Correlations in non-Hermitian systems and diagram techniques for the steady state

Johan Carlström

Department of Physics, Stockholm University, 106 91 Stockholm, Sweden

(Received 25 October 2019; published 24 January 2020)

I construct a diagrammatic technique for non-Hermitian fermionic systems which allows addressing correlation effects in the steady state by systematic expansion. Applying this method to exceptional points or rings, I find that nodal objects in non-Hermitian systems are generically displaced in momentum-space due to interactions. This is a consequence of the fact that exceptional points invariably break a class of orthonormal symmetries that are generally present for nodal points in Hermitian systems and which protect the integrity of the node at low energy scales.

DOI: [10.1103/PhysRevResearch.2.013078](https://doi.org/10.1103/PhysRevResearch.2.013078)

I. INTRODUCTION

Topological semimetals have become the focal point of current research due to a breadth of new forms of quantum matter that includes emergent quasiparticles in the form of Weyl and Dirac fermions that were originally considered in the context of high-energy physics, as well as the realization of their concomitant quantum anomalies [1,2]. Invoking symmetry arguments, these band structures can furthermore be generalized to line-nodes [3], knotted band touchings [4], higher-order Weyl nodes [5,6], and even Weyl semimetals that exhibit analogues of non-Abelian particle statistics [7,8].

Recently, the experimental realization of topological phases in optical wave guides with dissipation [9–11] has motivated an intense effort to generalize central ideas of topological band-theory to the case of non-Hermitian Hamiltonians [12–14]. Non-Hermitian models can furthermore be derived from open/dissipative quantum systems [15], which undergo nonunitary time-evolution. While a complete understanding of this topic is still lacking, a number of striking differences from the Hermitian case have already been established. Notable examples of these include a modified bulk-boundary correspondence [16–23], reduced co-dimension of band touching points [24], bulk Fermi arcs in 2D [25] and Fermi ribbons in 3D [26–28], which are direct manifestations of non-Abelian statistics in the non-Hermitian regime [29].

Technological applications based on non-Hermitian band topology are currently focused on sensors and detectors that exploit the sublinear dispersion in the proximity of exceptional points to enhance signal response. Presently, implementations utilizing optical microcavities already exist [30], while proposals based on ultracold atomic gases were put forth recently [31,32]. In the latter implementation, a central idea is to exploit particle correlations to realize higher-order

exceptional points with even steeper susceptibilities to a weak signal. While the particular models considered in Refs. [31,32] are exactly solvable, the extension of non-Hermitian physics to cold atomic gases and electronic systems out of equilibrium, where interactions are ubiquitous [33–38], represents a fundamental theoretical challenge since the current frameworks of quantum many-body theory either rely on unitary time-evolution, or are constructed at thermal equilibrium. Note that these works should not be confused with a number of recent reports on exceptional points in electronic systems at equilibrium that result from a complex self energy [39–42].

In this work I explore correlation effects in non-Hermitian systems, in particular, demonstrating that if the system possesses a steady state as a result of nonunitary time-evolution, then it can be described using diagrammatic techniques. Applying this method to simple nodal objects such as an exceptional point or ring, I find that these are generally more susceptible to interactions than band touching points in Hermitian systems. This in turn is related to the fact that exceptional points invariably break a class of orthonormal symmetries that are generally present for conventional nodal points.

II. PERTURBATION THEORY

In Hermitian systems, the perturbative expansion is generally organized in a diagrammatic series, which provides a systematic way of computing corrections to observables either in the ground state, or at thermal equilibrium [43]. Though dissipative or driven systems are by construction neither in the ground state or at equilibrium, it is still possible to conduct a perturbative expansion in the large-time limit, provided the existence of either a single steady state or a set of such states that are *macroscopically indistinguishable*.

More precisely, this condition may be understood as follows: Degeneracies of the steady state implies the existence of zero-decay modes for which the imaginary part of the energy vanishes exactly, that thus neither decays or grows. If the *i*-Fermi surface—i.e., the surface in momentum space which separates growing and decaying modes [26]—has a

nonzero codimension with respect to momentum-space, then the difference in occupancy between two steady states is nonextensive in the system size and can therefore not be registered by macroscopic observables. By contrast, if the *i*-Fermi surface has the same dimension as the system, then the difference in occupancy between steady states becomes extensive, which violates the condition. The latter scenario which can arise in systems that possess certain spectral symmetries [44,45] or due to fine-tuning must be treated separately and will not be considered hereafter.

To derive a description of the steady state, we start by assuming an initial state of our system at $t = 0$ that is denoted by

$$|\psi_0\rangle. \quad (1)$$

The time evolution of this state is then described by

$$|\psi_i(t)\rangle = \langle\psi_i|e^{itH^\dagger}, \quad |\psi_i(t)\rangle = e^{-itH}|\psi_i\rangle, \quad t \geq 0, \quad (2)$$

giving an expectation value of the operator $\hat{O}$ according to

$$\langle O\hat{t}(t) \rangle = \frac{\langle\psi_0|e^{itH^\dagger}\hat{O}e^{-itH}|\psi_0\rangle}{\langle\psi_0|e^{itH^\dagger}e^{-itH}|\psi_0\rangle}. \quad (3)$$

Since a non-Hermitian Hamiltonian generally lacks an orthonormal eigenbasis, it follows that the time-evolution operator is typically not diagonalizable. If there are no exceptional points in the theory, then it can still be expressed in terms of biorthogonal eigenvectors, though there is also a scenario when the eigenbasis of H does not even span the entire Hilbert space. When constructing the time-evolution operator, we must therefore distinguish between states that are within or outside the space of eigenvectors:

$$H|\alpha_i, R\rangle = \lambda_i|\alpha_i, R\rangle, \quad H|\beta_i, R\rangle = \lambda_{ij}|\psi_j, R\rangle, \quad (4)$$

where $\{|\alpha_i, R\rangle\}$ are the right eigenvectors of H , $\{|\beta_i, R\rangle\}$ spans the remainder of the Hilbert space, and $\{|\psi_i, R\rangle\} = \{|\alpha_i, R\rangle\} \cup \{|\beta_i, R\rangle\}$ forms a complete basis.

