Spontaneous thermal Hall conductance in superconductors with broken time-reversal symmetry

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The thermal Hall conductivities (THCs) κ_{ij} have extensively been studied in recent condensed matter experiments. THCs can spontaneously become nonzero for a time-reversal-symmetry-broken system, and have a contribution from topologically protected edge states. Here, we focus on an additional bulk effect, the impurity mechanism in superconductors (SCs). Previously, the THCs were calculated for the chiral p -wave [S. K. Yip, *Supercond. Sci. Technol.* **29**, 085006 (2016); B. Arfi *et al.*, *Phys. Rev. B* **39**, 8959 (1989)] SCs for point impurities. Generalizing this finding to even-pairing states, we calculate thermal transport coefficients for finite-size impurities using Keldysh quasiclassical Green's functions. We find that for d -wave pairing, the impurity mechanism is dominant in κ_{yx} when compared to the topological contribution except at very low temperatures. This finding is shown to be valid irrespective of the impurity density of the bulk. There are two experimental signatures of the impurity mechanism on κ_{yx} : a nonmonotonic temperature dependence and a possible sign change as a function of temperature depending on the scattering phase shifts and the corresponding d -wave representation.

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Broken time-reversal symmetry (TRS) in the context of superconductors (SCs) is possible with a complex-valued momentum-dependent order parameter. In such a situation, the generalized BCS pairs can acquire a finite angular momentum. Candidate TRS-broken SCs include Sr_2RuO_4 [1], UPt_3 [2], URu_2Si_2 [3], and SrPtAs [4] as summarized in a recent paper [5].

In addition, a typical order parameter can admit nodal points and lines (or at least a suppressed energy gap), which allows the gapless Bogoliubov quasiparticles (BQs).

Broken TRS in SCs can be investigated by a few methods. In μ -spin relaxation techniques [6], incident spin-polarized muons are precessed by local magnetic fields created by the TRS-broken phase of a SC. Second, the polar Kerr angle method [7–9] detects changes in the polarization of a polarized light incident on the surface of a superconductor. As it is proportional to the nonzero components of the electric Hall conductivity, σ_{xy} , Kerr rotation is an indication of the broken TRS. A recent experiment detects, via small-angle

neutron scattering, a difference in vortex lattice Bragg peak splitting dependent on applied magnetic field history [10], while another one looked for an asymmetry in dI/dV curves of edge states [11] to detect this broken TRS. An additional method, which also constitutes the main interest of this article, is the thermal Hall conductivity, κ_{ij} .

There has been strong interest in the thermal Hall coefficients, which have been discussed in magnetic systems [12–15] as well as superconductors in the vortex phase [16,17]. Recently, κ_{yx} has received even more attention due to the search for anyonic and fractionalized excitations in condensed matter systems [18–20], where the quantized thermal Hall conductivity has been proposed as a method to detect the topologically protected edge states in fractional quantum Hall states [19,20], Kitaev magnets [18,21], or topological superconducting systems [19,22–24]. Half-integer quantized thermal Hall conductance has already been reported for the $\nu = 5/2$ fractional quantum Hall state [25] and the α - RuCl_3 Kitaev quantum spin liquid [26].

Thermal Hall conductivities (THCs) can spontaneously become nonzero for TRS-broken SCs. Though the nonzero THCs can have a contribution from topologically protected edge states [24,27–29], here we investigate an additional effect, the impurity mechanism (IM) for κ_{yx} [30,31].

Impurities are present in almost all real materials. In the normal state, they scatter the Fermi quasiparticles resulting in the decrease of the conductivities. Conventional

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superconductors are insensitive to nonmagnetic impurities as the impurity scattering does not change the sign of the s -wave gap seen by the electrons. Consequently, there are no new BQs which alter the qualitative properties of the system. Interestingly, unconventional superconductors are sensitive to even a small number of impurities [32,33]. Compared to conventional superconductors at low temperatures, the formation of impurity bands changes the transport properties dramatically. On one hand, impurities scatter the existing BQs and reduce the transport; on the other, they break Cooper pairs and create new BQs which enhance the thermal transport [34–37]. In this way, impurities hold a double role in thermal transport processes.

The simplest nontrivial superconductor having finite THCs is a p -wave superconductor. Inspired by Sr_2RuO_4 , the low-temperature THCs are analytically calculated [30] in the presence of point impurities. It is shown that κ_{yx} by the IM in a p -wave SC for a quasi-2D cylindrical Fermi surface with vertical line nodes is around two orders of magnitude larger than the topological contribution [28]. Motivated by the possible generality of this effect, we investigated the role of IM TRS-broken superconductors for even-pairing states. However, THCs for the first nontrivial d -wave pairing states are zero in the presence of point impurities [31]. In this work, hence, κ_{ij} are calculated for (i) finite-size impurities [38–41] and (ii) finite temperatures numerically and in the $T \rightarrow 0$ limit analytically.

We consider two possible irreducible representations of d -wave pairing based on the materials named above. We find that, under a temperature gradient, $\hat{x} = \frac{dT}{dx}$, the spontaneous κ_{yx}^{imp} due to the IM is nonzero. At finite temperatures, κ_{yx}^{imp} is much larger than the topological contribution, $\kappa_{yx}^{\text{topo}}$, within reasonable values of the scattering s - and p -wave phase shifts ($\delta_s, \delta_p \neq \{0, \pi/2\}$) independently of the impurity density, n_i . However, κ_{yx}^{imp} is much smaller than the topological contribution at very low temperatures.

This work consists of four sections. Section I is a brief introduction to the quantum transport equation (QTE) for Fermi liquids. It summarizes the quasiclassical approach and how to obtain single-particle observables, which is necessary for this work using quasiclassical Green's functions. In Sec. II, we describe how to obtain the self-energies for the impurity scattering processes in the t -matrix approximation. In Sec. III, the thermal transport coefficients are calculated for two d -wave pairing states, the E_{1g} and E_{2g} representations (Secs. III A and III B), based on the theory given in the previous two sections. Section III C compares these results with the topological contribution. Section IV is devoted to κ_{yx} , the impurity density Γ_u relation. At small impurity density limit the connection between the QTE and the quantum Boltzmann equation is reestablished.

I. QUASICLASSICAL QUANTUM TRANSPORT FORMALISM

The quasiclassical approach is an effective description of the dynamics of fermions. It is a linearized theory [42] that is capable of describing both the static and dynamical properties of quasiparticles. In this work, THCs are calculated by using

the nonequilibrium quasiclassical (QC) theory of a Fermi liquid [42–44]. After ξ_k integration, the quasiclassical Green's functions (GFs) are obtained with the following $\tilde{\tau}_3$ convention, $\check{g} \sim \int_{\xi_k} \tilde{\tau}_3 \check{G}$, where $\check{G}$ is the full GF in the Keldysh space. $\check{g}(\hat{k}, \vec{R}, \varepsilon, T)$ has four arguments: the direction of the Fermi momentum is $\hat{k} = \hat{k}_i \hat{i}$, where $\hat{k}_i$ is the i th scalar component of the Fermi unit vector, $\vec{R}$ is the center-of-mass coordinate (COM), ε is the energy with respect to the Fermi level, and T is the center-of-mass time. We drop the symbols for $\vec{R}$ and T for simplicity and T is used to denote the temperature in the rest of the paper.

The quasiclassical GF is guided by a quantum transport equation similar to a quantum Boltzmann equation and also inherits microscopic properties such as spin, electron, and holes within a classical transport formalism. In this respect, it is much easier to obtain system observables including density, currents, magnetization, etc. In addition, the Keldysh QC GFs expand the capability of this theory to nonequilibrium behavior [45]. In this work, a nonequilibrium bulk thermal Hall current is generated by a temperature gradient, $\vec{\nabla}T(\vec{R}) = \hat{x} \frac{dT}{dx}$.

The kinetic equation obeyed by $\check{g}(\hat{k}, \vec{R}, \varepsilon, T)$ is obtained by a systematic expansion of the QTE in gradients of COM, $\vec{\nabla}_{\vec{R}}$, to first order. Note that the Green-Kubo formalism is equivalent to this theory in the first order, as long as it is renormalized. The transport equation and the form of the GFs in Keldysh space are given as

$$[\varepsilon \tilde{\tau}_3 - \hat{\Delta} - \check{\sigma}, \check{g}] + i \vec{v}_f \cdot \vec{\nabla} \check{g} = 0, \quad \check{x} = \begin{bmatrix} \hat{x}^R & \hat{x}^K \\ 0 & \hat{x}^A \end{bmatrix}, \quad (1)$$

with $\check{x} \in \{\check{g}, \check{\sigma}, \check{t}\}$ and the normalization condition $\check{g}^2 = -\pi^2$. The symbol $\check{\cdot}$ represents Keldysh space functions $\check{x}$ whose elements $\hat{x}^R, \hat{x}^A, \hat{x}^K$ are 4×4 Nambu space matrices. $\vec{v}_f$ is the Fermi velocity, $\hat{\Delta}$ is the unitary pairing gap function [$\hat{\Delta}^2 = -|\Delta(\hat{k})|^2$], $\check{\sigma}$ is the impurity self-energy, $\check{t}$ is the t matrix, and $\tilde{\tau}_3 = \tau_3 \sigma_0$. Note that R, A , and K correspond to retarded, advanced, and Keldysh components. In addition, $(\hat{g}^X)_{nm}$ denotes the $\tau_n \sigma_m$ component in the extended Nambu space. Let us expand all terms up to first order in the COM gradients,

$$\check{g} = \check{g}_0 + \check{g}_1, \quad \hat{\Delta} = \hat{\Delta}_0 + \hat{\Delta}_1, \quad \check{\sigma} = \check{\sigma}_0 + \check{\sigma}_1. \quad (2)$$

First-order gradient terms are due to the applied temperature gradient in slow spatial coordinates.

The idea is, starting from the well-known equilibrium-limit GFs $\check{g}_0^{R,A,K}$, to calculate the self-energies at each order and feed them back to the QTE to obtain all the GFs. Focusing on the retarded part, the clean-limit, zeroth-order equilibrium GF is

$$\check{g}_0^R(\hat{k}) = -\pi \frac{\varepsilon \tilde{\tau}_3 - \hat{\Delta}(\hat{k})}{D^R(\hat{k})}; \quad (3)$$

$D^R(\hat{k}) = \sqrt{|\Delta(\hat{k})|^2 - (\varepsilon + i0^+)^2}$. The anomalous function in particle-hole space is $\hat{\Delta}(\hat{k}) = [\Delta(\hat{k})\tau_+ + \Delta^*(\hat{k})\tau_-]i\sigma_2$. Inspired by the possible d -wave pairing superconductors discussed in the first section, we choose two irreducible representations for the d -wave pairing gap function as $\Delta_{E_{1g}}(\hat{k}) = \Delta_0 \hat{k}_z(\hat{k}_x + i\hat{k}_y)$ and $\Delta_{E_{2g}}(\hat{k}) = \Delta_0(\hat{k}_x + i\hat{k}_y)^2$. For

practical reasons, we consider a model for the maximum gap size $\Delta_0(T)$ as in [31]. In the presence of the impurities, the gap size and the energy are renormalized as $\Delta_0 \rightarrow \Delta_0^R$, $\varepsilon \rightarrow \varepsilon^R$, and the denominator of $\hat{g}_0^R$ becomes $D^R(\hat{k}) = [|\Delta^R(\hat{k})|^2 - (\varepsilon^R)^2]^{1/2}$. The definitions above are also valid for the advanced GFs, $R \rightarrow A$.

The first-order Keldysh GF, $\hat{g}_1^K$, can be separated into the regular and the anomalous parts,

$$\hat{g}_1^K = (\hat{g}_1^R - \hat{g}_1^A) \tanh \frac{\varepsilon}{2T} + \hat{g}_{1a}^K. \quad (4)$$

The regular part corresponds to the contribution due to the modified spectral density but with the equilibrium distribution function. The anomalous part [44], $\hat{g}_{1a}^K$, denotes the rest and reflects the nonequilibrium distribution. The energy current density, $J_i(\hat{k})$, can be determined as the phase space sum of $\hat{g}^K$ as

$$J_i = 2N_f \int \frac{d\hat{k}}{4\pi} v_{f,i} \int_{-\infty}^{\infty} \frac{d\varepsilon}{4\pi i} \varepsilon (\hat{g}^K)_{00}. \quad (5)$$

In this expression, the spin degeneracy imposes the factor of two. N_f is the density of states per spin at the Fermi level and $v_{f,i}$ are the components of the Fermi velocity. One can verify explicitly by inserting $\hat{g}^K$ into Eq. (5) that the only nonzero contribution comes from the anomalous part, $\hat{g}_{1a}^K$. The reason is that the rest of the terms are even in ε . As soon as J_i are obtained, the phenomenological relation provides the thermal transport coefficients as $J_i = \kappa_{ij} \frac{-dT}{dx_j}$.

The first goal this work is to obtain the anomalous Keldysh GF, $\hat{g}_{1a}^K$. It consists of two parts, $\hat{g}_{1a}^K = \hat{g}_{1a}^{K,ns} + \hat{g}_{1a}^{K,V}$, where the superscripts ns and V stand for the non-self-consistent and the vertex correction GFs. The first term, $\hat{g}_{1a}^{K,ns}$, captures the response by neglecting the nonequilibrium changes in self-energies. It has the following form,

$$\hat{g}_{1a}^{K,ns}(\hat{k}, \varepsilon) = \hat{N}^R (\hat{g}_0^R - \hat{g}_0^A) \left(\frac{-\varepsilon}{2T^2} \text{sech}^2 \frac{\varepsilon}{2T} \right) i \vec{v}_f \cdot \vec{\nabla} T, \quad (6)$$

where

$$\hat{N}^R = \frac{(D^R + D^A) \hat{g}_0^R / (-\pi) + (\sigma_0^R)_{00} - (\sigma_0^A)_{00}}{(D^R + D^A)^2 + [(\sigma_0^R)_{00} - (\sigma_0^A)_{00}]^2}. \quad (7)$$

$\hat{g}_{1a}^{K,ns}$ is proportional to the gradient of the thermal distribution $\vec{v}_f \cdot \nabla \tanh \frac{\varepsilon}{2T}$. The factor $\hat{N}^R$ arises from inversion of Eq. (1). We remark here that the self-energy difference $(\sigma_0^R)_{00} - (\sigma_0^A)_{00}$ represents the difference between electron and hole lifetimes (see the last paragraph in Appendix D 2). We shall see that this quantity is particularly important for the THCs at finite temperatures. Beware that $\hat{g}_{1a}^{K,ns}$ only has an odd k_x term originating from the temperature gradient $\vec{v}_f \cdot \vec{\nabla} T$ and hence leads to $J_y = 0$ when plugged into Eq. (5).

The second term, the vertex correction anomalous GF, $\hat{g}_{1a}^{K,V}$, is

$$\hat{g}_{1a}^{K,V}(\hat{k}, \varepsilon) = \hat{N}^R (\hat{g}_0^R \hat{\sigma}_{1a}^K - \hat{\sigma}_{1a}^K \hat{g}_0^A). \quad (8)$$

It is proportional to the anomalous self-energy $\hat{\sigma}_{1a}^K(\hat{k})$ and it implies the additional scattering events in the presence of the external field. $\hat{g}_{1a}^{K,V}$ is the crucial GF to lead a finite thermal Hall current.

The self-energies ($\hat{\sigma}^{R,A,K}$) and the RA GFs ($\hat{g}^{R,A}$) are inputs to the equations for the Keldysh GF $\hat{g}^K$. All the equations must be solved self-consistently at each order. Note that $\Delta(\hat{k})$, T_c , the impurity concentration, and the scattering phase shifts (to be discussed below) are sufficient to calculate the thermal transport coefficients. In the next section, the self-energies are obtained by the t -matrix method.

II. IMPURITY t MATRIX AND THE SELF-ENERGIES

In the literature, impurities are generally modeled as points with a momentum-independent scattering potential $v(\hat{k}, \hat{k}') = v_0$ (in units of $T_c \times \text{volume}$), which greatly reduces the algebraic load. This approach is generally adequate to describe the scattering lifetime and other single-particle modifications. The finite-size impurities allow a realistic description of scattering events where the scattering depends on the momentum direction and become crucial to a nonzero κ_{yx} in d -wave SCs. The first nontrivial contribution to a finite-size spherical impurity is a p -wave scattering part in the scattering potential, $v(\hat{k}, \hat{k}') = v_0 + 3v_p \hat{k} \cdot \hat{k}'$. For reasons that we shall explain below, we consider a more general form,

$$v(\hat{k}, \hat{k}') = v_0 + 3v_s(\hat{k}_x \hat{k}'_x + \hat{k}_y \hat{k}'_y) + 3v_c \hat{k}_z \hat{k}'_z. \quad (9)$$

The scattering potential becomes spherically symmetric when $v_s = v_c$. The components are represented interchangeably as follows: $\hat{k}_x = \sin \theta \cos \phi$, $\hat{k}_y = \sin \theta \sin \phi$, $\hat{k}_z = \cos \theta$. We shall see that v_c and v_s play different physical roles for $\Delta_{E_{1g}}$ and $\Delta_{E_{2g}}$. The coefficients v_0 , v_s , and v_c are inputs to the transport equation derived from the normal limit and related to the scattering phase shifts by $\cot \delta_0 = \frac{-1}{\pi N_f v_0}$, $\cot \delta_s = \frac{-1}{\pi N_f v_s}$, $\cot \delta_c = \frac{-1}{\pi N_f v_c}$. The subscripts 0, c , and s represent the constant, $\cos \theta$, and $\sin \theta$ components of $v(\hat{k}, \hat{k}')$, respectively.

