmetry factor is 2:

$$\Gamma_{Z_1} + \tilde{\Gamma}_{Z_1} = 2\Gamma_{Z_1}. \quad (\text{H18})$$

4. Vector vertex $\bar{u} \gamma^\rho u$: Both legs are gauge propagators

Replace **1** by γ_ρ to obtain the vector counterparts

$$\Gamma_{Z_\rho} = \left(\frac{-ig}{\sqrt{N_f}} \right)^4 \left(\frac{v}{c} \right)^{4-\delta_{\mu 0}-\delta_{\nu 0}-\delta_{\alpha 0}-\delta_{\beta 0}} \times (-1) \times (N_f) \int \frac{d^3 k d^3 q}{(2\pi)^3 (2\pi)^3} \bar{u}(p_3) \gamma_\beta S_F(p_1 - k) \gamma_\mu u(p_1) \\ \times G_{\mu\nu}(k) \text{Tr}[\gamma_\nu S_F(q - p_1 + p_3) \gamma_\rho S_F(q) \gamma_\alpha S_F(k + q - p_1 + p_3)] G_{\alpha\beta}(k - p_1 + p_3), \quad (\text{H19})$$

$$\tilde{\Gamma}_{Z_\rho} = \left(\frac{-ig}{\sqrt{N_f}} \right)^4 \left(\frac{v}{c} \right)^{4-\delta_{\mu 0}-\delta_{\nu 0}-\delta_{\alpha 0}-\delta_{\beta 0}} \times (-1) \times (N_f) \int \frac{d^3 k d^3 q}{(2\pi)^3 (2\pi)^3} \bar{u}(p_3) \gamma_\beta S_F(p_1 - k) \gamma_\mu u(p_1) \\ \times G_{\mu\nu}(k) \text{Tr}[\gamma_\nu S_F(-k + q + p_1 - p_3) \gamma_\alpha S_F(q) \gamma_\rho S_F(q + p_1 - p_3)] G_{\alpha\beta}(k - p_1 + p_3). \quad (\text{H20})$$

By the same argument as before, there are six γ 's in the trace, so Γ_{Z_ρ} and $\tilde{\Gamma}_{Z_\rho}$ cancel one another:

$$\Gamma_{Z_\rho} + \tilde{\Gamma}_{Z_\rho} = 0. \quad (\text{H21})$$

APPENDIX I: THREE-LOOP CORRECTIONS OF DISORDERS Δ_0 and Δ_j

1. With the $D_{\alpha\beta}^{\text{dis}}$ propagator

$$\pi_1^{\mu\nu} = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^2 Q dQ_0}{(2\pi)^2} \left(\frac{-ig}{\sqrt{N_f}} \right)^4 \left(\frac{v}{c} \right)^{4-\delta_{\mu 0}-\delta_{\nu 0}-\delta_{\alpha 0}-\delta_{\beta 0}} (-1)^2 (N_f)^2 \text{Tr}[\gamma_\mu S_F(k - p) \mathbf{1} S_F(k - Q) \gamma_\alpha S_F(k)] \\ \times g_m \delta(p_0 - Q_0) \delta(Q_0) D_{\alpha\beta}^{\text{dis}}(\mathbf{Q}, Q_0 = 0) \times \text{Tr}[\gamma_\nu S_F(q) \gamma_\beta S_F(q - Q) \mathbf{1} S_F(q - p)] \quad (\text{I1})$$

(flipping the signs for k and q variables)

$$= \int \frac{d^2 Q}{(2\pi)^2} \left[\int \frac{d^3 k}{(2\pi)^3} \left(\frac{v}{c} \right)^{2-\delta_{\mu 0}-\delta_{\alpha 0}} \text{Tr}[\gamma_\mu S_F(k + p) \mathbf{1} S_F(k + Q) \gamma_\alpha S_F(k)] \right] \\ \times \left[\int \frac{d^3 q}{(2\pi)^3} \left(\frac{v}{c} \right)^{2-\delta_{\nu 0}-\delta_{\beta 0}} \text{Tr}[\gamma_\nu S_F(q) \gamma_\beta S_F(q + Q) \mathbf{1} S_F(q + p)] \right] \left[g_m \left(\frac{-ig}{\sqrt{N_f}} \right)^4 N_f^2 D_{\alpha\beta}^{\text{dis}}(\mathbf{Q}, Q_0 = 0) \delta(p_0 = 0) \right]. \quad (\text{I2})$$

Naively evaluating this diagram is problematic because the Feynman parameter integrals are not doable. To extract the divergence, we Taylor expand the expression to second order in p . First, we define

$$T^{\mu\alpha}(Q, p) = \left[\int \frac{d^3k}{(2\pi)^3} \left(\frac{v}{c} \right)^{2-\delta_{\mu 0}-\delta_{\alpha 0}} \text{Tr}[\gamma_\mu S_F(k+p) \mathbf{1} S_F(k+Q) \gamma_\alpha S_F(k)] \right]. \quad (13)$$

