Conservative and Radiative Dynamics of Spinning Bodies at Third Post-Minkowskian Order Using Worldline Quantum Field Theory

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Using the spinning worldline quantum field theory formalism we calculate the quadratic-in-spin momentum impulse Δp_i^μ and spin kick Δa_i^μ from a scattering of two arbitrarily oriented spinning massive bodies (black holes or neutron stars) in a weak gravitational background up to third post-Minkowskian (PM) order (G^3). Two-loop Feynman integrals are performed in the potential region, yielding conservative results. For spins aligned to the orbital angular momentum we find a conservative scattering angle that is fully consistent with state-of-the-art post-Newtonian results. Using the 2PM radiated angular momentum previously obtained by Plefka, Steinhoff, and the present authors, we generalize the angle to include radiation-reaction effects, in which case it avoids divergences in the high-energy limit.

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Recent detections by the LIGO and Virgo Collaborations of gravitational waves emitted by binary black hole and neutron star mergers [1–5] have driven demand for high-precision gravitational waveform templates. In the early stage these inspirals typically run over many cycles, making them difficult to model using numerical techniques [6–8]; yet, as the gravitational field is weak, this regime is well tackled using perturbation theory. Often this is done in a post-Newtonian (PN) expansion in both G (Newton's constant) and c (the speed of light); however, methods involving the post-Minkowskian (PM) expansion in G are gaining prominence.

The crucial insight driving this shift is that bound orbits are closely related to unbound scattering events, the latter more naturally handled in the PM expansion. A well-studied approach to the bound problem in gravity is reverse engineering a gravitational potential from scattering data [9–15], which can in turn be used to describe bound orbits. More recent techniques such as the bound-to-boundary (B2B) correspondence directly relate bound with unbound observables [16–18]; scattering observables may also be used as direct input for an effective one-body description of the bound dynamics [19–22], also of spinning black holes or neutron stars [23–28].

To this end an enormous effort is now underway to apply techniques used to calculate scattering amplitudes in quantum field theory (QFT) to the bound-state problem in gravity. The technologies involved for both constructing integrands and performing loop integrals are well honed [29–43], and gauge-invariant scattering observables can now be obtained directly [44–47] without introducing a gravitational potential. Some impressive results have been achieved: at 3PM (two-loop) order [48–52] including radiation-reaction corrections [53–59], tidal effects [60–62], and most recently also at 4PM order [63,64]. A closely related approach is heavy-particle effective field theory (EFT) [65–70].

However, QFT-based methods suffer a drawback: the need to suppress terms that ultimately disappear in the classical $\hbar \to 0$ limit. While the classical limit is now well understood in the nonspinning case as a soft limit [20,44,45,71–73], the situation is further complicated by the need to reinterpret quantized spin degrees of freedom in a classical setting [74–77]. Nevertheless, these obstacles have been successfully overcome at 2PM order [26,78,79] up to quartic order in spin [80]; other studies of higher-spin amplitudes in this context have been done [81–91].

In this regard the worldline EFT framework is more economical [92–97], avoiding quantum corrections from the outset. Partial results for the gravitational potential are now available up to 6PN order [98–103]; in the PM expansion recent progress has closely followed the QFT program [104–108] including at 4PM order [109,110]. To handle spin a local corotating frame is often introduced [111–113]: quadratic-in-spin results are available up to 5PN (N^3LO) [114–122] and 2PM orders [123]—until now at 4PN order the former have remained unchecked.

The recently developed worldline QFT (WQFT) formalism [124–127] innovates over these approaches by quantizing worldline degrees of freedom. This leads to a highly streamlined PM setup wherein classical scattering observables are directly computed as sums of tree-level Feynman diagrams. The use of an $\mathcal{N}=2$ supersymmetric extension to the point-particle action to encapsulate spin degrees of freedom [126,127] circumvents the need for a local corotating frame. Recent work on the WQFT has included the double copy [128] and applications to light bending [129]; other closely related approaches involve directly solving the classical equations of motion [130] and Wilson line operators [131].

In this Letter we realize the spinning WQFT's full potential with a state-of-the-art calculation: deriving the quadratic-in-spin conservative momentum impulse Δp_i^μ and spin kick Δa_i^μ in a scattering encounter between massive bodies at 3PM order, including finite-size effects. Specializing to aligned spins yields the conservative scattering angle $\theta_{\rm cons}$, which we generalize to include dissipative effects using the linear response relation [132–134].