The effect of impurities is included by the single-impurity t -matrix approximation with an impurity-averaging approach. $\check{t}(\hat{k}, \hat{k}')$ is the t matrix connecting incoming and outgoing Fermi quasiparticles and quasiholes. Neglecting the correlation between the impurities, each impurity independently modifies the system. The contribution from such impurity-impurity correlation is expected to be an order of magnitude smaller in the “small” expansion parameter, $(k_f l)^{-1}$, where l is the typical impurity scattering length. The averaging procedure preserves the lattice symmetries whereas the system variables are modified. The recurrence relation of the t matrix is

$$\check{t} = v + N_f v \check{g} \check{t}. \quad (10)$$

The multiplicative part is a shorthand notation for the averaging over the Fermi unit sphere. Expanding Eq. (10), the R (or A) component of the t matrix is obtained as

$$\hat{t}^R = v + N_f \langle v \hat{g}^R \hat{t}^R \rangle, \quad (11)$$

while the Keldysh t matrix has the following recurrence relation,

$$\hat{t}^K = N_f \langle \hat{t}^R \hat{g}^K \hat{t}^A \rangle = (\hat{t}^R - \hat{t}^A) \tanh \frac{\varepsilon}{2T} + \hat{t}_{1a}^K. \quad (12)$$

Notice that the form of the latter is identical to $\hat{g}^K$. The last term, $\hat{t}_{1a}^K$, is the anomalous Keldysh t matrix,

$$\hat{t}_{1a}^K(\hat{k}, \hat{k}') = \pi N_f \left\langle \hat{t}_0^R(\hat{k}, \hat{k}_1) \frac{\hat{g}_{1a}^K(\hat{k}_1)}{\pi} \hat{t}_0^A(\hat{k}_1, \hat{k}') \right\rangle_{\hat{k}_1}. \quad (13)$$

The diagonal momentum components of $\hat{t}_{1a}^K$ are proportional to the self-energy as $\check{\sigma}(\hat{k}) = n_i \check{t}(\hat{k}, \hat{k})$. In this respect, the effect of many impurities is introduced by n_i , the impurity density.

For a d -wave SC, the zeroth-order GFs, $\hat{g}_0^{R,A,K}$, are even in $\hat{k}$ while the first-order GFs are odd, e.g., $\hat{g}_{1a}^K$. Then, a nonzero average in Eq. (13) is possible when even-odd combinations of $\hat{t}_0^R, \hat{t}_0^A$ are matched. However, a finite anomalous self-energy, $\hat{\sigma}_{1a}^K(\hat{k}) = n_i \hat{t}_{1a}^K(\hat{k}, \hat{k})$, does not automatically imply a finite κ_{yx} , as it must also have an odd momentum component in the $\hat{y}$ direction.

III. THERMAL CONDUCTIVITIES

A. E_{1g} order parameter, $\Delta_k = \Delta_0 \cos \theta \sin \theta e^{i\phi}$

The energy gap function $\Delta_{E_{1g}}(\hat{k})$ admits a nodal line around the equator and two point nodes at both poles of the Fermi level. For the impurity scattering potential in Eq. (9), one can argue that the BQs around the equator not only are much larger in numbers but also have much larger transverse momentum components, which would lead to a greater κ_{yx} compared to the BQs around the poles. Naturally, the main contribution to new BQs around the equator should be dominated by an effective potential, $v(\hat{k}, \hat{k}') \rightarrow v_0 + 3v_s(\hat{k}_x \hat{k}'_x + \hat{k}_y \hat{k}'_y)$. However, in this limit, the form of the anomalous vertex GFs and the intermediate steps become intractable. For the sake of clarity, in the E_{1g} case, we focus on only nonzero v_c along with the isotropic part, $v(\hat{k}, \hat{k}') = v_0 + 3v_c \hat{k}_z \hat{k}'_z$. The reader is encouraged to consult Appendix C 1 for the discussion where both v_s and v_c are finite. The contribution from the scatterers in the transverse momentum plane, v_s , is considered for the E_{2g} case in the next subsection.

Expanding the Born series in Eq. (10) and recollecting the common momentum components, the form of the full t matrix is found to be

$$\begin{aligned} \hat{t}_0^R(\hat{k}, \hat{k}') &= \hat{t}_{00}^R + \hat{t}_{cc}^R \hat{k}_z \hat{k}'_z \\ &= \frac{-1}{\pi N_f} \left(\frac{\cot \delta_0 + i \tilde{\gamma}_0 \hat{\tau}_3}{C_0} + 3 \frac{\cot \delta_c + i \tilde{\gamma}_c \hat{\tau}_3}{C_c} \hat{k}_z \hat{k}'_z \right), \end{aligned} \quad (14)$$

where $C_v = \cot^2 \delta_v + (\tilde{\gamma}_v)^2$ with $v \in \{0, c, s\}$. $\tilde{\gamma}_0 = \langle \frac{-ie^R}{D^R} \rangle$ and $\tilde{\gamma}_c = 3 \langle \frac{-ie^R}{D^R} \hat{k}_z^2 \rangle$ are the momentum averages of $(\hat{g}_0^R)_{30}$. $\hat{t}_0^R$ is diagonal in the particle-hole space and hence does not contain particle $\rightarrow$ hole type scattering events. The s -wave scattering part of the t matrix, $\hat{t}_{00}^R$, connects all incoming ($\hat{k}'$) and the outgoing ($\hat{k}$) particles (holes) with equal probabilities, whereas $\hat{t}_{cc}^R$ favors the scattering of BQs between the poles, $N \rightarrow N, N \rightarrow S, S \rightarrow N, S \rightarrow S$.

The zeroth-order self-energy is $\hat{\sigma}_0^R(\hat{k}) = n_i \hat{t}_0^R(\hat{k}, \hat{k})$. When plugged back into Eq. (1) $\hat{g}_0^R$ is obtained self-consistently. The diagonal form of $\hat{\sigma}_0^R$ does not renormalize the model gap func-

tion $\Delta(\hat{k})$ while modifying the energy as $\varepsilon^R = \varepsilon - (\hat{\sigma}_0^R)_{30}$. In order to calculate the Keldysh GFs, the advanced components are also necessary. The symmetry relations between the retarded and the advanced terms provide them automatically, $\hat{x}^A(\hat{k}) = \hat{\tau}_3 [\hat{x}^R(\hat{k})]^\dagger \hat{\tau}_3$ [42], where $\hat{x} \in \{\hat{\sigma}, \hat{g}\}$, and $\hat{t}^A(\hat{k}, \hat{k}') = \hat{\tau}_3 [\hat{t}^R(\hat{k}', \hat{k})]^\dagger \hat{\tau}_3$.

The strategy to obtain the anomalous components of Keldysh GFs is as follows: The non-self-consistent part, $\hat{g}^{K,ns}$, in Eq. (6) is obtained using the zeroth-order GFs and self-energies; then the vertex correction GF is calculated. $\hat{g}^{K,ns}$ has the same form for both E_{1g} and E_{2g} except how the impurities renormalize the components ε and Δ_0 . The $\tau_0 \sigma_0$ component of $\hat{g}^{K,ns}$ is

$$\frac{(\hat{g}_{1a}^{K,ns})_{00}}{\pi b(\varepsilon, T)} = \frac{2 \text{Re} D^R}{D_1} \left(1 + \frac{|\varepsilon^R|^2 - |\Delta(\hat{k})|^2}{|D^R|^2} \right) \hat{k}_x, \quad (15)$$

where

$$\begin{aligned} D_1 &= 4 \{ (\text{Re} D^R)^2 - [\text{Im}(\sigma_0^R)_{00}]^2 \}, \\ b(\varepsilon, T) &= i v_f \frac{-dT}{dx} \frac{\varepsilon}{2T^2} \text{sech}^2 \frac{\varepsilon}{2T}. \end{aligned} \quad (16)$$

The full form of $\hat{g}_{1a}^{K,ns}$ is given in Eq. (C5). In the clean limit, at each $(\hat{k}, \varepsilon)$, the expression within the large parentheses in Eq. (15) is equal to zero when ε is less than $|\Delta(\hat{k})|$ and equal to 2 otherwise. This is clearly due to the absence of Bogoliubov excitations within the gap. Yet, impurities modify the energy and the new BQs emerging within the gap also contribute to the heat current. The density-of-states plots in Fig. 1 shows the effect of impurity density compared to the clean limit. The finite density of new BQs within the gap is visible.

Plugging $\hat{g}_{1a}^{K,ns}$ and $\hat{t}_0^{R,A}$ into the anomalous t -matrix relation in Eq. (13), the anomalous self-energy is determined as

$$\hat{\sigma}_{1a}^{K,ns} = -\Gamma_u \hat{G} b(\varepsilon, T) \hat{k}_z \tau_1 i \sigma_2, \quad (17)$$

where $\hat{G} = G_0 + iG_3 \tau_3$ and $\Gamma_u = n_i / \pi N_f$. Note that the real coefficients G_0 and G_3 are even functions of ε and $\hat{k}$ and are found as

$$\begin{bmatrix} G_0 \\ G_3 \end{bmatrix} = [(\hat{t}_{0c}^-)_{00} \beta_0 + (i\hat{t}_{0c}^-)_{30} \beta_3 (\hat{t}_{0c}^-)_{00} \beta_3 - (i\hat{t}_{0c}^-)_{30} \beta_0], \quad (18)$$

where $\hat{t}_{0c}^- \equiv (\pi N_f)^2 (\hat{t}_{00}^R \hat{t}_{cc}^{A-} + \hat{t}_{cc}^R \hat{t}_{00}^{A-})/2$ is a shorthand notation for the coupled t -matrix scattering channels and $\hat{t}_{0c}^-$ is indicated with its Nambu space components as $(\hat{t}_{0c}^-)_{00} + (\hat{t}_{0c}^-)_{30} \hat{\tau}_3$. Also unlike $\hat{t}_{0c}^-$, “ $-$ ” superscripts of the t -matrix components, $\hat{t}_{00}^{A-}, \hat{t}_{cc}^{A-}$, indicate the negative τ_3 component. The two averages, β_0, β_3 , are obtained from $\hat{g}_{1a}^{K,ns}$ as

$$\begin{aligned} \beta_0(\varepsilon) &= \left\langle \frac{4 \text{Im} \sigma_{00}^R}{D_1} \text{Im} \frac{\Delta_0}{D^R} (\hat{k}_x^2 + \hat{k}_y^2) \hat{k}_z^2 \right\rangle, \\ \beta_3(\varepsilon) &= \left\langle \frac{4 \text{Re} D^R}{D_1} \frac{\text{Im} \varepsilon^R \Delta_0}{|D^R|^2} (\hat{k}_x^2 + \hat{k}_y^2) \hat{k}_z^2 \right\rangle \\ &= \left\langle \frac{4 \text{Re} D^R}{D_1} \frac{\text{Im} \varepsilon^R |\Delta(\hat{k})|^2}{|D^R|^2 \Delta_0} \right\rangle. \end{aligned} \quad (19)$$

They can be generalized to the E_{2g} case if written as in the last line above.

FIG. 1. Density of states for d -wave states $\Delta^{E_{1g}}$ and $\Delta^{E_{2g}}$ gap functions in the space of $\{\cos \theta - \varepsilon\}$, where θ is the polar angle. The plots (a) and (b) are the density of states for $\Delta^{E_{1g}}$ in the clean limit ($\Gamma_u = 0.001T_c$) and the finite impurity ($\Gamma_u = 0.06T_c$), where $T = 0.4T_c$. Plots (c) and (d), similarly, are density of states for $\Delta^{E_{2g}}$ in the clean limit and the finite impurity density with the same Γ_u values. The phase shifts are equal, $\delta_0 = \delta_s = \delta_c = \pi/3$.

There are two important aspects of $\hat{\sigma}_{1a}^{K,ns}$: the odd $\hat{k}_z$ component and the off-diagonal form in Nambu space. An odd $\hat{k}_z$ component is required in Eq. (8) for a nonvanishing thermal Hall current, J_y . Also, the off-diagonal form of $\hat{\sigma}_{1a}^{K,ns}$ provides two different sources to generate the transverse thermal Hall current: First, it matches the necessary $\hat{k}_z$ component within the parentheses of Eq. (8) from the off-diagonal components of $\hat{g}_0^{R,A}$. When multiplied by $\text{Im}\sigma_{00}^R$ within $\hat{N}^R$, the anomalous Hall current contribution becomes finite. Second, $\hat{\sigma}_{1a}^{K,ns}$ matches with the off-diagonal components of $\hat{g}_0^R$ within $\hat{N}^R$. More generally, the contributions to the thermal Hall current coming from the average, $\langle(\dots)_{\hat{k}}\hat{\Delta}(\hat{k})\vec{v}_f \cdot \vec{\nabla}T\rangle$, appearing in $\hat{\sigma}_{1a}^K$ of Eq. (13), where $(\dots)_{\hat{k}}$ is odd in $\hat{k}$ and represents the rest of the terms. The average is nonzero when $(\dots)_{\hat{k}}$ have components along both $\vec{\nabla}T$ and $\hat{\Delta}(\hat{k})$.

We find the $\tau_0\sigma_0$ component of $\hat{g}_{1a}^{K,V}(\hat{k}, \varepsilon)$ [see Eq. (C11)] as

$$(\hat{g}_{1a}^{K,V})_{00} = (X_{c,0}\hat{k}_x + X_{s,0}\hat{k}_y)\pi\Gamma_u b(\varepsilon, T), \quad (20)$$

where

$$\begin{bmatrix} X_{c,0} \\ X_{s,0} \end{bmatrix} = \frac{4\hat{k}_z^2\Delta_0}{D_1} \begin{bmatrix} \text{Im}\sigma_{00}^R \text{Im}1/D^R & \frac{-\text{Re}D^R \text{Im}\varepsilon^R}{|D^R|^2} \\ \frac{\text{Re}D^R \text{Im}\varepsilon^R}{|D^R|^2} & \text{Im}\sigma_{00}^R \text{Im}1/D^R \end{bmatrix} \begin{bmatrix} G_0 \\ G_3 \end{bmatrix}. \quad (21)$$

Clearly, the vertex correction coefficients vanish in the absence of superconductivity.

Actually, $\hat{\sigma}_{1a}^{K,ns}$ in Eq. (17) is the anomalous self-energy when only $\hat{g}_{1a}^{K,ns}$ is included. Plugging $\hat{g}_{1a}^{K,V}(\hat{k}, \varepsilon)$ back into the same relation, we can reveal the self-consistency condition that provides the full self-consistent solution and renormalized $\hat{G} \rightarrow \hat{\hat{G}}$ as well as the components of $\hat{g}_{1a}^{K,V}$ in Eq. (20).

All the necessary GFs are obtained for the E_{1g} representation and the thermal current density can now easily be calculated. We first focus on the contribution from $\hat{g}_{1a}^{K,ns}$. It creates only a longitudinal current, J_x :

$$J_x^{ns} = \frac{-dT}{dx} N_f v_f^2 \int_{-\infty}^{\infty} d\varepsilon \frac{\varepsilon^2}{2T^2} \text{sech}^2 \frac{\varepsilon}{2T} \Omega_0(\varepsilon). \quad (22)$$

Here $\tau = \varepsilon/2T$ is the dimensionless energy, T_c is the critical temperature, and the angular average is

$$\Omega_0(\varepsilon) = \left\langle \frac{4\text{Re}D^R}{D_1} \left(1 + \frac{|\varepsilon^R|^2 - |\Delta(\hat{k})|^2}{|D^R|^2} \right) \hat{k}_x^2 \right\rangle, \quad (23)$$

and it is in units of $\frac{1}{E}$. Then, κ_{xx}^{ns} is found as

$$\frac{\kappa_{xx}^{ns} T_c}{\frac{\pi^2}{3} N_f v_f^2 T} = \frac{6T_c}{\pi^2} \int_{-\infty}^{\infty} \frac{d\tau \tau^2}{\cosh^2(\tau)} \Omega_0(2\tau T). \quad (24)$$

Similarly for the vertex correction Keldysh GF, $\hat{g}_{1a}^{K,V}$, the thermal transport coefficients are

$$\frac{[\kappa_{yx}^V] T_c}{\frac{\pi^2}{3} N_f v_f^2 T} = \frac{6\Gamma_u T_c}{\pi^2} \int_{-\infty}^{\infty} \frac{d\tau \tau^2}{\cosh^2(\tau)} \left[\langle \hat{k}_x^2 \tilde{X}_{c,0} \rangle \right]. \quad (25)$$

Even though we are now calculating the thermal conductivities for E_{1g} , the integral expressions for κ_{xx}^{ns} in Eq. (24) and κ_{ij}^V in Eq. (25) are valid for both representations. The only difference lies in the exact forms of the averages.