Let us now return to the initial assumption that the system is described by a set of steady states which contains elements that are more rapidly growing (or slowly decaying) than the remaining states, and which are sufficiently similar that they may not be discriminated between by macroscopic observables. In the large-time limit we may drop subleading terms, which gives a time-evolution operator of the form

$$\lim_{t \rightarrow \infty} U(t) = \sum_j e^{-it\epsilon_j} |\alpha_j, R\rangle \langle\alpha_j, L|, \quad (5)$$

$$\lim_{t \rightarrow \infty} U^\dagger(t) = \sum_j e^{it\epsilon_j^*} |\alpha_j, L\rangle \langle\alpha_j, R|,$$

which projects onto a subspace where $\Im(\epsilon_j) = c$ for all j . Here, c is a constant that describes the imaginary part of the energy of the states in the subspace. Since the states in Eq. (5) are macroscopically indistinguishable, we may in principle give preference to a specific state by lifting the degeneracy in Eq. (5) without affecting macroscopic observables. This can be achieved by adding a convergence factor of $\pm i\eta$ to the energy of the modes situated exactly on the *i*-Fermi surface. After taking $t \rightarrow \infty$ we proceed to take $\eta \rightarrow 0$, so that we project on a single state which has definite particle occupation number.

Returning to the expectation value Eq. (3), we now recognize that we can write the particle density in the steady state according to

$$\lim_{t \rightarrow \infty} \langle \hat{n}(t, \mathbf{k}) \rangle = \frac{\langle \alpha_0, R | \hat{n}(\mathbf{k}) | \alpha_0, R \rangle}{\langle \alpha_0, R | \alpha_0, R \rangle}, \quad (6)$$

where $|\alpha_0, R\rangle$ is the sole steady state once we have lifted the degeneracy in Eq. (5). Since this state has definite particle occupation number, it follows that it is an eigenstate of the number operator $\hat{n}(\mathbf{k})$. Using Eq. (5) we can thus write Eq. (6) in the form

$$\frac{\langle \alpha_0, R | \hat{n}(\mathbf{k}) | \alpha_0, R \rangle}{\langle \alpha_0, R | \alpha_0, R \rangle} = n_{\alpha_0, R}(\mathbf{k}) \frac{\langle \alpha_0, R | \alpha_0, R \rangle}{\langle \alpha_0, R | \alpha_0, R \rangle} \quad (7)$$

$$= n_{\alpha_0, R}(\mathbf{k}) \frac{\langle \alpha_0, L | \alpha_0, R \rangle}{\langle \alpha_0, L | \alpha_0, R \rangle} = \frac{\langle \psi_0 | e^{-itH} \hat{n}(\mathbf{k}) e^{-itH} | \psi_0 \rangle}{\langle \psi_0 | e^{-itH} e^{-itH} | \psi_0 \rangle}, \quad (8)$$

where $t \rightarrow \infty$, and we have assumed that ψ_0 is not orthogonal to the steady states, a scenario which in principle would require retaining subleading terms of Eq. (5). Expressing $\hat{n}$ in terms of field operators and conducting a time translation, we obtain two-point correlators of the form

$$\langle \Psi^\dagger(t_2) \Psi(t_1) \rangle = \lim_{t \rightarrow \infty} \quad (9)$$

$$\times \frac{\langle \Psi_0 | e^{-i(t-t_2)H} \Psi^\dagger e^{-i(t_2-t_1)H} \Psi e^{-(t+t_1)} | \Psi_0 \rangle}{\langle \Psi_0 | e^{2t} | \Psi_0 \rangle}. \quad (10)$$

At this point we will make a second assumption, namely, that the Hamiltonian can be decomposed into a bilinear part H_0 and an interaction part H_1 ,

$$H = H_0 + H_1, \quad (11)$$

which in principle allows us to make contact with conventional zero temperature diagrammatics [43]. Expanding Eq. (10) in H_1 we obtain the following series expansion for the full Greens function:

$$iG(x_1, x_0) = \frac{1}{Z} \sum_n \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \dots dt_n \langle \psi_0 | e^{-itH_0} T_i \times [H_1(t_1) \dots H_1(t_n) \Psi(x_1) \Psi^\dagger(x_0)] e^{-itH_0} | \psi_0 \rangle, \quad (12)$$

where T_i denotes time-ordering of the operators, and we take with $t \rightarrow \infty$. Equation (15) can be expressed in terms of time-dependent field operators of the form

$$\Psi(t) = e^{itH_0} \Psi e^{-itH_0}. \quad (13)$$

While the operators Eq. (13) are described by nonunitary time-evolution, bilinearity of H_0 implies that this is only reflected in the presence of complex exponents, so that, for example,