Spinning WQFT formalism.—The dynamics of Kerr black holes with masses m_i and positions $x_i^\mu(\tau)$ on a curved D-dimensional background metric $g_{\mu\nu}$ are described up to quadratic order in spin by the $\mathcal{N}=2$ supersymmetric worldline action [135,136]:

$$\frac{S^{(i)}}{m_i}=-\int d\tau\left[\frac{1}{2}g_{\mu\nu}\dot{x}_i^\mu\dot{x}_i^\nu+i\bar{\psi}_{i,a}\frac{D\psi_i^a}{D\tau}+\frac{1}{2}R_{abcd}\bar{\psi}_i^a\psi_i^b\bar{\psi}_i^c\psi_i^d\right]. \tag{1}$$

The complex Grassmann-valued vectors $\psi_i^a(\tau)$, defined in a local frame e_μ^a with $g_{\mu\nu}=e_\mu^a e_\nu^b\eta_{ab}$ and $(D\psi_i^a/D\tau)=\dot{\psi}_i^a+\dot{x}^\mu\omega_\mu{}^{ab}\psi_{i,b}$, encode spin degrees of freedom (we use the mostly minus metric). The spin tensors $S_i^{\mu\nu}$ and Pauli-Lubanski spin vectors a_i^μ are composite fields:

$$S_i^{\mu\nu}(\tau)=-2ie_a^\mu e_b^\nu\bar{\psi}_i^{[a}\psi_i^{b]},\quad a_i^\mu(\tau)=\frac{1}{2m_i}\epsilon^\mu{}_{\nu\rho\sigma}S_i^{\nu\rho}p_i^\sigma, \tag{2}$$

where $p_{i,\mu}=m_i g_{\mu\nu}\dot{x}_i^\nu$ (referred to as $\pi_{i,\mu}$ in Ref. [127]).

Reparametrization invariance in τ and U(1) symmetry on the Grassmann vectors respectively imply conservation of p_i^2 and $\bar{\psi}_i\cdot\psi_i$. Global $\mathcal{N}=2$ supersymmetry provides two additional fermionic charges: $p_i\cdot\psi_i$ and $p_i\cdot\bar{\psi}_i$, which when set to zero together imply the Tulczyjew-Dixon spin-supplementary condition (SSC) $p_{i,\mu}S_i^{\mu\nu}=0$ [137,138]. The action (1) extends naturally to include finite-size objects like neutron stars by also including

$$S_E^{(i)}\coloneqq-m_iC_{E,i}\int d\tau R_{a\mu b\nu}\dot{x}_i^\mu\dot{x}_i^\nu\bar{\psi}_i^a\psi_i^b\tilde{P}_{cd}\bar{\psi}_i^c\psi_i^d, \tag{3}$$

with projector $\tilde{P}_{ab}\coloneqq\eta_{ab}-e_{a\mu}e_{b\nu}\dot{x}^\mu\dot{x}^\nu/\dot{x}^2$ and Wilson coefficients $C_{E,i}$, where $C_{E,i}=0$ for black holes. The projector ensures supersymmetry for terms up to $\mathcal{O}(S^2)$, enough to maintain the SSC and preserve lengths of the spin vectors.

The WQFT's distinguishing feature is quantization of both bulk and worldline degrees of freedom. In a weak gravitational field with $\kappa=\sqrt{32\pi G}$, we expand $g_{\mu\nu}(x)=\eta_{\mu\nu}+\kappa h_{\mu\nu}(x)$ with the vielbein $e_\mu^a=\eta^{a\nu}[\eta_{\mu\nu}+(\kappa/2)h_{\mu\nu}-(\kappa^2/8)h_{\mu\rho}h^\rho{}_\nu+\cdots]$. Thereafter we no longer distinguish between spacetime $\mu,\nu,\ldots$ and local frame $a,b,\ldots$ indices. The worldline fields are similarly expanded around their background values:

$$x_i^\mu(\tau)=b_i^\mu+v_i^\mu\tau+z_i^\mu(\tau),\quad \psi_i^\mu(\tau)=\Psi_i^\mu+\psi_i'^\mu(\tau),$$
$$S_i^{\mu\nu}(\tau)=\mathcal{S}_i^{\mu\nu}+S_i'^{\mu\nu}(\tau),\qquad a_i^\mu(\tau)=a_{i0}^\mu+a_i'^\mu(\tau), \tag{4}$$

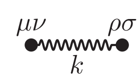
where $\mathcal{S}_i^{\mu\nu}=-2i\bar{\Psi}_i^{[\mu}\Psi_i^{\nu]}$ and $a_{i0}^\mu=\frac{1}{2}\epsilon^\mu{}_{\nu\rho\sigma}\mathcal{S}_i^{\nu\rho}v_i^\sigma$. Vanishing of the supercharges implies $v_i\cdot\Psi_i=v_i\cdot\bar{\Psi}=0$, so $v_{i,\mu}\mathcal{S}_i^{\mu\nu}=0$; using τ-reparametrization invariance on each worldline we fix $b\cdot v_i=0$, where $b^\mu=b_2^\mu-b_1^\mu$. We also define the Lorentz factor $\gamma=v_1\cdot v_2$ and the relative velocity $v=\sqrt{\gamma^2-1}/\gamma$.

