Origin of the slow growth of entanglement entropy in long-range interacting spin systems

Alessio Lerose^{1,2} and Silvia Pappalardi^{1,3}¹SISSA—International School for Advanced Studies, via Bonomea 265, I-34136 Trieste, Italy²INFN—Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, I-34136 Trieste, Italy³Abdus Salam ICTP—International Center for Theoretical Physics, Strada Costiera 11, I-34151 Trieste, Italy

(Received 3 June 2019; revised manuscript received 8 January 2020; accepted 10 January 2020; published 19 February 2020)

Long-range interactions allow far-distance quantum correlations to build up very fast. Nevertheless, numerical simulations demonstrated a dramatic slowdown of entanglement entropy growth after a sudden quench. In this work, we unveil the general mechanism underlying this counterintuitive phenomenon for d -dimensional quantum spin systems with slowly decaying interactions. We demonstrate that the semiclassical rate of collective spin squeezing governs the dynamics of entanglement, leading to a universal logarithmic growth in the absence of semiclassical chaos. In fact, the standard quasiparticle contribution is shown to get suppressed as the interaction range is sufficiently increased. All our analytical results agree with numerical computations for quantum Ising chains with long-range couplings. Our findings thus identify a qualitative change in the entanglement production induced by long-range interactions, and they are experimentally relevant for accessing entanglement in highly controllable platforms, including trapped ions, atomic condensates, and cavity-QED systems.

DOI: [10.1103/PhysRevResearch.2.012041](https://doi.org/10.1103/PhysRevResearch.2.012041)

Introduction. Entanglement entropy has been established as a fundamental tool for understanding the out-of-equilibrium dynamics of quantum many-body systems, their thermalization properties, and the complexity of their numerical simulations [1–11]. Significant progress has been achieved for systems with local interactions, where the finite speed of correlation spreading generically determines a linear growth in time of bipartite entanglement entropy [2–4,12–15], unless instances of ergodicity breaking induced by localization or glassy behavior cause a distinguished logarithmic slowdown [6,16–22].

On the other hand, long-range interacting systems—ubiquitous in quantum atomic, molecular, and optical experiments [23–42] and currently the focus of a great deal of attention in quantum many-body theory [43–77]—pose a conceptually different and challenging problem. Despite the fact that their nonlocal interactions allow quantum correlations between distant degrees of freedom to build up very quickly [78–88], several numerical simulations have shown that entanglement entropy growth after a quench features a dramatic counterintuitive slowdown as the range of interactions is increased: It becomes as slow as logarithmic when the couplings decay algebraically with the distances r_{ij} as $r_{ij}^{-\alpha}$, with α smaller than the spatial dimensionality d , even in the absence of disorder [89–91].

In this work, we identify the qualitative change in the mechanism that governs the growth of entanglement in quan-

tum spin systems as the interaction range is increased. By means of a systematic bosonization of spin excitations, we show that the standard quasiparticle contribution to the von

FIG. 1. Entanglement dynamics and collective spin squeezing in long-range quantum spin systems. (a) The system is partitioned into two blocks of N_A and N_B spins-1/2, initially fully polarized. (b) Collective spins of the two blocks. (c) Collective spin in the factorized initial state, represented on a sphere of radius $N/2$. The shaded area represents the quantum uncertainty of transverse components. (d) Nonlinear interactions determine spin squeezing, which makes the two blocks increasingly correlated (entangled). The slow rate of squeezing after noncritical quenches determines the slow growth of entanglement. (e) The analytical formula for spin-squeezing dynamics derived in this work captures the growth of entanglement entropy until saturation (here, quantum quench in a fully connected quantum Ising model).

Neumann entanglement entropy $S(t)$ is suppressed for $\alpha \leq d$ in a prethermal regime. The growth of $S(t)$ is determined by collective spin excitations, directly related to *spin squeezing* [92–98] for spin-1/2 systems, as schematically illustrated in Fig. 1. We demonstrate how a slow dynamical rate of spin squeezing after a quench generically leads to a universal logarithmic growth of $S(t)$ as reported in previous numerical studies. The theory further predicts fast entanglement growth for critical quenches. Analytical results are supported by numerical simulations of long-range quantum Ising chains via exact diagonalization (ED) and the matrix-product-state time-dependent variational principle (MPS-TDVP) [99,100].

Entanglement entropy in infinite-range spin systems. To isolate the contribution of collective degrees of freedom to entanglement dynamics, we start by considering spin-1/2 models with full permutational symmetry. In this limiting situation, the only dynamical spin excitations are collective, i.e., uniform, and quasiparticles at all possible nonvanishing momenta are completely suppressed. This is realized with arbitrary all-to-all multibody interactions, described by a Hamiltonian of the form

$$\hat{H} = - \sum_{p=1,2,\dots} \sum_{\mu_1,\dots,\mu_p} \frac{J_{\mu_1,\dots,\mu_p}}{N^{p-1}} \sum_{j_1 \neq \dots \neq j_p} \hat{s}_{j_1}^{\mu_1} \dots \hat{s}_{j_p}^{\mu_p}, \quad (1)$$

where $\hat{s}_i$, $i = 1, \dots, N$ are quantum spins-1/2 (or qubits). The rescaling factor $1/N^{p-1}$ ensures that the energy contribution of all p -body interactions is extensive. These Hamiltonians can be written in terms of the collective spin of the system $\hat{\mathbf{S}} = \sum_{i=1}^N \hat{\mathbf{s}}_i$ only, whose magnitude $|\mathbf{S}| = \sqrt{S(S+1)}$ is extensive and conserved, and generically maximal ($S = N/2$) in the ground state [101,102], leading to an effective $\hbar_{\text{eff}} \sim \hbar/N$ [103]. The behavior of the system (in and out of equilibrium) can thus be understood via systematic semiclassical expansions in quantum fluctuations around the average collective spin [104,105].

