Emergent Einstein Equation in p -adic Conformal Field Theory Tensor Networks

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We take the tensor network describing explicit p -adic conformal field theory partition functions proposed in [L.-Y. Hung *et al.*, *J. High Energy Phys.* **04** (2019) 170], and consider boundary conditions of the network describing a deformed Bruhat-Tits (BT) tree geometry. We demonstrate that this geometry satisfies an emergent graph Einstein equation in a *unique* way that is consistent with the bulk effective matter action encoding the same correlation function as the tensor network, at least in the perturbative limit order by order away from the *pure* BT tree. Moreover, the (perturbative) definition of the graph curvature in the mathematics [Y. Lin and S.-T. Yau, *Tohoku Math. J.* **63**, 605 (2011); Y. Ollivier, *J. Funct. Anal.* **256**, 810 (2009)] and physics [S. S. Gubser *et al.*, *J. High Energy Phys.* **06** (2017) 157] literature naturally emerges from the consistency requirements of the emergent Einstein equation. This could provide new insights into the understanding of gravitational dynamics potentially encoded in more general tensor networks.

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The AdS/CFT correspondence [1] provided deep insights of (quantum) gravity. One extremely important breakthrough inspired by the Ryu-Takayanagi entanglement formula [2] is the realization that (semiclassical) geometries as solutions of the gravity theory are basically geometrization of the entanglement structure of wave functions of the dual conformal field theory (CFT). (See for example Ref. [3] for a review and references therein.) Tensor networks (TN) widely used to construct many-body wave functions are also geometrizations of patterns of entanglement of many-body wave functions. This led to proposals that TNs similar to the multiscale entanglement renormalization ansatz capture the microscopic mechanism behind the AdS/CFT correspondence [4]. Toy models have been constructed [5,6] that recreate many aspects of the AdS/CFT correspondence, most notably the Ryu-Takayanagi formula and the error correcting property [7]. TN also provides deep insight in the surface-state correspondence [8], kinematic space [9], the complexity of the wave functions and their evolution [10–14], and the island formula that is probably the key to the black hole information paradox [15,16]. Nonetheless, reconstruction of gravitational dynamics using TN remains a tremendous challenge. There is some progress by considering relative entropies [17,18] and also complexity

optimization [12–14], although it is not completely clear how bulk matter or generic time dependence can be included. (Some progress was recently made in Ref. [19].)

In this paper, we present the emergence of a *graph* Einstein equation based on the proposed TN [20] of the p -adic AdS/CFT [21,22]. Our strategy is to parametrize the form of curvatures and bulk stress tensor using an ansatz that is based only on locality and symmetry. We will show that *if* an Einstein equation exists at all the self-consistency constraints are stringent enough to return a *unique* one from the TN. This is perhaps the first such quantitative demonstration involving both matter and time [23], and where both the CFT and the TN can be explicitly defined, taking advantage of the simplicity of p -adic CFT, where descendent operators do not feature in conformal blocks.

Lightning review of p -adic CFTs.—A one-dimensional p -adic CFT lives in the p -adic number field $\mathbb{Q}_p$ for any given prime number p , i.e., coordinates $x \in \mathbb{Q}_p$. p -adic numbers $\mathbb{Q}_p$ are field extensions of the rational numbers alternative to the reals $\mathbb{R}$. This can be readily generalized to an n -dimensional p -adic CFT by considering the field extension of $\mathbb{Q}_p$ to $\mathbb{Q}_{p^n}$ [21,22]. To avoid clutter we will take $n = 1$ although all the expressions can be generalized for generic n , basically by replacing $p \rightarrow p^n$. A p -adic number x can be uniquely expressed as an infinite series

$$x = p^v \sum_{i=0}^{\infty} a_i p^i; \quad v, a_i \in \mathbb{Z}; \quad 0 \leq a_i \leq p-1; \quad a_0 \neq 0. \quad (1)$$

The p -adic norm is defined as

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$$|x|_p = p^{-v}, \quad (2)$$

which satisfies various axioms of norms [24]. Conformal symmetry is defined as the transformation

$$x \rightarrow x' = \frac{ax+b}{cx+d}, \quad a, b, c, d \in \mathbb{Q}_p, \quad ad-bc \neq 0. \quad (3)$$