The WQFT is defined by a path integral, with physical observables calculated as operator expectation values:

$$\langle\mathcal{O}\rangle\coloneqq\int\mathcal{D}[h_{\mu\nu},z_i^\mu,\psi_i'^\mu]e^{i(S_{\rm EH}+S_{\rm gf}+\sum_{i=1}^2 S^{(i)}+S_E^{(i)})}\mathcal{O}. \tag{5}$$

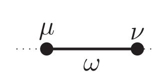
We have included the D-dimensional Einstein-Hilbert action $S_{\rm EH}$ and gauge-fixing term $S_{\rm gf}$ to enforce $\partial_\nu h^{\mu\nu}=\frac{1}{2}\partial^\mu h^\nu{}_\nu$. The stationary phase of the path integral is dominated by solutions to the physically relevant Einstein and Mathisson-Papapetrou-Dixon equations of motion [139–141]. This highlights the WQFT's main advantage when studying classical physics: the classical $\hbar\to 0$ limit is identified with the sum of tree-level Feynman diagrams.

The WQFT Feynman rules are most naturally expressed in the Fourier domain: $h_{\mu\nu}(x)=\int_k e^{-ik\cdot x}h_{\mu\nu}(k)$ and $z_i^\mu(\tau)=\int_\omega e^{-i\omega\tau}z_i^\mu(\omega)$, $\psi_i'^\mu(\tau)=\int_\omega e^{-i\omega\tau}\psi_i'^\mu(\omega)$, where $\int_k\coloneqq\int[d^Dk/(2\pi)^D]$ and $\int_\omega\coloneqq\int(d\omega/2\pi)$. Feynman rules for the graviton $h_{\mu\nu}$ originating from the bulk Einstein-Hilbert action are conventional, with propagator

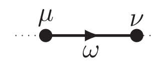


$$=i\frac{P_{\mu\nu;\rho\sigma}}{k^2+\text{sgn}(k^0)i0^+} \tag{6}$$

and $P_{\mu\nu;\rho\sigma}\coloneqq\eta_{\mu(\rho}\eta_{\sigma)\nu}-[1/(D-2)]\eta_{\mu\nu}\eta_{\rho\sigma}$. Given our current focus on conservative scattering, the retarded $i0^+$ pole displacement here plays no role upon integration, so we hide it in the following. The $i0^+$ prescription is, however, significant for the worldline propagators associated with z_i^μ and $\psi_i'^\mu$ which are, respectively,



$$=-i\frac{\eta^{\mu\nu}}{m_i(\omega+i0^+)^2}, \tag{7a}$$



$$=-i\frac{\eta^{\mu\nu}}{m_i(\omega+i0^+)}. \tag{7b}$$

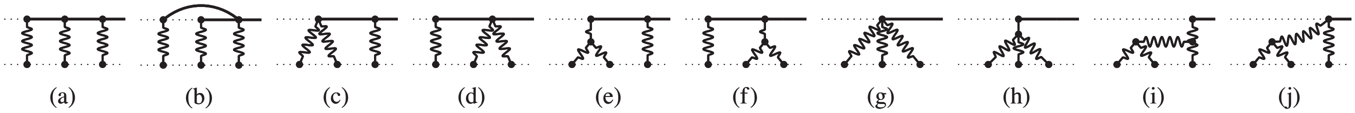


FIG. 1. The ten types of diagrams contributing to the $m_1 m_2^3$ components of $\Delta p_1^{(3)\mu}$ and the m_2^3 components of $\Delta\psi_1^{(3)\mu}$, involving $\mathcal{I}^{(1;\pm)}$-type integrals (12). In the test-body limit $m_1 \ll m_2$ these are the only surviving contributions. All graphs should be considered trees—the dotted lines represent the worldlines on which energy is conserved, instead of momentum.

The background parameters b_i^μ, v_i^μ, and Ψ_i^μ (and therefore $\mathcal{S}_i^{\mu\nu}$, a_{i0}^μ) are identified with the far past: $x_i^\mu(\tau) \overset{\tau\to-\infty}{\longrightarrow} b_i^\mu + \tau v_i^\mu$ and $\psi_i^\mu(\tau) \overset{\tau\to-\infty}{\longrightarrow} \Psi_i^\mu$. Expressions for Feynman vertices on the worldline were provided in Refs. [126,127]: they are distinguished by energy conservation and carry $m_i e^{ik\cdot b_i}\delta\!\!\!{}^-(k\cdot v_i + \sum_j \omega_j)$, where k is the total momentum of all outgoing gravitons and ω_j are the energies of emitted z_i^μ, $\psi_i'^\mu$, and $\bar\psi_i'^\mu$ modes.