$$\Psi^\dagger(\mathbf{k}, t) = e^{itH_0} \Psi^\dagger(\mathbf{k}) e^{-itH_0} = \Psi^\dagger(\mathbf{k}) e^{it\epsilon_{\mathbf{k}}}, \quad \epsilon_{\mathbf{k}} \in \mathbb{C}, \quad (14)$$

where we recall that the energy $\epsilon_{\mathbf{k}}$ is by contrast real in Hermitian systems. Correspondingly, under the assumption that H_0 is bilinear, we can treat Eq. (12) with Wicks theorem to produce an expansion in terms of connected diagrams that relates the full Greens function to its bare counterpart, which

is given by

$$iG_0(t_1 - t_2, \mathbf{k}) = \frac{\langle \psi_0 | e^{-it_1 H_0} T_t \Psi(\mathbf{k}, t_1) \Psi^\dagger(\mathbf{k}, t_2) e^{-it_2 H_0} | \psi_0 \rangle}{\langle \psi_0 | e^{-i2t H_0} | \psi_0 \rangle}, \quad (15)$$

where $t \rightarrow \infty$. For a more extensive discussion of Wicks theorem and contractions in a non-Hermitian system, see the Appendix. Notably, Eq. (15) is independent of the initial state as long as it has a nonzero overlap with the steady state of H_0 . Under this assumption, the state which is acted upon by the field operators is either vacant or occupied, depending on whether it is characterized by gain or loss at the level of bilinear theory. Specifically, we obtain

$$\begin{aligned} iG_0(t, \mathbf{k}) &= -\theta(\Im[\epsilon(\mathbf{k})])e^{-it\epsilon(\mathbf{k})}, \quad t < 0, \\ iG_0(t, \mathbf{k}) &= \theta(-\Im[\epsilon(\mathbf{k})])e^{-it\epsilon(\mathbf{k})}, \quad t > 0. \end{aligned} \quad (16)$$

Here $\Im(\epsilon) = 0$ —at the level of bilinear theory—defines the i-Fermi surface, i.e., the set of momenta for which the imaginary part of the energy vanishes. Notably, this surface separates the states with gain and loss, respectively. Since the energy is complex, the Greens function is exponentially localized in time, and so the time integral convergences naturally without a convergence factor $\sim i\eta$. In this respect, the expansion is reminiscent of Matsubara formalism in the zero temperature limit. Taking the Fourier transform of Eq. (16), we obtain the bare frequency-Greens function,

$$G_0(\omega, \mathbf{k}) = \frac{1}{\omega - H_0(\mathbf{k})}. \quad (17)$$

While superficially similar to the zero-temperature Greens function of a Hermitian system, Eq. (17) generally exhibits poles at a finite distance from the real axis that reflect the dissipative nature of non-Hermitian systems. In the limit $t \rightarrow -0$, the full Greens function Eq. (12) reproduces the particle density of the steady state in accordance with Eq. (8).

III. FOCK THEORY

Having established a diagrammatic framework, we can now return to the prelude question of how correlations affect nodal points in the non-Hermitian regime. At the lowest order, we obtain two types of diagrams, namely, Hartree and Fock type corrections. Of these, the former typically only lead to a renormalization of the chemical potential, while the Fock term gives a nontrivial correction to the self energy of the form

$$\begin{aligned} \Sigma_{\text{Fock}}(\mathbf{k}) &= \int d\mathbf{q} d\omega V(\mathbf{q}) iG^0(\omega, \mathbf{k} - \mathbf{q}) \\ &= \int d\mathbf{q} d\omega V(\mathbf{q}) \frac{i}{\omega - H_0(\mathbf{k} - \mathbf{q})} \\ &= \int d\mathbf{q} d\omega V(\mathbf{q}) i \frac{\omega + H_0}{\omega^2 - H_0^2}, \end{aligned} \quad (18)$$

where we have absorbed a factor $(2\pi)^{-D}$ into V . Dropping the part which is odd in frequency and assuming a particle-hole symmetric two-band model we obtain

$$i \int d\omega \frac{H_0}{\omega^2 - H_0^2} = i \int d\omega \frac{H_0}{(\omega - \Delta)(\omega + \Delta)}, \quad (19)$$

where we have defined $\Delta^2 \sigma_0 = H_0^2$. If we choose to define Δ such that $\Im(\Delta) > 0$, then we obtain a pole in the upper half-plane corresponding to $\omega = \Delta$. This gives a Fock integrand of the form

$$\xi(\mathbf{k}) = iH_0(\mathbf{k}) \frac{i2\pi}{2\Delta(\mathbf{k})} = -\pi \frac{H_0(\mathbf{k})}{\Delta(\mathbf{k})}, \quad (20)$$

with

$$\Sigma(\mathbf{k}) = \int d\mathbf{q} V(\mathbf{q}) \xi(\mathbf{k} - \mathbf{q}). \quad (21)$$

Thus, we see that the Fock integrand, despite being non-Hermitian, possesses a real spectrum. It should be stressed however that it is still generally accompanied by a nonorthogonal eigenbasis which it shares with H_0 .

