Mapping activation of resources among upper division physics students in non-Cartesian coordinate systems: A case study

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An essential skill for success in upper-division physics curricula is the ability to work with and apply mathematical and physical concepts through Cartesian and non-Cartesian coordinate systems. These skills are most notably necessary in electricity and magnetism, wherein students must build and solve integrals and perform vector derivatives in Cartesian, spherical, and cylindrical coordinate systems. While limited in scope, early investigations into students understanding of coordinate systems has shown that students have considerable difficulty. Through the analysis of four one-on-one semistructured interviews, students were observed to activate numerous productive resources, despite frequently incorrect answers. This paper will revisit our previously identified student resource clusters for unit vectors. We also identify additional resources that cast additional light on students' thinking about position vectors, and velocity vectors in non-Cartesian coordinate systems. Through detailed analysis of a single student, we identify several groupings of resources that will serve as a baseline for future analysis of additional students and provide initial insight into potential curriculum development.

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I. INTRODUCTION

A. Background and purpose

Using non-Cartesian coordinate systems continues to be difficult for undergraduate physics students [1–3] even in upper-division physics courses (e.g., classical mechanics, electromagnetism, thermodynamics, quantum mechanics, statistical mechanics) where application of these mathematical ideas plays a more significant role [4–7]. As part of a larger initiative into exploring the math-physics interface in upper-division physics courses [8,9], our current collaboration has initiated the development of a research-based curriculum for a mathematical methods course for undergraduate physics majors. This paper discusses a portion of that effort by focusing on the identification of cognitive resources students activate in the context of basis unit vectors, position vectors, and

velocity vectors in plane polar and spherical polar coordinate systems.

The polar coordinate system, both planar and spherical, is commonly used in a number of upper-division physics courses, including mechanics, electrodynamics, and quantum mechanics [4–7,10]. Despite this focus, research has shown that students struggle using these coordinate systems. Sayre and Wittmann [11] found that students did not always recognize when applying polar coordinates would be advantageous. They also found that while Cartesian coordinate understanding is typically quite solid, polar coordinate understanding is less robust and still in the formation process. Paoletti *et al.* [3] found that students spend the greatest proportion of their math training using Cartesian coordinates, and apply Cartesian coordinate system conventions to polar coordinates. Their findings are consistent with findings by Montiel and various colleagues [1,2] who claim mathematical meaning is often tied to the coordinate system in which a concept is learned. As an example, Montiel *et al.* describe students attempting to apply the vertical line test to determine if an expression is a function in the polar coordinate system. The vertical line test uses a vertical line to sweep across a coordinate plane. If the curve represented on the given plane intersects this

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vertical line at one and only one point for every x value, the curve represents a function. The vertical line test works for Cartesian—the coordinate system in which functions and the vertical line test are initially taught—but frequently fails in polar coordinates. Hinrichs [12] observed similar results while studying spherical unit vectors in the context of position vectors noting students “pattern-matched” position vector notation in spherical coordinates to match that of Cartesian coordinates. This pattern matching resulted in the inclusion of unnecessary unit vectors, which frequently appeared as students writing spherical position vectors in the incorrect (but Cartesian-imitating) form $\vec{r} = r\hat{r} + \theta\hat{\theta} + \phi\hat{\phi}$.

Our initial work on students’ thinking on unit vectors in non-Cartesian coordinate systems categorized student thinking into “clusters” of similar but not identical ways that students thought about unit vectors [13]. Specifically, we found that students conflated *unit vectors at a point* with *position vectors to a point*. Additionally, there was a tendency for students to conflate ideas about the *direction of a unit vector* having to do with either the *location of an object at a point* or with the *direction of motion of an object at a point*. Vega *et al.* [13] further describe a “Unit Vector Cluster” that has essential resources to fit a canonical definition of a unit vector: a unit vector is a vector; has a length of one unit; points in the direction of increasing coordinate; and is dimensionless. The naming of these resources is done for completeness and from an expert perspective, and not all of these resources were identified by the students in the interview sample.

The research presented in this paper used think-aloud interviews to gain additional insight into student thinking about a wider range of vector concepts in non-Cartesian—and specifically planar polar, spherical, and, at times, cylindrical coordinate systems. There are many other non-Cartesian coordinate systems, however, hereafter, when we refer to non-Cartesian coordinates, unit vectors, and coordinate systems we imply only these three. Two interview protocols were developed that included questions about unit vectors, position vectors, and velocity vectors in polar and spherical coordinates and polar and spherical unit vectors. The second protocol included cylindrical coordinates and unit vectors, as well. Four interviews were conducted using the first protocol. Two of the four interview subjects were senior undergraduate physics majors, one was a first-year physics graduate student, and one was a third-year physics graduate student. Using the second protocol, three subjects were interviewed; one subject a junior undergraduate physics major, one a second-year physics graduate student, and the third a physics faculty member. Across the two protocols, a total of three of the seven subjects responded to unit vector questions in a manner consistent with the resource clusters described in our initial work [13] that were not productive for these particular questions. The remaining four subjects answered

the initial unit vector questions in our protocol successfully. Later in their interviews, these four subjects struggled when attempting to apply their productive thinking about non-Cartesian unit vectors in the context of position and velocity vectors.

The responses from the interviewees were nuanced and, in subtle ways, were different across individuals. Responses demonstrated activation of ideas that were likely previously useful for students, and some of these ideas had commonalities across interview subjects, making them suitable for analysis using a resources framework (Sec. 1C). All seven subjects, regardless of their activating productive or nonproductive unit vector resources initially, responded in a similar fashion to questions on position vectors using non-Cartesian coordinates and unit vectors. For example, all subjects included $\hat{\phi}$ and/or $\hat{\theta}$ terms in algebraic expressions for polar and spherical position vectors where only $\hat{r}$ terms are needed. Given the consistency with which all the subjects included such unnecessary terms, identifying resources that students activated when answering questions on position and velocity vectors was determined to be a useful framing for better understanding students’ vector thinking. The work presented here is a case study of a single subject from our interview pool that will be used to present and discuss some of the subject’s thinking and identify that thinking as cognitive resources. Presentation of this case study allows us to build on our prior work for unit vectors, as well as explore the extent to which previously identified resources are appropriate for the context of position and velocity vectors. This essential step builds toward the analysis of our remaining interview sample, wherein we may be able to identify resource clusters for positions vectors and velocity vectors.

B. Mathematics discussion

Non-Cartesian coordinates, e.g., plane polar coordinates (which we will call polar coordinates), polar spherical coordinates (spherical coordinates), and polar cylindrical coordinates, leverage the geometry of a given situation to make mathematical modeling and calculations more tractable, and in some cases to make such calculations possible. Such situations include rotational motion and rotating reference frames [4–6] and electromagnetic fields around variously shaped charge distributions [7]. As such, being able to communicate mathematically about such phenomena in the language of non-Cartesian coordinates and unit vectors is an essential skill for upper-division physics students, as well as professional physicists.

Given the often-nuanced descriptions of the mathematics used in this paper, it is essential to have a clear definition of unit and position vectors in non-Cartesian coordinate systems to establish an operational definition. First, the position vector is defined because it contains many of the mathematical concepts studied in this research. Consider a particle moving in space. At time $t_0 = 0$, the particle is at

FIG. 1. A displacement vector $\vec{d}$ is the directed quantity from some initial point A to some other point B .

point A , and at some time $t > 0$ the particle is at some other point B . A displacement vector, commonly denoted $\vec{d}$, is then defined as the directed quantity from point A to point B (Fig. 1).