Momentum impulse and spin kick.—We derive two physical observables in an unbound scattering event: the momentum impulse $\Delta p_i^\mu \coloneqq [p_i^\mu]_{\tau=-\infty}^{\tau=+\infty}$ and change in spin vectors $\Delta a_i^\mu \coloneqq [a_i^\mu]_{\tau=-\infty}^{\tau=+\infty}$—the "spin kick." In the PM expansion with $\Delta X = \sum_n G^n \Delta X^{(n)}$ we focus on the 3PM components $\Delta p_1^{(3)\mu}$ and $\Delta a_1^{(3)\mu}$. The latter we recover from both Δp_1^μ and $\Delta S_1^{\mu\nu}$ using Eq. (2):

$$\Delta a_i^\mu = \frac{1}{2m_i}\epsilon^\mu{}_{\nu\rho\sigma}(\mathcal{S}_i^{\nu\rho}\Delta p_i^\sigma + m_i \Delta S_i^{\nu\rho} v_i^\sigma + \Delta S_i^{\nu\rho}\Delta p_i^\sigma), \tag{8}$$

selecting the G^3 component on both sides. Meanwhile, $\Delta S_i^{\mu\nu}$ we derive from $\Delta\psi_i^\mu \coloneqq [\psi_i^\mu]_{\tau=-\infty}^{\tau=+\infty}$ and $\Delta\bar\psi_i^\mu$:

$$\Delta S_i^{\mu\nu} = -2i(\bar\Psi_i^{[\mu}\Delta\psi_i^{\nu]} + \Delta\bar\psi_i^{[\mu}\Psi_i^{\nu]} + \Delta\bar\psi_i^{[\mu}\Delta\psi_i^{\nu]}). \tag{9}$$

In the WQFT formalism these quantities are considered observables:

$$\begin{aligned} \Delta p_i^\mu &= m_i \int_{-\infty}^{\infty} d\tau \left\langle \frac{d^2 x_i^\mu(\tau)}{d\tau^2} \right\rangle = -m_i \omega^2 \langle z_i^\mu(\omega)\rangle|_{\omega=0}, \\ \Delta\psi_i^\mu &= \int_{-\infty}^{\infty} d\tau \left\langle \frac{d\psi_i^\mu(\tau)}{d\tau} \right\rangle = -i\omega\langle \psi_i'^\mu(\omega)\rangle|_{\omega=0}. \end{aligned} \tag{10}$$

Diagrammatically this amounts to drawing all tree-level diagrams with a single cut external z_i^μ or $\psi_i'^\mu$ line.

The diagrams required to calculate both $\Delta p_1^{(3)\mu}$ and $\Delta\psi_1^{(3)\mu}$ are divided into three categories, the first two of which are illustrated schematically in Figs. 1 and 2. As the diagrams involved in $\Delta p_1^{(3)\mu}$ and $\Delta\psi_1^{(3)\mu}$ differ only by the cut outgoing line, we display them together. For additional brevity we use only solid lines to represent propagating worldline modes z_i^μ, $\psi_i'^\mu$, and $\bar\psi_i'^\mu$; however, it should be assumed that each internal worldline mode could be of all three types (with expressions adjusted accordingly). The third set of diagrams (not drawn) consists simply of mirrored versions of the graphs in Fig. 1 through a horizontal plane, but with the external cut line still on the first (upper) worldline. For the impulse we avoid calculating these contributions directly, instead making use of momentum conservation $\Delta p_2^{(3)\mu} = -\Delta p_1^{(3)\mu}$ (for conservative scattering).

We assemble expressions using the WQFT Feynman rules in $D = 4 - 2\epsilon$ spacetime dimensions, with the later intention of recovering four-dimensional results in the $\epsilon \to 0$ limit. Each retarded graviton (6) and worldline (7) propagator points toward the outgoing line: from cause to effect. As diagrams belonging to each of the three categories carry common overall factors of the masses $m_1^\alpha m_2^\beta$, the categories themselves are separately gauge invariant. This helpfully breaks the calculation up into gauge-invariant subcomponents. Diagrams in Fig. 1 carry the maximum allowed power of m_2, and represent the test-body limit $m_1 \ll m_2$. Integrals are performed over the energies (on the worldlines $\int_\omega$) or momenta (in the bulk $\int_k$) of all internal lines.