To examine the correction from Eq. (20) to an exceptional point, we consider a minimalistic model of the form

$$H_0 = \kappa + \gamma, \quad \kappa = \sigma_x + i\sigma_z, \quad \gamma = \frac{k_x}{2}\sigma_x + \frac{k_z}{2}\sigma_z, \quad (22)$$

which possesses an exceptional point in $k_x = k_z = 0$, where γ vanishes. To leading order, this gives a gap which can be expressed on polar form according to

$$\Delta = \sqrt{k_x + ik_z} = \chi e^{i\phi/2}, \quad \chi = |k_x + ik_z|^{1/2} = \sqrt{k}. \quad (23)$$

For a contact interaction we have $V(\mathbf{q}) = V$, implying that Eq. (21) corresponds to an integral of the Fock integrand $\xi(\mathbf{q})$ over $\mathbf{q}$. In this scenario we can work in polar coordinates, and symmetrize the integrand with respect to the angle ϕ . Decomposing ξ into terms $\sim \kappa$ and $\sim \gamma$, we obtain

$$\xi_\kappa^s = \int_0^{2\pi} k d\phi \left(-\pi \kappa \frac{e^{-i\phi/2}}{\chi} \right) = 4i\pi \kappa \sqrt{k}, \quad (24)$$

$$\xi_\gamma^s = \int_0^{2\pi} k d\phi \left(-\pi \gamma \frac{e^{-i\phi/2}}{\chi} \right) = -\frac{2\pi k^{3/2}}{3} (i\sigma_x + 2\sigma_z). \quad (25)$$

In the next stage we integrate over $k = \sqrt{k_x^2 + k_z^2}$ to obtain

$$J_\kappa = V \int dk 4i\pi \sqrt{k}, \quad J_\gamma = -V \int dk \frac{2\pi k^{3/2}}{3}, \quad (26)$$

which gives a renormalized dispersion of the form

$$H_0 + \Sigma = (1 + J_\kappa)\kappa + \sigma_x \frac{k_x + 2iJ_\gamma}{2} + \sigma_z \frac{k_z + 4J_\gamma}{2}. \quad (27)$$

Computing the spectrum of Eq. (27) we find a node in

$$k_x = 0, \quad k_z = -6J_\gamma, \quad (28)$$

so that the exceptional point is translated in momentum space along the k_z -axis by a distance that is linear in the interaction strength V .

In principle the finding that correlation effects lead to a translation of nodes in momentum space also holds for an exceptional ring that results when adding a non-Hermitian perturbation to a Weyl point. To the lowest order we then obtain a dispersion of the form

$$H_0 = \mathbf{k} \cdot \boldsymbol{\sigma} + i\sigma_z, \quad (29)$$

with a gap that is given by

$$\Delta = \frac{k_z}{|k_z|} \sqrt{k^2 - 1 + 2ik_z}. \quad (30)$$

Similar to the preceding scenario, the integral of ξ over $\mathbf{q}$ is simplified by symmetrization of the integrand. To this end we thus note the following properties of the dispersion and the gap

$$\frac{H_0(\mathbf{k}_{xy}, k_z) + H_0(-\mathbf{k}_{xy}, k_z)}{2} = (k_z + i)\sigma_z, \quad (31)$$

$$\Delta(\mathbf{k}_{xy}, k_z) = \Delta(-\mathbf{k}_{xy}, k_z), \quad (32)$$

$$\Delta^*(\mathbf{k}_{xy}, k_z) = -\Delta(\pm\mathbf{k}_{xy}, -k_z). \quad (33)$$

Using the properties Eqs. (31), (32), and (33) we can construct a Fock integrand that is symmetrized with respect to a π rotation around the z axis according to

$$\xi_\pi = \frac{\xi(\mathbf{k}_{xy}, k_z) + \xi(-\mathbf{k}_{xy}, k_z)}{2} = -\frac{(k_z + i)\sigma_z}{\Delta(\pm\mathbf{k}_{xy}, k_z)}, \quad (34)$$

which is thus diagonal, but features a complex spectrum. In the next stage, we use Eq. (33) and symmetrize with respect to a reflection in the xy -plane. We begin by noting the relationship

$$\xi_\pi(\mathbf{k}_{xy}, -k_z) = -\frac{(-k_z + i)\sigma_z}{-\Delta^*(\mathbf{k}_{xy}, k_z)} = \xi_\pi^\dagger(\mathbf{k}_{xy}, k_z), \quad (35)$$

which allows us to define the rotation and reflection symmetrized Fock integrand according to

$$\xi_s(\mathbf{k}) = \frac{\xi_\pi(\mathbf{k}_{xy}, k_z) + \xi_\pi(\mathbf{k}_{xy}, -k_z)}{2} = \frac{\xi_\pi(\mathbf{k}) + \xi_\pi^\dagger(\mathbf{k})}{2}. \quad (36)$$

The implication of Eqs. (34) and (36) is that $\xi_s(\mathbf{k})$ is a diagonal matrix with real eigenvalues of the form

$$\xi_s(\mathbf{k}) = f(\mathbf{k})\sigma_z, \quad f \in \mathbb{R}, \quad f(\mathbf{k}) < 0 \quad \forall \mathbf{k}. \quad (37)$$

Taking $V(\mathbf{q})$ to be a contact interaction, inserting Eq. (37) into Eq. (21) and integrating over $\mathbf{k}$, we find that the exceptional ring is translated along the k_z -axis by

$$\Delta k_z = -V \int d\mathbf{k} f(\mathbf{k}), \quad (38)$$

which is orthogonal to the ring.