Focusing back onto the E_{1g} case, the explicit forms of $\tilde{X}_{c,0}$ and $\tilde{X}_{s,0}$ are inserted into Eq. (25) and the further simplifications provide a physically relatable expression for κ_{ij}^V as

$$\approx \Gamma_u \int_{\tau} (\dots) \frac{\beta_0^2 + \beta_3^2}{2} \text{Re} \left[\frac{(\cot \delta_0 \cot \delta_c - \tilde{\gamma}_0^* \tilde{\gamma}_c)/C_0^* C_c}{(\cot \delta_0 \tilde{\gamma}_c + \cot \delta_c \tilde{\gamma}_0^*)/C_0^* C_c} \right]. \quad (26)$$

$(\dots)$ simply includes rest of the terms on the right-hand side of Eq. (25). κ_{ij} are kept up to the first order in Γ_u . The thermal transport coefficients are the sum of the ns and V contributions as $\kappa_{ij} = \kappa_{ij}^{ns} + \kappa_{ij}^V$. The vertex contributions are dependent on the averages β_0, β_3 , and as shown in Table I, these averages could dominate one another at two limits of the temperature and the impurity density.

The longitudinal coefficient, $\kappa_{xx} = \kappa_{xx}^{ns} + \kappa_{xx}^V$, can only fluctuate around the universal value on the order of impurity density and it depends on the phase shifts in $\delta_0, \delta_c \in [0, \pi/2]$. The transverse thermal transport coefficient, $\kappa_{yx} =$

TABLE I. The comparison of the magnitudes of the four averages $\Omega_{0,3}, \beta_{0,3}$ for E_{1g} and E_{2g} at two limits of the temperature and the impurity density.

	$T \rightarrow 0$	$T \approx \Delta_0/2$
E_{1g} :		
$\Gamma_u \rightarrow 0$		$\{\Omega_0, \beta_0\} \gg \beta_3$
$\Gamma_u \approx \frac{\Delta_0}{20}$	$\{\Omega_0, \beta_3\} \neq 0, \beta_0 \rightarrow 0$	$\{\Omega_0, \beta_0, \beta_3\} \neq 0$
E_{2g} :		
$\Gamma_u \rightarrow 0$		$\{\Omega_0, \beta_0\} \gg \{\Omega_3, \beta_3\}$
$\Gamma_u \approx \frac{\Delta_0}{20}$	$\{\Omega_0, \beta_3\} \neq 0, \{\Omega_3, \beta_0\} \rightarrow 0$	$\{\Omega_3, \Omega_0, \beta_0, \beta_3\} \neq 0$

FIG. 2. The dimensionless κ_{yx} as a function of temperature (in units of T_c). It is calculated both for E_{1g} (dotted lines) and for E_{2g} (solid lines) for three different δ_c (or δ_s). The s -wave scattering phase shifts for E_{1g} and E_{2g} are $\delta_0 = \pi/6$ and $\Gamma_u = 0.04T_c$.

κ_{yx}^V , roughly follows the sign of $\cot \delta_0 + \cot \delta_c$ (except for a small region of phase shifts where $\tilde{\gamma}_0$ and $\tilde{\gamma}_c$ greatly differ from each other; see Fig. 5 for the full κ_{yx} in the space of δ_0 - δ_c).

κ_{ij} can further be evaluated at two energy scales. The low-temperature limit provides an analytical form whereas the finite-temperature limit can only be obtained numerically.

In the low-temperature limit, the BQs populate only the vicinity of the Fermi level; the relevant energy scale becomes the impurity bandwidth. Taking $\varepsilon \rightarrow 0$, the residual energy scale becomes $\varepsilon^{R,A} \rightarrow \pm i\gamma(\theta) = i(\gamma_0 + \gamma_c \hat{k}_z^2)$ with γ_0, γ_c having positive-definite real values. With the help of explicit τ integration in Eq. (22), $\frac{6}{\pi^2} \int d\tau \tau^2 \text{sech}^2 \tau = 1$, the energy integral leads to $\frac{6}{\pi^2} \int_{-\infty}^{\infty} d\tau (\dots) \rightarrow \Omega_0(0) = 2(\frac{\gamma^2}{D^3} \hat{k}_x^2)_{\theta}$. The momentum averages are evaluated in the asymptotic limit, $T, \varepsilon \rightarrow 0 \sim 1/\Delta_0$. Then, Ω_0 is on the order of $\sim 1/\Delta_0$, where $D = [|\Delta(\hat{k})|^2 + \gamma^2]^{1/2}$. Then, κ_{xx}^{ns} simplifies to

$$\frac{\kappa_{xx}^{\text{ns}} T_c}{\frac{\pi^2}{3} N_f v_f^2 T} \sim T_c / \Delta_0, \quad (27)$$

with a universal value [35]. The vertex correction coefficients κ_{ji}^V , unlike κ_{xx}^{ns} , depend on the impurity density and a combination of phase shifts,

$$\approx \frac{\Gamma_u T_c \beta_3^2(0)}{2C_0 C_c} \left[\cot \delta_0 \cot \delta_c - \tilde{\gamma}_0 \tilde{\gamma}_c \right]. \quad (28)$$

It is crucial to determine why the average $\beta_3 = 2\langle \frac{\gamma(\theta)\Delta_0}{D^3} \hat{k}_z^2 \hat{k}_x^2 \rangle$ is finite at low temperatures. This finite value is realized by the match of odd $\hat{k}_x, \hat{k}_z$ components in $\Delta(\hat{k})^{E_{1g}} \propto \hat{k}_z(\hat{k}_x + i\hat{k}_y)$ with one $\hat{k}_z$ coming from the potential v_c and the other $\hat{k}_x$ from the temperature gradient. It is evaluated as $\beta_3(0) \sim \frac{\gamma_0}{\Delta_0^2} \ln \frac{\Delta_0}{\gamma_0}$ as $T \rightarrow 0$. Also, $\tilde{\gamma}_0 = \langle \frac{\gamma(\theta)}{D} \rangle \sim \frac{\gamma_0}{\Delta_0} \ln \frac{\Delta_0}{\gamma_0}$, $\tilde{\gamma}_c = 3\langle \frac{\gamma(\theta)}{D} \hat{k}_z^2 \rangle \sim \frac{\gamma_0}{\Delta_0}$. Equation (28) shows that both δ_0 and δ_c must be nonzero for a finite κ_{yx} . Interestingly, even in the unitary limit of s -wave scattering phase shift $\delta_0 = \pi/2$, a finite κ_{yx} is achieved for $\delta_c \in (0, \pi/2)$, and vice versa. The vertex correction

coefficients in Eq. (28) are simplified as

$$\sim \frac{T_c}{\Delta_0} \frac{\gamma_0^2 \ln \frac{\Delta_0}{\gamma_0}}{\Delta_0^2} \left[\cot \delta_0 \cot \delta_c - \frac{\gamma_0^2}{\Delta_0^2} \left(\ln \frac{\Delta_0}{\gamma_0} \right)^2 \right] \frac{1}{C_c}. \quad (29)$$

For nonzero and finite values of $\cot \delta_0$ and $\cot \delta_c$, the sign of κ_{yx} is the same with $\cot \delta_0 + \cot \delta_c$. In addition, the unitless form of κ_{yx} is significantly a small number.

At finite temperatures, in Fig. 2, we numerically calculate κ_{yx} as a function of temperature for two sets of phase shifts $\{(\delta_0, \delta_c)\}$ and Γ_u . The peaks are within the range 0.1–0.6 T_c , where it depends on the impurity density and the scattering phase shifts. Similarly to this, the THC's were predicted previously for p -wave pairing to have their maximum value around $T \sim \Delta_0/2$ [31].

At not too small temperatures in Table I, not only β_3 but another significant contribution to κ_{yx} comes from the β_0 term defined in Eq. (C7). In this regime, thermal transport in general is dominated by excitations above the (angular dependent) gap, where D^R can acquire an imaginary part $\propto \sqrt{\varepsilon^2 - |\Delta(\hat{k})|^2}$ even without impurity self-energies, whereas $\text{Im} \varepsilon^R$ is finite only in the presence of such a contribution. The contribution from β_0 increases with T due to the excitations above the gap.

B. E_{2g} order parameter, $\Delta_k = \Delta_0 \sin^2 \theta e^{i2\phi}$

The energy gap function $\Delta_{E_{2g}}$ has two point nodes at the poles. Even if the v_c part of $v(\hat{k}, \hat{k}')$ in Eq. (9) is dominant for BQs around the poles, it can be shown that it generates no thermal current in the y direction. It is because the integral average in Eq. (13) vanishes due to the orthogonal components in momentum space, $\hat{\sigma}_{1a}^K \sim \langle (\dots)_{\text{even}} k_x k_z \rangle \sim 0$, then $\kappa_{yx} \rightarrow 0$. Note that $(\dots)_{\text{even}}$ corresponds to the rest of the irrelevant terms in Eq. (13). Therefore, we focus on the

equator scattering potential $v(\hat{k}, \hat{k}') = v_0 + 3v_s(\hat{k}_x\hat{k}'_x + \hat{k}_y\hat{k}'_y)$. In this limit, the t matrix is

$$\hat{t}_0^R(\hat{k}, \hat{k}') = \hat{t}_{00}^R + \left\{ \hat{t}_{ssc}^R \left[\cos(\phi - \phi') - \frac{1}{2} \alpha e^{i(\phi - \phi')\hat{\tau}_3} \right] - \frac{3\alpha_0}{2\pi N_f} e^{i(\phi + \phi')\hat{\tau}_3} \hat{\tau}_1 i \hat{\sigma}_2 \right\} \sin \theta \sin \theta', \quad (30)$$

$$\hat{t}_{ssc}^R = \frac{-3}{\pi N_f} \frac{\cot \delta_s + i \tilde{\gamma}_s \hat{\tau}_3}{C_s}. \quad (31)$$

Note $\hat{t}_{00}^R$ has the same form with Eq. (14), $\alpha_0 = \tilde{\gamma}^1/C_{s2}$, $\alpha = \tilde{\gamma}^1\alpha_0$, $C_{s2} = \cot^2 \delta_s + (\tilde{\gamma}_s)^2 + (\tilde{\gamma}^1)^2$, where $\tilde{\gamma}^1 = \langle \frac{3}{2} \frac{\tilde{\Delta}_0 \sin^4 \theta}{D^R} \rangle$ and $\tilde{\gamma}_s = \langle \frac{3}{2} \frac{-ie^R \sin^2 \theta}{D^R} \rangle$. $\tilde{\Delta}_0 = \frac{\Delta_0}{d}$ is the renormalized gap size with $d = |1 + \frac{3}{2} \Gamma_u \frac{\tilde{\gamma}^1}{C_{s2}}|^2$. For the sake of simplicity, $\Delta_0^{R,A}$ are real and equal to each other when the sufficiently small $\pm$ imaginary parts are omitted.

In Eq. (30), the term with the coefficient α clearly shows the asymmetry in the particle-hole space. It is a spontaneously generated skew-scattering effect [46] not present in the E_{1g} case (unless v_s is included) and leads to an additional contribution to κ_{yx} in the superconducting state ($\alpha \propto \tilde{\gamma}^1 \neq 0$). The other sources of κ_{yx} are coming from $\langle (\dots)_{\hat{k}} \hat{\Delta}(\hat{k}) v_{f,y} \rangle$ in Eq. (5) as previously discussed for E_{1g} . Each term contributes to the κ_{yx} through the anomalous self-energy $\hat{\sigma}_{1a}^K$. Note that $\hat{g}_{1a}^{K,ns}$ and J_x^{ns} have the same form as in the E_{1g} case and $\hat{\sigma}_{1a}^K$ can be obtained though acquiring a more complicated form,

$$\hat{\sigma}_{1a}^{K,ns}(\hat{k}) = \Gamma_u [\hat{P}\hat{k}_x + \hat{S}\hat{k}_y + \hat{G}(\hat{k}_x + i\hat{k}_y\tau_3)\hat{\tau}_1 i\hat{\sigma}_2] b(\varepsilon, T). \quad (32)$$

The Keldysh GFs and self-energies obey the symmetry relation $\hat{x}^K(\hat{k}) = -\tau_3 [\hat{x}^K(\hat{k})]^\dagger \tau_3$. This means that each of the coefficients of $\hat{P}$, $\hat{S}$, $\hat{G}$, which are $\hat{P} = P_0 + P_3\tau_3$, $\hat{G} = S_0 + S_3\tau_3$, and $\hat{G} = G_0 + iG_3\tau_3$, must be real valued. The full form of each of the coefficients is not important [see Eq. (D11)] but it is important to know that they are linear superpositions of the averages Ω_0 , Ω_3 , β_0 , and β_3 , where each average is multiplied by one of the combined s - and p -wave scattering channels, e.g., $\hat{t}_{00}^R \hat{t}_{ssc}^R$, $\alpha_0 \hat{t}_{00}^R$, etc. The latter is significant because the vertex correction is possible only when even and odd scattering channels are combined. The forms of the Ω_0 , β_0 , β_3 were already given before and $\Omega_3(\varepsilon)$ is defined as

$$\Omega_3(\varepsilon) = \left\langle \frac{8\text{Im}(\sigma_0^R)_{00} \text{Im} \frac{\varepsilon^R}{D^R} \hat{k}_x^2}{D_1} \right\rangle. \quad (33)$$

Plugging $\hat{\sigma}_{1a}^{K,ns}(\hat{k}, \varepsilon)$ into $\hat{g}_{1a}^{K,V}(\hat{k}, \varepsilon)$, one can see that the anomalous vertex GF has the same form with Eq. (20). The only difference lies in the explicit expressions of $\hat{X}_c$, $\hat{X}_s$ [see Eq. (D15)]. The vertex correction to the thermal transport coefficients in Eq. (25) is evaluated as

$$\frac{[\kappa_{yx}^V]_{E_{2g}}}{\frac{\pi^2}{3} N_f v_f^2 \frac{T}{T_c}} = \Gamma_u \int_{\tau} (\dots) \left[\Omega_0 \tilde{P}_0 - \beta_3 \tilde{G}_3 + \Omega_3 \tilde{P}_3 + \beta_0 \tilde{G}_0 \right]. \quad (34)$$

Depending on the coefficients $P_{0,3}$, $S_{0,3}$, $G_{0,3}$, the κ_{ij} are the energy integral of the linear combination of the pairs $\Omega_i\Omega_j$, $\beta_i\beta_j$, and $\Omega_i\beta_j$. Note that the sign of $\{\Omega_0, \beta_3\}$ is positive while $\{\Omega_3, \beta_0\}$ is negative for $\delta_0, \delta_s \in (0, \pi/2)$. The magnitude of

the averages varies with respect to the temperature and the impurity density. Therefore it is crucial to determine the dominant average(s) at each limit of T and Γ_u . This is shown in Table I and we refer to one limit in the rest of this subsection.

κ_{yx} is numerically calculated in Fig. 2 at finite temperatures for three different pairs of phase shifts, $\delta_0 = \pi/6$ and $\delta_s = \pi/9, \pi/3, -\pi/3$, with a small impurity density $\Gamma_u = 0.04T_c$. The center of the peak for κ_{yx} varies as a function of phase shift. The sign of κ_{yx} is negative when both of the phase shifts are in the first quadrant, $\delta_0, \delta_s \in (0, \pi/2)$. A reversal of all the phase shifts also means the reversal of the thermal Hall current.