The integrals involved in both $\Delta p_1^{(3)\mu}$ and $\Delta\psi_1^{(3)\mu}$ are Fourier transforms of two-loop Feynman integrals:

$$\int_q e^{iq\cdot b}\delta\!\!\!{}^-(q\cdot v_1)\delta\!\!\!{}^-(q\cdot v_2)|q|^\alpha \mathcal{I}^{(i;\pm)}_{n_1,n_2,\ldots,n_7}, \quad i = 1,2,3, \tag{11}$$

where $\delta\!\!\!{}^-(\omega) \coloneqq 2\pi\delta(\omega)$, q^μ is the total momentum exchanged from the second to the first worldline, and α is an arbitrary power of $|q| \coloneqq \sqrt{-q\cdot q}$. The two-loop integral families are

$$\begin{aligned} &\mathcal{I}^{(1,2;\pm)}_{n_1,\ldots,n_7}[\ell_1^{\mu_1}\cdots\ell_1^{\mu_n}\ell_2^{\nu_1}\cdots\ell_2^{\nu_m}] \\ &\quad\coloneqq \int_{\ell_1,\ell_2} \frac{\delta\!\!\!{}^-(\ell_1\cdot v_2)\delta\!\!\!{}^-(\ell_2\cdot v_{2,1})\ell_1^{\mu_1}\cdots\ell_1^{\mu_n}\ell_2^{\nu_1}\cdots\ell_2^{\nu_m}}{D_1^{n_1}D_2^{n_2}\cdots D_7^{n_7}}, \\ &D_1 = \ell_1\cdot v_1 + i0^+, \qquad D_2 = \pm\ell_2\cdot v_{1,2} + i0^+, \\ &D_3 = \ell_1^2, \qquad D_4 = \ell_2^2, \\ &D_5 = (\ell_1+\ell_2-q)^2, \qquad D_6 = (\ell_1-q)^2, \\ &D_7 = (\ell_2-q)^2, \end{aligned} \tag{12}$$

and $\mathcal{I}^{(3;\pm)}_{n_1,\ldots,n_7} = \mathcal{I}^{(1;\pm)}_{n_1,\ldots,n_7}|_{v_1\leftrightarrow v_2}$. Each pair $(\pm)$ is associated with one of the three categories of diagrams. To achieve these representations one must first integrate on the energies carried by any internal deflection z_i^μ or spin $\psi_i'^\mu$, $\bar\psi_i'^\mu$ modes on the worldlines.

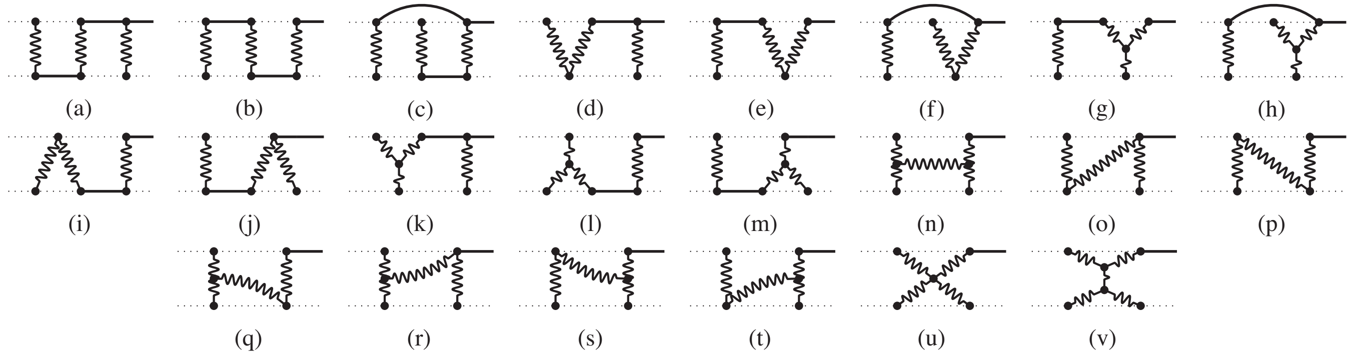


FIG. 2. The 22 types of diagrams contributing to the $m_1^2m_2^2$ components of $\Delta p_1^{(3)\mu}$ and the $m_1m_2^2$ components of $\Delta\psi_1^{(3)\mu}$, involving $\mathcal{I}^{(2;\pm)}$-type integrals (12). We exclude "mushroom graphs" that integrate to zero in the potential region.