It furnishes the matrix group $\text{PGL}(2, \mathbb{Q}_p)$, the direct analogue of $\text{SL}(2, \mathbb{R})$ in 1D conformal transformation in real space-time. There are two pieces of algebraic data required to specify completely a p -adic CFT [25]: (i) the spectrum of (quasi) primary operators $\mathcal{O}_a$ with conformal dimensions Δ_a , which transform under conformal symmetry as

$$\mathcal{O}_a(x) \rightarrow \tilde{\mathcal{O}}_a(x') = \left| \frac{ad-bc}{(cx+d)^2} \right|_p^{-\Delta_a} \mathcal{O}_a(x) \quad (4)$$

and (ii) operator product expansion (OPE) coefficients C^{abc} defined as [26]

$$\mathcal{O}_a(x_1) \mathcal{O}_b(x_2) = \sum_c C^{abc} |x_1 - x_2|_p^{\Delta_c - \Delta_a - \Delta_b} \mathcal{O}_c(x_2). \quad (5)$$

The OPE coefficients define an associative operator fusion, i.e., $\sum_c C^{abc} C^{cde} = \sum_c C^{bdc} C^{cae}$. There is a unique identity operator $\mathbb{I}$ so that $C^{1ab} = C^{a1b} = \delta_{ab}$, and that there is a unique dual of a which we denote as a^* satisfying $C^{ab1} = \delta_{ba^*}$. Assuming $a^* = a$ and $C^{ab1} = \delta_{ab}$, the edges need not be oriented, and the notations are less cluttered. This assumption will be taken in the rest of the paper, but it is readily generalized to cases where $a \neq a^*$ [20].

TN on the Bruhat-Tits tree and the p -adic AdS/CFT.—The partition function of a generic p -adic CFT can be constructed explicitly in the form of a TN covering the Bruhat-Tits tree [20], the latter of which is a discrete $p+1$ valent tree graph. The BT tree is the analog of the AdS_2 space whose isometry is the corresponding conformal symmetry group. The TN covers the tree, such that at each vertex of the tree sits a $p+1$ index tensor $T^{a_1 \dots a_{p+1}}$. The tensor is given by a fusion tree of $p+1$ operators, expressed in terms of the OPE coefficient of the p -adic CFT [20].

$$T^{a_1 \dots a_{p+1}} = \sum_{b_1 \dots b_{p-2}} C^{a_1 a_2 b_1} C^{b_1 a_3 b_2} \dots C^{b_{p-2} a_p a_{p+1}}. \quad (6)$$

In the special case where $p=2$, $T^{abc} = C^{abc}$. Two tensors at two vertices connected by an edge are contracted with the edge index weighted by $p^{-\Delta_a}$, where Δ_a is the conformal dimension of the corresponding primary labeled a . The asymptotic boundary of the tree is $\mathbb{Q}_p$, analogous to the real line being the asymptotic boundary of the AdS_2 space. The partition function is defined by setting appropriate

boundary conditions at the cut-off surface which is then taken to approach the asymptotic boundary. Specifically, the dangling legs at the cut-off surface Λ are projected along the vector $|V_f\rangle \equiv \sum_a \delta_1^a |a\rangle = |1\rangle$ [20]. To compute correlation functions, one inserts operator $\mathcal{O}_a(x_b)$ by projecting the boundary leg at x_b along $|a\rangle$ instead. The TN reproduces the correct correlation function in the form of Witten-like diagrams [20]. The boundary insertions of $\mathcal{O}_a(x_b)$ source these $p^{-\Delta_a}$ weighted paths in the TN, which coincides with “bulk-boundary propagators” $G_a(x_b, v_i)$ on the tree, until they meet at some vertices v_i in the bulk. These bulk-boundary propagators are solutions of the graph Klein-Gordon equation

$$(\square_v + m_a^2) G_a(u, v) = \delta_{u,v}, \quad G_a(u, v) = \zeta_p(2\Delta_a) \frac{p^{-\Delta_a d(u,v)}}{p^{\Delta_a}}, \quad (7)$$

$$m_a^2 = -\frac{1}{\zeta_p(\Delta_a - 1) \zeta_p(-\Delta)}, \quad \zeta_p(s) \equiv \frac{1}{1 - p^{-s}} \quad (8)$$

$$\square_u \phi(u) \equiv \sum_{\langle uv \rangle, v \sim u} [\phi(u) - \phi(v)], \quad u \sim v \equiv \text{nearest neighbor}. \quad (9)$$

Here $\langle uv \rangle$ denotes an edge ending at vertices u, v , and $d(u, v)$ is the distance between u, v . Here, every link has unit length $d_e = 1$. Operators labeled $a, b, \dots$ traversing different paths can meet at a vertex if they can fuse to the identity operator. The emergence of Witten-diagrams suggests that the TN is recreating the p -adic AdS/CFT correspondence proposed in [21,22], connecting a p -adic CFT and a dual bulk theory containing some bulk fields ϕ^a living on the BT tree in 1-1 correspondence with the primary operators. In fact, we can define bulk field $\phi^a(x)$ insertion by fusing an extra a leg to the bulk vertex x . The bulk correlation functions $\langle \phi^a(x) \phi^b(y) \dots \rangle$ are thus defined as evaluation of the tensor network with the extra legs inserted at the appropriate vertices (see Fig. 1).