By reversing the trace order, we have

$$\left[\int \frac{d^3q}{(2\pi)^3} \left(\frac{v}{c} \right)^{2-\delta_{\nu 0}-\delta_{\beta 0}} \text{Tr}[\gamma_\nu S_F(q) \gamma_\beta S_F(q+Q) \mathbf{1} S_F(q+p)] \right] \quad (14)$$

$$= (-1)^5 \left[\int \frac{d^3q}{(2\pi)^3} \left(\frac{v}{c} \right)^{2-\delta_{\nu 0}-\delta_{\beta 0}} \text{Tr}[\gamma_\nu S_F(q+p) S(q+Q) \gamma_\beta S_F(q)] \right] = -T^{\nu\beta}(Q, p). \quad (15)$$

Let

$$\pi_1^{\mu\nu} = \int \frac{d^2Q}{(2\pi)^2} [T^{\mu\alpha}(Q, p)] [-T^{\nu\beta}(Q, p)] \times \left[g_m \left(\frac{-ig}{\sqrt{N_f}} \right)^4 N_f^2 D_{\alpha\beta}^{\text{dis}}(Q, Q_0=0) \delta(p_0=0) \right] \quad (16)$$

and

$$T_2(Q, p) \equiv T^{\mu\alpha} T^{\nu\beta}, \quad (17)$$

$$T_2(Q, p) = T_2(Q, 0) + \frac{\partial T_2}{\partial p_x} p_x + \frac{\partial T_2}{\partial p_y} p_y + \frac{1}{2} \left[\frac{\partial^2 T_2}{\partial p_x^2} p_x^2 + \frac{\partial^2 T_2}{\partial p_y^2} p_y^2 + 2 \frac{\partial^2 T_2}{\partial p_x \partial p_y} p_x p_y \right] + \mathcal{O}(p^3) \quad (18)$$

$$= \left(\frac{\partial T^{\mu\alpha}}{\partial p_x} \frac{\partial T^{\nu\beta}}{\partial p_x} \right) \Big|_{p=0} p_x^2 + \left(\frac{\partial T^{\mu\alpha}}{\partial p_y} \frac{\partial T^{\nu\beta}}{\partial p_y} \right) \Big|_{p=0} p_y^2 + \left(\frac{\partial T^{\mu\alpha}}{\partial p_x} \frac{\partial T^{\nu\beta}}{\partial p_y} + \frac{\partial T^{\mu\alpha}}{\partial p_y} \frac{\partial T^{\nu\beta}}{\partial p_x} \right) \Big|_{p=0} p_x p_y + \mathcal{O}(p^3). \quad (19)$$

Straightforward calculation gives

$$T^{\mu\alpha}(Q, p=0) = 0. \quad (110)$$

For first-order derivatives, we can also obtain (after lengthy algebra)

$$\frac{\partial T^{\mu\alpha}}{\partial p_j} = \left(\frac{v}{c} \right)^{2-\delta_{\mu 0}-\delta_{\alpha 0}} \frac{1}{v^2} \frac{i(-i)^3}{32|\bar{Q}|} \left(\epsilon^{\mu\alpha j} + \frac{1}{\bar{Q}^2} [\epsilon^{\mu\alpha\tau} \bar{Q}_\tau \bar{Q}_j + \epsilon^{\alpha j\tau} \bar{Q}_\tau \bar{Q}_\mu - \epsilon^{j\mu\tau} \bar{Q}_\tau \bar{Q}_\alpha] \right). \quad (111)$$

Notice that this result is true in three dimensions with general temporal component Q_0 .

Plugging into Eq. (12) and taking the four diagrams into consideration (each triangle has either clockwise or counterclockwise flowing momenta), the total result is

$$\pi_{\mu\nu}^{\text{tot}} = 4\pi_{\mu\nu}, \quad (112)$$

where

$$\pi_{11} = (-1) \frac{g_m(g_1^2 \Delta_0 + v^2 \Delta_j \kappa^2)}{1024\pi v^4 \epsilon (g_1^2 + g_1 \frac{w_x}{v} + \kappa^2)^2} (\bar{p}_y^2 + \bar{p}_x^2), \quad (113)$$

$$\pi_{ij} = (+1) \frac{g_m(g_1^2 v^2 \Delta_j + 2g_1 v w_x \Delta_j + w_x^2 \Delta_j + \Delta_0 \kappa^2)}{512\pi v^4 \epsilon (g_1^2 + g_1 \frac{w_x}{v} + \kappa^2)^2} (\delta_{ij} \bar{p}^2 - \bar{p}_i \bar{p}_j), \quad (114)$$

and $\bar{p}_i = v p_i$. π_{11} renormalizes Δ_j and π_{ij} renormalizes Δ_0 . This diagram scales as $1/N_f^2$ if g_m , Δ_0 , and Δ_j scale as $1/N_f$.