A position vector, commonly denoted $\vec{r}$, is defined as a displacement vector where point A is coincident with the origin or pole of some coordinate system (Fig. 2). Thus, the position vector is a vector with its tail at the origin and tip at the coordinates of the point of interest. In Cartesian coordinates, a position vector is typically written in the algebraic form $\vec{r} = a\hat{x} + b\hat{y} + c\hat{z}$ or $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ where a , b , and c are the magnitudes of the x , y , and z components, respectively, and the vectors with circumflexes above them connote unit vectors. A unit vector is a dimensionless vector of length one that points in the direction of the increasing coordinate it connotes.

Position vectors in polar, spherical, and cylindrical coordinates operate differently from position vectors in Cartesian coordinates. Spherical unit vectors are illustrated in Fig. 3 [7].

In the polar coordinate system, the algebraic expression for the position vector $\vec{r}$ terminating at some point P a distance R from the pole would be $\vec{r} = R\hat{r}$, where $\hat{r}$ is a vector of length one that points radially outward from the pole in the direction of increasing radius. Thus, the symbol $\hat{r}$ does not have a static meaning like $\hat{x}$, $\hat{y}$, and $\hat{z}$ and algebraic expressions of polar position vectors are not symbolically unique (Fig. 4). This dynamic behavior of

FIG. 2. A position vector $\vec{r}$ is a displacement vector with tail at the origin and terminal point at the point of interest.

FIG. 3. An illustration of the spherical basis unit vectors from Griffiths' EM textbook [7]. This diagram uses the physics convention of ϕ as the azimuthal angle and θ as the polar angle (the mathematics convention reverses these symbols).

polar unit vectors results from the bases of non-Cartesian coordinate systems being location dependent. Furthermore, the expression $R\hat{r}$ is sufficient to describe a polar position vector because both its magnitude R and its direction $\hat{r}$ are given. Despite this difference in the directional natures of Cartesian and non-Cartesian unit vectors, there is a conversion of Cartesian unit vectors to polar with the transformations,

$$\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}, \quad (1)$$

$$\hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}, \quad (2)$$

where it can be seen the angular dependence accounts for the nonstatic directionality of $\hat{r}$. For spherical coordinates these transformations (with the physics convention of θ as the polar angle and ϕ as the azimuthal) are

$$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}, \quad (3)$$

$$\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}, \quad (4)$$

FIG. 4. In polar coordinates $\vec{r}_A$ and $\vec{r}_B$ have identical-looking algebraic expressions on the right-hand sides of their equations: $\vec{r}_A = 2\hat{r}$, $\vec{r}_B = 2\hat{r}$. But, because their respective $\hat{r}$ unit vectors point in different directions, $\vec{r}_A \neq \vec{r}_B$.

$$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}. \quad (5)$$

These transformations often represent some of students' first exposure to polar and spherical coordinates and their corresponding unit vectors [1–3].

The nonuniqueness of the algebraic expressions for polar position vectors does not mean that $\hat{r}$ rotates about the origin so it can match its associated position vector, it means there is an infinite number of $\hat{r}$ unit vectors for any given polar coordinate system. Such algebraic ambiguity can lead to some confusion. It is tempting to say that when using polar basis-vector notation in polar coordinates that $\vec{r} = R\hat{r}$ defines a circle since there is no way of knowing which $\hat{r}$ is being used without any other information. However, such thinking would imply that a vector and a surface are equal, which is mathematically incorrect. It would be reasonable and correct to say the set of all position vectors $\{R\hat{r}\}$ defines a circle, however, the reality is, upon seeing just the algebraic expression $R\hat{r}$, there is no way to determine which specific vector is being referenced without additional information. Thus, for the purposes of this paper, we claim that $R\hat{r}$ without additional information refers to a single, unique but unknown polar or spherical position vector.

C. Theoretical framework

The initial data of student written responses were analyzed using the theoretical framework of identifying student difficulties [14–16]. Within this framework student difficulties and successes are identified through written or interview responses and are utilized to develop curricular materials that use a variety of strategies, including cognitive conflict, to guide students to realize inconsistencies within their own thinking and lead them to correct understanding. The analysis of initial written data uncovered a number of student difficulties, however only a handful of student responses fell into common categories and among those responses there were a variety of ideas. As a result, the analysis did not disentangle specific and consistent difficulties with sufficient clarity to be of use as the foundation for a targeted curriculum. Some students would report correct answers but with unclear or incomplete reasoning. Some incorrect answers appeared superficially similar but presented slight differences that made it unclear what the underlying difficulty might be [13]. Very often, it simply wasn't clear what the student was trying to do, and a considerable amount of interpretation and inference on the part of the researcher would have been required to categorize a response into any category. Such varied responses are inconsistent with the description of persistent, firmly held, and consistently applied incorrect ideas, or difficulties, associated with a student difficulties framework. Despite these variations and the limited sample size, a handful of thematic patterns of thinking were observable

and useful for designing additional questions for further exploration guided by a different theoretical framework.

The theoretical framework of resources and framing was selected to analyze students' in-the-moment thinking [17–19]. Within this framework, a cognitive resource refers to a content specific, fine-grained idea that students unconsciously use, or sometimes do not use, to address a question or problem posed to them. These resources may be closely associated or connected to other resources within a frame or schema that individual students have assembled and are activated when presented with some stimulus. Stated more generally and according to Hammer (private communication), a resource is a nugget of information or a nugget of an idea that a student has found useful at some time in the past. The resources framework affords the opportunity to dissect student reasoning into pieces that can be identified, understood, labeled, and described; increasing opportunities for the development of targeted curricular materials. It is worth noting that the precise naming of individual resources does not need to be “correct” in order to be productive to the researcher [18]. Meaning, if the chosen nomenclature for individual resources was somehow incorrect or non-intuitive, the claims of the research are unaffected because the utility of the framework is in its ability to assign working labels to individual resources which allows the use—or lack of use—of those resources to be tracked and analyzed.

II. METHODOLOGY

A. Interview protocol

Using the written data described in the previous study [13], a think-aloud interview protocol was developed to further probe student thinking and reasoning regarding non-Cartesian unit, position, and velocity vectors. The interview sample was described in the introduction. This paper focuses on results from the first set of interviews. Interview subjects completed the interview tasks in front of a whiteboard. Images for the interview tasks were projected onto the whiteboard so that interview subjects could draw and write answers directly on top of projected images.

Our interview protocol first presents interview subjects with a two-dimensional spiral with two points, *A* and *B* (Fig. 5) [13]. Subjects are asked to draw the directions of the polar unit vectors $\hat{r}$ (radial) and $\hat{\theta}$ (azimuthal) originating at those two points, and to explain their reasoning as to why they drew them in their illustrated directions. This question will henceforth be referred to as spiral question A (SQA). A correct student response is shown in Fig. 6, where the $\hat{r}$ vectors are drawn pointing radially outward from the pole and the $\hat{\theta}$ unit vectors are drawn (approximately) tangentially to the circles of radii equal to the distances the points are from the pole. This drawing was done by this paper's case study subject.

FIG. 5. First interview question designed to establish students' initial level of understanding of polar unit vectors.