As two-loop integrals of this kind are now well studied—see, e.g., Refs. [49,56,142,143]—we relegate full details of how to perform them to the Supplemental Material [144]. The $\mathcal{I}^{(1;\pm)}_{n_1,\ldots,n_7}$ integrals—associated with the test-body diagrams in Fig. 1—are more straightforward, being naturally evaluated in the rest frame $v_2^\mu=(1,\mathbf{0})$. The more involved $\mathcal{I}^{(2;\pm)}_{n_1,\ldots,n_7}$ integrals—associated with the diagrams in Fig. 2—contain the $\mathrm{arccosh}\gamma$ function. To fix boundary conditions we adopt the *potential region* of integration, which ignores radiation-reaction contributions and may be interpreted as a resummation of the terms arising from a conservative PN expansion $(v/c)\ll 1$. We have therefore excluded certain graphs from Fig. 2—the so-called "mushroom graphs"—which integrate to zero within this regime.

Our final results for $\Delta p_1^{(3)\mu}$ and $\Delta a_1^{(3)\mu}$ are presented together with the corresponding 1PM and 2PM results in the Supplemental Material [144]. They have the schematic form

$$\Delta p_1^{(3)\mu}=\sum_{s=0}^{2}\frac{m_1^2m_2^2}{|b|^{3+s}}\left[c_0^{(s)\mu}\mathrm{arccosh}\gamma+\sum_{n=1}^{3}\left(\frac{m_1}{m_2}\right)^{n-2}c_n^{(s)\mu}\right],\tag{13a}$$

$$\Delta a_1^{(3)\mu}=\sum_{s=1}^{2}\frac{m_1m_2^2}{|b|^{2+s}}\left[d_0^{(s)\mu}\mathrm{arccosh}\gamma+\sum_{n=1}^{3}\left(\frac{m_1}{m_2}\right)^{n-2}d_n^{(s)\mu}\right].\tag{13b}$$

The coefficients $c_i^{(s)\mu}$ and $d_i^{(s)\mu}$ are rational functions of v_i^μ, the initial spin vectors a_{i0}^μ, and the unit-normalized impact parameter $\hat{b}^\mu\coloneqq b^\mu/|b|$, where $|b|\coloneqq\sqrt{-b\cdot b}$. We have performed several consistency checks. Firstly, all poles in $\epsilon=2-(D/2)$ arising from the dimensionally regularized two-loop integrals (12) are seen to cancel, thus ensuring finiteness of our results in the limit $D\to 4$. Secondly, conservation of p_i^2, $\bar{\psi}_i\cdot\psi_i$ and the fermionic supercharge $p_i\cdot\psi_i$ between initial and final states implies a set of consistency requirements:

$$\begin{aligned}&0=m_1v_1\cdot\Delta p_1^{(3)}+\Delta p_1^{(1)}\cdot\Delta p_1^{(2)},\\&0=\bar{\Psi}_1\cdot\Delta\psi_1^{(3)}+\Delta\bar{\psi}_1^{(3)}\cdot\Psi_1+\Delta\bar{\psi}_1^{(1)}\cdot\Delta\psi_1^{(2)}+\Delta\bar{\psi}_1^{(2)}\cdot\Delta\psi_1^{(1)},\\&0=m_1v_1\cdot\Delta\psi_1^{(3)}+\Delta p_1^{(3)}\cdot\Psi_1+\Delta p_1^{(1)}\cdot\Delta\psi_1^{(2)}\\&\quad+\Delta p_1^{(2)}\cdot\Delta\psi_1^{(1)}.\end{aligned}\tag{14}$$

All three of these checks are highly nontrivial: for instance, the third compares parts of $\Delta\psi_1^{(3)\mu}$ containing $\mathrm{arccosh}\gamma$ with $\Delta p_1^{(3)\mu}$ at different orders in spin.

Scattering angle.—We now specialize to spin vectors a_i^μ aligned with the orbital angular momentum: $a_i^\mu=s_il^\mu$, where $l^\mu\coloneqq\epsilon^\mu{}_{\nu\rho\sigma}\hat{b}^\nu v_1^\rho v_2^\sigma/(\gamma v)$, confining the motion to a plane. The conservative part of the scattering angle is then given by (see, e.g., Ref. [104])

$$\sin\left(\frac{\theta_{\rm cons}}{2}\right)=\frac{|\Delta p_1|}{2p_\infty},\tag{15}$$

with the full scattering angle (including radiative corrections) given by $\theta=\theta_{\rm cons}+\theta_{\rm rad}$. The center-of-mass momentum is $p_\infty=\mu\gamma v/\Gamma$, where $\mu=M\nu=m_1m_2/M$ is the symmetric mass, $M=m_1+m_2$ is the total mass, and $\Gamma=E/M=\sqrt{1+2\nu(\gamma-1)}$, E being the total energy. We decompose the scattering angle as