2. With the $W_{\alpha\beta}$ propagator

Replacing the internal propagator with $W_{\alpha\beta}$, the remaining calculations are the same:

$$\tilde{\pi}_{\mu\nu}^{\text{tot}} = 4\tilde{\pi}_{\mu\nu}, \quad (\text{I15})$$

where

$$\tilde{\pi}_{11} = (-1) \frac{g_m g_j N_f}{1024\pi v^4 \epsilon} (\bar{p}_y^2 + \bar{p}_x^2) + \phi_1 \frac{g_1^2 g_m N_f (g_1^2 g_j + 2g_1 g_j \frac{w_x}{v} + (2g_j - g_0)\kappa^2)}{1024\pi v^4 \epsilon (g_1^2 + g_1 \frac{w_x}{v} + \kappa^2)^2} (\bar{p}_y^2 + \bar{p}_x^2), \quad (\text{I16})$$

$$\tilde{\pi}_{ij} = (+1) \frac{g_0 g_m N_f}{512\pi v^4 \epsilon} (\delta_{ij} \bar{\mathbf{p}}^2 - \bar{p}_i \bar{p}_j) + \phi_1 \frac{g_1 g_m N_f [g_1 g_j \kappa^2 - g_0 (g_1 + \frac{w_x}{v}) (g_1^2 + g_1 \frac{w_x}{v} + 2\kappa^2)]}{512\pi v^4 \epsilon (g_1^2 + g_1 \frac{w_x}{v} + \kappa^2)^2} (\delta_{ij} \bar{\mathbf{p}}^2 - \bar{p}_i \bar{p}_j), \quad (\text{I17})$$

and $\bar{p}_i = v p_i$. $\tilde{\pi}_{11}$ renormalizes Δ_j , and $\tilde{\pi}_{ij}$ renormalizes Δ_0 . This diagram scales as $1/N_f$ if g_m, g_0 , and g_j scale as $1/N_f$.

3. With gauge propagator $G_{\alpha\beta}$

By dimensional analysis, this term should be UV finite:

$$\approx \int d^2 Q \frac{1}{Q} \frac{1}{Q} p^2 \frac{1}{Q} \Big|_{Q_0=p_0}.$$

APPENDIX J: SUMMARY

$$\bar{\delta}_1 p_0 \gamma^0 + \bar{\delta}_2 v p_j \gamma^j = \Sigma_d + \Sigma_g + \Sigma_b, \quad (\text{J1})$$

$$\bar{\delta}_{g_m} (\bar{\psi} \mathbf{1} \psi) (\bar{\psi} \mathbf{1} \psi) = \text{box}_{11} + 2(\Gamma_1^m + \Gamma_3^m + \Gamma_4^m + 4\Gamma_{x_1} + 2\Gamma_{z_1}) + 2\Gamma_2^m, \quad (\text{J2})$$

$$\bar{\delta}_{g_0} (\bar{\psi} \gamma^0 \psi) (\bar{\psi} \gamma^0 \psi) = \text{box}_{\gamma_0 \gamma_0} + 2(\Gamma_1^\mu + \Gamma_3^\mu + \Gamma_4^\mu)_{\mu=1} \left(\frac{-ig}{\sqrt{N_f}} \right)^{-1} + 2\Gamma_2^{\mu=1} \left(\frac{-ig}{\sqrt{N_f}} \right)^{-1}, \quad (\text{J3})$$

$$\bar{\delta}_{g_j} (\bar{\psi} i \gamma^j \psi) (\bar{\psi} i \gamma^j \psi) = \text{box}_{\gamma_j \gamma_j} + 2(\Gamma_1^\mu + \Gamma_3^\mu + \Gamma_4^\mu)_{\mu=j} \left(\frac{-ig}{\sqrt{N_f}} v \right)^{-1} + 2\Gamma_2^{\mu=j} \left(\frac{-ig}{\sqrt{N_f}} v \right)^{-1}, \quad (\text{J4})$$

where Σ_b , Γ_2^m , $\Gamma_2^{\mu=1}$, and $\Gamma_2^{\mu=j}$ are the sub-leading-order terms in the above expressions:

$$\bar{\delta}_{\Delta_0} (\delta_{ij} \mathbf{p}^2 - p_i p_j) = 4\pi_{ij} + 4\tilde{\pi}_{ij}, \quad (\text{J5})$$

$$\bar{\delta}_{\Delta_j} (p_x^2 + p_y^2) = 4\pi_{11} + 4\tilde{\pi}_{11}. \quad (\text{J6})$$

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