We aim at understanding entanglement dynamics in spin systems described by Eq. (1). For a composite system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ in a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, the entanglement between subsystems A and B can be quantified by the Von Neumann entropy $S_A = -\text{Tr}[\hat{\rho}_A \ln \hat{\rho}_A]$ of the reduced density matrix $\hat{\rho}_A = \text{Tr}_B \hat{\rho}$. In this light, we consider a bipartition of the system into two subsystems A and B with $N_A = f_A N$ and $N_B = N - N_A = f_B N$ spins, respectively. The collective spin $\hat{\mathbf{S}}$ can be correspondingly decomposed as $\hat{\mathbf{S}} = \hat{\mathbf{S}}_A + \hat{\mathbf{S}}_B$ (see Fig. 1). We extend to the nonequilibrium setting the two-boson formalism developed in Refs. [106,107] (see also Ref. [108]) for equilibrium problems. Preparing the system in a fully polarized state, both the instantaneous collective spin configuration $\langle \hat{\mathbf{S}}(t) \rangle$ and its quantum fluctuations evolve in time. This can be described by letting the $\mathbf{Z}$ -axis follow the evolution of $\langle \hat{\mathbf{S}}(t) \rangle$ [109,110]. Quantum fluctuations of the two spins $\hat{\mathbf{S}}_{A,B}$ are bosonized via Holstein-Primakoff transformations around their instantaneous average polarization, and the entanglement between A and B is encoded in the entanglement between these two bosons ($\hat{q}_A, \hat{p}_A$) and ($\hat{q}_B, \hat{p}_B$). The resulting bosonic Hamiltonian governs the dynamics of the transverse quantum fluctuations ($\hat{Q}, \hat{P}$) of the global spin $\hat{\mathbf{S}}$ around its direction $\mathbf{Z}(t)$. The latter encodes the bosonic excitations of $\hat{\mathbf{S}}_A, \hat{\mathbf{S}}_B$ via the relations $\hat{Q} = \sqrt{f_A} \hat{q}_A + \sqrt{f_B} \hat{q}_B$ and $\hat{P} =$

$\sqrt{f_A} \hat{p}_A + \sqrt{f_B} \hat{p}_B$. (See Supplemental Material [104] for details.) We may now safely perform a quadratic approximation, as the spin fluctuations are subextensive in ground states [105] as well as out of equilibrium until the so-called *Ehrenfest time scale*—which diverges with N ; see below. The time evolution of quantum fluctuations ($\hat{Q}, \hat{P}$) is then formally equal to the classical evolution of displacements away from the trajectory of the initial state under consideration.

Within this formalism, the Von Neumann entanglement entropy between the two subsystems A and B is computed with standard Gaussian bosonic techniques [104,111,112], obtaining

$$S_A = \sqrt{1 + 4f_A f_B \langle \hat{n}_{\text{exc}} \rangle} \text{arccoth}(\sqrt{1 + 4f_A f_B \langle \hat{n}_{\text{exc}} \rangle}) + \frac{1}{2} \ln(f_A f_B \langle \hat{n}_{\text{exc}} \rangle), \quad (2)$$

where $\hat{n}_{\text{exc}} = (\hat{Q}^2 + \hat{P}^2 - 1)/2$ represents the number of bosonic excitations of the collective spin $\hat{\mathbf{S}}$. The function on the right-hand side vanishes for $\langle \hat{n}_{\text{exc}} \rangle \rightarrow 0$ and grows as $\frac{1}{2} \ln \langle \hat{n}_{\text{exc}} \rangle$ for $\langle \hat{n}_{\text{exc}} \rangle \gg 1$. Hence, Eq. (2) clarifies that the state of subsystem A (or B) is pure only if $\langle \hat{n}_{\text{exc}} \rangle = 0$, i.e., if the spin system is fully polarized (coherent), as occurs in the absence of interactions. Conversely, the state is entangled in the presence of collective spin excitations. This comes via *spin squeezing*, a concept first introduced in Ref. [113] and usually quantified by the minimal transverse variance of collective spin fluctuations [114,115] as $\xi^2 \equiv \min_{|\mathbf{u}|=1, \mathbf{u} \perp \mathbf{Z}} \langle (\mathbf{u} \cdot \hat{\mathbf{S}})^2 \rangle / (N/4)$. It is equal to 1 for coherent states, and smaller for squeezed states (see, e.g., Refs. [113,115]). At the level of Gaussian fluctuations relevant here, one derives that [104,112]

$$\xi^2 = 1 + 2\langle \hat{n}_{\text{exc}} \rangle - 2\sqrt{\langle \hat{n}_{\text{exc}} \rangle (1 + \langle \hat{n}_{\text{exc}} \rangle)}. \quad (3)$$

It has long been known [93,94,98] that collective spin squeezing is a witness of many-body quantum entanglement. Equations (2) and (3) express the quantitative link—pictorially illustrated in Fig. 1—between the entanglement entropy S_A in collective spin models in and out of equilibrium and the spin-squeezing parameter ξ .

The growth of entanglement entropy out of equilibrium is completely determined by the dynamical generation of the collective excitations $\langle \hat{n}_{\text{exc}}(t) \rangle$, which is in turn induced by the interactions. This rate depends on the nature of the underlying classical dynamics. In fact, the time-dependent quadratic Hamiltonian that governs the *quantum* evolution of fluctuations ($\hat{Q}, \hat{P}$) describes as well the *classical* evolution of small displacements away from it in the linear approximation [116]; see Fig. 2(a) for an illustration. In classical integrable systems, trajectories originating from nearby initial conditions separate linearly in time, which implies $\hat{Q}(t), \hat{P}(t) \sim t$. Since $\hat{n}_{\text{exc}}$ is quadratic in the spin fluctuations $\hat{Q}, \hat{P}$, the collective excitations grow polynomially in time $\langle \hat{n}_{\text{exc}}(t) \rangle \sim t^2$ after a generic quench [103]. This leads to

$$S_A(t) \sim \ln t \quad (4)$$

by Eq. (2). A more rigorous analysis [104] further predicts periodic oscillations of $S_A(t)$ superimposed to the logarithmic growth, the period being that of the underlying classical trajectory.