There is a sign change in κ_{yx} at relatively small temperatures when δ_0 and δ_s have opposite signs. Neglecting the mathematical background for now, this sign change is due to the competition between the attractive and repulsive parts of two phase shifts, $\delta_{0,s}$, but is not as simple as s - and p -wave channels lead to two opposite heat currents. It is hidden within two groups of contributions to κ_{yx} with opposite signs, $\{\Omega_0, \beta_3\}$ and $\{\Omega_3, \beta_0\}$ (see Table I) for $\delta_{0,s} \in \{0, \pi/2\}$. The contributions are possible only when a combination of both s - and p -wave channels is present as discussed in the last paragraph of Sec. II. At small temperatures, only $\{\Omega_0, \beta_3\}$ are finite and lead to $\kappa_{yx} \leq 0$ as in Eq. (36). In addition, around $T \sim \Delta_0/2$, $\{\Omega_3, \beta_0\}$ also makes a significant contribution to the heat current. The contributions from $\{\Omega_3, \beta_0\}$ are large because they are proportional to $[\text{Im}(\hat{\sigma}_0^R)_{00}]^2 \geq 0$. A nonzero $\text{Im}(\hat{\sigma}_0^R)_{00}$ creates an asymmetry in the lifetime for electrons and holes, which results in a contribution to the directional choice of the thermal Hall current.

In the low-temperature limit, $\varepsilon^R \rightarrow i(\gamma_0 + \gamma_s \sin^2 \theta)$, where $\gamma_0, \gamma_s \geq 0$ are the impurity bandwidths,

$$\kappa_{xx}^{ns} / \left(\frac{\pi^2}{3} N_f v_f^2 \frac{T}{T_c} \right) = T_c \Omega_0 \sim \frac{T_c}{\Delta_0} \frac{\gamma_0}{\Delta_0}. \quad (35)$$

The explicit forms of the vertex correction thermal transport coefficients even in the low-temperature limit are cumbersome and we direct the interested reader to Eq. (D21). For nonvanishing and nondivergent values of $\cot \delta_0$ and $\cot \delta_s$, the vertex corrections in Eq. (34) are estimated as

$$\frac{[\kappa_{yx}^V]_{T_c}}{\frac{\pi^2}{3} N_f v_f^2 T} \sim -\frac{T_c}{\Delta_0} \frac{\gamma_0^2}{\tilde{\Delta}_0^2} \ln \frac{\tilde{\Delta}_0}{\gamma_0} \left[\frac{\text{Re} \alpha \cot \delta_0}{(1 - \text{Re} \alpha) \frac{\gamma_0}{\tilde{\Delta}_0}} \right] \frac{\cot \delta_s}{C_s}. \quad (36)$$

Note that $\text{Re} \alpha \sim \frac{1}{\cot^2 \delta_s + (\tilde{\gamma}^1)^2} > 0$. The sign of κ_{yx} and $\cot \delta_0$ is the same as $T \rightarrow 0$, and it can also be verified in Fig. 2. The unitless form of κ_{yx} is again a very small number.

C. Comparison with the topological contribution

We find that even a very small anisotropic phase shift can lead to very large κ_{yx}^{imp} at finite temperatures. The unit that is used to nondimensionalize κ_{yx}^{imp} is $N_f v_f^2 / T_c$ and it can be reexpressed as $\sim \frac{E_f}{T_c} k_f$, where typically $\frac{E_f}{T_c} \sim 10^2 - 10^3$. Therefore, κ_{yx}^{imp} can be an order of magnitude larger than the topological contribution $\kappa_{yx}^{\text{topo}}$, which is of the order k_f [28] (Boltzmann constant, $k_B = 1$), except for very low

FIG. 3. $\kappa_{yx}(T)/\kappa_{xx}^{ns}(T)$ vs Γ_u/T_c , thermal Hall conductivity normalized to the longitudinal conductivity as a function of the impurity density, Γ_u , in units of the superconducting critical temperature T_c . Plot (a) is for the E_{1g} case and (b) is for the E_{2g} case. The quantity is plotted for two different temperatures $T = \{0.01, 0.4\}T_c$ and two different anisotropic phase shifts as $\delta_{c,s} = \{\pi/9, \pi/3\}$ where $\delta_0 = \pi/9$ for E_{1g} and $\delta_0 = -\pi/9$ for E_{2g} . The presence of κ_{xx} avoids a possible divergence occurring in the $\Gamma_u \rightarrow 0$ limit.

temperatures. In addition, at small impurity densities, the bulk contribution to κ_{yx}^{imp} is even larger than $\kappa_{yx}^{\text{topo}}$. We also observe a sign change in κ_{yx}^{imp} as a function of both temperature and the impurity density. These two nonmonotonic behaviors are distinct signatures of the impurity mechanism.

IV. κ_{yx} DEPENDENCE ON THE IMPURITY DENSITY, Γ_u

In the literature, to our knowledge, there is no discussion on the κ_{yx} vs Γ_u dependence. In the clean limit $\Gamma_u \rightarrow 0$, the impurity contribution dominates over the topological part due to the longer scattering lifetime τ for the Bogoliubov quasiparticles. The decay rate is calculated as

$$\tau^{-1} = \Gamma_u \frac{\cot \delta_0^2 + |\tilde{\gamma}_0|^2}{|\cot \delta_0^2 + (\tilde{\gamma}_0)^2|} \text{Re} \tilde{\gamma}_0, \quad (37)$$

where the anisotropic part is neglected for simpler expressions. This result can be understood by examining the averages Ω_0 , Ω_3 , β_0 , and β_3 . They are evaluated as

$$\begin{aligned} \beta_0(\varepsilon) &\sim 1/\Gamma_u, & \Omega_0(\varepsilon) &\sim 1/\Gamma_u, \\ \beta_3(\varepsilon) &\sim \Gamma_u, & \Omega_0(\varepsilon) &\sim \Gamma_u. \end{aligned}$$

At finite temperatures, κ_{xx}^{ns} diverges along with Ω_0 integral and κ_{yx} also diverges with β_0 or Ω_0 depending on the representation. Therefore we focus on their ratio, κ_{yx}/κ_{xx} , which is proportional to the thermal Hall angle:

$$\kappa_{xx}^{ns} \sim \int_{\Delta_0}^{\infty} d\varepsilon (\dots) \Omega_0(\varepsilon), \quad (38)$$

$$\kappa_{yx}^{E_{1g}} \sim \Gamma_u \int_0^{\Delta_0} d\varepsilon (\dots) \beta_0^2(\varepsilon), \quad (39)$$

$$\kappa_{yx}^{E_{2g}} \sim \Gamma_u \int_{\Delta_0}^{\infty} d\varepsilon (\dots) \Omega_0^2(\varepsilon), \quad (40)$$

$$\kappa_{yx}/\kappa_{xx}^{ns} \sim \text{finite}. \quad (41)$$

The expression $(\dots)$ is the rest of the uninteresting terms that determine the finite scales in the corresponding Eqs. (24), (26), and (34).

However, in the extremely clean limit, the ballistic regime will be reached and the approach of the present paper does not apply. Our discussion in this section assumes that the ballistic

regime has not been reached and we are able to fully rederive the quantum Boltzmann transport results [31].

In Fig. 3, we plot the thermal Hall conductivity normalized to κ_{xx}^{ns} as a function of the impurity density, Γ_u (in units of T_c). The ratio $\kappa_{yx}/\kappa_{xx}^{ns}$ increases for smaller impurity densities for E_{1g} , whereas it is highly dependent on the phase shifts and Γ_u in the E_{2g} case. The sign change occurs as Γ_u decreases because the dominant averages $\Omega_3 < 0$, $\beta_0 < 0$ are replaced with $\Omega_0 > 0$, $\beta_0 < 0$.

V. CONCLUSION AND OUTLOOK

Our results are not only specific to superconductors but should be considered as an example of a more general effect in TRS-broken systems which host the emergent excitations as the heat carriers. The impurity-generated contribution to THCs due to emergent excitations such as broken BCS pairs, phonons, magnons, and fractional excitations could be significantly important along with the topological contribution. We believe this point of view could enrich the discussions on the recent thermal Hall conductivity measurements cited above. It still requires further investigations to pose a quantitative universality of such a result.

We recently become aware of a recent work carrying out a similar calculation [47].

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APPENDIX A: A FEW NOTES ON THE QUASICLASSICAL TRANSPORT METHOD

$\hat{\Delta}(\hat{k})$ is self-consistently determined with the following sum:

$$\Delta(\hat{k}) = \int \frac{d\hat{k}'}{4\pi} \mathcal{V}(\hat{k}, \hat{k}') f^K(\hat{k}'). \quad (A1)$$

Note that the first order off-diagonal component of the Keldysh propagator, $f^K(\hat{k}')$, is odd in $\hat{k}'$ and $\mathcal{V}(\hat{k}, \hat{k}')$ is even; therefore $\hat{\Delta}_1 = 0$. In addition, for practical reasons, we

consider a model for the maximum gap size $\Delta_0(T)$ as in [2]. More technically, we use an effective theory and the true nature of the unconventional (probably spin) interaction is not known.

Plugging the expanded terms into Eq. (1), the equations for the RAK components at each order can be obtained. The zeroth-order transport equations for RA GFs are

$$[\varepsilon \hat{\tau}_3 - \hat{\Delta}_0 - \hat{\sigma}_0, \hat{g}_0^{R,A}] = 0, \quad (\hat{g}_0^{R,A})^2 = -\pi^2. \quad (\text{A2})$$

The $\hat{\tau}_0$ component of $\hat{\sigma}_0$ commutes with $\hat{g}_0^{R,A}$ and it should be dropped in the zeroth order. The clean-limit, zeroth-order equilibrium GF is

$$\hat{g}_0^R(\hat{k}) = -\pi \frac{\varepsilon \hat{\tau}_3 - \hat{\Delta}(\hat{k})}{D^R(\hat{k})}, \quad (\text{A3})$$

where $D^R(\hat{k}) = \sqrt{|\Delta(\hat{k})|^2 - (\varepsilon + i0^+)^2}$. The effect of impurity scattering is introduced through the self-energy term, $\hat{\sigma}^R(\hat{k})$. It modifies the elements of the equilibrium GFs as $\varepsilon^R = \varepsilon - (\sigma_0^R)_{30}$ and $\hat{\Delta}^R = \Delta(\hat{k}) + (\sigma_0^R)_{\text{off-diag}}$.

Expanding the Keldysh component of Eq. (1), the zeroth-order Keldysh GF can be expressed as $\hat{g}_0^K = (\hat{g}_0^R - \hat{g}_0^A) \tanh \frac{\varepsilon}{2T}$. The retarded and the advanced components have the information on the spectral density of the system while the Keldysh component reveals how these states are occupied.

In the first order after a few algebraic manipulations, the first-order GFs for the RA components are obtained as

$$\hat{g}_1^{R,A}(\hat{k}, \varepsilon) = -\frac{\hat{M}^{R,A}}{2(D^{R,A})^2} \times (i\vec{v}_f \cdot \vec{\nabla}_{\hat{R}} \hat{g}_0^{R,A} - [\hat{\sigma}_1^{R,A}, \hat{g}_0^{R,A}]),$$

where $\hat{M}^R = \varepsilon^{R,A} \hat{\tau}_3 - \Delta^{R,A}(\hat{k})$. We do not have to calculate $\hat{g}_1^{R,A}$ explicitly as the main concern of this article is to obtain the first-order Keldysh GF. $\hat{g}_1^K$ has the equilibrium and the anomalous parts as follows,

$$\hat{g}_1^{K,\text{ns}}(\hat{k}, \varepsilon) = \hat{N}^R(\hat{g}_0^R - \hat{g}_0^A) \left(\frac{-\varepsilon}{2T^2} \text{sech}^2 \frac{\varepsilon}{2T} \right) i\vec{v}_f \cdot \vec{\nabla} T, \quad (\text{A4})$$

where

$$\hat{N}^R = \frac{(D^R + D^A)\hat{g}_0^R/(-\pi) + (\sigma_0^R)_{00} - (\sigma_0^A)_{00}}{(D^R + D^A)^2 + [(\sigma_0^R)_{00} - (\sigma_0^A)_{00}]^2}. \quad (\text{A5})$$

The vertex correction anomalous GF is

$$\hat{g}_{1a}^{K,V}(\hat{k}, \varepsilon) = \hat{N}^R(\hat{g}_0^R \hat{\sigma}_{1a}^K - \hat{\sigma}_{1a}^K \hat{g}_0^A). \quad (\text{A6})$$

APPENDIX B: NORMAL STATE LIMIT THERMAL TRANSPORT COEFFICIENTS

The coefficients of the scattering potential in Eq. (9), v_0 , v_s , and v_c , are inputs to the transport equation. They are obtained from the normal state Fermi liquid limit. Therefore, the normal state limit has a fundamental role determining the inputs of the transport equation in the superconducting limit. The physical observables are the phase shifts in a typical scattering experiment; the coefficients of the scattering potential, v_0 , v_s , v_c , are parametrized in terms of the phase shifts of partial waves, $\cot \delta_0 = \frac{-1}{\pi N_f v_0}$, $\cot \delta_s = \frac{-1}{\pi N_f v_s}$, $\cot \delta_c = \frac{-1}{\pi N_f v_c}$.

In the normal state limit, $\Delta_0 \rightarrow 0$, the RA components of the GFs are $\hat{g}_0^{R,A} = \mp i\pi \hat{\tau}_3$ and the impurity averages are 1,

$\tilde{\gamma}_0 = \tilde{\gamma}_c = \tilde{\gamma}_s = 1$. The t -matrix recurrence relation $\hat{t}^R = v + N_f \langle v \hat{g}^R \hat{t}^R \rangle$ in the zeroth order gives $\hat{t}_0^R$ as

$$\begin{aligned} \hat{t}_{0,N}^R &= \frac{-1}{\pi N_f} \left[\frac{\cot \delta_0 + i\hat{\tau}_3}{\cot^2 \delta_0 + 1} + \frac{\cot \delta_s + i\hat{\tau}_3}{\cot^2 \delta_s + 1} \sin \theta \sin \theta' \right. \\ &\quad \times \cos(\phi - \phi') + \frac{\cot \delta_c + i\hat{\tau}_3}{\cot^2 \delta_c + 1} \cos \theta \cos \theta' \left. \right] \\ &= \frac{-1}{\pi N_f} [\sin \delta_0 e^{i\tau_3 \delta_0} + 3 \sin \delta_s e^{i\tau_3 \delta_s} \sin \theta \sin \theta' \\ &\quad \times \cos(\phi - \phi') + 3 \sin \delta_c e^{i\tau_3 \delta_c} \cos \theta \cos \theta']. \end{aligned}$$

The most general zeroth-order t matrix in the normal state can be simplified into the partial waves with the corresponding phase shifts as

$$\begin{aligned} \hat{t}_{0,N}^R &= \frac{-4\pi}{\pi N_f} \sum_{l=0}^{\infty} \sum_{m=-l}^l \sin \delta_{lm} e^{i\delta_{lm} \hat{\tau}_3} Y_l^{m*}(\hat{k}') Y_l^m(\hat{k}) \\ &= \frac{-1}{\pi N_f} [\sin \delta_{00} e^{i\delta_{00} \hat{\tau}_3} + 3 \sin \delta_{11} e^{i\delta_{11} \hat{\tau}_3} \sin \theta \sin \theta' \\ &\quad \times \cos(\phi - \phi') + 3 \sin \delta_{10} e^{i\delta_{10} \hat{\tau}_3} \cos \theta \cos \theta']. \end{aligned}$$

Comparing each term (in our case $l_{\text{max}} = 1$) and setting $\delta_{00} = \delta_0$, $\delta_{1,-1} = \delta_{11} = \delta_s$, $\delta_{10} = \delta_c$, it is verified that δ_i are actually the normal state phase shifts, which are also valid in the superconductor state.

The normal state self-energy is $\hat{\sigma}_{0,N}^R = n_i \hat{t}_{0,N}^R$. We can now also calculate the normal state GFs, $\hat{g}_{1a}^K$. First, the non-self-consistent part becomes

$$\hat{g}_{N,1a}^{K,\text{ns}} = \frac{\pi}{\text{Im} \varepsilon^R} \hat{k}_x b(\varepsilon, T).$$

Then, the anomalous self-energy only has the $\tau_0 \sigma_0$ component,

$$\sigma_{N,1a}^K = \Gamma_u b(\varepsilon, T) 6 \sin \delta_0 \sin \delta_s \cos(\delta_s - \delta_0) \left\langle \frac{\hat{k}_x'^2}{\text{Im} \varepsilon^R} \right\rangle \hat{k}_x. \quad (\text{B1})$$

The vertex GF is a scaled version of the non-self-consistent GF,

$$\hat{g}_{1a}^{K,V} \xrightarrow{\hat{g}_{N,1a}^K \rightarrow \hat{g}_{N,1a}^{K,\text{ns}}} \Gamma_u \left(6 \sin \delta_0 \sin \delta_s \cos(\delta_s - \delta_0) \left\langle \frac{\hat{k}_x'^2}{\text{Im} \varepsilon^R} \right\rangle \right) \hat{g}_{N,1a}^{K,\text{ns}}.$$

However, the full self-consistency demands a correction on the order of Γ_u . For this purpose let us define $Q_0 \equiv 6 \sin \delta_0 \sin \delta_s \cos(\delta_s - \delta_0)$, where $\hat{g}_{N,1a}^{K,V} = \Gamma_u Q \frac{\pi}{\text{Im} \varepsilon^R} \hat{k}_x b(\varepsilon, T)$ is the fully self-consistent GF. In the non-self-consistent limit, $Q \rightarrow Q_0 \equiv 6 \sin \delta_0 \sin \delta_s \cos(\delta_s - \delta_0)$ naturally. In this setting, if $\hat{g}_{N,1a}^{K,V}$ is plugged back into Eq. (C3), Q is obtained self-consistently in terms of Q_0 :

$$Q = \frac{Q_0}{1 - \Gamma_u}. \quad (\text{B2})$$

FIG. 4. Longitudinal thermal conductivity, κ_{xx} , as a function of temperature (in units of T_c), for the non-self-consistent part $\kappa_{xx}^{ns}(T)/\kappa_{xx}^{ns}(T_c)$ (dashed lines) only and for $\kappa_{xx}(T)/\kappa_{xx}(T_c)$ (solid lines) where the vertex correction is included. The upper two pairs of lines are for E_{1g} and the lower two pairs are for E_{2g} . For both cases $\delta_0 = \pi/3$ and $\delta_s \in \{\pi/9, \pi/3\}$. The unity limit is reproduced at $T = T_c$. The vertex correction is significant only for nonzero δ_s , in this case for E_{2g} .