One can readily show that these results are consistent with an emergent bulk matter field theory living on the BT tree [20,21], with action in the large mass limit [27]

$$\begin{aligned} S_m = & \sum_{\langle xy \rangle} \frac{1}{2} \left\{ \sum_a [\phi^a(x) - \phi^a(y)]^2 \right. \\ & + \frac{m_a^2}{p+1} [\phi^a(x)^2 + \phi^a(y)^2] \Big\} \\ & + \sum_{a,b,c} \frac{\tilde{C}^{abc}}{3!(p+1)} [\phi^a(x) \phi^b(x) \phi^c(x) \\ & + \phi^a(y) \phi^b(y) \phi^c(y)] + \dots, \\ \tilde{C}^{abc} \equiv & C^{abc} \sqrt{\frac{p^{\Delta_a + \Delta_b + \Delta_c}}{\zeta_p(2\Delta_a) \zeta_p(2\Delta_b) \zeta_p(2\Delta_c)}}, \end{aligned} \quad (10)$$

FIG. 1. The tensor network representation of a $p = 2$ -adic CFT. The diagram depicts three bulk operator insertion. Bulk insertions pushed to the asymptotic boundary asymptotes to boundary insertions. The boundary condition V_{Λ_i} , defined in (11), are chosen to be the fixed point tensor $V_f^a = \delta_1^a$. Each vertex tensor is C^{abc} and each edge of the tensor with index a is weighted by $p^{-\Delta_a}$.

where $\dots$ corresponds to higher point interaction terms. The effective action truncates at the ϕ^{p+1} level, which reduces to cubic vertices at $p = 2$ [20]. We deliberately present the action as a sum over edges anticipating coupling the theory to a dynamical *metric* shortly.

Distances and curvatures in a tensor network.—The tensor network constructed appears to be describing a *pure* BT space (analog of pure AdS) when the boundary edges are projected to the identity vector apart from locations where operators are inserted. As in the usual story of AdS/CFT, a most natural way to deform the background geometry is to change the boundary conditions which would drive an renormalization group flow in the CFT. Specifically, each boundary link i at the cutoff surface Λ is projected to a generic vector $|V_{\Lambda_i}\rangle$ [20]. Where there is translation invariance, we can take $|V_{\Lambda_i}\rangle = |V_{\Lambda}\rangle$ for all boundary legs i . Explicitly the vector $|V_{\Lambda}\rangle$ is parametrized as

$$|V_{\Lambda}\rangle = \sum_a V_{\Lambda}^a |a\rangle. \quad (11)$$

When the vectors $|V_{\Lambda}\rangle$ are contracted with the p dangling legs of a tensor $T^{a_1 \dots a_{p+1}}$ at the cut-off surface, it would generate a new vector $|V_{\Lambda-1}\rangle$

$$\begin{aligned} |V_{\Lambda-1}\rangle &\equiv \sum_a V_{\Lambda-1}^a |a\rangle \\ &= \sum_{a_1 \dots a_{p+1}} V_{\Lambda}^{a_1} p^{-\Delta_{a_1}} \dots V_{\Lambda}^{a_p} p^{-\Delta_{a_p}} T^{a_1 \dots a_p a_{p+1}} |a_{p+1}\rangle \end{aligned} \quad (12)$$

which is fed into the next layer of tensors recursively. The flow of these vectors suggests that the geometry described by the tensor network is deformed from the pure BT background. We note that the original boundary condition describing the undeformed CFT partition function

FIG. 2. Vector V_{xy}^a and $\tilde{V}_{xy}^a$ following from the contraction of tensors *above* (colored red) and *below* (colored green) the edge $\langle xy \rangle$ respectively. Boundary conditions V_{Λ_i} differing from the fixed point vector drives an renormalization group flow. The curvature of the patch centered at x depends on the edge lengths $d_{\langle xy_i \rangle}$ symmetrically.

$$V_f^a = \delta_1^a \quad (13)$$

is indeed a fixed point vector under this flow, which recovers the original pure BT space supposedly dual to the undeformed p -adic CFT partition function. In this case, all the edges contribute equally to the partition function, and we can assign unit length to every edge, i.e., $d_e = 1$.