FIG. 2. Schematic illustration of spin-squeezing dynamics. The classical phase space of a ferromagnetic fully connected Ising model [cf. Eq. (7) with $\alpha = 0$] is pictured on the left. It features ferromagnetic (green) and paramagnetic (blue) periodic trajectories, separated by a critical trajectory (*separatrix*, red). Initial spin-coherent (nonsqueezed) states (a) and (b) at $t = 0$ are represented by points surrounded by small gray circles representing the quantum fluctuations of transverse spin components. Due to nonlinear interactions, the spin state undergoes squeezing, quantified by the parameter $\xi(t)$ [cf. Eq. (3)]. The rate of squeezing is governed by the separation of nearby semiclassical trajectories, and by Eq. (2) it determines the rate of growth of entanglement entropy. Right panels: (a) For generic (noncritical) quenches, nearby trajectories separate linearly in time, leading to a polynomially fast squeezing. (b) For a critical quench, the collective spin lies on the stable manifold of an unstable fixed point in phase space. In this case, nearby trajectories separate exponentially fast in time at a rate λ set by the eigenvalue of the linearized flow.

In the (nongeneric) case of quenches to dynamical critical points [103], the collective spin lies on unstable trajectories (*separatrices*) in the classical phase space, around which displacements grow exponentially in time, with a rate set by the positive eigenvalue λ of the corresponding unstable fixed point on which they terminate; see Fig. 2(b). The out-of-equilibrium generation of collective excitations is thus exponentially fast, $\langle \hat{n}_{\text{exc}}(t) \rangle \sim e^{2\lambda t}$, leading to a linear growth of entanglement entropy with a predicted slope $S(t) \sim \lambda t$ and without superimposed oscillations. (See also the related discussion in Refs. [117–119].) In all cases, entanglement entropy saturates $S_A \sim \ln N_A$ at the Ehrenfest time scale defined by $\langle \hat{n}_{\text{exc}}(t_{\text{Ehr}}) \rangle \sim N$.

Spatially decaying interactions. We are now in a position to understand the effects of having slowly decaying interactions on entanglement dynamics. For the sake of definiteness, we focus on periodic d -dimensional lattices of spins with arbitrary *two-body* long-range interactions, described by a Hamiltonian as in Eq. (1) with $p \leq 2$, where now $j = 1, \dots, N = L^d$ label lattice sites at positions denoted $\mathbf{r}_j$, and the uniform couplings $J_{\mu\nu}$, with $\mu, \nu = x, y, z$, are replaced by $J_{\mu\nu}/|\mathbf{r}_i - \mathbf{r}_j|^\alpha$ [120]. The exponent $\alpha \geq 0$ characterizes the algebraic decay of spin-spin interactions. A Kac rescaling factor $1/\mathcal{N}_{\alpha,N}$ with $\mathcal{N}_{\alpha,N} = \sum_{i \neq j} |\mathbf{r}_i - \mathbf{r}_j|^{-\alpha}/N$ replaces the $1/N$ factor in Eq. (1), ensuring the extensivity of the Hamiltonian for $\alpha \leq d$ [121]. The fully connected limit is recovered by letting $\alpha \rightarrow 0$.

When interactions decay with the distance between spins, the full permutational symmetry of the infinite-range

Hamiltonian (1) is broken and the finite-wavelength spin modes participate in the dynamics. These excitations now allow the system to explore the full Hilbert space beyond the Dicke manifold, i.e., inside the Bloch sphere, and the system may be expected to thermalize by accumulating extensive entanglement entropy [5]. However, we demonstrate that these quasiparticles are weakly excited in typical quenches, and hence yield only a bounded contribution to entanglement growth for a long temporal regime. In this case, the analysis of dynamical spin squeezing captures the leading behavior of $S(t)$ even for $0 < \alpha < d$.

The two-boson approach to entanglement dynamics described above was strictly based on the conservation of the collective spin magnitude. To treat a truly many-body problem, one can refine the approach as follows. We rewrite the Hamiltonian in terms of the Fourier spin modes $\hat{s}_{\mathbf{k}}^{x,y,z} = \sum_j e^{-i\mathbf{k} \cdot \mathbf{r}_j} \hat{s}_j^{x,y,z}$, with $\mathbf{k} = (2\pi/L)(n_1, \dots, n_d)$, $n_\mu = 0, 1, \dots, L-1$ varying in the d -dimensional Brillouin zone, and bosonize individual spin excitations $\hat{s}_j \rightarrow (\hat{q}_j, \hat{p}_j)$ around the instantaneous direction $\mathbf{Z}(t)$ of the collective spin $\langle \hat{\mathbf{S}}(t) \rangle$ via Holstein-Primakoff transformations [104,109,110]. One thus obtains a *spin-wave* expansion of the time-dependent frame Hamiltonian $\tilde{H}(t) = \hat{H} - \omega(t) \cdot \hat{\mathbf{S}}$ in terms of $\tilde{q}_{\mathbf{k}} = L^{-d/2} \sum_j e^{-i\mathbf{k} \cdot \mathbf{r}_j} \hat{q}_j$ and $\tilde{p}_{\mathbf{k}} = L^{-d/2} \sum_j e^{-i\mathbf{k} \cdot \mathbf{r}_j} \hat{p}_j$ with all possible momenta $\mathbf{k}$, expressed up to $O(1/\sqrt{N})$ terms as [122]

$$\begin{aligned} \tilde{H}(t) \simeq \tilde{H}_0(t) - \sum_{\mathbf{k} \neq 0} \tilde{f}_{\alpha,\mathbf{k}} \left[J_{qq}(t) \frac{\tilde{q}_{\mathbf{k}} \tilde{q}_{-\mathbf{k}}}{2} + J_{pp}(t) \frac{\tilde{p}_{\mathbf{k}} \tilde{p}_{-\mathbf{k}}}{2} \right. \\ \left. + J_{qp}(t) \frac{\tilde{q}_{\mathbf{k}} \tilde{p}_{-\mathbf{k}} + \tilde{p}_{\mathbf{k}} \tilde{q}_{-\mathbf{k}}}{2} \right], \end{aligned} \quad (5)$$