Part B of the spiral question (SQB) instructs subjects to assume the magnitude of the radial distance of the particle from the origin is $r = (R_0 - b\theta)$ and the angle $\theta = \omega t$, where R_0 , b , and ω are constants, and then asks them to identify the units of each constant and then to use those constants to write a polar position vector $\vec{r}$ as a function of time. The correct position vector is $\vec{r} = (R_0 - b\theta)\hat{r}$. Subjects are then asked to write a velocity vector $\vec{v}$ describing the situation. As with SQA, subjects are asked to explain their thinking verbally. The velocity vector is the first time derivative of the position vector and because $\hat{r}$ is nonstatic and changes with time, a product rule operation is required in the differentiation, resulting in $\vec{v} = -b\omega\hat{r} + (R_0 - b\omega t)\omega\hat{\theta}$. This differentiation is shown in the Appendix. While it is possible that an interview subject could recall that $d\hat{r}/dt = \omega\hat{\theta}$, where ω is angular speed, the researchers assumed it is more likely that such

FIG. 6. A correct student response to spiral question A. The $\hat{r}$ unit vectors point in the direction of increasing radius and the $\hat{\theta}$ unit vectors point in the direction of increasing azimuthal angle. (This drawing was done by this paper's case study subject).

a result would be obtained from transforming to the Cartesian components of $\hat{r}$, differentiating those components, then retransform the resulting expression into polar coordinates. Such a task is not trivial. This spiral question was thus not an attempt to probe the ability to do basis transformations, but to probe the understanding of the consequences of non-Cartesian bases being location dependent, i.e., the basis unit vectors being directionally nonstatic. An interview subject attempting to differentiate or stating that $\hat{r}$ needs to be differentiated would be sufficient evidence of such understanding. Thus, for the purposes of our analysis, any indication from the interviewees that the time derivative of $\hat{r}$ was nonzero would have been evidence of productive thinking that non-Cartesian unit vectors have nontrivial time derivatives.

The next interview question is based on a question developed by Hinrichs [12] and presents students with a three-dimensional Cartesian coordinate system with four points, labeled A, B, C, and D, respectively, with one point each along the positive x, y, and z axes and one along the negative y axis (Fig. 7). For each point, subjects are asked to write a spherical polar position vector in terms of $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ that defines that point and to explain their reasoning as they do so. Henceforth this question will be referred to as the 3D question (3DQ). The correct position vectors are $\vec{r}_A = 4\hat{r}$, $\vec{r}_B = 3\hat{r}$, $\vec{r}_C = 3\hat{r}$, and $\vec{r}_D = 2\hat{r}$.

The interview protocol includes additional questions. However, this paper focuses on only the questions described above. The interviews took 50–70 min for students to complete.

For each of the points A, B, C, and D in the figure above, write the position vector $\vec{r}$ in terms of the unit vectors $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$. Please explain your reasoning for why this is the position vector.

FIG. 7. Interview question (3DQ) used to determine students' understanding of position vectors in 3D space using spherical coordinates and unit vectors.

B. Case study

The variance seen in student responses to the written questions described in the theoretical framework section (Sec. IC) were also present in the interview results. Each interview subject responded with a unique combination of productive and nonproductive ideas. There was not a consistent underlying thinking present that the authors could identify as a “student difficulty.” For example, while the four interview subjects would often produce similar answers consistent with “pattern matching” as defined by Hinrichs [12], identical pattern-matched answers across interviews were quite rare. Through repeated viewings of the recorded interview videos, it was observed that there were some fine-grain ideas that seemed to be activated in most, if not all, of the students’ thinking. We determined that the clearest way forward was to present a careful analysis of a single interview. Doing so allows for the meticulous task of identifying and justifying specific resources for this student and developing a model of how those resources seem to be connected and organized. Once completed, we will further elaborate on these resources and their possible organization and structure in subsequent papers with the remaining interview sample.

The subject of the case study will be known by the pseudonym Mark. Mark is an undergraduate physics major who was enrolled in both classical and quantum mechanics and had completed upper-division electromagnetism, modern physics, and all math credits necessary for a physics degree (Calculus I, II, III, Differential Equations, Linear Algebra). We selected Mark’s interview for detailed analysis because his description of the mathematics was similar to those of the other interviewees and those responses seen in previously analyzed written data. Such similarities include both mathematically correct and incorrect responses, several of which demonstrate what Hinrichs describes as pattern-matching [12]. Mark demonstrates a firm understanding of the polar unit vectors on SQA, however his thinking about these unit vectors is challenged as he progresses through the interview. Through the lens of our previous work [13], Mark productively activated many of the resources in the *Unit Vectors Cluster* described and did not activate resources from the other clusters such as *Motion Cluster*, the *Coordinate Cluster*, or the *Location Dependent Cluster*. Herein, we expand upon our previous analysis, and analyze Mark’s activation of unit vector resources and how he used them in the context of position and velocity vectors. Mark was also chosen for this analysis due to the clear articulation of his thought processes during his interviews. He provides clear descriptions of his thinking which enable us to confidently identify resources he was activating. Overall, Mark is one of the more successful students from our interview sample in terms of answering questions correctly.

In this paper, the results of the interview and the discussion and analysis of those results will be presented

in separate sections. This presentation not only mirrors the process in which the research team collected and analyzed the data, it also allows for a more holistic analysis. A goal of analyzing this case study was to develop a vocabulary and framework for analyzing the other interviews. Thus, it is not only important to identify individual, fine-grain ideas and name them as resources, it is also informative to identify and map how groups of such resources activate, or don’t activate, together. Presenting a small portion of Mark’s responses as evidence for the activation of an individual resource together could inadvertently shift the focus of the analysis more toward the naming individual resources rather than the simultaneous goals of naming individual resources and the mapping of the activation of groups of resources that are thematically connected in some way.

III. RESULTS

A. Unit vector reasoning

Initially on SQA, Mark gave accurate and complete definitions for $\hat{r}$ and $\hat{\theta}$, drawing $\hat{r}$ vectors in the same direction as the corresponding $\vec{r}$ vectors (Fig. 6) and stating the following two justifications respectively (M# are Mark’s statements):

M1 “ $[\hat{r}]$ points away from the origin”

M2 “ $[\hat{\theta}]$ is in the direction of increasing θ where θ is like your azimuthal.”

Mark later stated that regarding point A the direction of the $\hat{r}$ vector is

M3 “...from the origin pointing toward A.”

Mark clearly stated and illustrated the correct definitions of the polar unit vectors, with no evidence of any confusion.

B. Position vector reasoning: Initial conflict and resolution

When asked to write the general polar position vector on SQB Mark responded with $\vec{r} = (R_0 - b\theta)\hat{r} + \theta\hat{\theta}$. The interviewer asked Mark why he included both $\hat{r}$ and $\hat{\theta}$ terms. Mark paused for a moment, and stated:

M4 “so the way I previously described my description to $\hat{r}$ you wouldn’t necessarily need $\hat{\theta}$ ’cause like what I said for $\hat{r}$ earlier was that it was measured from the origin to whatever point you’re talking about, but now I’m believing that that was not right.”

Then after a long pause, the interviewer encouraged him to verbalize his thoughts, and the following exchange occurred (I# are interviewer statements or questions):

M5 “I’m thinking that $\hat{r}$ thing is right, ‘cause otherwise what other direction would $\hat{r}$ be in?”

I1 “So your answer way back in part A?”

M6 “So I’m thinking that’s still right. I’m thinking this extra θ term is not supposed to be here.”

I2 “OK. And why are you now thinking that?”

M7 “Well, because given the definition that I gave of $\hat{r}$, [$\vec{r} = (R_0 - b\theta)\hat{r}$] tells you everything you need to know about your position vector.”

Mark further stated that $\hat{\theta}$ would be needed when describing a change in position:

M8 “Oh right, yeah. It’ll only be relevant in the next question when you talk about the change in position.”

Mark used correct reasoning based on his understanding of the unit vectors to inform his thinking about this polar position vector. Such understanding appeared to include the realization that polar unit vectors are not directionally static. To confirm this, the interviewer asked what the Cartesian unit vectors $\hat{x}$ and $\hat{y}$ would look like at points A and B. Mark drew them pointing in the correct directions at both points.