When we depart from pure BT background, one needs to assign a general length d_e to each edge on the graph. We note that there are two characteristic vectors at an edge $e = \langle xy \rangle$ bounded by vertices x and y . One is $|V_{xy}\rangle$ introduced above, corresponding to repeating the flow described in Eq. (12), contracting all the tensors from the cut-off surface all the way down to the vertex x . The other vector $|\tilde{V}_{xy}\rangle$ follows from analogously contracting all the tensors *below* the vertex y . This misleading notion of being “above” or “below” is illustrated in Fig. 2.

The distinction between $|V_{xy}\rangle$ and $|\tilde{V}_{xy}\rangle$ comes from boundary conditions very far away, and so we expect the edge length $d_{\langle xy \rangle}$ to depend on them symmetrically. It is also natural to expect that $|V_{xy}\rangle$ and $|\tilde{V}_{xy}\rangle$ fully determine $d_{\langle xy \rangle}$. To have better analytic control, we consider perturbing around the undeformed CFT by perturbing the boundary conditions $|V_{\Lambda}\rangle$ around the fixed point tensor:

$$V_{\Lambda_i}^a = \delta_1^a + \lambda v_{\Lambda_i}^a, \quad (14)$$

where $\lambda \ll 1$, but $v_{\Lambda_i}^a$ is completely arbitrary. The boundary conditions are not assumed to respect any translation symmetries. In the perturbative limit, the flowed vector in the interior would admit the general expansion

$$V_{xy}^a = \delta_1^a + \omega_{xy}^a, \quad \omega_{xy}^a \equiv \lambda_{xy}^{(1)a} + \lambda_{xy}^{(2)a} + \dots, \quad (15)$$

$$\tilde{V}_{xy}^a = \delta_1^a + \tilde{\omega}_{xy}^a, \quad \tilde{\omega}_{xy}^a \equiv \tilde{\lambda}_{xy}^{(1)a} + \tilde{\lambda}_{xy}^{(2)a} + \dots, \quad (16)$$

where $\lambda^{(n)}$, $\tilde{\lambda}^{(n)}$ are of order λ^n . The edge length $d_{e=\langle xy \rangle}(V_e^a, \tilde{V}_e^a)$ would also admit an expansion about the pure BT space as $d_e = 1 + j_e$, where

$$\begin{aligned} j_e = & A^a(\omega_e^a + \tilde{\omega}_e^a) + B^{ab}(\omega_e^a \omega_e^b + \tilde{\omega}_e^a \tilde{\omega}_e^b) + C^{ab} \omega_e^a \tilde{\omega}_e^b \\ & + D^{abc}(\omega_e^a \omega_e^b \omega_e^c + \tilde{\omega}_e^a \tilde{\omega}_e^b \tilde{\omega}_e^c) \\ & + E^{abc}(\omega_e^a \omega_e^b \tilde{\omega}_e^c + \tilde{\omega}_e^a \tilde{\omega}_e^b \omega_e^c) + \mathcal{O}(\omega^4), \end{aligned} \quad (17)$$

for some constants $A^a, B^{ab}, C^{ab} \dots$. Having assigned edge lengths, one can compute the curvature of this graph. There are various proposals, e.g., Refs. [28–30]. Consider the curvature R_x of a patch surrounding a vertex x . The graph curvature should be a symmetric function of lengths $d_{\langle xy_i=1, \dots, p+1 \rangle}$ of the $p+1$ edges connected to x . Generally therefore, in the small j_e limit we expect

$$\begin{aligned} R_x = & a_0 + a_1 \sum_i j_{xy_i} + b \sum_i j_{xy_i}^2 + c \sum_{i \neq k} j_{xy_i} j_{xy_k} \\ & + \mathcal{O}(j^3), \end{aligned} \quad (18)$$

again for constants $a_{0,1}, b, c \dots$.

Emergent action and Einstein equation.—In the above, we have defined distances and curvatures which are, up to some undetermined coefficients, determined by the TN. On the other hand, the TN encodes a bulk scalar field theory where the expectation values of the fields $\phi^a(x)$ can be readily computed. We would therefore like to inquire if the geometry and the expectation values of $\phi^a(x)$, both read off from the TN, can be related by some graph Einstein equation.