where $\tilde{f}_{\alpha,\mathbf{k}} = \frac{1}{\mathcal{N}_{\alpha,N}} \sum_{j(\neq i)} \frac{e^{-i\mathbf{k} \cdot (\mathbf{r}_j - \mathbf{r}_i)}}{|\mathbf{r}_j - \mathbf{r}_i|^\alpha}$ is proportional to the Fourier transform of the interactions, J 's are coefficients depending on the collective spin trajectory, and the collective-mode Hamiltonian $\tilde{H}_0(t)$ accounts for the infinite-range part $\tilde{f}_{\alpha,0} \delta_{\mathbf{k},0} \equiv \delta_{\mathbf{k},0}$ of the interaction $\tilde{f}_{\alpha,\mathbf{k}}$. $\tilde{H}_0(t)$ is independent of α and describes the dynamics of collective spin fluctuations $\tilde{Q} \equiv \tilde{q}_0$ and $\tilde{P} \equiv \tilde{p}_0$ as detailed above, and conserves the bosonic occupation numbers $\hat{n}_{\mathbf{k} \neq 0} \equiv (\tilde{q}_{\mathbf{k}} \tilde{q}_{-\mathbf{k}} + \tilde{p}_{\mathbf{k}} \tilde{p}_{-\mathbf{k}} - 1)/2$ of all the spin-wave modes with finite wavelength [123]. The dynamical excitation of spin waves with finite wavelengths for $\alpha > 0$ is responsible for modifications to the spin-squeezing-induced entanglement entropy growth. As is evident in Eq. (5), their impact is controlled by the strength of the finite-range part $\tilde{f}_{\alpha,\mathbf{k} \neq 0}$ of the interaction. In fact, the following estimate can be derived for $\alpha < d$ [104]: $|\tilde{f}_{\alpha,\mathbf{k} \neq 0}| \leq \text{const} \times \frac{1}{(|\mathbf{k}|L)^\beta}$, with $\beta \equiv \min(d - \alpha, (d + 1)/2)$ (for $\alpha = d$ the power law is replaced by a logarithm). This bound implies that for all fixed $\mathbf{k} \neq 0$, the coupling $\tilde{f}_{\alpha,\mathbf{k}}$ is vanishingly small in the thermodynamic limit $L \rightarrow \infty$ whenever $\alpha \leq d$, and hence the associated number of bosonic excitations is an approximate constant of motion,

$$|[\hat{n}_{\mathbf{k} \neq 0}, \tilde{H}(t)]| \leq \frac{\text{const}}{(|\mathbf{k}|L)^\beta}. \quad (6)$$

Therefore, there exists a long time scale $T_{\text{sw}} \sim N^{\beta/d}$, during which the dynamical excitation of spin waves with finite wavelengths is suppressed [124] (note the interesting relation

to the *prethermalization* time in Ref. [125]). On the other hand, permutational symmetry may severely break over large length scales via excitations with $|\mathbf{k}| \propto 1/L$. Their dynamics governed by the Hamiltonian (5) is equivalent to a *discrete* set of periodically driven quantum oscillators, the drive being induced by the precession of the collective spin. From a stability analysis, we find that for typical quenches these long-wavelength spin-wave modes are nonresonantly driven and hence weakly excited. Their resulting contribution to entanglement dynamics amounts to bounded oscillations on top of the dominant, spin-squeezing-induced logarithmic growth. Near dynamical criticality, however, resonant excitation of these modes may lead to exponentially growing quantum fluctuations (cf. Ref. [125]) and hence a linear increase of the entanglement entropy (see Supplemental Material [104] for details). We thus conclude that long-range interacting spin-1/2 systems with $\alpha < d$ typically exhibit logarithmic growth of entanglement entropy.

Numerical simulations. We test our analytical understanding in paradigmatic one-dimensional long-range quantum Ising chains, described by the Hamiltonian

$$\hat{H} = -\frac{J}{\mathcal{N}_{\alpha,N}} \sum_{i < j}^N \frac{\hat{\sigma}_i^x \hat{\sigma}_j^x}{|i - j|^\alpha} - h \sum_i^N \hat{\sigma}_i^z, \quad (7)$$

where $i, j = 1, \dots, N$, $\hat{\sigma}_i^{x,z}$ are Pauli matrices, h is a global transverse magnetic field, and $\mathcal{N}_{\alpha,N}$ is the Kac rescaling factor introduced above.

We compare the numerical computations of entanglement entropy evolution at finite N with the analytical calculation of the spin-squeezing contribution [Eq. (2)] and with the full spin-wave calculation, obtained from Eq. (5) via standard bosonic techniques [104,111,112,119]. For the sake of illustration, we focus here on the initial state $|\psi_0\rangle = |\rightarrow \rightarrow \dots \rightarrow\rangle$, i.e., on quenches in the transverse field from $h_0 = 0$ to h_f . As Figs. 1 and 3 show, in all cases the numerical data are captured by the corresponding analytical curves for $t \lesssim t_{\text{Ehr}}(N)$. In the fully connected limit $\alpha \rightarrow 0$, equivalent to the Lipkin-Meshov-Glick model [126], Eq. (2) is exact in the thermodynamic limit and the finite-size ED data perfectly match it before saturation at the Ehrenfest time, $t_{\text{Ehr}} \sim \sqrt{N}$ for generic quenches [$h_0 = 0 \rightarrow h_f = 2J$ in Fig. 1 (bottom) and $h_f = 0.2J$ in Fig. 3(a)] and $t_{\text{Ehr}} \sim \ln N$ for the critical quench [$h_f = h_c \equiv J/2$ in Fig. 3(b), cf. the red line in Fig. 2(b)], corresponding to the *dynamical phase transition* of the model [35,103,127,128]. For spatially decaying interactions with $0 < \alpha < 1$, we employ the MPS-TDVP [99,100] with periodic boundary conditions (see Supplemental Material [104] for details). Upon increasing N , the TDVP data approach the full spin-wave entanglement entropy, for all considered values of α and quench parameters, as shown in the examples in Figs. 3(c) and 3(d). This analysis confirms that the growth of $S(t)$ is logarithmic for typical initial configurations. For further discussion, including varying initial states, bipartition sizes, and details on the spin-wave analysis, see Supplemental Material [104].