Finally, the interviewer asked Mark to draw the position vector for point A on the spiral. Mark drew a vector with its tail at the origin and arrowhead terminating at point A, which is consistent with the definition of a position vector discussed in Sec. IB, and as can be seen in part of Fig. 8. In his Fig. 8 sketch, Mark initially labeled the position vector as “ $\vec{r}$ ” and subsequently added a subscript “i” to the label while answering SQB.

C. Velocity vector reasoning: Attempting to resolve two competing ideas

When asked to write the velocity vector for SQB, Mark’s initial response was to write $\vec{v} = \Delta\vec{r}$. He then drew the velocity vector as the difference between initial and final position vectors, which can be seen in Fig. 8. He further wrote $\vec{v} = \Delta\vec{r}/\Delta t$, and, with some leading questions by the

FIG. 8. Mark’s initial approach to $\vec{v}$ was to show that $\vec{v}$ is the change in position vectors. He was able to follow this line of thinking to eventually reach taking the first time derivative of the position vector.

interviewer, realized he needed to take the first time derivative of his previous position vector to get the velocity vector. At this point, Mark stated that $\hat{\theta}$ was needed in the velocity vector and should come out in the result of differentiating the position vector, $\vec{r}$. However, when he attempted this differentiation he missed the necessary product rule operation (see the Appendix) and wrote $\dot{\vec{r}} = -b\omega\hat{r}$. Mark said nothing about the time derivative of $\hat{r}$, which is the only unit vector in his correct position vector expression, being non-zero and thereby requiring the use of the product rule.

While contemplating these ideas, Mark questioned his initial and correct definitions of the polar unit vectors:

M9 “So what I originally came up with [pointing to his correct position vector], this isn’t right. This needs to have some θ dependence of some sort because, um, it should depend on $\hat{r}$ —it should have dependence on r direction because in this case [$\vec{r}$ is] getting shorter, it spirals inwards. But it should also have θ —it should also have $\hat{\theta}$ dependence...”

M10 “I wonder if I defined my definition wrong, or something, with $\hat{r}$.”

I3 “Well, you’ve already questioned yourself on that. Why are you questioning your questioning?”

M11 “I feel like this [pointing to velocity vector] should definitely not just be one term. It should have θ in here.”

I4 “So we didn’t—you did not have it in your position vector—the $\hat{\theta}$ dependence—now you want to say that it shows up again in velocity?”

M12 “If it doesn’t show up [in $\vec{r}$] it definitely won’t show up [in $\vec{v}$], but I’m, I’m feeling like it should.”

I5 “OK. Just tell me why you think it should.”

M13 “Because, um, I mean when you think about $\hat{\theta}$, $\hat{\theta}$ we said is like this direction [pointing in direction of increasing θ] sorta...um, and in this case the higher that—or the larger values of b and ω take on, the faster conceivably we’ll be changing the θ . So I mean, these [pointing to b and ω] should have some bearing on what θ is going to be doing. I’m just totally blanking on...[long pause]”

I6 “So if I could summarize, you do think that you need to include a θ term somehow in your velocity vector but you are just not sure how to do that?”

M14 “Yes... why am I struggling with this? I feel like it should be really easy.”

D. Changing unit vector definitions

After saying he thought it should be easy to get $\hat{\theta}$ out of differentiating $\vec{r}$ and struggling to do so, Mark questioned his original definitions. He stated:

M15 “So, I defined $\hat{r}$ incorrectly... OK, so $\hat{r}$ should not, it does not have any actual bearing on where the point is, the position vector is what matters there.”

Mark then developed a new strategy so he could include a $\hat{\theta}$ term in the position vector. He determined how $\hat{r}$ would behave if it were static, that is, always pointing in the same direction like a Cartesian unit vector. He explained this static nature by making $\hat{r}$ an explicit transformation of $\hat{x}$ and $\hat{y}$ and a scaling factor (as he called it) to result in unit length. He wrote $\hat{r} = \sqrt{\hat{x} + \hat{y}}$ as a representation of his thinking. His illustration accompanying his explanation (Fig. 9) clearly shows $\hat{x}$ along the x axis and $\hat{y}$ along the y axis resulting in a $\hat{r}$ that makes a 45° angle between $\hat{x}$ and $\hat{y}$. His verbal explanation explicitly demonstrates a return to a Cartesian way of thinking:

M16 “...now that I’m thinking about this so, because like what I was talking about earlier [his initially correct definition of $\hat{r}$], it was like totally nonsense because think about it like parallel to Cartesian coordinates; like $\hat{x}$ is just here [as he drew $\hat{x}$] and $\hat{y}$ is just here [as he drew $\hat{y}$]. Like it doesn’t matter what point I’m looking at, right? So looking at A, $\hat{r}$ should always be right there [as he drew $\hat{r}$ at a 45° angle between $\hat{x}$ and $\hat{y}$].”

Mark flipped from his initial, nonstatic definition of $\hat{r}$ to a static definition of $\hat{r}$. After talking through all of this, he remained uncertain as to which of his two definitions was better or which one was correct. He described this uncertainty as a “moral dilemma.” When asked to elaborate on this moral dilemma, he explained,

M17 “I think when I was defining $\hat{r}$ before [his original, correct definition] I was defining $\hat{r}$ to be the position vector $\vec{r}$ over the magnitude of $\vec{r}$, which is not the same thing that you’re talking about in polar coordinates, I don’t believe... I’m not thinking that’s right because I’m thinking [$\hat{r}$] should just be something that is a strict transformation of $\hat{x}$ and $\hat{y}$ so it would be pointing like [$\hat{r}$ at a 45° angle between $\hat{x}$ and $\hat{y}$].”

On the surface, this explanation of his moral dilemma appeared to be a complete change of thinking from his correct definitions to a more Cartesian-like definition.

FIG. 9. Mark’s illustration used during his attempt to explain $\hat{r}$ as a transformation of $\hat{x}$ and $\hat{y}$, and therefore directionally static.

E. Questioning the efficacy of new unit vector definitions

Before Mark gave any answers to the 3DQ, the interviewer asked him which of his two stated definitions of $\hat{r}$ he would use to answer the question. After spending a few moments in thought, he replied

M18 “I think it would be easier using my first one.”

As he prepared to write his answers to the position vectors on the board, he further stated

M19 “OK, so, now I’m thinking I’m gonna go contradict everything I just said [his second definition].”

F. Returning to and attempting to apply initially productive unit vector definitions

Mark’s initial responses to three of the four points on the 3DQ are examples of Hinrichs’s description of pattern matching. Mark’s four position vectors were $\vec{r}_A = 4\hat{r} + (\pi/2)\hat{\phi}$, $\vec{r}_B = 3\hat{r}$, $\vec{r}_C = 3\hat{r} + (\pi/2)\hat{\theta}$, and $\vec{r}_D = 2\hat{r} + (3\pi/2)\hat{\phi}$. (Of note, Mark made the decision to use the physics convention of ϕ as the azimuthal angle and θ as the polar angle.) As he answered $\vec{r}_A$, he counted out the number of intervals point A was from the origin and said that the radial term should be $4\hat{r}$ and then stated

M20 “So then $\vec{r}_A$ would be $4\hat{r}$ plus—it would be π in the $\hat{\theta}$ direction—’cause this [while pointing to $+x$ axis] would be like our zero ϕ . ϕ would be equal to zero along here [still pointing to $+x$ axis].”