$$\frac{\theta}{\Gamma}=\sum_n\left(\frac{GM}{|b|}\right)^n\theta^{(n)},\qquad\theta^{(n)}=\sum_m\frac{\theta^{(n,m)}}{|b|^m},\tag{16}$$

with n and m counting the PM and spin orders, respectively. At 3PM order using our results,

$$\begin{aligned}\theta_{\rm cons}^{(3,0)}=&\,2\frac{64\gamma^6-120\gamma^4+60\gamma^2-5}{3(\gamma^2-1)^3}\Gamma^2-\frac{8\nu\gamma(14\gamma^2+25)}{3(\gamma^2-1)}\\&-8\nu\frac{4\gamma^4-12\gamma^2-3}{(\gamma^2-1)^{3/2}}\mathrm{arccosh}\gamma,\end{aligned}\tag{17a}$$

$$\theta^{(3,1)}_{\rm cons} = 2\gamma\frac{16\gamma^4-20\gamma^2+5}{(\gamma^2-1)^{5/2}}(5\Gamma^2 s_+ - \delta s_-) - 4\nu s_+\left(\frac{44\gamma^4+100\gamma^2+41}{(\gamma^2-1)^{3/2}} + 12\gamma\frac{(\gamma^2-6)(2\gamma^2+1)}{(\gamma^2-1)^2}\operatorname{arccosh}\gamma\right), \tag{17b}$$

$$\begin{aligned}\theta^{(3,2)}_{\rm cons} =& \frac{4\Gamma^2}{(\gamma^2-1)^3}\left((96\gamma^6-160\gamma^4+70\gamma^2-5)s_+^2 - \frac{1772\gamma^6-2946\gamma^4+1346\gamma^2-137}{35}s^2_{E,+}\right) - 8\delta\left(\frac{16\gamma^4-12\gamma^2+1}{(\gamma^2-1)^2}s_-s_+\right.\\ &\left. - \frac{214\gamma^4-223\gamma^2+44}{35(\gamma^2-1)^2}s^2_{E,-}\right) + 8\nu\gamma\left[\frac{2\gamma^4+86\gamma^2+87}{5(\gamma^2-1)^2}s_-^2 - \frac{298\gamma^4+834\gamma^2+853}{5(\gamma^2-1)^2}s_+^2 + \frac{3244\gamma^4+7972\gamma^2+4639}{105(\gamma^2-1)^2}s^2_{E,+}\right.\\ &\left. - \left(3s_-^2(4\gamma^4+7\gamma^2+1) + 3s_+^2(8\gamma^6-68\gamma^4-63\gamma^2-9) - 2s^2_{E,+}(8\gamma^6-56\gamma^4-24\gamma^2-3)\right)\frac{\operatorname{arccosh}\gamma}{\gamma(\gamma^2-1)^{5/2}}\right], \end{aligned}\tag{17c}$$

where we have defined $\delta = (m_2 - m_1)/M$ as well as $s_\pm = s_1 \pm s_2$ and $s^2_{E,\pm} = C_{E,1}s_1^2 \pm C_{E,2}s_2^2$. We have checked $\theta^{(3)}_{\rm cons}$ both in the test-body limit $\nu \to 0$ and up to 4PN order (N^{2}LO) for comparable masses against Refs. [26,145]. For aligned spins, this provides a first check on the complete quadratic-in-spin conservative dynamics of compact binaries at 4PN order [117,118] together with recent work in the worldline EFT formalism [122].

As explained by Bini and Damour [132–134], the conservative scattering angle is generalized to include radiation using the linear response relation:

$$\theta_{\rm rad} = -\frac{1}{2}\frac{\partial\theta_{\rm cons}}{\partial E}E_{\rm rad} - \frac{1}{2}\frac{\partial\theta_{\rm cons}}{\partial J}J_{\rm rad}. \tag{18}$$

Here J is the total angular momentum in the center-of-mass frame: the derivative is equivalent to one with respect to the orbital angular momentum $L = p_\infty|b|$. It has recently been clarified [146] that Eq. (18) applies only using an "intrinsic" gauge choice with respect to Bondi-Metzner-Sachs symmetry, wherein the radiated angular momentum $J_{\rm rad}$ begins at $\mathcal{O}(G^2)$. With $E_{\rm rad}$ starting at $\mathcal{O}(G^3)$ to deduce $\theta^{(3)}_{\rm rad}$ we need only $J^{(2)}_{\rm rad}$, which was provided by Plefka, Steinhoff, and the present authors for arbitrary spin orientations in Ref. [126]. For aligned spins,

$$\begin{aligned}\frac{J^{(2)}_{\rm rad}}{L} =& \left(1 + \frac{2vs_+}{|b|(1+v^2)} + \frac{s_+^2 - s^2_{E,+}}{|b|^2}\right)\\ &\times\frac{4m_1m_2}{|b|^2}\frac{(2\gamma^2-1)}{\sqrt{\gamma^2-1}}\left(-\frac{8}{3} + \frac{1}{v^2} + \frac{(3v^2-1)}{v^3}\operatorname{arccosh}\gamma\right).\end{aligned} \tag{19}$$