IV. ORTHONORMAL SYMMETRY BREAKING

The nontrivial response of exceptional points to many-body effects is strikingly different from conventional nodal points, which are at the lowest energy scales generally protected by an orthonormal symmetry. Specifically, expanding the dispersion of a Hermitian system to the lowest order around a band touching point, we typically find an effective description which is odd under some orthonormal map according to

$$H_0(T\mathbf{k}) = -H_0(\mathbf{k}), \quad |T\mathbf{k}| = |\mathbf{k}|. \quad (39)$$

Notable examples of this include Weyl points, where the dispersion is odd under inversion,

$$H_0(\mathbf{k}) = k_i v_{ij} \sigma_j, \quad H_0(\mathbf{k}) = -H_0(-\mathbf{k}), \quad (40)$$

and higher-order Weyl semimetals of the form

$$H_0(\mathbf{k}) = (k_x + ik_y)^n \sigma_+ + (k_x - ik_y)^n \sigma_- + k_z \sigma_z, \quad (41)$$

where the orthonormal map corresponds to rotation around the z axis and reflection in the xy plane. Further examples arise in single and multilayer graphene.

For two-body interactions, the property Eq. (39) implies a symmetry of the self energy given by

$$\Sigma(\omega, \mathbf{k} = 0) = -\Sigma(-\omega, \mathbf{k} = 0), \quad (42)$$

which in turn results in a solution to Dyson's equation at zero energy [46]. Thus, not only is the semimetallic phase protected, but the position of the node in momentum-space is preserved by the symmetry Eq. (39). This fact has also been verified by diagrammatic Monte Carlo simulations [47]. Once higher-order terms in the dispersion are included, the orthonormal symmetry is generally broken, but the fact that Eq. (39) holds to the lowest order implies that interaction effects are significantly diminished in Hermitian nodal systems.

In contrast to nodal points in Hermitian systems, exceptional points are irreconcilable with the orthonormal symmetry Eq. (39), since a vanishing (or diagonal) H_0 necessarily possesses an eigenbasis that spans the Hilbert space. The implication of this discrepancy is that exceptional points already at the level of linearized theory are shifted in momentum-space due to many-body corrections. In full lattice models this suggests that non-Hermitian systems are far more susceptible to interactions than their Hermitian counterparts, particularly if the interaction is long ranged in real-space and thus rapidly decaying in momentum space.

V. DISCUSSION

In this work I have constructed a diagram technique for non-Hermitian systems that allows systematic and unbiased calculation of correlation effects in the steady state. The corrections that transpire from this approach are formally similar to those of conventional zero-temperature formalism with the distinct difference that the bare Greens function possesses poles at a finite distance from the real axis. This feature reflects the dissipative nature of non-Hermitian physics and implies that the Greens function becomes exponentially localized in the time-domain. The closest analog of this technique is arguably Keldysh formalism, though this rather complex approach is not primarily aimed at the steady state [48]. The prerequisites for applicability of this technique are that the system has an i -Fermi surface that is of lower dimension than the system itself, and that the Hamiltonian can be described in terms of a bilinear part plus many-body interactions.

The existence of a diagrammatic expansion for observables in the non-Hermitian regime implies that these can be systematically computed using diagrammatic simulation techniques [49]. Recently it has been demonstrated that these methods can be applied even to strongly interacting quantum matter [50,51].

Since the particle energies are complex, the time-evolution operator takes the form of a projection onto a subspace in the limit of large t , effectively destroying some of the information about the initial state. Therefore the assumption of macroscopic indistinguishability of the set of steady states is equivalent to the requirement that all information about the initial state that can be extracted by macroscopic observables is lost in the infinite time limit. This also explains

why the expansion is independent of the initial state as long as it has a nonzero overlap with a steady state. If the assumption is violated and the *i*-Fermi surface has the same dimension as momentum-space, then the time-evolution operator fails to erase measurable thermodynamic properties that are encoded in the initial state. In this scenario, additional selection rules based on the real part of the spectrum or a specific choice of the initial state are necessary to compute relevant observables.

Finally we note that displacement of nodal points are ubiquitous in interacting non-Hermitian systems except in the case of very particular inter-particle forces. This is a consequence of breaking of an orthonormal symmetry and raises the question of whether these corrections can be connected systematically to topological characteristics of the nodal objects [14,52,53].

ACKNOWLEDGMENTS

This work was supported by the the Swedish Research Council (VR). The author thanks Emil J. Bergholtz for important input and discussions.

APPENDIX: WICKS THEOREM FOR A NON-HERMITIAN SYSTEM

The derivation of Wicks theorem is in itself not dependent on hermiticity: It is only in the evaluation of contractions that this aspect becomes important. Here, a standard proof of the theorem based on Ref. [43] will be outlined. Thereafter, the questions of nonhermiticity will be discussed:

In principle we want to evaluate expressions of the form

$$\langle \bar{L} | ABCD... | \bar{R} \rangle, \quad | \bar{R} \rangle = \lim_{t \rightarrow \infty} e^{-iH_0 t} | \psi_0 \rangle. \quad (\text{A1})$$