When the interviewer asked him if ϕ was the angle between the x and y axes he said it was, and drew ϕ as the angle between them. After he wrote all the position vectors, the interviewer asked him why there was ϕ and/or θ dependence for points A, C, and D but not for B. He replied

M21 “Well, I mean there is a dependence per se, right? I mean you could write all these out and just say that [missing θ term] is you know plus 0 times $\hat{\theta}$ but uh, for B, just the way it, and I think you’re, basically when I think you’re using this you’re free to define ϕ equals 0 to be anywhere, but I’ve sorta defined it to be like, ya know, you’re measuring it with respect to like, [points to $+x$ axis] sort of, $\hat{\phi}$ yeah, ϕ would be like you measure it with respect to [$+x$] so [$+x$ axis] is just 0 because it lies on that ϕ equals 0 line,”

indicating that he saw a need for explicit ϕ and/or θ dependence but only wrote such terms when he believed those dependencies were nonzero.

IV. DISCUSSION

A. Identifying Mark's resources in the context of our previous resource clusters

As noted in the results section, some of Mark's responses to position and velocity vector questions can be described using resources that are consistent with our previous description of the *Unit Vector Cluster* [13]. Additional responses share similarities with the previously identified *Location Dependent Cluster*; however, the ideas described in that cluster do not map well to Mark's responses, particularly in the context of position and velocity vector questions. Mark's initial (and later) idea that non-Cartesian unit vectors change direction with location and his use of the spiral-shaped path to inform his construction of a velocity vector are examples of location informing his thinking about unit vectors. However, in the context of position and velocity vectors, Mark appeared to activate resources from both of these previously identified clusters and combine those activated resources in patterns dependent on whether he was answering a position vector question or a velocity vector question. For clarity in the presentation of Mark's thinking, we are first naming individual resources Mark activates and then grouping those activated resources by the context of the question he was answering, specifically, the "*Position Vector Resources* group" and the "*Velocity Vector Resources* group." Grouping Mark's resources based on the context of the question allows for a particular coherent mapping of Mark's thinking which we find useful. We acknowledge that this is only one of many possible groupings, and that alternative groupings of resources could illuminate other essential features of Mark's thinking.

In this paper, we have chosen to use the term "*group*" in part to reflect that we are also sorting resources into thematic categories. But, we also use *group* as opposed to "*Cluster*" to reflect that this approach to thematic categorization is different in nature than that used in our previous work [13]. In the Vega *et al.* paper, the clusters were themes of thinking that emerged from responses from multiple interview subjects and exclusively within the context of unit vectors. In this present paper, these new resource groups align with the context of questions about additional vector types and emerge from the responses of a single subject. Additionally, this present work seeks to name individual resources, whereas the work of Vega *et al.* sought to name groups of ideas in thematic clusters. We use similar naming structures, i.e., "[*Theme Title*] *Cluster*" in Vega *et al.* and "[*Theme Title*] *group*" in this paper to reflect that the goal of understanding and describing student thinking about vector and coordinate-system concepts remains the same across individual studies within the broader project. Finally, it is not our intention to add the term *group* to the vocabulary of Resource Theory, only to use the term as a tool of categorization in the context of this research.

To more clearly communicate the flow of Mark's thinking and introduce our new resource nomenclature, we developed two toy models. The first consists of resource graphs [20] organized by the context of each question—one each for unit vectors, position vectors, and velocity vectors (Fig. 10). The second (Fig. 11) is a combination of resource graphs and a box-and-arrow model that presents a chronological map of Mark's thinking and an expanded *Velocity Vector Resources* group to illustrate how his activated resources are connected within it. Within each group, a solid-color bubble indicates a resource that is coordinate system invariant, i.e., productive in all coordinate systems and not specific to any individual coordinate system. A cross-hatched background in a bubble indicates a resource that is productive in the context in which Mark is working but not necessarily productive in all coordinate systems. The solid white background bubble for a resource in the *Velocity Vector Resources* group is explained in detail in Sec. IV D.

B. Mark's unit vector resources

At the beginning of the interview, Mark clearly and correctly illustrates and defines both polar and Cartesian unit vectors. While answering SQA, he draws all four requested unit vector arrows in the correct directions and provides clear and correct reasoning for why he drew them in those directions, as seen in statements M1-3 and his drawing in Fig. 6. Based on previous work, this is not a trivial task [13]. The combination of statements M1 and M2, Fig. 6, and Mark's illustration of the Cartesian unit vectors in their correct directions suggests that Mark activates productive resources for unit vectors pointing in the direction of increasing coordinate, having unit length, and being orthogonal within their respective coordinate systems. "*Increasing coordinate direction*" is a more refined version of our previously identified *Location Dependent Cluster* and is a statement of the general behavior of unit vectors in all coordinate systems. Previous work identified a nonproductive activation of this resource for Cartesian unit vectors, however Mark activates this in a productive fashion and the model is more clear when using the description of *Increasing coordinate direction*. Working together these resources form at least a portion of a resource group we have called the "*Unit Vector Resources* group" (Fig. 10). This *Unit Vector Resources* group contains three of the four resources identified previously from Vega *et al.* [13], and, in this case, is identified by evidence from a single student, as opposed to our previous work where we identified them from an expert perspective. The only missing resource from our previous expert-like unit vector cluster is one where unit vectors are identified as dimensionless, which is not directly probed by this question. It should be noted that Mark did not explicitly state that the basis unit vectors are orthogonal in his responses to SQA, SQB, and 3DQ.

FIG. 10. Resource groups identified from Mark's statements and drawings, organized by the content of the question. A solid-color background indicates a resource that is coordinate system invariant. A cross-hatched bubble background indicates a resource that is productive in the coordinate system Mark is currently working. In the *Velocity Vector Resources* group, the $d\hat{r}/dt \neq 0$ resource bubble has a solid white background because it is an expert-identified resource that Mark does not activate.

However, Mark draws the unit vectors perpendicular to each other whenever he draws them together. Furthermore, in a later interview question not analyzed in-depth for this paper, Mark stated the following:

M22 "...as long as $\hat{\theta}$ and $\hat{\phi}$ — $\hat{\theta}$ is perpendicular to $\hat{r}$ and $\hat{\phi}$ is perpendicular to both of them it should be ok."

This statement appears to confirm the thinking behind his consistent orthogonal illustrations. The fourth resource in our new *Unit Vector Resources* group seen in Fig. 10 is discussed in Sec. IV E. We did not find evidence of Mark's activation of resources associated with the previously identified *Coordinate Cluster*, or *Motion Cluster* [13]. Despite Mark's productive activation of the three expert-identified resources, Mark struggled to activate them

successfully when faced with problems using position and velocity vectors that triggered additional resources.

C. Mark's position vector resources

On several occasions, Mark talked about and drew pictures of position vectors when answering questions. The ideas he presented seemed distinct from his thinking about unit vectors and therefore we categorized them separately. Resources associated with Mark's thinking about position vectors are summarized here and are represented in Fig. 10's *Position Vector Resources* group.