To look for such a relation, we need an emergent covariant effective action that describes the bulk theory, and obtains the equations of motion following from it, before we could even check if such an equation is satisfied by the TN. The matter part of the covariant action should reduce to Eq. (10) in the pure BT limit to be consistent with the correlation functions. We need to upgrade Eq. (10) by coupling it to the background geometry via the edge lengths d_e . This is attempted in Ref. [30], although it has made a choice of treating the mass term as a local term blind to d_e . This is unlike in continuous field theories where every term contains at least the volume form and are thus always sensitive to the metric. Rather than making such choices, we write down a more general ansatz that mimics the continuous covariant action more closely,

$$\begin{aligned} S_m^{\text{cov}} = & S_2^{\text{cov}} + S_3^{\text{cov}} + \dots \\ S_2^{\text{cov}} = & \sum_{\langle xy \rangle} d_{\langle xy \rangle}^k (\phi_x^a - \phi_y^a)^2 + \sum_{\langle xy \rangle} \frac{d_{\langle xy \rangle}}{p+1} m_a^2 [(\phi_x^a)^2 + (\phi_y^a)^2] \\ S_3^{\text{cov}} = & \sum_{\langle xy \rangle} (h(d_{\langle xy \rangle}) H^{abc} (\phi_x^a \phi_x^b \phi_x^c + \phi_y^a \phi_y^b \phi_y^c) \\ & + r(d_{\langle xy \rangle}) R^{abc} (\phi_x^a \phi_x^b \phi_y^c + \phi_y^a \phi_y^b \phi_x^c)), \end{aligned} \quad (19)$$

where k is a constant to be determined. For generality we considered more general cubic interactions other than the exactly local term with coupling H^{abc} , and allowed also for nearest neighbor interaction with coupling R^{abc} . The functions $r(d_e)$ and $h(d_e)$ should be regular in the pure BT background. Therefore in the perturbative limit they can be expanded as

$$h(d_{xy}) = h_0 + h_1 j_{xy} + h_2 j_{xy}^2 + \dots, \quad (20)$$

$$r(d_{xy}) = r_0 + r_1 j_{xy} + r_2 j_{xy}^2 + \dots \quad (21)$$

We introduce the graph Einstein Hilbert action making use of the graph curvature introduced in Eq. (18).

$$S_{\text{EH}} = \sum_x R_x(d_{xy_1}, d_{xy_2}, \dots, d_{xy_{p+1}}) + \sum_{\langle xy \rangle} d_{xy} \Lambda, \quad (22)$$

where we introduced also the cosmological constant term analogous to $\int d^d x \sqrt{g} \Lambda$. The total test effective action is thus $S_{\text{tot}} = S_{\text{EH}} + S_m^{\text{cov}}$. Now we are ready to vary these actions with respect to each edge length $d_{\langle xy \rangle}$ to obtain the graph Einstein equation. Varying S_{EH} we obtain the *graph Einstein tensor* G :

$$\begin{aligned} G_{xy} \equiv & \frac{\delta S_{\text{EH}}}{\delta d_{xy}} \\ = & \Lambda + 2a_1 + 4b j_{xy} + c \left(\sum_{(y_i \neq y)} j_{xy_i} + \sum_{(x_i \neq x)} j_{x_i y} \right) \\ & + \mathcal{O}(j^2). \end{aligned} \quad (23)$$

Varying S_m^{cov} we obtain

$$\begin{aligned} T_{xy} \equiv & \frac{\delta S_m^{\text{cov}}}{\delta d_{xy}} \\ = & \frac{k}{2} (\phi_x^a - \phi_y^a)^2 + \frac{m_a^2 [(\phi_x^a)^2 + (\phi_y^a)^2]}{2(p+1)} \\ & + [h_1 H^{abc} (\phi_x^a \phi_x^b \phi_x^c + \phi_y^a \phi_y^b \phi_y^c) + r_1 R^{abc} (\phi_x^a \phi_x^b \phi_y^c \\ & + \phi_y^a \phi_y^b \phi_x^c)] + \dots \end{aligned} \quad (24)$$