Outlook. The mechanism unveiled in this work complements the available paradigms for entanglement dynamics characterizing systems with local interactions [2–4,12–15] and, at the same time, improves the current understanding

FIG. 3. Comparison between theory and numerical results for long-range quantum Ising chains. Quenches in the transverse field $h_0 = 0 \rightarrow h_f > 0$ are considered, and the evolution of the half-system entanglement entropy $S_{N/2}(t)$ is shown. Top: fully connected limit with $\alpha = 0$. Analytical results (black lines) are compared with ED data for increasing system sizes $N = 20$ –800. (a) For a noncritical quench $h_0 = 0 \rightarrow h_f = 0.2J$, the growth of $S_{N/2}(t)$ is logarithmic up to saturation around $1/2 \ln N$ at $t_{\text{Ehr}} \sim \sqrt{N}$. (b) For the critical quench $h_f = h_c = J/2$, the growth of $S_{N/2}(t)$ is linear until $t_{\text{Ehr}} \sim \ln N$, with a slope $\lambda_{h_c} = J$. Bottom: deep quench with $h_f = 2J$ in long-range interacting chains with $\alpha > 0$. The contribution due to collective spin squeezing [Eq. (2)] and the full spin-wave calculation of the time-dependent entanglement (see the main text) are compared with MPS-TDVP data for $N = 20$ –80 converged with bond dimension $D = 128$, for $\alpha = 0.1$ (c) and $\alpha = 0.7$ (d). As α increases, finite corrections due to long-wavelength spin waves appear on top of the dominant spin-squeezing-induced logarithmic growth; see the inset.

of the efficiency of “classical” simulations of quantum long-range interacting spin chains with matrix-product-state techniques [99,100]. Similar approaches can be used to characterize entanglement entropy dynamics in more general many-body systems [103], including higher-spin systems, whose semiclassical dynamics may display chaos and hence fast entanglement growth for generic quenches. In connection with known instances of ergodicity breaking stemming from long-range interactions [31,44,45,73,76,77,89,90,125,128–130], the suppression of spin-wave excitations clarifies the role of long-range interactions in constraining quantum dynamics to small portions of the many-body Hilbert space in the relevant time regime. Concerning the slow entanglement growth, this is reminiscent of localized or glassy dynamics [18–22].

The bridge between bipartite entanglement entropy and other measures of multiparticle entanglement, most notably spin squeezing [92–98,131,132], may help to elucidate the relation between bipartite and multiparticle entanglement,

making the latter an equivalent metrologically useful measure for systems with long-range interactions. Most importantly, this connection has direct experimental relevance for the detection of entanglement and its dynamics via measurements of collective quantities [133,134], the experimental accessibility of which is well-established with standard techniques and tools of quantum atomic experiments, ranging from spinor condensates and cavity-QED systems to trapped ions [132,135,136].

Acknowledgments. We gratefully thank P. Calabrese, M. Dalmonte, R. Fazio, and J. Goold for insightful comments on the manuscript, G. Giudici for sharing with us his TDVP code, and A. De Luca, J. De Nardis, A. V. Gorshkov, P. Hauke, J. G. Hirsch, J. Marino, G. Morigi, C. Murthy, G. Pagano, S. Parameswaran, A. M. Rey, L. F. Santos, A. Silva, P. Titum, Q. Zhuang, and B. Žunkovič for interesting discussions on the subject matter of this work.