This yields the radiative part of the scattering angle:

$$\begin{aligned}\theta^{(3)}_{\rm rad} =& \frac{4\nu(2\gamma^2-1)^2}{(\gamma^2-1)^{3/2}}\left(-\frac{8}{3} + \frac{1}{v^2} + \frac{(3v^2-1)}{v^3}\operatorname{arccosh}\gamma\right)\\ &\times\left[1 + \frac{6\gamma^2 v}{(2\gamma^2-1)}\frac{s_+}{|b|} + 4\left(\frac{6\gamma^4-6\gamma^2+1}{(2\gamma^2-1)^2}\frac{s_+^2}{|b|^2} - \frac{s^2_{E,+}}{|b|^2}\right)\right].\end{aligned} \tag{20}$$

The nonspinning part of $\theta^{(3)}_{\rm rad}$ has also been confirmed without reference to Eq. (18); see, e.g., Refs. [53,54].

A key criterion of $\theta^{(3)}_{\rm rad}$ is that the total scattering angle should remain finite in the high-energy limit. We write $\theta(E, \nu, |b|, \gamma, s_i)$ in terms of the energy, symmetric mass ratio, impact parameter, Lorentz factor, and spin magnitudes, and let $\gamma \to \infty$, in which case the individual masses are negligible. In this limit,

$$\begin{aligned}\theta =& 4\frac{GE}{|b|}\left(1 + \frac{s_+}{|b|} + \frac{s_+^2 - s^2_{E,+}}{|b|^2}\right) + \frac{32}{3}\left(\frac{GE}{|b|}\right)^3\left[1 + 3\frac{s_+}{|b|}\right.\\ &\left. + \frac{3}{20}\frac{41s_+^2 + s_-^2 - 16s^2_{E,+}}{|b|^2}\right] + \mathcal{O}(G^4, \gamma^{-1/2}).\end{aligned} \tag{21}$$

While we know of no spinning extension to the results of Amati *et al.* [147] to compare with in the high-energy limit, we do see that a logarithmic divergence appearing in the conservative part of the angle (17) is canceled by the radiative correction (20).

Discussion.—We conclude with a brief discussion of bound observables. Using the B2B dictionary [16–18] one may, for instance, recover the aligned-spin periastron advance $\Delta\Phi$ from our scattering angle:

$$\Delta\Phi = \theta(E, L, m_i, s_i) + \theta(E, -L, m_i, -s_i). \tag{22}$$

Similarly one may relate the unbound and bound radial actions, from which the scattering angle and periastron advance are respectively given by a derivative with respect to L. At 3PM order $\theta^{(3)}$ cancels in Eq. (22); nevertheless, from $\theta^{(3)}$ one may reconstruct the leading-PN parts of $\theta^{(4)}$ and $\theta^{(6)}$ (and similarly for the radial action) [145]. This suffices for a comparison with bound quadratic-in-spin results at N^{2}LO: for example, we have reproduced the quadratic-in-spin N^{2}LO binding energy for circular orbits [117,118], as was also very recently done in Ref. [122].

For arbitrarily aligned spins there is currently no extension of the B2B map (22). An alternative would therefore be to make an ansatz for a conservative two-body Hamiltonian—for example, building on that used at 2PM order [78,79]—and solve Hamilton's equations for comparison with Δp_1^μ

and Δa_1^μ, thus extending those results to 3PM. On the other hand, we are hopeful that direct maps between unbound and bound gauge-invariant observables for arbitrary spins will be discovered in the near future. In that spirit, all information is captured by the impulse and spin kick.

There remains much work to be done: for example, extending $\Delta p_1^{(3)\mu}$ and $\Delta a_1^{(3)\mu}$ to incorporate radiation-reaction effects, as we have already done for the scattering angle $\theta_{\mathrm{rad}}^{(3)}$ (20). This requires us to upgrade our two-loop master integrals to account for the retarded pole displacement on the graviton propagator (6) and restore the mushroom graphs to Fig. 2. We are also interested in the eikonal phase, which was computed in Ref. [127] at 2PM order as the free energy of the WQFT, and captures both the impulse and spin kick. Nevertheless, for the time being we believe that we have effectively showcased the spinning WQFT's utility and efficiency.

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