Mark consistently constructed position vector expressions and drew position vector sketches that had a vector beginning at the origin or pole of the coordinate system and terminating at the point of interest. Examples of such can be

FIG. 11. A chronological box-and-arrow plot that illustrates the connections between Mark's resources and resource groups. He starts with productive unit vector resources then activates a nonproductive position vector resource. He then compares that nonproductive resource to his productive unit vector resources, namely, the *Increasing coordinate direction* resource, and then seems to continue using that productive resource. But, Mark's productive thinking was then challenged when he moved to a velocity vector question. The chronological plot is then expanded into a "zoomed-in" representation of the *Velocity Vector Resources* group. The individual resources Mark activated in that group are then linked by how he appeared to activate them chronologically during the interview. There are two paths of thinking in this expanded group view that lead him to the same two resources which he is ultimately unable to reconcile with one another. This unsuccessful reconciliation could possibly be resolved with the nonactivated $d\hat{r}/dt \neq 0$ resource. After unsuccessfully reconciling those two resources, Mark activates a new resource we call $\hat{r}$ depends on $\hat{x}$ and $\hat{y}$ that we sort into the *Unit Vector Resources* group because he is questioning his unit vector definitions when he activates this new resource. However, Mark does not recall the actual unit vector transformations, which leads him to challenge his previously stated productive unit vectors resources and develop a model in which polar unit vectors behave more like Cartesian ones.

seen in his drawings for Fig. 8 as well as his statement M20. Thus, we have included a resource in the *Position Vector Resources* group graph of Fig. 10 called "From origin to point." This resource is invariant across coordinate systems, and as such, has a solid background in Fig. 10. Similar to identified resources in the *Unit Vector Resources* group, this is a resource that aligns with an expert definition of a position vector (see Sec. 1B).

Mark's first attempt at writing a polar position vector on SQB was $\vec{r} = (R_0 - b\theta)\hat{r} + \theta\hat{\theta}$. Later in the interview, Mark's written response to 3DQ and his statements, M20 and M21, demonstrate his propensity to write algebraic expressions for polar or spherical position vectors that include azimuthal and/or polar terms. It is challenging to identify a single fine-grained resource from Mark's data exclusively, however, drawing from Hinrichs's work, we

identify " $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ " as a resource to denote Mark's activation of an idea that position vector expressions have the general form $\vec{r} = a\hat{a} + b\hat{b}$, with a $c\hat{c}$ added in three-dimensional cases. As identified, this $\vec{r} = a\hat{a} + b\hat{b}$ resource is both productive in Cartesian coordinates—i.e., mapping a point's coordinates to a vector expression—and consistent with what Hinrichs calls "pattern matching" [12]. Mathematics education research literature indicates this type of response is not surprising given the primacy of Cartesian coordinates in the education system. Students spend the vast majority of their time learning about and applying the Cartesian coordinate system [1–3]. Note the cross-hatched background of this resource in Fig. 10 denoting its lack of universal utility across coordinate systems. A deeper look into the thinking of the study's remaining interview subjects may reveal that there is more

going on in the position vector resources group than Mark's data alone demonstrates. We aim to further explore these "pattern-matched" responses [12] and expand upon the position vector resources group in a future paper.

After his initial response to SQB, Mark recalled his previous description of $\hat{r}$ and through his statements M4-M7, ultimately decided that " $\hat{r}$ tells you everything you need to know," and dropped the extra $\theta\hat{\theta}$ term. Based on the direct cues in his statements, we attribute this reasoning to Mark activating his *Unit Vector Resources* group, and activating a new resource, that the position vector in polar coordinates only has an $\hat{r}$ term which we identify as $\vec{r}$ has only $\hat{r}$. The activation of these resources is represented in the timeline graphic at the top of Fig. 11, as the red boxes (unit vector resources) and green circle ($\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ from the *Position Vector Resources* group).

D. Mark's velocity vector resources

A combination of Mark's verbal and written responses, when considering the velocity vector question in SQB, present a nuanced look into Mark's thinking about velocity vectors. Because of the context Mark is considering when they are activated, we define these resources to be within a grouping of *Velocity Vector Resources*. The *Velocity Vector Resources* group (Fig. 10) contains five resources activated by Mark (solid or cross-hatched blue background) and one resource (white background) that is not activated by Mark. In Fig. 11, Mark's velocity vector resources group is connected via arrows to highlight the order in which Mark activated them. This expanded look into Mark's velocity vector resources group is embedded in the timeline at the top of Fig. 11, which identifies the order in which Mark activates specific resources, and/or groups of resources. All three resource groupings were included into Fig. 10 for completeness; however, all information for the velocity vector resources group in Fig. 10 is presented in Fig. 11 with additional chronological information. As such, we refer only to Fig. 11 henceforth, unless otherwise noted.

When Mark first defined the velocity vector $\vec{v}$ in his response to SQB, he stated that $\vec{v} = \Delta\vec{r}$. He later expanded that to $\vec{v} = \Delta\vec{r}/\Delta t$ and drew what can be seen in Fig. 8. Synthesizing these very similar ideas it is clear Mark has a nugget of knowledge that $\vec{v}$ and $\Delta\vec{r}$ are closely related. Therefore, we identify a resource " $\vec{v} \rightleftharpoons \Delta\vec{r}$ " in the velocity vector resources group (Fig. 11). Note the placement of this resource on the far left of the cluster signifying it being activated first chronologically.

While contemplating the problem, Mark sketched the geometric representation shown in Fig. 8 without any prompt. We identify this new representation as a distinct resource. It is possible that this is an activation of the *From origin to point* resource within the *Position Vector Resources* group, however Mark appears to be using a

position vector resource within a different context, to reason about what the "change" in position vectors would look like. Therefore, the sketch is included as one of Mark's resources in a different grouping in Fig. 10 and as one of the two paths (colored black to denote an unprompted activation) emanating from $\vec{v} \rightleftharpoons \Delta\vec{r}$ in Fig. 11.

The interview protocol was written with the expectation that interview subjects would attempt to determine the velocity vector by differentiating the position vector with respect to time. When Mark appeared stuck and had not yet performed this differentiation, the interviewer asked him gently leading questions in an attempt to elicit this idea. Mark came up with this relationship almost immediately, and we identify it as the " $\vec{v} = \dot{\vec{r}}$ " resource in Fig. 11, connected by an orange arrow to indicate a prompted activation. It is possible that relating the velocity vector to some kind of change in the position vector and specifically equating the velocity vector to the time rate of change of the position vector are similar, and perhaps even the same resource. For Mark, however, he does not make an explicit connection between the two resources as being the same or a very similar idea. Therefore, for the purposes of this analysis, we consider them as distinct resources.

Before answering SQB, in statement M8, Mark notes that $\hat{\theta}$ would be a necessary term to "talk about the change in position." Mark activates this idea again when responding to SQB in statement M9: "This needs to have some θ dependence of some sort..." We identify this resource as $\vec{v} \xrightarrow{\text{needs}} \hat{\theta}$ and connect it to the resource of the drawing of two position vectors. He also reasons that the velocity vector will have an $\hat{r}$ component because it is "spiraling inward" (from statement M9).

Mark performs the differentiation of his final (and correct) answer for the position vector [$\vec{r} = (R_0 - b\theta)\hat{r}$] and gets an expression with only an $\hat{r}$ term. From each of these resources independently and his correct observation that both the radial and angular positions change as an object moves along the spiral (statement M9), he correctly reasoned $\vec{v}$ needed both $\hat{r}$ and $\hat{\theta}$ terms; hence the resources $\vec{v} \xrightarrow{\text{needs}} \hat{r}$ and $\vec{v} \xrightarrow{\text{needs}} \hat{\theta}$ inside the *Velocity Vector Resources* group and the connections to each from the spiral sketch and $\vec{v} = \dot{\vec{r}}$ in Fig. 11. This thinking is markedly different from our previously reported *Motion Cluster*, which articulated student ideas associated with the unit vector $\hat{r}$ pointing in the direction of motion or tangential to the path as the definition of $\hat{r}$ [13]. Mark is not asserting that $\hat{r}$ points in the direction of motion, but rather that the velocity vector should have components of both $\hat{r}$ and $\hat{\theta}$. His dialogue in statements M10-M14 demonstrate more of his thinking around these ideas, but his only conclusion at that time is that "So what I originally came up with [pointing to his correct position vector], this isn't right."