-
- [1] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, *Rev. Mod. Phys.* **80**, 517 (2008).
  - [2] P. Calabrese and J. Cardy, *J. Stat. Mech.: Theor. Exp.* (2004) P06002.
  - [3] P. Calabrese and J. Cardy, *J. Stat. Mech.: Theor. Exp.* (2005) P04010.
  - [4] V. Alba and P. Calabrese, *Sci. Post. Phys.* **4**, 017 (2018).
  - [5] V. Alba and P. Calabrese, *Proc. Natl. Acad. Sci. (USA)* **114**, 7947 (2017).
  - [6] M. Serbyn, Z. Papić, and D. A. Abanin, *Phys. Rev. Lett.* **110**, 260601 (2013).
  - [7] D. A. Huse, R. Nandkishore, and V. Oganesyan, *Phys. Rev. B* **90**, 174202 (2014).
  - [8] G. Vidal, *Phys. Rev. Lett.* **91**, 147902 (2003).
  - [9] U. Schollwöck and S. R. White, in *Effective Models for Low-Dimensional Strongly Correlated Systems*, edited by G. G. Batrouni and D. Poilblanc, AIP Conf. Proc. No. 816 (AIP, New York, 2006), pp. 155–185.
  - [10] F. Verstraete, V. Murg, and J. I. Cirac, *Adv. Phys.* **57**, 143 (2008).
  - [11] M. C. Bañuls, M. B. Hastings, F. Verstraete, and J. I. Cirac, *Phys. Rev. Lett.* **102**, 240603 (2009).
  - [12] M. Fagotti and P. Calabrese, *Phys. Rev. A* **78**, 010306(R) (2008).
  - [13] H. Kim and D. A. Huse, *Phys. Rev. Lett.* **111**, 127205 (2013).
  - [14] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, *Phys. Rev. X* **7**, 031016 (2017).
  - [15] A. Chan, A. De Luca, and J. T. Chalker, *Phys. Rev. X* **8**, 041019 (2018).
  - [16] M. Žnidarič, T. Prosen, and P. Prelovšek, *Phys. Rev. B* **77**, 064426 (2008).
  - [17] J. H. Bardarson, F. Pollmann, and J. E. Moore, *Phys. Rev. Lett.* **109**, 017202 (2012).
  - [18] N. Y. Yao, C. R. Laumann, J. I. Cirac, M. D. Lukin, and J. E. Moore, *Phys. Rev. Lett.* **117**, 240601 (2016).
  - [19] A. A. Michailidis, M. Žnidarič, M. Medvedyeva, D. A. Abanin, T. Prosen, and Z. Papić, *Phys. Rev. B* **97**, 104307 (2018).
  - [20] A. Smith, J. Knolle, R. Moessner, and D. L. Kovrizhin, *Phys. Rev. Lett.* **119**, 176601 (2017).
  - [21] A. Smith, J. Knolle, R. Moessner, and D. L. Kovrizhin, *Phys. Rev. B* **97**, 245137 (2018).
  - [22] M. Brenes, M. Dalmonte, M. Heyl, and A. Scardicchio, *Phys. Rev. Lett.* **120**, 030601 (2018).
  - [23] L. Béguin, A. Vernier, R. Chicireanu, T. Lahaye, and A. Browaeys, *Phys. Rev. Lett.* **110**, 263201 (2013).
  - [24] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, *et al.*, *Nature (London)* **551**, 579 (2017).
  - [25] A. Browaeys, D. Barredo, and T. Lahaye, *J. Phys. B* **49**, 152001 (2016).
  - [26] H. Labuhn, D. Barredo, S. Ravets, S. de Léséleuc, T. Macrì, T. Lahaye, and A. Browaeys, *Nature (London)* **534**, 667 (2016).
  - [27] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, *Phys. Rev. Lett.* **95**, 010402 (2005).
  - [28] H. Häffner, C. F. Roos, and R. Blatt, *Phys. Rep.* **469**, 155 (2008).
  - [29] P. Jurcevic, H. Shen, P. Hauke, C. Maier, T. Brydges, C. Hempel, B. P. Lanyon, M. Heyl, R. Blatt, and C. F. Roos, *Phys. Rev. Lett.* **119**, 080501 (2017).
  - [30] R. Blatt and C. F. Roos, *Nat. Phys.* **8**, 277 (2012).
  - [31] B. Neyenhuis, J. Zhang, P. W. Hess, J. Smith, A. C. Lee, P. Richerme, Z. Gong, A. V. Gorshkov, and C. Monroe, *Sci. Adv.* **3**, e1700672 (2017).
  - [32] J. W. Britton, B. C. Sawyer, A. C. Keith, C.-C. J. Wang, J. K. Freericks, H. Uys, M. J. Biercuk, and J. J. Bollinger, *Nature (London)* **484**, 489 (2012).
  - [33] P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakakis, A. V. Gorshkov, and C. Monroe, *Nature (London)* **511**, 198 (2014).
  - [34] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, and C. F. Roos, *Nature (London)* **511**, 202 (2014).
  - [35] J. Zhang, G. Pagano, P. W. Hess, A. Kyprianidis, P. Becker, H. Kaplan, A. V. Gorshkov, Z.-X. Gong, and C. Monroe, *Nature (London)* **551**, 601 (2017).
  - [36] G. Pagano, P. W. Hess, H. B. Kaplan, W. L. Tan, P. Richerme, P. Becker, A. Kyprianidis, J. Zhang, E. Birkelbaw, M. R. Hernandez, Y. Wu, and C. Monroe, *Quantum Sci. Technol.* **4**, 014004 (2018).
  - [37] K.-K. Ni, S. Ospelkaus, M. H. G. de Miranda, A. Pe'er, B. Neyenhuis, J. J. Zirbel, S. Kotochigova, P. S. Julienne, D. S. Jin, and J. Ye, *Science* **322**, 231 (2008).
  - [38] S. A. Moses, J. P. Covey, M. T. Miecnikowski, D. S. Jin, and J. Ye, *Nat. Phys.* **13**, 13 (2016).
  - [39] G. Balasubramanian *et al.*, *Nat. Mater.* **8**, 383 (2009).
  - [40] A. de Paz, A. Sharma, A. Chotia, E. Maréchal, J. H. Huckans, P. Pedri, L. Santos, O. Gorceix, L. Vernac, and B. Laburthe-Tolra, *Phys. Rev. Lett.* **111**, 185305 (2013).
  - [41] A. T. Black, H. W. Chan, and V. Vuletić, *Phys. Rev. Lett.* **91**, 203001 (2003).
  - [42] I. D. Leroux, M. H. Schleier-Smith, and V. Vuletić, *Phys. Rev. Lett.* **104**, 073602 (2010).