We would like to substitute the expectation values of $\phi^a(x)$ into T . We provide detailed expressions of $\phi^a(x)$ in the Supplemental Material [31]. Clearly $\phi^a(x) \sim \lambda$, and the omitted terms in Eq. (24) are thus of order λ^4 and beyond. The graph Einstein equation is thus given by

$$G_{xy} + T_{xy} = 0. \quad (25)$$

We substitute the geometrical data and expectation value of the bulk stress tensor determined by the TN into Eq. (25). Requiring that Eq. (25) is satisfied order by order in λ turns

into constraints of the undetermined parameters we have introduced. These constraints turn out to be very powerful because G_{xy} and T_{xy} are nontrivial functions of $\lambda_{xy_i}^{(n)a}$, $\tilde{\lambda}_{xy_i}^{(n)a}$ and $\lambda_{yx_i}^{(n)a}$, $\tilde{\lambda}_{yx_i}^{(n)a}$. These edges $\langle xy_i \rangle$, $\langle yx_i \rangle$ are marked in Fig. 2 for a given pair of connected vertices x , y . These variables $\lambda_{xy_i}^{(n)a}$, $\tilde{\lambda}_{xy_i}^{(n)a}$ are virtually independent because they are distinct functions of generic boundary conditions arbitrarily far away. Therefore, we can isolate the coefficient of each of these independent monomials of $\lambda_{xy_i}^{(n)a}$, $\tilde{\lambda}_{xy_i}^{(n)a}$, and require that it vanishes separately. Up to some overall normalization, it fixes a *unique* perturbative dependence of the graph curvature on the edges, the edge lengths, and coupling of matter to the edge lengths such that Eq. (25) can in fact be satisfied. Solving constraints up to order λ^2 , we have

$$\frac{2b}{c} = -p, \quad A^a = 0, \quad \Lambda + 2a_1 = 0, \quad (26)$$

$$k = 1. \quad (27)$$

Substituting into G_{xy} the linear j term is equivalent to the perturbative graph curvature defined in the literature, specifically the form emerging in Ref. [30], up to an overall normalization $-c$. We note that the relation in Eq. (8) has been used when solving the constraints. B^{ab} and C^{ab} introduced in Eq. (17) are in turn determined by b , c , and Δ_a . Constraints at order λ^3 again lead to unique expressions for the matter couplings H^{abc} , R^{abc} and edge length expansion coefficients D^{abc} , E^{abc} up to an overall undetermined normalization. More detailed expressions and explanations are relegated to the Supplemental Material [31]. We are interested in the limit $m_a^2 \rightarrow \infty \Rightarrow \Delta_a \rightarrow \infty$. This is when the bulk effective action S_m exactly reproduces $\phi^a(x)$ correlation functions of the TN. In this limit, satisfying only the local term survives with an overall undetermined normalization,

$$\lim_{\Delta_a \rightarrow \infty, d_e \rightarrow 1} S_3^{\text{cov}} = -\frac{h_0}{4h_1} \sum_{\langle xy \rangle} \frac{\tilde{C}^{abc}}{1+p} \phi_x^a \phi_x^b \phi_x^c, \quad (28)$$

but otherwise in exact agreement with the effective action [Eq. (10)].

Summary and discussion.—In this paper, we demonstrated that there is, up to some overall normalization, a *unique* way of assigning lengths to the p -adic TN so that the geometry read off from the TN satisfies a graph Einstein equation that is consistent with the bulk effective action that reproduces bulk correlation functions encoded by the TN, in the perturbative limit away from pure BT geometry. We have made minimal assumptions other than locality in our ansatz for the action and also the dependence of edge lengths on the TN data. We note that in retrospect, the edge

distance $d_{\langle xy \rangle} = 1 + j_{xy}$ defined in Eq. (17) can be written as

$$d_{\langle xy \rangle} = 1 - \langle u_x | u_y \rangle, \quad (29)$$

whose complete expression is in the Supplemental Material [31]. In the limit $\Delta_a \rightarrow \infty$,

$$|u_x\rangle = \frac{1}{\sqrt{2(p+1)}} \sum_a \tilde{\phi}_x^a |a\rangle. \quad (30)$$

Up to the overall factor in front, this is simply the state corresponding to one dangling leg inserted at a bulk vertex x in the TN. Indeed, $d_{\langle xy \rangle}$ ends up being a Fisher information metric between these vertex states. This is a rare quantitative demonstration of an emergent Einstein equation from a TN that couples to matter, albeit in a simplified setting of p -adic CFTs. There are more patterns in the emergent graph Einstein equation that we will report in a forthcoming accompanying paper. We believe this provides further evidence of geometry being molded by the correlation of matter, supporting the TN as the microscopic mechanism behind the AdS/CFT correspondence. We believe some ideas and methodology discussed here should admit generalization to TN describing more realistic CFTs.

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