In line with the discussion in the main article, we assume that the *i*-Fermi surface—that is, the set of momenta for which the imaginary part of the energy vanishes—has a nonzero codimension with **k** space. This means that there is in momentum space a sharp cutoff between populated and unpopulated states. Correspondingly, we can divide the real-space field operators into parts that act on states that are filled or vacant in the unperturbed steady state:

$$\Psi(x) = \sum_{\mathbf{k}, \Im(\epsilon_k) < 0} e^{i(\mathbf{k} \cdot \mathbf{x} - \epsilon_k t)} \Psi(\mathbf{k}) + \sum_{\mathbf{k}, \Im(\epsilon_k) > 0} e^{i(\mathbf{k} \cdot \mathbf{x} - \epsilon_k t)} \Psi(\mathbf{k}) \quad (\text{A2})$$

$$= \Psi_v(x) + \Psi_o(x). \quad (\text{A3})$$

Here, $\Psi_v(x)$ acts on states that are vacant in the (unperturbed) steady state, while $\Psi_o(x)$ acts on states that would be filled. Correspondingly, we have

$$\Psi_v(x) | \bar{R} \rangle = \Psi_o^\dagger(x) | \bar{R} \rangle = 0, \quad (\text{A4})$$

$$\Psi_o(x) | \bar{R} \rangle \neq 0, \quad \Psi_v^\dagger(x) | \bar{R} \rangle \neq 0, \quad (\text{A5})$$

since one cannot annihilate a particle in a vacant state, or create a particle in an occupied state. Here we also note that a product of field operators can always be decomposed into a sum of products where all operators are either of the form Eq. (A4) or of the form Eq. (A5). Thus, it is sufficient to derive the theorem for such operators.

Next, we introduce the time ordered product, which orders the operators in time, with a sign impaired by every commutation. That is, assuming $t_A > t_B > t_C \dots$ we have

$$T(ABCD...) = ABCD..., \quad (\text{A6})$$

$$T(ABCD...) = (-1)T(CABD...), \quad (\text{A7})$$

such that the *T* product always returns a set of time ordered operators up to some sign.

In the same way, we may introduce the normal ordering, where operators that vanish when acting on the steady state, i.e., Eq. (A4), are placed to the right. Here we also take

$$N(ABCD...) = (-1)N(ABCD...). \quad (\text{A8})$$

Finally, we define the contraction of two operators according to

$$\hat{U} \hat{V} = T(\hat{U} \hat{V}) - N(\hat{U} \hat{V}). \quad (\text{A9})$$

Here, we note the following: If time and normal ordering coincide, then the contraction vanishes. If both operators are of the type Eq. (A4), or both of type Eq. (A5), then the normal ordered product is ill defined, but can be chosen such that $\hat{U} \hat{V}$ is zero. We can thus summarize the properties of the contraction Eq. (A9):

$$\Psi_v^\dagger(x) \Psi_v^\dagger(y) = \{\Psi_v^\dagger(x), \Psi_v^\dagger(y)\}, \quad t_x > t_y, \quad (\text{A10})$$

$$\Psi_o^\dagger(x) \Psi_o^\dagger(y) = -\{\Psi_v^\dagger(x), \Psi_v^\dagger(y)\}, \quad t_y > t_x. \quad (\text{A11})$$

The remaining contractions vanish.

Since the normal ordered product places the operators Eq. (A4) on the right, it follows that it vanishes when acting on $| \bar{R} \rangle$. Therefore, the expectation value of a set of operators can be evaluated by considering all anti-commutations when bringing them into normal order: Lets then assume that *B* is of type Eq. (A4) and should be moved to the right: This gives us

$$ABCD... = A(\{B, C\} - CB)D..., \quad (\text{A12})$$

$$= \{B, C\}AD... - \{B, D\}AC... + ACD...B. \quad (\text{A13})$$

The last term vanishes when acting on the steady state and we are left with products of an anti-commutator and a set of remaining operators. The anticommutators may be written

$$\{B, O\} = B'O'. \quad (\text{A14})$$

Contractions of *B* with operators to its left vanish due to the definition Eq. (A9). Hence, $A'B = 0$ for example. Therefore, we may write Eq. (A13) as

$$A'B'CD... + AB'C'D... - AB'CD'... + BCD...A, \quad (\text{A15})$$

such that *B* is formally contracted with all other operators, though contractions with operators to its left trivially vanishes.

Applying Eq. (A15) recursively we find that the expectation value Eq. (A1) is equal to the set of all contractions of the product $ABCD...$ evaluated with respect to $| \bar{R} \rangle$, with a fermionic sign that is related to the number of commutations required to bring the contracted operators into contact. So far this argument is identical to standard textbooks [43]. The principal challenge that emerges now is evaluation of contractions with respect to a non-Hermitian theory.

In momentum space, the contractions Eq. (A9) are given by

$$\langle \bar{L} | T_i \Psi_i^\dagger(t_1, \mathbf{k}) \Psi_j(t_2, \mathbf{k}) | \bar{R} \rangle, \quad (\text{A16})$$

which thus depends on nonunitary time-evolution. If the dispersion is nonexceptional, then it can be expressed as

$$H_0(\mathbf{k}) = \sum_{\alpha} |R_0, \alpha\rangle \epsilon_{\alpha}^0 \langle L_0, \alpha|, \quad (\text{A17})$$

where R_0, L_0 are right and left eigenvectors of H_0 . The bilinear theory can thus be written explicitly as

$$\Psi_j^\dagger H_{ij}^0 \Psi_j = \Psi_i^\dagger \langle i | R_0, \alpha \rangle \langle L_0, \alpha | H_0 | R_0, \beta \rangle \langle L_0, \beta | i \rangle \Psi_i \quad (\text{A18})$$

$$= \Psi_{\alpha}^\dagger \epsilon_{\alpha}^0 \Psi_{\alpha}, \quad \Psi_{\alpha}^\dagger = \Psi_i^\dagger \langle i | R_0, \alpha \rangle, \quad \Psi_{\alpha} = \langle L_0, \alpha | i \rangle \Psi_i, \quad (\text{A19})$$