The thermal current is nonzero only for the longitudinal component, κ_{xx} :

$$\kappa_{xx}^N = \kappa_{xx}^{N,ns} + \kappa_{xx}^{N,V},$$

$$\frac{\kappa_{xx}^N}{\frac{\pi^2}{3} N_f v_f^2 T} = \frac{6}{\pi^2} \int d\tau \tau^2 \text{sech}^2 \tau \left[1 + \Gamma_u Q \left\langle \frac{\hat{k}_x^2}{\text{Im} \varepsilon^R} \right\rangle \right] \left\langle \frac{\hat{k}_x^2}{\text{Im} \varepsilon^R} \right\rangle, \quad (\text{B3})$$

where $\tau = \varepsilon/2T$ is the dimensionless energy, $\text{Im} \varepsilon^R = -\text{Im}(\sigma_0^R)_{30} = \Gamma_u (\sin^2 \delta_0 + 3 \sin^2 \delta_s \sin^2 \theta + 3 \sin^2 \delta_c \cos^2 \theta)$. The latter Γ_u -dependent expression is obtained by the self-consistency relations for the averages $\tilde{\gamma}_{0,s,c}$ in Eqs. (D6) and the two equations of (C2). More importantly, it shows that $\langle \frac{\hat{k}_x^2}{\text{Im} \varepsilon^R} \rangle$ is independent of the energy as expected in the normal state. The τ integral, then, goes to unity as $\frac{6}{\pi^2} \int d\tau \tau^2 \text{sech}^2 \tau = 1$ and we obtain the final form of the normal state longitudinal conductivity,

$$\frac{\kappa_{xx}^N}{\frac{\pi^2}{3} N_f v_f^2 T} = \left[1 + \Gamma_u Q \left\langle \frac{\hat{k}_x^2}{\text{Im} \varepsilon^R} \right\rangle \right] \left\langle \frac{\hat{k}_x^2}{\text{Im} \varepsilon^R} \right\rangle. \quad (\text{B4})$$

In Fig. 4, we plot the $\kappa_{xx}/\kappa_{xx}^N$ as a function of temperature. At $T/T_c = 1$, $\kappa_{xx}/\kappa_{xx}^N \rightarrow 1$, where the exact normal state limits are reproduced both for E_{1g} and E_{2g} as given in Eq. (B3). Keep in mind that either δ_c or δ_s are dropped in each representation. The integral averages multiplied with Γ_u are evaluated as

$$E_{1g}(\delta_s \rightarrow 0): \quad \Gamma_u \left\langle \frac{\hat{k}_x'^2}{\text{Im} \varepsilon^R} \right\rangle$$

$$= \frac{1}{2} \left[\left(1 + \frac{1}{c} \right) \frac{\arctan \sqrt{\frac{c}{a}}}{\sqrt{ac}} - \frac{1}{c} \right] \equiv \langle \dots \rangle_{E_{1g}}, \quad Q = 0, \quad (\text{B5})$$

$$E_{2g}(\delta_c \rightarrow 0): \quad \Gamma_u \left\langle \frac{\hat{k}_x'^2}{\text{Im} \varepsilon^R} \right\rangle$$

$$= \frac{1}{2} \left[\left(1 - \frac{1}{b} \right) \frac{\text{arctanh} \sqrt{\frac{b}{a+b}}}{\sqrt{(a+b)b}} + \frac{1}{b} \right] \equiv \langle \dots \rangle_{E_{2g}}, \quad (\text{B6})$$

where $(a, b, c) \equiv (\sin^2 \delta_0, 3 \sin^2 \delta_s, 3 \sin^2 \delta_c)$. The averages $\langle \dots \rangle_{E_{1g}}$ and $\langle \dots \rangle_{E_{2g}}$, hence, are independent of the impurity density.

APPENDIX C: DETAILED DERIVATION OF THERMAL TRANSPORT COEFFICIENTS FOR E_{1g}

1. E_{1g} , the zeroth-order t matrix, and the self-energy

Expanding the Born series and collecting the common terms for E_{1g} , the general form of the $\hat{t}_0^R$ matrix can be fully obtained by inspection as

$$\hat{t}_0^R(\hat{k}, \hat{k}') = \hat{t}_{00}^R + [\hat{t}_{ssc}^R \cos(\phi - \phi') + \hat{t}_+^R e^{i(\phi - \phi')\tau_3}] \sin \theta \sin \theta'$$

$$+ \hat{t}_{cc}^R \cos \theta \cos \theta' + (\hat{A}_1 \cos \theta \sin \theta' e^{i\phi'\tau_3} + \hat{A}_2 \cos \theta' \sin \theta e^{i\phi\tau_3}) \hat{t}_1 \hat{\sigma}_2. \quad (\text{C1})$$

Plugging $\hat{t}_0^R(\hat{k}, \hat{k}')$ in Eq. (C1) back into the recurrence relation $\hat{t} = v + N_f \langle v \hat{g}^R \hat{t}^R \rangle$ and matching the terms with respect to the angular functions and Nambu space components $\hat{t}_i \hat{\sigma}_j$, we obtain the unknown elements of the

$\hat{t}_0^R(\hat{k}, \hat{k}')$ as

$$\begin{aligned}\hat{t}_{00}^R &= \frac{-1}{\pi N_f} \frac{\cot \delta_0 + i\tilde{\gamma}_0 \hat{\tau}_3}{C_0}, \quad \hat{t}_{ssc}^R = \frac{-3}{\pi N_f} \frac{\cot \delta_s + i\tilde{\gamma}_s \hat{\tau}_3}{C_s}, \\ \hat{t}_{cc}^R &= \frac{-3}{\pi N_f} \left[\cot \delta_c - i\tilde{\gamma}_c \hat{\tau}_3 + \frac{2(\tilde{\gamma}^1)^2}{\cot \delta_s + i\tilde{\gamma}_s \hat{\tau}_3} \right]^{-1}, \\ \hat{t}_+^R &= \frac{-3}{\pi N_f} \frac{\tilde{\gamma}^1 \cot \delta_c}{(\cot \delta_c + i\tilde{\gamma}_c \hat{\tau}_3)(\cot \delta_s - i\tilde{\gamma}_s \hat{\tau}_3) \cot \delta_c \cot \delta_s + 2(\tilde{\gamma}^1)^2}, \\ \hat{A}_1 &= -(\cot \delta_s + i\tilde{\gamma}_s \hat{\tau}_3) \hat{t}_+^{R,-}, \quad \hat{A}_2 = \frac{\pi N_f \tilde{\gamma}^1}{3} \hat{t}_{ssc}^R \hat{t}_{cc}^R,\end{aligned}$$

where $C_0 = \cot^2 \delta_0 + (\tilde{\gamma}_0)^2$,
 $C_s = \cot^2 \delta_s + (\tilde{\gamma}_s)^2$,
 $C_c = \cot^2 \delta_c + (\tilde{\gamma}_c)^2$,
 $\hat{t}_+^{R,-} \equiv (\hat{t}_+^R)_{00} - (\hat{t}_+^R)_{30} \hat{\tau}_3$.

The integrals are different momentum averages of the GFs:

$$\begin{aligned}\tilde{\gamma}_0 &= \left\langle \frac{-i\varepsilon^R}{D^R} \right\rangle, \quad \tilde{\gamma}^1 = \frac{3}{2} \left\langle \frac{\Delta_0}{D^R} (\hat{k}_x^2 + \hat{k}_y^2) \hat{k}_z^2 \right\rangle, \\ \tilde{\gamma}_c &= 3 \left\langle \frac{-i\varepsilon^R}{D^R} \hat{k}_z^2 \right\rangle, \quad \tilde{\gamma}_s = \frac{3}{2} \left\langle \frac{-i\varepsilon^R}{D^R} (\hat{k}_x^2 + \hat{k}_y^2) \right\rangle. \quad (C2)\end{aligned}$$

Then, the zeroth-order self-energy is

$$\begin{aligned}\hat{\sigma}_0^R(\hat{k}) &= n_i \hat{t}^R(\hat{k}, \hat{k}) \\ &= \pi N_f \Gamma_u [\hat{t}_{00}^R + (\hat{t}_{ssc}^R + \hat{t}_+^R) \sin^2 \theta + \hat{t}_{cc}^R \cos^2 \theta \\ &\quad + \cos \theta \sin \theta (\hat{A}_1 + \hat{A}_2) e^{i\phi \tau_3} \tau_1 i \hat{\sigma}_2],\end{aligned}$$

where it is an even function of $\hat{k}$. Using $\hat{\sigma}_0^R(\hat{k})$ above, the self-consistency relation for the diagonal element is found to be

$$\begin{aligned}\varepsilon^R &= \varepsilon - \pi N_f \Gamma_u ((\hat{t}_{00}^R)_{30} + [(\hat{t}_{ssc}^R)_{30} + (\hat{t}_+^R)_{30}] \sin^2 \theta \\ &\quad + (\hat{t}_{cc}^R)_{30} \cos^2 \theta).\end{aligned}$$

The second self-consistency relation is for the off-diagonal terms, $\Delta_0^R = \Delta_0 + \pi N_f \Gamma_u (A_1 + A_2)$.

Inspecting the t matrix, it should be emphasized that δ_c plays the most important role. Along with δ_s , they lead to a spontaneous skew scattering [46] channel, $\hat{t}_+^R$, which contributes both the diagonal and off-diagonal parts of the t matrix. If it goes to zero, the t matrix is reduced to two channels $\hat{t}_{00}^R, \hat{t}_{ssc}^R$, which is inadequate to produce a finite Hall current. Actually, it is possible to retain the finite κ_{yx} only with the δ_c component as will be shown at the beginning of next subsection. Even if δ_s is introduced back, it only renormalizes the contributions due to δ_c .

2. E_{1g} , the anomalous t matrix, and the self-energy

The anomalous t matrix $\hat{t}_{1a}^{K,ns}$ can be calculated with a known initial condition $\hat{g}_{1a}^{K,ns}$, and is given as follows:

$$\hat{t}_{1a}^K(\hat{k}, \hat{k}') = \pi N_f \left\langle \hat{t}_0^R(\hat{k}, \hat{k}_1) \frac{\hat{g}_{1a}^K(\hat{k}_1)}{\pi} \hat{t}_0^A(\hat{k}_1, \hat{k}') \right\rangle_{\hat{k}_1}. \quad (C3)$$

It vanishes in the absence of the external field, the temperature gradient. The whole procedure including the self-consistent equation to calculate the vertex correction GF is straightforward. In the first step, $\hat{g}_{1a}^K \rightarrow \hat{g}_{1a}^{K,ns}$, as the initial condition. It is called the non-self-consistent solution, corresponding to $\hat{t}_{1a}^{K,ns}$. In the second step, plug $\hat{g}_{1a}^K = \hat{g}_{1a}^{K,ns}$. Then, the resulting t matrix is $\hat{t}_{1a}^K = \hat{t}_{1a}^K - \hat{t}_{1a}^{K,ns}$. Comparing the coefficients of the self-energies $\hat{\sigma}_{1a}^K$ and $\hat{\sigma}_{1a}^{K,ns}$, the self-consistency relations

can be solved for the fully self-consistent GFs, $\hat{g}_{1a}^K$, as $\hat{\sigma}_{1a}^K = \hat{\sigma}_{1a}^K - \hat{\sigma}_{1a}^{K,ns} = n_i N_f (\hat{t}_0^R \hat{g}_{1a}^{K,V} \hat{t}_0^A)$.

Plugging $\hat{t}_0^R(\hat{k}, \hat{k}') = \hat{\tau}_3 [\hat{t}_0^R(\hat{k}', \hat{k})]^\dagger \hat{\tau}_3$, into Eq. (C3), the self-energy has the following form:

$$\begin{aligned}\hat{\sigma}_{1a}^K &\sim \langle \hat{t}_0^R \hat{g}_{1a}^{K,ns} \hat{t}_0^A \rangle \\ &\sim (\hat{t}_0^R)_{\text{even}} \langle (\hat{g}_{1a}^{K,ns})_{\text{diag}} (\hat{t}_0^A)_{\text{odd}} \rangle + \langle (\hat{t}_0^R)_{\text{odd}} (\hat{g}_{1a}^{K,ns})_{\text{diag}} \rangle (\hat{t}_0^A)_{\text{even}} \\ &\quad + \hat{t}_{00}^R / (\hat{g}_{1a}^{K,ns})_{\text{off-diag}} \hat{k}_z \hat{t}_{cc}^A \hat{k}_z + \hat{t}_{cc}^R / (\hat{g}_{1a}^{K,ns})_{\text{off-diag}} \hat{k}_z \hat{t}_{00}^A \hat{k}_z \\ &= \Gamma_u (\hat{\cdot} \hat{\cdot})_{\text{even}} \hat{k}_x + (\hat{\cdot} \hat{\cdot})_{\text{even}} \hat{k}_y b(\varepsilon, T).\end{aligned}$$

The even and odd parts of the t matrix are $(\hat{t}_0^{R,A})_{\text{even}} = \hat{t}_{00}^{R,A}$ and $(\hat{t}_0^{R,A})_{\text{odd}} = \hat{t}_0^{R,A} - \hat{t}_{00}^{R,A}$. Using Eq. (A6) the anomalous GF has the form

$$\begin{aligned}\hat{g}_{1a}^{K,V} &= \Gamma_u \hat{N}_R (\hat{g}_0^R (\hat{\cdot} \hat{\cdot})_{\text{even}} - (\hat{\cdot} \hat{\cdot})_{\text{even}} \hat{g}_0^A) \hat{k}_x b(\varepsilon, T) \\ &\sim \Gamma_u [(\hat{\cdot} \hat{\cdot})_{\text{even}} \hat{k}_x + (\hat{\cdot} \hat{\cdot})_{\text{even}} \hat{k}_y] b(\varepsilon, T).\end{aligned}$$

As seen clearly, the thermal Hall current has a few contributions and $J_y \sim \int d\hat{k} v_{f,y} k_y (\dots) \neq 0$. However, the form of the $(\dots)$ is complicated for a physical intuition.

Below, we therefore consider only v_0, v_c . The polar scattering potential $v(k, k') = v_0 + 3v_c \cos \theta \cos \theta'$ is considered as the dominant scattering process. It gives rise to finite κ_{yx} . It preserves the rotational symmetry of the system and does not break any extra symmetry even when the equator scattering part is dropped. The t matrix becomes

$$\begin{aligned}\hat{t}_0^R(\hat{k}, \hat{k}') &= \hat{t}_{00}^R + \hat{t}_{cc}^R \hat{k}_z \hat{k}_z' \\ &= \frac{-1}{\pi N_f} \left(\frac{\cot \delta_0 + i\tilde{\gamma}_0 \hat{\tau}_3}{C_0} + 3 \frac{\cot \delta_c + i\tilde{\gamma}_c \hat{\tau}_3}{C_c} \hat{k}_z \hat{k}_z' \right).\end{aligned} \quad (C4)$$

Note that $\hat{t}_0^A(\hat{k}, \hat{k}') = \hat{\tau}_3 [\hat{t}_0^R(\hat{k}', \hat{k})]^\dagger \hat{\tau}_3$.