E. Attempting to resolve productive resources leads to a shift toward Cartesian thinking

Mark's activated resources are not sufficient to resolve his issue of the velocity vector needing both $\hat{r}$ and $\hat{\theta}$ terms, and he believes one of his previous answers is problematic and requires revision. Differentiating his expression for $\vec{r}$ should result in both $\hat{r}$ and $\hat{\theta}$ terms. As mentioned in Sec. IB, a challenging aspect of this differentiation is the recognition that the time derivative of $\hat{r}$ is nonzero. As discussed at the end of the Sec. II, if Mark had mentioned that $\hat{r}$ was dependent on time, or that $\hat{r}$ itself needed to be differentiated, or attempted to differentiate it while differentiating $\vec{r}$, there would be evidence to suggest that he activated productive resources regarding the behavior of $\hat{r}$. However, Mark first came to the realization that the velocity vector needed an azimuthal term through physical reasoning—using the motion of the particle—not from the differentiation of the position vector.

While Mark has several productive resources that are helpful in deriving $\vec{v}$, Mark appears to be missing or simply not activating one that would connect his productive resources: $d\hat{r}/dt \neq 0$ (shown with a white bubble background in Figs. 10 and 11). Remarkably, this nonactivated resource combined with his not reconciling the resources $\vec{v} \xrightarrow{\text{needs}} \hat{r}$ and $\vec{v} \xrightarrow{\text{needs}} \hat{\theta}$ led him to question and attempt to redefine the productive resources from the *Unit Vector Resources* group he had already activated. His thinking is illustrated in Fig. 11 with the arrows from $\vec{v} \xrightarrow{\text{needs}} \hat{r}$ and $\vec{v} \xrightarrow{\text{needs}} \hat{\theta}$ bypassing the nonactivated $d\hat{r}/dt \neq 0$ resource and pointing to activation of a new resource that we attribute to the *Unit Vector Resources* group.

Mark's new analysis resulted in his statement that $\hat{r}$ is an “explicit transformation of $\hat{x}$ and $\hat{y}$,” and we name this new resource as “ $\hat{r}$ depends on $\hat{x}$ and $\hat{y}$.” This resource has the potential to be productive, since $\hat{r}$ is an explicit transformation of $\hat{x}$ and $\hat{y}$ [see Eq. (1) in Sec. IB]. However, Mark wrote down the relationship as $\hat{r} = \sqrt{\hat{x} + \hat{y}}$. This relationship is possibly consistent with some vector operations in using the Pythagorean theorem while calculating the magnitude of a vector in Cartesian coordinates, i.e., $|\vec{r}| = \sqrt{x^2 + y^2}$. Further evidence of this Pythagorean- and vector-operation-like thinking can be seen in Fig. 9, where, along with his reasoning in M16, $\hat{x}$ and $\hat{y}$ are drawn as components of $\hat{r}$. This drawing appears to be consistent with the position vector, in Cartesian coordinates, being the hypotenuse of the right triangle formed by the vector and its x and y components arranged tip-to-tale. Within our analysis, this $\hat{r} = \sqrt{\hat{x} + \hat{y}}$ relationship straddles the *Unit Vector Resources* group—by providing a definition for $\hat{r}$ —and the position vector resources group—by using reasoning that is consistent with position vector and Pythagorean theorem rules or conventions. During initial analysis of Mark's thinking, we had identified this as a resource;

however, our thinking has evolved. We now think it is unlikely that this relationship is a resource, i.e., a “nugget of useful information,” that Mark has found useful in the past. Rather, we suggest that it can be viewed as a “conceptual blend” [21,22] of two separate resources. Bing and Redish [22] specifically used conceptual blending to make sense of what appeared to be newly generated student ideas at the math-physics interface. This framework could provide tremendous insight into future analysis of our data.

During Mark's attempt to transform from Cartesian unit vectors to polar unit vectors, he attempted to make $\hat{r}$ static, likening its behavior to that of Cartesian unit vectors:

(A portion of M16) “...now that I'm thinking about this so, because like what I was talking about earlier [his initially correct definition of $\hat{r}$], it was like totally nonsense because think about it like parallel to Cartesian coordinates [emphasis added]”

From an expert perspective, it seems that Mark attempted to develop a justification for why $\hat{r}$ is a static quantity. While taking the time derivative of the position vector, his math is consistent with treating it as a static quantity, and then he articulated that $\hat{r}$ depends on $\hat{x}$ and $\hat{y}$. Thus, it seems that he attempted to develop a model where $\hat{r}$ is static because $\hat{x}$ and $\hat{y}$ are static, using Cartesian-coordinate behavior as a guide (statement M16). This new thinking demonstrates a distinct shift away from his *Increasing coordinate direction* resource.

It is possible that a more complete recollection of how the polar unit vectors can be derived from the Cartesian unit vectors would have helped Mark see that $\hat{r}$ needs to be differentiated. However, the interview did not prompt Mark to do so, nor did it explicitly probe Mark's ability to convert from Cartesian to polar coordinates. Mark made no mention that $\hat{r}$ would need to be differentiated despite his prior productive activation of the *Increasing coordinate direction* resource and illustrations of a directionally non-static $\hat{r}$ (Fig. 6). While having access to the correct mathematical transformations could have helped him, there was a connection that was not made between his productively stated behavior of $\hat{r}$ and the mathematical consequences of that behavior. Further probing thinking concerning the conversion of Cartesian to spherical unit vectors is an area of future research interest. It also seems likely that the nonactivated $d\hat{r}/dt \neq 0$ resource could serve as a point for curricular emphasis, at least for Mark.

Despite these new ideas, when Mark continued on to 3DQ, he returned to his first definition of $\hat{r}$ as pointing in the *Increasing coordinate direction* and abandoned his newly activated $\hat{r}$ depends on $\hat{x}$ and $\hat{y}$ resource. Statements M18–19 show he recognized his initial definitions of polar unit vectors are more convenient for answering the 3DQ. However, Mark wrote the position vectors to 3DQ in the same form as he did initially in SQB, with additional terms

of $\hat{\theta}$ and/or $\hat{\phi}$. We claim Mark is activating the same resource as he had previously, $\vec{r}$ has form $a\hat{x} + b\hat{y}$ with an additional $c\hat{z}$ term for the third dimension. Unlike his response to SQB, Mark did not compare his responses with his *Increasing coordinate direction* resource, even after probing questions by the interviewer. In Statement M21, Mark used the measures of the angles ϕ and θ of the angular coordinates from (r, ϕ, θ) as the “magnitudes” of the $\hat{\phi}$ and $\hat{\theta}$ terms. As was previously mentioned in Sec. IV C, this $\vec{r} = a\hat{a} + b\hat{b} + c\hat{c}$ form is likely based on the Cartesian notation for position vectors, and is a result consistent with findings from math education research literature [1–3] and Hinrichs [12].

There is also evidence Mark expected the algebraic notation of a position vector to define a unique position in space and that he did not trust that correct polar or spherical notation did so. While constructing his spherical position vectors in 3DQ he continually and consistently gestured to show how his algebraic expression would lead a reader to the exact point his expression attempted to define. He explicitly gestured and stated that azimuthal angles are measured from the x -axis and polar angles from the z axis (statements M20–M21). In each case, he showed how the $\hat{\phi}$ and/or $\hat{\theta}$ terms he wrote corresponded to a direction one should take if one were attempting to move from the origin to the point of interest. Moreover, for point B , where he wrote the correct position vector, he further explained there is a dependence on $\hat{\phi}$ and $\hat{\theta}$, but those dependencies are 0, making them unnecessary to write (statement M21). He did not mention, as he had done on SQB, that $\hat{r}$ “tells you everything you need to know.” These statements also seem to indicate he believed the Cartesian form of a position vector is the complete form and that the spherical form does not communicate enough information.