where summation over i is assumed. This defines a set of bi-orthogonal field operators with anticommutation relations given by

$$\{\Psi_{\alpha}, \Psi_{\beta}^\dagger\} = \sum_{ij} \{\langle L_0, \alpha | i \rangle \Psi_i, \Psi_i^\dagger \langle j | R_0, \beta \rangle\} \quad (\text{A20})$$

$$= \sum_i \langle L_0, \alpha | i \rangle \langle i | R_0, \beta \rangle = \delta_{\alpha\beta}, \quad (\text{A21})$$

just as we would expect for an orthogonal system. The time-dependence of the field operators follows from Eq. (A17), and is given by

$$\Psi_{\alpha}^\dagger(t) = e^{it\epsilon_{\alpha}} \Psi_{\alpha}^\dagger, \quad \Psi_{\alpha}(t) = e^{-it\epsilon_{\alpha}} \Psi_{\alpha}. \quad (\text{A22})$$

Expressing Eq. (A16) in bilinear operators, we obtain

$$G_{\alpha}^0(t_1 - t_2, \mathbf{k}) = \frac{\langle \bar{L} | T_i \Psi_{\alpha}(t_1, \mathbf{k}) \Psi_{\alpha}^\dagger(t_2, \mathbf{k}) | \bar{R} \rangle}{\langle \bar{L} | \bar{R} \rangle} \quad (\text{A23})$$

$$= e^{-i\epsilon_{\alpha}(t_1 - t_2)} \frac{\langle \bar{L} | T_i \Psi_{\alpha}(\mathbf{k}) \Psi_{\alpha}^\dagger(\mathbf{k}) | \bar{R} \rangle}{\langle \bar{L} | \bar{R} \rangle}, \quad (\text{A24})$$

while off-diagonal terms vanish due to Eq. (A21).

Whether the state $\alpha, \mathbf{k}$ is occupied or vacant in the unperturbed steady state $\bar{R}$ depends on if the imaginary part of the energy dictates gain or loss. Specifically, we find

$$iG_{\alpha}^0(t, \mathbf{k}) = -\theta(\Im[\epsilon_{\alpha}(\mathbf{k})])e^{-it\epsilon_{\alpha}(\mathbf{k})}, \quad t < 0, \\ iG_{\alpha}^0(t, \mathbf{k}) = \theta(-\Im[\epsilon_{\alpha}(\mathbf{k})])e^{-it\epsilon_{\alpha}(\mathbf{k})}, \quad t > 0. \quad (\text{A25})$$

Taking the Fourier transform of Eq. (A25) we find the bare Greens function expressed in the biorthogonal basis:

$$G^0(\omega, \mathbf{k}) = |L, \alpha, \mathbf{k}\rangle \frac{1}{\omega - \epsilon_{\alpha}(\mathbf{k})} \langle R_0, \alpha, \mathbf{k}| = \frac{1}{\omega - H_0(\mathbf{k})}. \quad (\text{A26})$$

In exceptional points $\mathbf{k}_e$, the Greens function is defined by taking the limit

$$G^0(\omega, \mathbf{k}) = \lim_{\mathbf{q} \rightarrow \mathbf{q}_e} \frac{1}{\omega - H_0(\mathbf{q})} = \frac{1}{\omega - H_0(\mathbf{k}_e)}, \quad (\text{A27})$$

where $\mathbf{q}(\mathbf{k})$ is a set of model parameters (such as matrix elements) describing H_0 , and $\mathbf{q}(\mathbf{k}_e) = \mathbf{q}_e$. Thus, we conclude

$$G^0(\omega, \mathbf{k}) = \frac{1}{\omega - H_0(\mathbf{k})}. \quad (\text{A28})$$