- [43] Z.-X. Gong, M. Foss-Feig, F. G. S. L. Brandão, and A. V. Gorshkov, *Phys. Rev. Lett.* **119**, 050501 (2017).
- [44] T. Koffel, M. Lewenstein, and L. Tagliacozzo, *Phys. Rev. Lett.* **109**, 267203 (2012).
- [45] D. Vodola, L. Lepori, E. Ercolessi, A. V. Gorshkov, and G. Pupillo, *Phys. Rev. Lett.* **113**, 156402 (2014).
- [46] I. Frérot, P. Naldesi, and T. Roscilde, *Phys. Rev. B* **95**, 245111 (2017).
- [47] M. L. Wall, A. Safavi-Naini, and A. M. Rey, *Phys. Rev. A* **95**, 013602 (2017).
- [48] D. Dylewsky, J. K. Freericks, M. L. Wall, A. M. Rey, and M. Foss-Feig, *Phys. Rev. A* **93**, 013415 (2016).
- [49] N. Defenu, A. Trombettoni, and S. Ruffo, *Phys. Rev. B* **96**, 104432 (2017).
- [50] S. Fey and K. P. Schmidt, *Phys. Rev. B* **94**, 075156 (2016).
- [51] D. Jaschke, K. Maeda, J. D. Whalen, M. L. Wall, and L. D. Carr, *New J. Phys.* **19**, 033032 (2017).
- [52] L. Vanderstraeten, M. Van Damme, H. P. Büchler, and F. Verstraete, *Phys. Rev. Lett.* **121**, 090603 (2018).
- [53] N. Defenu, T. Enss, M. Kastner, and G. Morigi, *Phys. Rev. Lett.* **121**, 240403 (2018).
- [54] N. Shammah, S. Ahmed, N. Lambert, S. De Liberato, and F. Nori, *Phys. Rev. A* **98**, 063815 (2018).
- [55] F. Machado, G. D. Kahanamoku-Meyer, D. V. Else, C. Nayak, and N. Y. Yao, *Phys. Rev. Res.* **1**, 033202 (2019).
- [56] K. R. Fratus and M. Srednicki, [arXiv:1611.03992](https://arxiv.org/abs/1611.03992).
- [57] O. Viyuela, L. Fu, and M. A. Martin-Delgado, *Phys. Rev. Lett.* **120**, 017001 (2018).
- [58] A. Lerose, J. Marino, A. Gambassi, and A. Silva, *Phys. Rev. B* **100**, 104306 (2019).
- [59] N. Y. Yao, C. R. Laumann, S. Gopalakrishnan, M. Knap, M. Müller, E. A. Demler, and M. D. Lukin, *Phys. Rev. Lett.* **113**, 243002 (2014).
- [60] R. Singh, R. Moessner, and D. Roy, *Phys. Rev. B* **95**, 094205 (2017).
- [61] P. Hauke and M. Heyl, *Phys. Rev. B* **92**, 134204 (2015).
- [62] A. L. Burin, *Phys. Rev. B* **92**, 104428 (2015).
- [63] G. L. Celardo, R. Kaiser, and F. Borgonovi, *Phys. Rev. B* **94**, 144206 (2016).
- [64] K. S. Tikhonov and A. D. Mirlin, *Phys. Rev. B* **97**, 214205 (2018).
- [65] P. Sierant, K. Biedroń, G. Morigi, and J. Zakrzewski, *SciPost. Phys.* **7**, 008 (2019).
- [66] S. Roy and D. E. Logan, *SciPost. Phys.* **7**, 042 (2019).
- [67] S. Nag and A. Garg, *Phys. Rev. B* **99**, 224203 (2019).
- [68] R. Modak and T. Nag, [arXiv:1903.05099](https://arxiv.org/abs/1903.05099).
- [69] A. O. Maksymov and A. L. Burin, *Phys. Rev. B* **101**, 024201 (2020).
- [70] R. M. Nandkishore and S. L. Sondhi, *Phys. Rev. X* **7**, 041021 (2017).
- [71] A. Russomanno, F. Iemini, M. Dalmonte, and R. Fazio, *Phys. Rev. B* **95**, 214307 (2017).
- [72] I. Homrighausen, N. O. Abeling, V. Zauner-Stauber, and J. C. Halimeh, *Phys. Rev. B* **96**, 104436 (2017).
- [73] J. C. Halimeh and V. Zauner-Stauber, *Phys. Rev. B* **96**, 134427 (2017).
- [74] M. G. Nezhadhighi and M. A. Rajabpour, *Phys. Rev. B* **90**, 205438 (2014).
- [75] L. F. Santos, F. Borgonovi, and G. L. Celardo, *Phys. Rev. Lett.* **116**, 250402 (2016).
- [76] F. Liu, R. Lundgren, P. Titum, G. Pagano, J. Zhang, C. Monroe, and A. V. Gorshkov, *Phys. Rev. Lett.* **122**, 150601 (2019).
- [77] A. Lerose, B. Žunkovič, A. Silva, and A. Gambassi, *Phys. Rev. B* **99**, 121112(R) (2019).
- [78] Z.-X. Gong, M. Foss-Feig, S. Michalakakis, and A. V. Gorshkov, *Phys. Rev. Lett.* **113**, 030602 (2014).
- [79] M. Foss-Feig, Z.-X. Gong, C. W. Clark, and A. V. Gorshkov, *Phys. Rev. Lett.* **114**, 157201 (2015).
- [80] T. Matsuta, T. Koma, and S. Nakamura, *Ann. Henri Poincaré* **18**, 519 (2017).
- [81] D. V. Else, F. Machado, C. Nayak, and N. Y. Yao, [arXiv:1809.06369](https://arxiv.org/abs/1809.06369).
- [82] P. Hauke and L. Tagliacozzo, *Phys. Rev. Lett.* **111**, 207202 (2013).
- [83] J. Eisert, M. van den Worm, S. R. Manmana, and M. Kastner, *Phys. Rev. Lett.* **111**, 260401 (2013).
- [84] L. Lepori, A. Trombettoni, and D. Vodola, *J. Stat. Mech.: Theor. Exp.* (2017) 033102.
- [85] L. Cevolani, G. Carleo, and L. Sanchez-Palencia, *Phys. Rev. A* **92**, 041603(R) (2015).
- [86] L. Cevolani, G. Carleo, and L. Sanchez-Palencia, *New J. Phys.* **18**, 093002 (2016).
- [87] M. Van Regemortel, D. Sels, and M. Wouters, *Phys. Rev. A* **93**, 032311 (2016).
- [88] I. Frérot, P. Naldesi, and T. Roscilde, *Phys. Rev. Lett.* **120**, 050401 (2018).
- [89] J. Schachenmayer, B. P. Lanyon, C. F. Roos, and A. J. Daley, *Phys. Rev. X* **3**, 031015 (2013).
- [90] A. S. Buyskikh, M. Fagotti, J. Schachenmayer, F. Essler, and A. J. Daley, *Phys. Rev. A* **93**, 053620 (2016).
- [91] S. Pappalardi, A. Russomanno, B. Žunkovič, F. Iemini, A. Silva, and R. Fazio, *Phys. Rev. B* **98**, 134303 (2018).
- [92] M. Kitagawa and M. Ueda, *Phys. Rev. A* **47**, 5138 (1993).
- [93] A. S. Sørensen and K. Mølmer, *Phys. Rev. Lett.* **86**, 4431 (2001).
- [94] A. Sørensen, L.-M. Duan, J. I. Cirac, and P. Zoller, *Nature (London)* **409**, 63 (2001).
- [95] G. Tóth, C. Knapp, O. Gühne, and H. J. Briegel, *Phys. Rev. Lett.* **99**, 250405 (2007).
- [96] K. R. A. Hazzard, M. van den Worm, M. Foss-Feig, S. R. Manmana, E. G. Dalla Torre, T. Pfau, M. Kastner, and A. M. Rey, *Phys. Rev. A* **90**, 063622 (2014).
- [97] M. Foss-Feig, Z.-X. Gong, A. V. Gorshkov, and C. W. Clark, [arXiv:1612.07805](https://arxiv.org/abs/1612.07805).
- [98] L. Pezzé and A. Smerzi, *Phys. Rev. Lett.* **102**, 100401 (2009).
- [99] J. Haegeman, C. Lubich, I. Oseledets, B. Vandereycken, and F. Verstraete, *Phys. Rev. B* **94**, 165116 (2016).
- [100] J. Haegeman, J. I. Cirac, T. J. Osborne, I. Pižorn, H. Verschelde, and F. Verstraete, *Phys. Rev. Lett.* **107**, 070601 (2011).
- [101] J. Vidal, R. Mosseri, and J. Dukelsky, *Phys. Rev. A* **69**, 054101 (2004).
- [102] Even at quantum critical points: see, e.g., Refs. [137,138].
- [103] B. Sciolla and G. Biroli, *J. Stat. Mech.: Theor. Exp.* (2011) P11003.
- [104] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevResearch.2.012041> for details.