Step 1. The non-self-consistent anomalous GF, $\hat{g}_{1a}^{K,ns}$, has the following form,

$$\begin{aligned}\hat{g}_{1a}^{K,ns} &= \pi \left\{ \frac{2\text{Re} D^R}{D_1} \left[\left(1 + \frac{|\varepsilon^R|^2 - |\Delta(\hat{k})|^2}{|D^R|^2} \right) - 2i \frac{\text{Im} \varepsilon^R \hat{\tau}_3 \hat{\Delta}}{|D^R|^2} \right] \right. \\ &\quad \left. + \frac{4\text{Im} \sigma_{00}^R}{D_1} \left[\text{Im} \frac{\varepsilon^R}{D^R} \hat{\tau}_3 - \text{Im} \frac{1}{D^R} \hat{\Delta} \right] \right\} \hat{k}_x b(\varepsilon, T), \quad (C5)\end{aligned}$$

where $b(\varepsilon, T) = i v_f \frac{-dT}{dx} \frac{\varepsilon}{2T^2} \text{sech}^2 \frac{\varepsilon}{2T}$ and we omit the $(k_x + ik_y)^2$ -dependent part self-consistently as it does not contribute to any average in the E_{1g} case. Plugging $\hat{g}_{1a}^{K,ns}$ into the t -matrix

relation in Eq. (C3), the nonvanishing terms arise from the off-diagonal part of $\hat{g}_{1a}^{K,ns}$ as the integrals have the form $\sim \langle \hat{g}_{1a}^{K,ns} \hat{k}_z \rangle$ which is clarified below. Taking the $\hat{\sigma}_{1a,ns}^K = n_i \hat{t}_{1a}^K(\hat{k}, \hat{k})$ limit of the anomalous t matrix, $\hat{\sigma}_{1a,ns}^K$ is an odd function of $\hat{k}$ and proportional to $\hat{k}_z$,

$$\hat{\sigma}_{1a}^{K,ns}(\hat{k}) = -\Gamma_u \hat{t}_{0c}^-(\beta_0 + i\beta_3 \hat{\tau}_3) \hat{\tau}_1 i \hat{\sigma}_2 \hat{k}_z b(\varepsilon, T), \quad (C6)$$

where $\hat{t}_{0c}^- \equiv (\pi N_f)^2 (\hat{t}_{00}^R \hat{t}_{cc}^A - \hat{t}_{cc}^R \hat{t}_{00}^A)/2 = (\hat{t}_{0c}^-)_{00} + (\hat{t}_{0c}^-)_{30} \hat{\tau}_3$. The averages are the various integrals of the $\hat{\tau}_i \hat{\sigma}_j$ components of the non-self-consistent anomalous GFs,

$$\beta_0(\varepsilon) = \left\langle \frac{4\text{Im}\sigma_{00}^R}{D_1} \text{Im} \frac{\Delta_0}{D^R} (\hat{k}_x^2 + \hat{k}_y^2) \hat{k}_z^2 \right\rangle, \quad (C7)$$

$$\beta_3(\varepsilon) = \left\langle \frac{4\text{Re}D^R}{D_1} \frac{\text{Im}\varepsilon^R \Delta_0}{|D^R|^2} (\hat{k}_x^2 + \hat{k}_y^2) \hat{k}_z^2 \right\rangle. \quad (C8)$$

Note that $D_1 = 4(\text{Re}D^R)^2 + 4\text{Im}(\sigma_{00}^R)_{00}^2$ and $\hat{t}_{00}^{A,\pm} = (\hat{t}_{00}^A)_{00} \pm (\hat{t}_{00}^A)_{30} \tau_3$.

We simplify $\hat{\sigma}_{1a}^K$ by defining

$$\begin{bmatrix} G_0 \\ G_3 \end{bmatrix} = [(\hat{t}_{0c}^-)_{00} \beta_0 + (\hat{t}_{0c}^-)_{30} (\beta_3 \hat{t}_{0c}^-)_{00} \beta_3 - (\hat{t}_{0c}^-)_{30} \beta_0]. \quad (C9)$$

The self-energy becomes

$$\hat{\sigma}_{1a}^{K,ns}(\hat{k}) = -\Gamma_u \hat{G} \hat{\tau}_1 i \hat{\sigma}_2 \hat{k}_z b(\varepsilon, T), \quad (C10)$$

where $\hat{G} = G_0 + iG_3 \hat{\tau}_3$. Plugging $\hat{\sigma}_{1a}^K$ into $\hat{g}_{1a}^{K,V}$, we obtain the latter GF in terms of $\hat{G}$. We know that $\hat{G}$ should be replaced by $\hat{\tilde{G}}$ in the fully self-consistent case. Its explicit form is long and will not be given here in detail, but it can be cast into the following form:

$$\hat{g}_{1a}^{K,V}(\hat{k}, \varepsilon) = \pi \Gamma_u b(\varepsilon, T) [\hat{X}_c \hat{k}_x + \hat{X}_s \hat{k}_y + (\hat{X}_{2\phi} e^{i2\phi\tau_3} + \hat{Y} \hat{k}_z) \tau_1 i \sigma_2]. \quad (C11)$$

Note that all $\hat{X}_i$ are diagonal matrices, which can be written as linear combinations of G_0 and G_3 as follows:

$$\begin{bmatrix} X_{c,0} \\ X_{s,0} \end{bmatrix} = \frac{4\hat{k}_z^2 \Delta_0}{D_1} \begin{bmatrix} \text{Im}\sigma_{00}^R \text{Im}1/D^R & \frac{\text{Re}D^R \text{Im}\varepsilon^R}{|D^R|^2} \\ -\frac{\text{Re}D^R \text{Im}\varepsilon^R}{|D^R|^2} & \text{Im}\sigma_{00}^R \text{Im}1/D^R \end{bmatrix} \begin{bmatrix} G_0 \\ G_3 \end{bmatrix}. \quad (C12)$$

In addition, the vertex corrections modify these coefficients, let us denote them as $\tilde{G}_0, \tilde{G}_3$, and consequently the coefficients of the vertex correction GF are also modified.

Step 2. The same procedure can be repeated if we replace $\hat{g}_{1a}^K \rightarrow \hat{g}_{1a}^{K,V}$ in Eq. (C3). The corresponding t matrix is $\hat{t}_{1a}^K = \hat{t}_{1a}^K - \hat{t}_{1a}^{K,ns}$. The only nonvanishing average in the renormalization comes from the off-diagonal component of $\hat{g}_{1a}^K$ in Eq. (C11), $\hat{Y}$, which is given as

$$\hat{Y} = -2 \left[\frac{\text{Re}D^R}{D_1} \left(1 - \left| \frac{\varepsilon^R}{D^R} \right|^2 \right) - 2i \frac{\text{Im}\sigma_{00}^R}{D_1} \text{Re} \frac{\varepsilon^R}{D^R} \tau_3 \right] \hat{\tilde{G}}. \quad (C13)$$

In this way, the coefficients of the anomalous self-energy are obtained self-consistently,

$$\begin{bmatrix} \tilde{G}_0 \\ \tilde{G}_3 \end{bmatrix} = \frac{1}{\text{Det}} \begin{bmatrix} 1 - \Gamma_u F_0 & -\Gamma_u F_1 \\ \Gamma_u F_1 & 1 - \Gamma_u F_0 \end{bmatrix} \begin{bmatrix} G_0 \\ G_3 \end{bmatrix}, \quad (C14)$$

where $\text{Det} = (1 - \Gamma_u F_0)^2 + \Gamma_u^2 F_1^2$, and

$$\begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = \begin{bmatrix} \lambda_0 (t_{0c}^-)_{00} + \lambda_3 (it_{0c}^-)_{30} \\ \lambda_0 (-it_{0c}^-)_{30} - \lambda_3 (t_{0c}^-)_{00} \end{bmatrix}, \quad (C15)$$

where the integrals are defined as

$$\begin{bmatrix} \lambda_0(\varepsilon) \\ \lambda_3(\varepsilon) \end{bmatrix} = 2 \left\langle \frac{1}{D_1} \begin{bmatrix} \text{Re}D^R(1 - |\varepsilon^R/D^R|^2) \\ 2\text{Im}\sigma_{00}^R \text{Re}(\varepsilon^R/D^R) \end{bmatrix} \hat{k}_z^2 \right\rangle. \quad (C16)$$

We finally obtain all necessary terms to calculate the components of current density. The current density, J_i , is proportional to the $\tau_0 \sigma_0$ components of $\hat{g}_{1a}^K$. Note that all $X_{c,0}$ and $X_{s,0}$ must be replaced by their renormalized forms due to vertex correction.

The non-self-consistent part, $\hat{g}_{1a}^{K,ns}$, yields only the nonzero longitudinal conductivity,

$$\frac{\kappa_{xx}^{ns} T_c}{\frac{\pi^2}{3} N_f v_f^2 T} = T_c \frac{6}{\pi^2} \int_{-\infty}^{\infty} \frac{d\varepsilon}{2T} \frac{\varepsilon^2}{4T^2} \text{sech}^2\left(\frac{\varepsilon}{2T}\right) \Omega_0(\varepsilon). \quad (C17)$$

The integral average is defined as $\Omega_0(\varepsilon) = \langle \frac{2\text{Re}D^R}{D_1} (1 + \frac{|\varepsilon^R|^2 - |\Delta|^2}{|D^R|^2}) \sin^2 \theta \rangle_\theta$. The whole expression is dimensionless.

The thermal conductivities due to the vertex correction part can be calculated as

$$\begin{aligned} \frac{\begin{bmatrix} \kappa_{xx}^V \\ \kappa_{yx}^V \end{bmatrix}}{\frac{\pi^2}{3} N_f v_f^2 T} &= \Gamma_u \frac{6}{\pi^2} \int_{-\infty}^{\infty} \frac{d\varepsilon}{2T} \left(\frac{\varepsilon}{2T} \right)^2 \text{sech}^2\left(\frac{\varepsilon}{2T}\right) \\ &\quad \times \left\langle \hat{k}_x^2 \begin{bmatrix} \tilde{X}_{c,0}(\varepsilon) \\ \tilde{X}_{s,0}(\varepsilon) \end{bmatrix} \right\rangle \\ &= \Gamma_u \frac{6}{\pi^2} \int_{-\infty}^{\infty} d\tau \tau^2 \text{sech}^2(\tau) \frac{1}{2} \begin{bmatrix} \beta_0 \tilde{G}_0 + \beta_3 \tilde{G}_3 \\ -\beta_3 \tilde{G}_0 + \beta_0 \tilde{G}_3 \end{bmatrix} \\ &= \Gamma_u \frac{6}{\pi^2} \int_{-\infty}^{\infty} d\tau \tau^2 \text{sech}^2(\tau) \frac{\beta_0^2 + \beta_3^2}{2\text{Det}} \\ &\quad \times \left(\begin{bmatrix} (t_{0c}^-)_{00} \\ (-it_{0c}^-)_{30} \end{bmatrix} (1 - \Gamma_u F_0) + \Gamma_u F_1 \begin{bmatrix} (it_{0c}^-)_{30} \\ (t_{0c}^-)_{00} \end{bmatrix} \right) \\ &\approx \Gamma_u \frac{6}{\pi^2} \int_{-\infty}^{\infty} d\tau \tau^2 \text{sech}^2(\tau) \frac{\beta_0^2 + \beta_3^2}{2} \\ &\quad \times \text{Re} \left[\frac{(\cot \delta_0 \cot \delta_c - \tilde{\gamma}_0^* \tilde{\gamma}_c)/C_0^* C_c}{(\cot \delta_0 \tilde{\gamma}_c + \cot \delta_c \tilde{\gamma}_0^*)/C_0^* C_c} \right] + O(\Gamma_u^2). \quad (C18) \end{aligned}$$

Before focusing on the thermal transport coefficients, we emphasize that there are four averages ($\Omega_0, \Omega_3, \beta_0, \beta_3$) playing a decisive role in the magnitudes and the sign of κ_{ij} . Moreover, they are found to be dominating one another depending on the temperature and the impurity density. These limits have been summarized in Table I in text.

Here, we discuss the sign of the thermal Hall coefficient, κ_{yx} . It may not be easy to realize, but first note that it has two contributions, $(\text{Re}Nom \text{Re}Denom + \text{Im}Nom \text{Im}Denom)/|\cdot|^2$, where Nom is the nominator and $Denom$ is the denominator of terms within the square brackets. Yet, we should also recognize that the real part of the impurity averages $\gamma_{0,c}$ are even in ε and actually mean the angular average of the local density of states, $\rho(\varepsilon, \hat{k}) = \text{Im} \frac{\varepsilon^x}{D^x} = \text{Re} \frac{-i\varepsilon^x}{D^x}$, $X \in \{R, A\}$. Neglecting the angular component in γ_0 or γ_c , the clean-limit average, $\sim \int_{\hat{k}} \varepsilon / \sqrt{|\Delta(\hat{k})|^2 - \varepsilon^2}$, is real valued within the gap and

FIG. 5. Thermal Hall conductivity κ_{yx} (in units of $\frac{N_f v_f^2 T}{T_c}$) in the plane of phase shifts δ_0 - $\delta_{c,s}$ for E_{1g} as plot (a) and for E_{2g} as plot (b). The impurity density and the temperature are the same in both cases as $\Gamma_u = 0.04T_c$ and $T = 0.3T_c$. The sign reversal of the phase shifts, $\{\delta_0, \delta_s\} \rightarrow \{-\delta_0, -\delta_s\}$, effectively means the particle-hole transformation, and as can be seen, κ_{yx} is reversed in both cases, as expected. Inset: The sign of κ_{yx} for E_{1g} (left) and E_{2g} (right) in the δ_0 - $\delta_{c,s}$ plane.

imaginary valued otherwise, but the imaginary contributions are mainly suppressed by the thermal distribution (even in the presence of impurities due to small Γ_u). Therefore, one can simply consider the real nominator and the real denominator contributions as the dominant part. That function along with the various combinations coming from the denominator is roughly a positive function. One can verify it directly from κ_{yx} in the space of phase shifts in Fig. 5.

In the low-temperature limit, $\varepsilon^{R,A} \rightarrow i\gamma(\theta)$, $D^{R,A} = D \rightarrow \sqrt{|\Delta(\hat{k})|^2 + \gamma^2(\theta)}$, and the following two integrals vanish: $\beta_0(0), \lambda_3(0) \rightarrow 0$. Also, the zeroth-order impurity-averaged GFs become real values as follows,

$$\tilde{\gamma}_0 \sim \frac{\gamma_0}{\Delta_0} \ln \frac{\Delta_0}{\gamma_0}, \quad \tilde{\gamma}_c \sim \frac{\gamma_0}{\Delta_0} \ln \frac{\Delta_0}{\gamma_0}.$$

The first-order GF averages are

$$\begin{aligned} \Omega_0(0) &\rightarrow \left\langle \frac{\gamma^2}{D^3} \sin^2 \theta \right\rangle \sim \frac{\gamma_0}{\Delta_0^2}, \\ \beta_3(0) &\rightarrow \left\langle \frac{\gamma \Delta_0}{D^2} \cos^2 \theta \sin^2 \theta \right\rangle \sim \gamma_0 / \Delta_0^2, \\ \lambda_0(0) &= \left\langle \frac{|\Delta(\hat{k})|^2}{D^3} \cos^2 \theta \right\rangle \sim \frac{1}{\Delta_0} \ln \frac{\Delta_0}{\gamma_0}. \end{aligned}$$

The coupled scattering channels, t_{0c}^- , acquire a simpler form,

$$[(t_{0c}^-)_{00}(t_{0c}^-)_{30}] \rightarrow \frac{1}{C_0 C_c} \left[\cot \delta_0 \cot \delta_c - \tilde{\gamma}_0 \tilde{\gamma}_c \right].$$

The vertex correction renormalization coefficients F_0 and F_1 are

$$\begin{bmatrix} F_0 \\ F_1 \end{bmatrix} \rightarrow \frac{\lambda_0}{C_0 C_c} \begin{bmatrix} \cot \delta_0 \cot \delta_c - \tilde{\gamma}_0 \tilde{\gamma}_c \\ \cot \delta_0 \tilde{\gamma}_c + \cot \delta_c \tilde{\gamma}_0 \end{bmatrix}.$$

The scaling weight due to the vertex correction, Det, is simplified as

$$\begin{aligned} \text{Det} &\rightarrow \left(1 + \Gamma_u \lambda_0 \frac{\tilde{\gamma}_0 \tilde{\gamma}_c - \cot \delta_0 \cot \delta_c}{C_0 C_c} \right)^2 \\ &\quad + \Gamma_u^2 \lambda_0^2 \left(\frac{\cot \delta_0 \tilde{\gamma}_c + \cot \delta_c \tilde{\gamma}_0}{C_0 C_c} \right)^2. \end{aligned}$$

It is clear that the vertex renormalization for $G_0 \rightarrow \tilde{G}_0$ is provided by only a scaling vertex correction weight (Det) while $G_3 \rightarrow \tilde{G}_3$ has a negative-valued correction on top of this weight.