-
- [1] N. P. Armitage, E. J. Mele, and A. Vishwanath, *Rev. Mod. Phys.* **90**, 015001 (2018).
- [2] T. Wehling, A. Black-Schaffer, and A. Balatsky, *Adv. Phys.* **63**, 1 (2014).
- [3] T. Bzdusek, Q. Wu, A. Rüegg, M. Sigrist, and A. A. Soluyanov, *Nature* **538**, 75 (2016).
- [4] R. Bi, Z. Yan, L. Lu, and Z. Wang, *Phys. Rev. B* **96**, 201305(R) (2017).
- [5] C. Fang, M. J. Gilbert, X. Dai, and B. A. Bernevig, *Phys. Rev. Lett.* **108**, 266802 (2012).
- [6] S.-M. Huang, S.-Y. Xu, I. Belopolski, C.-C. Lee, G. Chang, T.-R. Chang, B. Wang, N. Alidoust, G. Bian, M. Neupane, D. Sanchez, H. Zheng, H.-T. Jeng, A. Bansil, T. Neupert, H. Lin, and M. Z. Hasan, *Proc. Natl. Acad. Sci.* **113**, 1180 (2016).
- [7] Q. Wu, A. A. Soluyanov, and T. Bzdušek, *Science* **365**, 1273 (2019).
- [8] A. Bouhon, R.-J. Slager, and T. Bzdušek, *arXiv:1907.10611*.
- [9] J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, *Phys. Rev. Lett.* **115**, 040402 (2015).
- [10] A. Cerjan, S. Huang, M. Wang, K. P. Chen, Y. Chong, and M. C. Rechtsman, *Nat. Photon.* **13**, 623 (2019).
- [11] S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K. G. Makris, M. Segev, M. C. Rechtsman, and A. Szameit, *Nat. Mater.* **16**, 433 (2016).
- [12] S. Yao, F. Song, and Z. Wang, *Phys. Rev. Lett.* **121**, 136802 (2018).
- [13] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, *Phys. Rev. X* **8**, 031079 (2018).
- [14] H. Zhou and J. Y. Lee, *Phys. Rev. B* **99**, 235112 (2019).
- [15] N. Shibata and H. Katsura, *Phys. Rev. B* **99**, 174303 (2019).
- [16] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, *Phys. Rev. Lett.* **121**, 026808 (2018).
- [17] F. K. Kunst and V. Dwivedi, *Phys. Rev. B* **99**, 245116 (2019).
- [18] S. Yao and Z. Wang, *Phys. Rev. Lett.* **121**, 086803 (2018).
- [19] Y. Xiong, *J. Phys. Commun.* **2**, 035043 (2018).
- [20] D. S. Borgnia, A. Jura Kruchkov, and R.-J. Slager, *arXiv:1902.07217*.
- [21] A. Ghatak, M. Brandenbourger, J. van Wezel, and C. Coulais, *arXiv:1907.11619*.
- [22] T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, T. Kiessling, L. W. Molenkamp, C. H. Lee, A. Szameit, M. Greiter, and R. Thomale, *arXiv:1907.11562*.
- [23] C. H. Lee and R. Thomale, *Phys. Rev. B* **99**, 201103(R) (2019).
- [24] M. V. Berry, *Czech. J. Phys.* **54**, 1039 (2004).
- [25] H. Zhou, C. Peng, Y. Yoon, C. W. Hsu, K. A. Nelson, L. Fu, J. D. Joannopoulos, M. Soljačić, and B. Zhen, *Science* **359**, 1009 (2018).
- [26] J. Carlström and E. J. Bergholtz, *Phys. Rev. A* **98**, 042114 (2018).

- [27] J. Carlström, M. Stålhammar, J. C. Budich, and E. J. Bergholtz, *Phys. Rev. B* **99**, 161115(R) (2019).
- [28] C. H. Lee, G. Li, Y. Liu, T. Tai, R. Thomale, and X. Zhang, [arXiv:1812.02011](#).
- [29] W. D. Heiss, *J. Phys. A: Math. Theor.* **45**, 444016 (2012).
- [30] W. Chen, S. Kaya Özdemir, G. Zhao, J. Wiersig, and L. Yang, *Nature* **548**, 192 (2017).
- [31] L. Pan, S. Chen, and X. Cui, *Phys. Rev. A* **99**, 011601(R) (2019).
- [32] L. Pan, S. Chen, and X. Cui, *Phys. Rev. A* **99**, 063616 (2019).
- [33] Y. Xu, S.-T. Wang, and L.-M. Duan, *Phys. Rev. Lett.* **118**, 045701 (2017).
- [34] J. Li, A. K. Harter, J. Liu, L. de Melo, Y. N. Joglekar, and L. Luo, *Nat. Commun.* **10**, 855 (2019).
- [35] T. Tomita, S. Nakajima, Y. Takasu, and Y. Takahashi, *Phys. Rev. A* **99**, 031601(R) (2019).
- [36] Y. Ashida, S. Furukawa, and M. Ueda, *Phys. Rev. A* **94**, 053615 (2016).
- [37] T. Tomita, S. Nakajima, I. Danshita, Y. Takasu, and Y. Takahashi, *Sci. Adv.* **3**, e1701513 (2017).
- [38] E. J. Bergholtz and J. C. Budich, *Phys. Rev. Research* **1**, 012003(R) (2019).
- [39] A. A. Zyuzin and A. Y. Zyuzin, *Phys. Rev. B* **97**, 041203(R) (2018).
- [40] K. Moors, A. A. Zyuzin, A. Y. Zyuzin, R. P. Tiwari, and T. L. Schmidt, *Phys. Rev. B* **99**, 041116(R) (2019).
- [41] T. Yoshida, R. Peters, and N. Kawakami, *Phys. Rev. B* **98**, 035141 (2018).
- [42] T. Yoshida, R. Peters, N. Kawakami, and Y. Hatsugai, *Phys. Rev. B* **99**, 121101(R) (2019).
- [43] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-particle Systems* (Dover Publications, Mineola, NY, 1971).
- [44] J. C. Budich, J. Carlström, F. K. Kunst, and E. J. Bergholtz, *Phys. Rev. B* **99**, 041406(R) (2019).
- [45] R. Okugawa and T. Yokoyama, *Phys. Rev. B* **99**, 041202(R) (2019).
- [46] J. Carlström and E. J. Bergholtz, *Phys. Rev. B* **97**, 161102(R) (2018).
- [47] J. Carlström and E. J. Bergholtz, *Phys. Rev. B* **98**, 241102(R) (2018).
- [48] L. V. Keldysh, *JETP* **20**, 1515 (1965).
- [49] K. V. Houcke, E. Kozik, N. Prokof'ev, and B. Svistunov, *Phys. Procedia* **6**, 95 (2010).
- [50] J. Carlström, *J. Phys.: Condens. Matter* **29**, 385602 (2017).
- [51] J. Carlström, *Phys. Rev. B* **97**, 075119 (2018).
- [52] K. Kawabata, T. Bessho, and M. Sato, *Phys. Rev. Lett.* **123**, 066405 (2019).
- [53] L. Li, C. H. Lee, and J. Gong, *Phys. Rev. B* **100**, 075403 (2019).