- [105] This applies to higher spin systems described by Eq. (1) as well. However, in this case, more general permutationally symmetric interactions give rise to a description in terms of a larger set of semiclassical variables. See the related comment in the conclusions of the paper.
- [106] S. Dusuel, J. Vidal, and T. Barthel, *J. Stat. Mech.: Theor. Exp.* (2007) P01015.
- [107] T. Barthel, S. Dusuel, and J. Vidal, *Phys. Rev. Lett.* **97**, 220402 (2006).
- [108] M. A. Metlitski and T. Grover, [arXiv:1112.5166](https://arxiv.org/abs/1112.5166).
- [109] A. Leroise, J. Marino, B. Žunkovič, A. Gambassi, and A. Silva, *Phys. Rev. Lett.* **120**, 130603 (2018).
- [110] A. Leroise, B. Žunkovič, J. Marino, A. Gambassi, and A. Silva, *Phys. Rev. B* **99**, 045128 (2019).
- [111] T. Barthel, M.-C. Chung, and U. Schollwöck, *Phys. Rev. A* **74**, 022329 (2006).
- [112] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, *Rev. Mod. Phys.* **84**, 621 (2012).
- [113] T. Hiroshima, *Phys. Rev. A* **63**, 022305 (2001).
- [114] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, *Phys. Rev. A* **46**, R6797 (1992).
- [115] J. Ma, X. Wang, C. Sun, and F. Nori, *Phys. Rep.* **509**, 89 (2011).
- [116] R. G. Littlejohn, *Phys. Rep.* **138**, 193 (1986).
- [117] W. H. Zurek and J. P. Paz, *Physica D* **83**, 300 (1995).
- [118] E. Bianchi, L. Hackl, and N. Yokomizo, *J. High Energy Phys.* **03** (2018) 025.
- [119] L. Hackl, E. Bianchi, R. Modak, and M. Rigol, *Phys. Rev. A* **97**, 032321 (2018).
- [120] The distance on the lattice $\mathbf{r}_i - \mathbf{r}_j$ is thus taken to be $\sqrt{\sum_{\mu=1}^d [\min(|r_i^\mu - r_j^\mu|, L - |r_i^\mu - r_j^\mu|)]^2}$.
- [121] M. Kac and C. J. Thompson, *J. Math. Phys.* **10**, 1373 (1969).
- [122] In this case, the dynamical generation of spin waves modifies the classical trajectory of the collective spin, due to the feedback from quantum fluctuations. This effect is mostly negligible here, except for critical quenches [109,110].
- [123] Note that this is rigorously true to all orders in the Holstein-Primakoff expansion.
- [124] Note that this justifies a posteriori the Holstein-Primakoff approach, as the density of spin waves remains small over a long time window.
- [125] T. Mori, *J. Phys. A: Math. Theor.* **52**, 054001 (2019).
- [126] H. Lipkin, N. Meshkov, and A. Glick, *Nucl. Phys.* **62**, 188 (1965).
- [127] B. Žunkovič, A. Silva, and M. Fabrizio, *Philos. Trans. R. Soc. A* **374**, 20150160 (2016).
- [128] B. Žunkovič, M. Heyl, M. Knap, and A. Silva, *Phys. Rev. Lett.* **120**, 130601 (2018).
- [129] D. Mukamel, S. Ruffo, and N. Schreiber, *Phys. Rev. Lett.* **95**, 240604 (2005).
- [130] Z.-X. Gong and L.-M. Duan, *New J. Phys.* **15**, 113051 (2013).
- [131] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wiczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, *Phys. Rev. A* **85**, 022321 (2012).
- [132] H. Strobel, W. Muessel, D. Linnemann, T. Zibold, D. B. Hume, L. Pezzé, A. Smerzi, and M. K. Oberthaler, *Science* **345**, 424 (2014).
- [133] I. Klich, G. Refael, and A. Silva, *Phys. Rev. A* **74**, 032306 (2006).
- [134] S. Ghose, R. Stock, P. Jessen, R. Lal, and A. Silberfarb, *Phys. Rev. A* **78**, 042318 (2008).
- [135] J. G. Bohnet, B. C. Sawyer, J. W. Britton, M. L. Wall, A. M. Rey, M. Foss-Feig, and J. J. Bollinger, *Science* **352**, 1297 (2016).
- [136] M. Gärttner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Bollinger, and A. M. Rey, *Nat. Phys.* **13**, 781 (2017).
- [137] S. Dusuel and J. Vidal, *Phys. Rev. Lett.* **93**, 237204 (2004).
- [138] S. Dusuel and J. Vidal, *Phys. Rev. B* **71**, 224420 (2005).