Then, the conductivities in Eq. (D19) are found to be

$$\frac{[\kappa_{xx}^V] T_c}{\frac{\pi^2}{3} N_f v_f^2 T} = \Gamma_u T_c \frac{\beta_3}{2 \text{Det}} \left[\begin{matrix} G_3 - \Gamma_u (F_0 G_3 - F_1 G_0) \\ -G_0 \end{matrix} \right] \quad (\text{C20})$$

$$\approx \Gamma_u \frac{T_c}{C_0 C_c} \left[\frac{\cot \delta_0 \cot \delta_c - \tilde{\gamma}_0 \tilde{\gamma}_c}{\cot \delta_0 \tilde{\gamma}_c + \cot \delta_c \tilde{\gamma}_0} \right] \frac{\beta_3^2}{2} + O(\Gamma_u^2) \quad (\text{C21})$$

$$\sim \frac{T_c}{\Delta_0} \frac{\gamma_0^2 \ln \frac{\Delta_0}{\gamma_0}}{\Delta_0^2} \left[\frac{\cot \delta_0 \cot \delta_c - \frac{\gamma_0^2}{\Delta_0^2} \left(\ln \frac{\Delta_0}{\gamma_0} \right)^2}{\frac{\gamma_0}{\Delta_0} \ln \frac{\Delta_0}{\gamma_0} (\cot \delta_0 + \cot \delta_c)} \right] \frac{1}{C_c}. \quad (\text{C22})$$

Note that the κ_{ij} in Eq. (C22) are obtained by utilizing the self-consistency relations in Eq. (C2).

To get more insight into the expressions for κ_{ij} in Eq. (C21), but if θ is fixed, $\theta = \theta_0$ and $\theta = \pi - \theta_0$, the order parameter becomes identical to a p -wave superconductor with s -wave scattering. The scattering potential takes a constant value, $V = v_0 + 3v_c \cos^2 \theta_0$ in the upper hemisphere (or lower hemisphere), while it does not scatter between θ_0 and $\pi - \theta_0$ if we choose $v_0 = 3v_c \cos^2 \theta_0$. Then the vertex corrections to the conductivities converge to the result

[30,31] as

$$\frac{\left[\frac{\kappa_{xx}^V}{\kappa_{yy}^V}\right]}{\frac{\pi^2}{3}N_f T} \propto \frac{\Gamma_u}{\text{Det}} \left[\frac{\cot^2 \delta_0 - (\tilde{\gamma}_0)^2}{2 \cot \delta_0 \tilde{\gamma}_0} \right] \frac{\gamma_0^2}{C_0^2} v_f^2 \left\langle \frac{|\Delta(\hat{k})|^2}{\Delta_0 D^3} \right\rangle. \quad (\text{C23})$$

For simplicity, we have assumed the impurity band to be constant, $\gamma(\theta) \approx \gamma_0$.

APPENDIX D: DETAILED DERIVATION OF THERMAL TRANSPORT COEFFICIENTS FOR E_{2g}

1. E_{2g} , the zeroth-order t matrix, and the self-energy

Expanding the Born series for $\Delta_k^{E_{2g}}$ gap symmetry, and by inspection, the zeroth-order t matrix takes the form

$$\begin{aligned} \hat{t}_0^R &= \hat{t}_{00}^R + \hat{t}_{ssc}^R \sin \theta \sin \theta' \left[\cos(\phi - \phi') - \frac{\alpha}{2} e^{i(\phi - \phi')\hat{\tau}_3} \right] \\ &\quad + \hat{t}_{cc}^R \cos \theta \cos \theta' - \frac{3\alpha_0}{2\pi N_f} e^{i(\phi + \phi')\hat{\tau}_3} \hat{\tau}_1 i \hat{\sigma}_2 \sin \theta \sin \theta'. \end{aligned}$$

Similarly to the E_{1g} case for nonzero δ_s , there is one distinct term in the t matrix with the coefficient, α . It indicates a skew scattering effect which spontaneously distinguishes the particles and the holes. Plugging these functions back into the recurrence relation for the t matrix and matching the terms with respect to momentum direction and $\hat{\tau}_i \hat{\sigma}_j$ matrix forms, we obtain the coefficients of $\hat{t}_0^R$ as

$$\hat{t}_{00}^R = \frac{-1}{\pi N_f} \frac{\cot \delta_0 + i \tilde{\gamma}_0 \hat{\tau}_3}{C_0}, \quad C_0 = \cot^2 \delta_0 + (\tilde{\gamma}_0)^2, \quad (\text{D1})$$

$$\hat{t}_{cc}^R = \frac{-1}{\pi N_f} \frac{\cot \delta_c + i \tilde{\gamma}_c \hat{\tau}_3}{C_c}, \quad C_c = \cot^2 \delta_c + (\tilde{\gamma}_c)^2, \quad (\text{D2})$$

$$\hat{t}_{ssc}^R = \frac{-3}{\pi N_f} \frac{\cot \delta_s + i \tilde{\gamma}_s \hat{\tau}_3}{C_s}, \quad C_s = \cot^2 \delta_s + (\tilde{\gamma}_s)^2, \quad (\text{D3})$$

$$\alpha_0 = \frac{\tilde{\gamma}^1}{C_{s2}}, \quad \alpha = \frac{(\tilde{\gamma}^1)^2}{C_{s2}}, \quad C_{s2} = C_s + (\tilde{\gamma}^1)^2. \quad (\text{D4})$$

In addition, the GF averages are same with Eq. (C2) except $\tilde{\gamma}^1 = \frac{3}{2} \langle \frac{\tilde{\Delta}_0 \sin^4 \theta}{D^R} \rangle$. The self-energy is found as

$$\begin{aligned} \hat{\sigma}_0^R(\hat{k}) &= \pi N_f \Gamma_u \left[\hat{t}_{00}^R + \hat{t}_{cc}^R \cos^2 \theta + \hat{t}_{ssc}^R \sin^2 \theta (1 - \alpha/2) \right. \\ &\quad \left. - \frac{3\alpha_0}{2\pi N_f} e^{i2\phi\hat{\tau}_3} \hat{\tau}_1 i \hat{\sigma}_2 \sin^2 \theta \right]. \end{aligned} \quad (\text{D5})$$

For simplicity, we neglected the imaginary part $\tilde{\Delta}_0^R$ self-consistently, and only keep the scaling part as $\tilde{\Delta}_0^R = \Delta_0/d$, where $d = |1 + \frac{9}{4} \Gamma_u \frac{(\sin^4 \theta / D^R)}{2C_{s2}}|^2$. Hence, $\tilde{\Delta}_0$ becomes identical for the retarded and the advanced part. Using $\hat{\sigma}_0^R(\hat{k})$ in Eq. (D5), another self-consistency relation of the diagonal elements is also found to be

$$\begin{aligned} \varepsilon^R &= \varepsilon - \pi N_f \Gamma_u \left[(\hat{t}_{00}^R)_{30} + (\hat{t}_{cc}^R)_{30} \cos^2 \theta \right. \\ &\quad \left. + (\hat{t}_{ssc}^R)_{30} \sin^2 \theta (1 - \alpha/2) \right]. \end{aligned} \quad (\text{D6})$$

2. E_{2g} , the anomalous t matrix, and the self-energy

$\hat{t}_{1a}^K$ vanishes in the polar scattering limit ($v_s, \delta_s \rightarrow 0$), $v(\hat{k}, \hat{k}') = v_0 + 3v_c \cos \theta \cos \theta'$, with the zeroth-order t matrix $\hat{t}_0^{R,A} = \hat{t}_{00}^{R,A} + \hat{t}_{cc}^{R,A} \cos \theta \cos \theta'$:

$$\begin{aligned} \hat{\sigma}_{1a}^K &\sim \langle \hat{t}_0^R \hat{g}_{1a}^{K,ns} \hat{t}_0^A \rangle \\ &\sim [(\hat{t}_0^R)_{\text{even}} \langle (\cdot)_{\text{even}} \sin \theta'' \cos \theta'' \cos \phi'' \rangle \\ &\quad + \langle (\cdot)_{\text{even}} \sin \theta'' \cos \theta'' \cos \phi'' \rangle (\hat{t}_0^A)_{\text{even}}] \sin \theta \\ &= 0. \end{aligned} \quad (\text{D7})$$

Both averages disappear and therefore the anomalous Keldysh self-energy is zero, $\hat{g}_{1a}^{K,V} \rightarrow 0$. The components of $(\cdot)_{\text{even}}$ and $(\cdot)_{\text{even}}$ have either $m = 0, 2$ angular momentum components; meanwhile the multiplicative integrand has an $m = 1$ component.

Below, we consider only v_0, v_s . The effective scattering potential is $v(\hat{k}, \hat{k}') = v_0 + 3v_s \sin \theta \sin \theta' \cos(\phi - \phi')$. In this limit, ($\delta_c \rightarrow 0$), the components of the k_z channel in the t matrix vanish, $\hat{t}_{cc}^R \rightarrow 0$. The form of $\hat{g}_{1a}^{K,ns}$ is given in Eq. (C5). The only difference is the order parameter and the momentum dependence of the modified particle-hole energies, $\varepsilon^{R,A}(\theta)$.

Step 1. Inserting $\hat{g}_{1a}^{K,ns}$ into the anomalous t -matrix equation, the anomalous self-energy is obtained:

$$\begin{aligned} t_{1a}^{K,ns} &= \pi N_f \left\langle \hat{t}_0^R \frac{\hat{g}_{1a}^{K,ns}}{\pi} \hat{t}_0^A \right\rangle \\ &= \pi N_f t_{0,00}^R \left\langle \frac{\hat{g}_{1a}^{K,ns}}{\pi} (\hat{t}_0^A)_{\text{odd}} \right\rangle + \pi N_f \left\langle (t_0^R)_{\text{odd}} \frac{\hat{g}_{1a}^{K,ns}}{\pi} \right\rangle \hat{t}_{0,00}^A \end{aligned} \quad (\text{D8})$$

$$\hat{\sigma}_{1a}^{K,ns}(\hat{k}) = \Gamma_u [\hat{P} \hat{k}_x + \hat{S} \hat{k}_y + \hat{G}(\hat{k}_x + i \hat{k}_y \tau_3) \hat{\tau}_1 i \hat{\sigma}_2] b(\varepsilon, T). \quad (\text{D10})$$

Note that $\hat{P} = P_0 + P_3 \hat{\tau}_3$, $\hat{S} = S_0 + S_3 \hat{\tau}_3$, $\hat{G} = G_0 + i G_3 \hat{\tau}_3$ are diagonal matrix coefficients. The general symmetry considerations of Keldysh GFs and the self-energies constrain these matrices as $\underline{P}^T = [P_0, P_3, S_0, S_3, G_0, G_3] \in \mathbb{R}^6$. The relation between each of the coefficients

and the integral averages, $\underline{\Omega}^\top = [\Omega_0, \Omega_3, \beta_0, \beta_3]$, is

$$\begin{bmatrix} \hat{P} \\ \hat{S} \\ \hat{G} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} [\hat{t}_{S0+}^+ (1 - \text{Re}\alpha) - i\hat{t}_{S0-}^+ \text{Im}\alpha] \hat{\Omega} - 3\pi N_f \{ \text{Re}\alpha_0 [\text{Re}(\hat{t}_{00}^R) \beta_0 - \text{Im}(\hat{t}_{00}^R) \beta_3 \tau_3] + \text{Im}\alpha_0 [\text{Im}(\hat{t}_{00}^R) \beta_0 + \text{Re}(\hat{t}_{00}^R) \beta_3 \tau_3] \} \\ (-i\hat{t}_{S0-}^+ \text{Re}\alpha + \hat{t}_{S0+}^+ \text{Im}\alpha) \hat{\Omega} \tau_3 - 3\pi N_f \{ \text{Re}\alpha_0 [\text{Im}(\hat{t}_{00}^R) \beta_0 + \text{Re}(\hat{t}_{00}^R) \beta_3 \tau_3] - \text{Im}\alpha_0 [\text{Re}(\hat{t}_{00}^R) \beta_0 - \text{Im}(\hat{t}_{00}^R) \beta_3 \tau_3] \} \tau_3 \\ \frac{1}{2} [\hat{t}_{S0+}^- (-1 + \text{Re}\alpha) + i\hat{t}_{S0-}^- \text{Im}\alpha] \hat{\beta} - 3\pi N_f [\text{Re}\alpha_0 (\hat{t}_{00}^+ \Omega_0 + \hat{t}_{00}^- \Omega_3 \tau_3) - i\text{Im}\alpha_0 (\hat{t}_{00}^+ \Omega_0 + \hat{t}_{00}^- \Omega_3 \tau_3)] \end{bmatrix}. \quad (\text{D11})$$

Note that $\hat{t}_{00}^\pm = (\hat{t}_{00}^R \pm \hat{t}_{00}^{R*})$ and the matrix $\hat{t}_{00}^{R*}$ indicates the same matrix $\hat{t}_{00}^R$ with negative τ_3 component. The integrals defined in the E_{1g} section, which are given in Eq. (C7), $\beta_0(\varepsilon)$, $\beta_3(\varepsilon)$, are modified by the replacement $\cos\theta \rightarrow \sin\theta$ as there is no $\cos\theta$ term in the E_{2g} order parameter. Moreover, the $\hat{\beta} = \beta_0 + \beta_3 \tau_3$ and $\hat{\Omega} = \Omega_0 + \Omega_3 \tau_3$ integrals can be grouped into two as follows:

$$\begin{bmatrix} \Omega_0(\varepsilon) \\ \beta_3(\varepsilon) \end{bmatrix} = \left\langle \frac{2\text{Re}D^R}{D_1} \left[1 + \frac{|\varepsilon^R|^2 - |\Delta(\hat{k})|^2}{|D^R|^2} \right] \sin^2\theta \right\rangle_\theta \quad (\text{D12})$$

is categorized as the first group, which dominates at low temperatures;

$$\begin{bmatrix} \Omega_3(\varepsilon) \\ \beta_0(\varepsilon) \end{bmatrix} = \left\langle \frac{4\text{Im}(\sigma_0^R)_{00}}{D_1} \left[\frac{\text{Im}\varepsilon^R}{D^R} \sin^2\theta \right] \right\rangle_\theta \quad (\text{D13})$$

is the second group, which dominates at finite temperatures.

$$\begin{bmatrix} X_{c,0} \\ X_{s,0} \\ X_{c,3} \\ X_{s,3} \\ Y_0 \\ Y_3 \end{bmatrix} = \begin{bmatrix} \frac{2\text{Re}D^R}{D_1} \left[\left(1 + \frac{|\varepsilon^R|^2 - |\Delta(\hat{k})|^2}{|D^R|^2} \right) P_0 - \frac{2\text{Im}(\varepsilon^R) \tilde{\Delta}_0 \sin^2\theta}{|D^R|^2} G_3 \right] + \frac{4\text{Im}\sigma_{00}^R}{D_1} [\text{Im}\frac{\varepsilon^R}{D^R} P_3 + \tilde{\Delta}_0 \text{Im}\frac{1}{D^R} G_0] \\ \frac{2\text{Re}D^R}{D_1} \left[\left(1 + \frac{|\varepsilon^R|^2 - |\Delta(\hat{k})|^2}{|D^R|^2} \right) S_0 - \frac{2\text{Im}(\varepsilon^R) \tilde{\Delta}_0 \sin^2\theta}{|D^R|^2} G_0 \right] + \frac{4\text{Im}\sigma_{00}^R}{D_1} [\text{Im}\frac{\varepsilon^R}{D^R} S_3 + \tilde{\Delta}_0 \text{Im}\frac{1}{D^R} G_3] \\ \frac{2\text{Re}D^R}{D_1} \left[\left(1 + \frac{|\varepsilon^R|^2 + |\tilde{\Delta}(\hat{k})|^2}{|D^R|^2} \right) P_3 + \frac{2\text{Re}(\varepsilon^R) \tilde{\Delta}_0 \sin^2\theta}{|D^R|^2} G_0 \right] + \frac{4\text{Im}\sigma_{00}^R}{D_1} \text{Im}\frac{\varepsilon^R}{D^R} [P_0 - \text{Re}\frac{\tilde{\Delta}_0 \sin^2\theta}{D^R} G_3] \\ \frac{2\text{Re}D^R}{D_1} \left[\left(1 + \frac{|\varepsilon^R|^2 + |\tilde{\Delta}(\hat{k})|^2}{|D^R|^2} \right) S_3 + \frac{2\text{Re}(\varepsilon^R) \tilde{\Delta}_0 \sin^2\theta}{|D^R|^2} G_3 \right] + \frac{4\text{Im}\sigma_{00}^R}{D_1} \text{Im}\frac{\varepsilon^R}{D^R} [S_0 + \text{Re}\frac{\tilde{\Delta}_0 \sin^2\theta}{D^R} G_0] \\ \frac{2\text{Re}D^R}{D_1} \left(\left[1 - \left| \frac{\varepsilon^R}{D^R} \right|^2 \right] G_0 - \frac{\tilde{\Delta}_0 \sin^2\theta}{|D^R|^2} [\text{Re}\varepsilon^R P_3 + \text{Im}\varepsilon^R S_0] \right) + \frac{4\text{Im}\sigma_{00}^R}{D_1} (\text{Re}(\frac{\varepsilon^R}{D^R}) G_3 - \frac{\tilde{\Delta}_0 \sin^2\theta}{2} [\text{Im}\frac{1}{D^R} P_0 - \text{Re}\frac{1}{D^R} S_3]) \\ \frac{2\text{Re}D^R}{D_1} \left(\left[1 - \left| \frac{\varepsilon^R}{D^R} \right|^2 \right] G_3 + \frac{\tilde{\Delta}_0 \sin^2\theta}{|D^R|^2} [\text{Re}\varepsilon^R S_3 - \text{Im}\varepsilon^R P_0] \right) + \frac{4\text{Im}\sigma_{00}^R}{D_1} (-\text{Re}(\frac{\varepsilon^R}{D^R}) G_0 + \frac{\tilde{\Delta}_0 \sin^2\theta}{2} [\text{Im}\frac{1}{D^R} S_0 + \text{Re}\frac{1}{D^R} P_3]) \end{bmatrix}. \quad (\text{D15})$$