Mark’s responses on the 3DQ are consistent with Hinrichs’s description of pattern-matching [12]. However, Hinrichs’s definition of pattern-matching spherical position vectors to the Cartesian form is purely descriptive and not rooted in a theoretical base. His work did not intend to provide a theoretical explanation for why pattern-matched responses are common. A theoretical framework of resources provides one possible explanation. Mark’s responses to the 3DQ potentially provide some evidence that the pattern-matching response occurs due to the activation of a related set of resources consistent with Cartesian conventions. Given that mathematical meaning is often tied to the coordinate system in which a concept is learned [1–3] and the primacy of Cartesian coordinates in mathematics instruction [3], it’s not surprising that Mark regularly activated Cartesian-like resources. Additionally, Mark is not the only interview subject in our interview sample to respond with pattern-matched position vectors. In fact, all of the interview subjects answered SQB and/or 3DQ with what Hinrichs would have identified as pattern-matched position

vectors in non-Cartesian coordinates [12]. A future paper will delve further into these responses and how the use of a resources framework might further elucidate students’ thinking. This new paper will likely also allow us to expand the number of named resources within the position vector resources group and possibly rename—or discover yet-simpler resources driving the activation of—the resources we named as equations in this paper.

F. Implications

Despite this paper presenting the initial step in a broader analysis, there are initial implications for instruction and future investigation. When looking at Mark’s ideas and the published literature, it seems that Cartesian-like thinking and some of the associated resources identified here may be a significant barrier to the use of non-Cartesian coordinates and unit vectors for solving math and physics problems. Even when a student activates resources appropriate for non-Cartesian coordinates and unit vectors, activation of resources that are productive—and likely formed [1–3]—in Cartesian coordinates can interfere with the thinking of and confuse that student. Returning to the flow of thinking modeled in Fig. 11 provides insight into one possible path for instruction. An instructional sequence could elicit ideas

about $\vec{v} \xrightarrow{\text{needs}} \hat{r}$ and $\vec{v} \xrightarrow{\text{needs}} \hat{\theta}$. It could further provide students opportunities to reflect on the definitions of the polar unit vectors and reconcile how, unlike Cartesian unit vectors, the time derivative of the polar unit vectors are nonzero. A more complete set of student resources will be identified using our remaining interview subjects and will expand the space of what we know about student thinking in this area. Going forward, additional interviews will attempt to probe student thinking with regard to their use of non-Cartesian coordinate systems in answering other physically relevant questions, such as magnetic fields, as well as in other non-Cartesian coordinate systems, i.e., cylindrical coordinates.

An instructional implication arises when considering the number of times Mark activated resources that conflict with one another. This implication also has immediate relevance for upper-division physics courses. On the surface, the task of writing a position vector for a point in space in spherical coordinates may seem to be a straightforward one. Yet, when asked to write such a position vector, a high-achieving student activated numerous resources. Some of these resources were productive for the coordinate system he was working in and some resources would be productive in other coordinate systems, but were not productive in the polar and spherical coordinate systems with which Mark was working. Given this dichotomy, it is not surprising Mark struggled to explain how these resources fit together. Mapping these varied activation patterns and interactions of resources is helpful for us as researchers as we investigate what ideas a student activates as such knowledge will inform eventual curricular development. For instance, an instructional sequence that supports students’ ideas about

mathematical definitions of unit vectors and students' physical ideas about the direction of a particular vector of interest—such as a velocity vector or a magnetic field vector—could guide students to reconcile differences in their thinking. Such tasks can be designed on models for curriculum design within a resources framework [23,24]. However, in the interim it is also worthwhile for instructors of upper-division physics courses to be aware that even some of their best students are going to enter their courses with wide-ranging ideas regarding the essential mathematics of those courses.

V. CONCLUSION

This paper is a crucial step in a larger research study aimed at developing research-based curricular materials for an undergraduate math methods course. Towards this goal, we conducted one-on-one interviews to probe students' thinking regarding Cartesian and non-Cartesian coordinate systems. We presented a detailed analysis of one student's work, Mark. Mark was selected due to his thoughtfulness and articulate descriptions of his thinking. Analysis was conducted through the theoretical lens of the resources framework. This theoretical framework provides us the opportunity to develop a coherent description of Mark's nuanced thinking, and in doing so, provides insight into analysis of additional student interviews in the future. It further provides potential paths for curriculum development based on our findings.

The analysis of Mark's thinking revealed he activated productive resources about the definitions of non-Cartesian unit vectors. Our previous work had focused on resource clusters of similar kinds of student reasoning on unit vector questions, but Mark demonstrated few of the nonproductive resources identified there [13]. We expanded our analysis to identify resources associated with questions on position and velocity vectors. When answering questions that required the use of those unit vectors in the context of position and velocity vectors, Mark activated additional resources, some of which are productive. Mark activated what we identified as the $\vec{v} \xrightarrow{\text{needs}} \hat{r}$ and $\vec{v} \xrightarrow{\text{needs}} \hat{\theta}$ resources. He did not activate a resource or provide evidence that he was or should be considering that the time derivative of polar or spherical unit vectors is nonzero. Mark reconciled these ideas by reevaluating his productive unit vector resources and led him to redefine polar or spherical unit vectors in terms of incorrectly applied Cartesian conventions.

On subsequent questions in the interview, these resources of the redefined unit vectors were not used when Mark realized that his initially activated unit vector resources would be more convenient in answering position vector questions. Despite this realization, his responses to position

vector questions included extra terms consistent with the Cartesian pattern matching described by Hinrichs [12]. Mark's responses are also consistent with the findings of Paoletti *et al.* [3] and Sayre and Wittman [11]; that students often have an emerging, but not complete, understanding of non-Cartesian coordinate systems. The findings herein provide the initial insights into a broader study into students' use of mathematics in the upper-division upon which additional investigation will continue to build.

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APPENDIX: TIME DERIVATIVE OF $\hat{r}$

Given polar coordinates (r, ϕ) , the polar basis unit vectors $\hat{r}$ and $\hat{\phi}$ in terms of the Cartesian unit vectors $\hat{x}$ and $\hat{y}$ are

$$\begin{aligned}\hat{r} &= \cos \phi \hat{x} + \sin \phi \hat{y}, \\ \hat{\phi} &= -\sin \phi \hat{x} + \cos \phi \hat{y}.\end{aligned}$$

When a system modeled in polar coordinates evolves in time, r and ϕ are both dependent on time, t . Therefore, the time derivative of $\hat{r}$ is

$$\begin{aligned}\frac{d\hat{r}}{dt} &= \frac{d}{dt}(\cos \phi \hat{x} + \sin \phi \hat{y}) \\ &= \frac{d}{dt} \cos \phi \hat{x} + \cos \phi \frac{d}{dt} \hat{x} + \frac{d}{dt} \sin \phi \hat{y} + \sin \phi \frac{d}{dt} \hat{y} \\ &= \frac{d}{dt} \cos \phi \hat{x} + \frac{d}{dt} \sin \phi \hat{y} \\ &= -\dot{\phi} \sin \phi \hat{x} + \dot{\phi} \cos \phi \hat{y} \\ &= \dot{\phi}(-\sin \phi \hat{x} + \cos \phi \hat{y}).\end{aligned}$$

Here we see that the right-hand side of the equation is equal to $\dot{\phi} \hat{\phi}$. Thus,

$$\frac{d\hat{r}}{dt} = \dot{\phi} \hat{\phi}.$$

In physics, $\dot{\phi} = \omega$, angular speed.

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