The coefficients of the vertex correction Keldysh GF are obtained. However, the coefficients are renormalized if the problem is treated with full self-consistency, $\underline{X} \rightarrow \tilde{\underline{X}}$. For full self-consistency, we first replace $\underline{P} \rightarrow \tilde{\underline{P}}$ in the above relation and second replace $\hat{g}_{1a}^{K,\text{ns}}(\hat{k}) \rightarrow \hat{g}_{1a}^{K,V}(\hat{k})$ in Eq. (D8). The anomalous t -matrix relation becomes

$$\hat{t}_{1a}^K - \hat{t}_{1a}^{K,\text{ns}} = \left[(\hat{t}_0^R)_{00} \left\langle \frac{\hat{g}_{1a}^{K,V}}{\pi} (\hat{t}_0^A)_{\text{odd}} \right\rangle + \left\langle (\hat{t}_0^R)_{\text{odd}} \frac{\hat{g}_{1a}^{K,V}}{\pi} \right\rangle (\hat{t}_0^A)_{00} \right]. \quad (\text{D16})$$

Finally, the matrices $\hat{t}_{S0(\pm)}^\pm$ are abbreviations for $\hat{t}_{S0(\pm)}^\pm \equiv (\pi N_f)^2 [\hat{t}_{ssc}^R \hat{t}_{00}^{\pm} (\pm) \hat{t}_{00}^{R*} \hat{t}_{ssc}^A] / 2$,

$$\begin{aligned} \hat{t}_{S0+}^+ / (\pi N_f)^2 &= \text{Re}[\hat{t}_{ssc}^R \hat{t}_{00}^{R*}] \\ &= \text{Re}[(\hat{t}_{ssc}^R)_{00} (\hat{t}_{00}^{R*})_{00} + (\hat{t}_{ssc}^R)_{30} (\hat{t}_{00}^{R*})_{30}] \\ &\quad + \text{Re}[(\hat{t}_{ssc}^R)_{00} (\hat{t}_{00}^{R*})_{30} + (\hat{t}_{ssc}^R)_{30} (\hat{t}_{00}^{R*})_{00}] \tau_3, \\ \hat{t}_{S0-}^+ / (\pi N_f)^2 &= i\text{Im}[\hat{t}_{ssc}^R \hat{t}_{00}^{R*}] \\ &= i\text{Im}[(\hat{t}_{ssc}^R)_{00} (\hat{t}_{00}^{R*})_{00} + (\hat{t}_{ssc}^R)_{30} (\hat{t}_{00}^{R*})_{30}] \\ &\quad + i\text{Im}[(\hat{t}_{ssc}^R)_{00} (\hat{t}_{00}^{R*})_{30} + (\hat{t}_{ssc}^R)_{30} (\hat{t}_{00}^{R*})_{00}] \tau_3, \\ \hat{t}_{S0+}^- / (\pi N_f)^2 &= \text{Re}[(\hat{t}_{ssc}^R)_{00} (\hat{t}_{00}^{R*})_{00} - (\hat{t}_{ssc}^R)_{30} (\hat{t}_{00}^{R*})_{30}] \\ &\quad - i\text{Im}[(\hat{t}_{ssc}^R)_{00} (\hat{t}_{00}^{R*})_{30} - (\hat{t}_{ssc}^R)_{30} (\hat{t}_{00}^{R*})_{00}] \tau_3, \\ \hat{t}_{S0-}^- / (\pi N_f)^2 &= i\text{Im}[(\hat{t}_{ssc}^R)_{00} (\hat{t}_{00}^{R*})_{00} - (\hat{t}_{ssc}^R)_{30} (\hat{t}_{00}^{R*})_{30}] \\ &\quad - \text{Re}[(\hat{t}_{ssc}^R)_{00} (\hat{t}_{00}^{R*})_{30} - (\hat{t}_{ssc}^R)_{30} (\hat{t}_{00}^{R*})_{00}] \tau_3. \end{aligned}$$

Step 2. Plugging $\hat{\sigma}_{1a}^{K,\text{ns}}(\hat{k})$ into $\hat{g}_{1a}^{K,V}(\hat{k})$, the explicit form of $\hat{g}_{1a}^{K,V}(\hat{k})$ is found to be

$$\hat{g}_{1a}^{K,V}(\hat{k}) = \Gamma_u [\hat{X}_c \hat{k}_x + \hat{X}_s \hat{k}_y + \hat{Y}(\hat{k}_x + i\hat{k}_y \tau_3) \hat{t}_1 i\hat{\sigma}_2] b(\varepsilon, T), \quad (\text{D14})$$

where the coefficients $\underline{X}^\top = [X_{c,0}, X_{c,3}, X_{s,0}, X_{s,3}, Y_{1,0}, Y_{1,3}] \in \mathbb{R}^6$ are related to $\underline{P}$, $\underline{X} = \underline{W}^X \underline{P}$,

Repeating the same procedure retains a self-consistency relation for the renormalized coefficients of $\hat{\sigma}_{1a}^K, \tilde{\underline{P}}$ by connecting the renormalized $\underline{X}$ and $\tilde{\underline{P}}$ with the relation $\tilde{\underline{X}} = \underline{\mathcal{W}}^{XP} \tilde{\underline{P}}$. Together, we obtain the following relations,

$$\tilde{\underline{P}} - \underline{P} = \Gamma_u \langle \sin^2\theta \underline{\mathcal{W}}^{PX} \underline{\mathcal{W}}^{XP} \tilde{\underline{P}} \rangle, \quad (\text{D17})$$

$$\tilde{\underline{P}} = (1 - \Gamma_u \underline{\mathcal{W}}^{PX} \langle \sin^2\theta \underline{\mathcal{W}}^{XP} \rangle)^{-1} \underline{P}. \quad (\text{D18})$$

We have obtained all the necessary terms to calculate the thermal conductivities for the E_{2g} case. As a reminder, in the

the non-self-consistent case, the thermal transport contribution κ_{xx}^{ns} has the same form in the E_{1g} case as in Eq. (C17). The average Ω_0 is modified as shown in Eq. (D12),

$$\frac{[\kappa_{xx}^V]}{\frac{\pi^2}{3} N_f v_f^2 T} = \Gamma_u \frac{6}{\pi^2} \int_{-\infty}^{\infty} \frac{d\varepsilon}{2T} \left(\frac{\varepsilon}{2T} \right)^2 \text{sech}^2 \left(\frac{\varepsilon}{2T} \right) \left\langle \hat{k}_x^2 \left[\tilde{X}_{c,0}(\varepsilon) \right] \right\rangle \quad (\text{D19})$$

$$= \Gamma_u \frac{6}{\pi^2} \int_{-\infty}^{\infty} d\tau \tau^2 \text{sech}^2(\tau) \frac{1}{2} \left[\Omega_0 \tilde{P}_0 - \beta_3 \tilde{G}_3 + \Omega_3 \tilde{P}_3 + \beta_0 \tilde{G}_0 \right]. \quad (\text{D20})$$

The coefficients $P_{0,3}$, $S_{0,3}$, $G_{0,3}$ are functions of the combined scattering channels $\hat{t}_{S0^\pm}^\pm$, $\alpha_0 \hat{t}_{00}^\pm$ which emerge as a combination of scattering s - and p -wave channels. This also verifies that

$$\frac{[\kappa_{xx}^V] T_c}{\frac{\pi^2}{3} N_f v_f^2 T} \approx \Gamma_u T_c \frac{3/4}{C_0 C_s} \left[(1 - \text{Re}\alpha)(\cot \delta_0 \cot \delta_s + \tilde{\gamma}_0 \tilde{\gamma}_s) \Omega_0^2 - \text{Re}\alpha(\cot \delta_0 \cot \delta_s - \tilde{\gamma}_0 \tilde{\gamma}_s) \beta_3^2 - 2 \frac{\tilde{\gamma}_0 \tilde{\gamma}_s C_s}{C_s^2} \Omega_0 \beta_3 \right]. \quad (\text{D21})$$

The interested reader is encouraged to include the vertex renormalization self-consistently, where we include it only in our numerical coding. In addition, $\tilde{\gamma}_0 = \langle \frac{\gamma}{D} \rangle \sim \frac{\gamma_0}{\Delta_0} \ln \frac{\Delta_0}{\gamma_0}$, $\tilde{\gamma}_s = \langle \frac{\gamma \sin^2 \theta}{D} \rangle \sim \frac{\gamma_0}{\Delta_0}$, and $\tilde{\gamma}^1 = \langle \frac{\tilde{\Delta}_0 \sin^4 \theta}{D} \rangle \sim \text{constant}$, where $\text{Re}\alpha_0$, $\text{Re}\alpha \sim \alpha \sim \frac{1}{C_{s2}} = (\cot^2 \delta_s + \frac{\gamma_0^2}{\Delta_0^2})^{-1}$.

Equation (D21) is a complicated expression, but the low-temperature integrals are estimated in orders of magnitudes above. For typical, nonvanishing, and nondiverging values of $\cot \delta_0$ and $\cot \delta_s$, the terms with $\beta_3^2 \cot \delta_0$ or $\Omega_0^2 \cot \delta_0$ have the dominant contribution for κ_{xx} , while it is only the terms with $\beta_3^2 \cot \delta_s$ for κ_{yx} . Evaluating the vertex corrections with the dominant terms along with $\Gamma_u/C_0 = \gamma_0/\tilde{\gamma}_0$, we obtain the following expression,

$$\sim - \frac{T_c}{\tilde{\Delta}_0} \frac{\gamma_0^2 \ln \frac{\tilde{\Delta}_0}{\gamma_0}}{\tilde{\Delta}_0^2} \frac{1}{C_s} \left[\frac{\alpha \cot \delta_0 \cot \delta_s}{\frac{\gamma_0}{\tilde{\Delta}_0} (\cot \delta_0 + \ln \frac{\tilde{\Delta}_0}{\gamma_0} \cot \delta_s) (1 - \alpha)} \right]. \quad (\text{D22})$$

At finite temperatures, the thermal Hall conductivity is dominated not only by Ω_0 , β_3 but also by Ω_3 , β_0 (with the vertex correction it is complicated pairs of all averages) due to the nonzero $\text{Im}(\sigma_0^R)_{00}$ term. Physically, if the bare GFs are to be used, the imaginary parts of $(\Sigma^{R,A})_{00}$ change the lifetime of electrons and holes. Overall, the lifetimes for electrons and holes differ, $\tau_{e,h}^{-1} \propto -\text{Im}[(\Sigma^R)_{30} \pm (\Sigma^R)_{00}]$. This effect is not visible directly in the quasiclassical approach as $\hat{g}^{R,A}$ and $(\sigma^{R,A})_{00}$ commute in the main quantum transport equation in Eq. (1) and $\hat{g}_0^{R,A}$ does not include the $\tau_0 \sigma_0$ component. Interestingly, it still modifies the nonequilibrium occupation because $(\sigma^{R,A})_{00}$ are explicitly present in Eq. (A5).

APPENDIX E: κ_{ij} AND THE LOW IMPURITY DENSITY LIMIT, $\Gamma \rightarrow 0$

For completeness, let us evaluate the averages for the low impurity density limit $\Gamma_u \rightarrow 0$. In this limit, the Bogoliubov

the thermal Hall current vanishes in the absence of even and odd scattering channels.

At low temperatures, $T \rightarrow 0$, the excitations are only possible in the vicinity of the nodes (poles) where $\varepsilon \rightarrow 0$. Then, $\varepsilon^R \rightarrow i(\gamma_0 + \gamma_s \sin^2 \theta)$, and γ_0 and γ_s are the bandwidths with positive-definite values. Two of the integral averages vanish in this limit, $\Omega_3 = \beta_0 \rightarrow 0$. The non-self-consistent contribution to the longitudinal conductivity is $\frac{\kappa_{xx}^{\text{ns}} T_c}{\frac{\pi^2}{3} N_f v_f^2 T} = \Omega_0$. The nonzero integrals, Ω_0 and β_3 , are evaluated as $\Omega_0(0) = \langle \frac{\gamma^2(\theta)}{D^3} \sin^2 \theta \rangle \sim \frac{\gamma_0}{\Delta_0^2}$, $\beta_3(0) = \langle \frac{\gamma(\theta) \Delta_0}{D^3} \sin^4 \theta \rangle \sim \gamma_0 / \Delta_0^2 \ln \frac{\tilde{\Delta}_0}{\gamma_0}$, and $\tilde{\gamma}^1 \approx \text{constant}$. The vertex corrections to the thermal coefficients, κ_{xx}^V , κ_{yx}^V , without the vertex renormalization are found to be

quasiparticle lifetime goes to infinity, $\tau \rightarrow \infty$. The averages that go with Γ_u^0 are negligible compared to the averages which are divergent on the order of Γ_u^{-1} . We present the results in terms of the BQ decay rate for the diverging parts as $\tau^{-1} = \Gamma_u \frac{\cot \delta_0^2 + |\tilde{\gamma}_0|^2}{|\cot \delta_0^2 + (\tilde{\gamma}_0)^2|^2} \text{Re} \tilde{\gamma}_0$. We omitted anisotropic parts of the impurity averages for clearer forms as they do not change the relevant scales here. Noting that $x = \frac{\varepsilon}{\Delta_0}$, the integrals are calculated as

$$\Omega_0(x) \rightarrow \begin{cases} \Gamma_u^0, & |x| < 1, \\ \tau \left(\frac{\sqrt{x^2 - f^2(\hat{k})}}{x} (\hat{k}_x^2 + \hat{k}_y^2) \right), & |x| > 1, \end{cases} \quad (\text{E1})$$

$$\beta_0(x) \rightarrow \begin{cases} \tau h(\varepsilon) \left(\frac{\sqrt{x^2 - f^2(\hat{k})} f^2(\hat{k})}{x^2 + [x^2 - f^2(\hat{k})] h^2(\varepsilon)} \right), & |x| < 1, \\ \Gamma_u^0, & |x| > 1, \end{cases} \quad (\text{E2})$$

$$\beta_3(x), \Omega_3(x) \rightarrow \Gamma_u^0. \quad (\text{E3})$$

The numerical value for the average changes for each order parameter and it is denoted by $f^2(\hat{k})$. $f^2(\hat{k})^{E_{1g}} = \hat{k}_z^2 (\hat{k}_x^2 + \hat{k}_y^2)$ and $f^2(\hat{k})^{E_{2g}} = (\hat{k}_x^2 + \hat{k}_y^2)^2$. Also, $h(\varepsilon)$ is independent of $\hat{k}$ and has an explicit dependence on the phase shifts, $h(\varepsilon) = \frac{2 \cot \delta_0}{\cot^2 \delta_0 + |\tilde{\gamma}_0|^2} \text{Im} \left\langle \frac{-ix}{\sqrt{f^2(\hat{k}) - x^2}} \right\rangle_{x^2 < f^2}$. As the BQ lifetime dominates, one can neglect the integral contribution below (above) the superconducting gap $\frac{\varepsilon}{\Delta_0} < 1$ (> 1). In any case, it is irrelevant to our case where both averages diverge as τ .

In this limit if one uses the clean-limit Green's functions at finite temperature, the thermal transport coefficients that are obtained from the quasiclassical quantum transport equation reduce to the same coefficients obtained in the quantum Boltzmann equation approach [31]. The low-temperature limit, on the contrary, can only be captured by the the quasiclassical quantum transport equation as it is possible to obtain the GFs within the impurity scattering energy scale, $\gamma(\theta)$ [48].

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